Katholieke Universiteit Leuven

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Universality in critical random matrix ensembles and pole-free solutions of Painlevé equations

Tom Claeys

Promotor:
Prof. dr. ir. A.B.J. Kuijlaars

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If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is.

J.L. von Neumann

Introduction

In the present thesis we use Riemann-Hilbert problems in order to obtain results in the theory of random matrices, orthogonal polynomials, and Painlevé equations.

The first technique we use is the Deift/Zhou steepest descent method applied to the Fokas-Its-Kitaev Riemann-Hilbert problem characterizing orthogonal polynomials. Using this method, we obtain universality of double scaling limits in certain critical random matrix ensembles. Furthermore we find asymptotics for recurrence coefficients of orthogonal polynomials which are related to the considered random matrix ensembles.

Secondly we use the technique of a vanishing lemma. This approach enables us to prove the solvability of certain Riemann-Hilbert problems associated with Painlevé equations. The solvability of those Riemann-Hilbert problems leads to the existence of real pole-free solutions of Painlevé equations.

Random matrices

Random matrices were first introduced in mathematical statistics by Wishart in 1928 [102]. In the 50's and 60's, mathematical physicists like Wigner, Dyson, and Mehta began to study random matrices [42, 82, 101]. During the last decades, random matrices have appeared in various branches of mathematics and physics.

- There are remarkable similarities between the eigenvalues of random matrices and the non-trivial zeros of the Riemann-zeta function [84, 58, 90].

- Partition functions of random matrix ensembles are generating functions for combinatorial quantities that are of interest in quantum field theory, see e.g. [19, 38, 56].

- There are connections between random matrices and Brownian motions, random permutations, random tilings, last passage percolation, polynuclear growth models, and other models in statistical physics [6, 7, 8, 9, 10, 64, 87].

- Certain bus systems and airplane boarding can be modelled using random matrices [5, 70, 4].

A random matrix ensemble consists of a space of matrices with a probability measure on it. The statistical properties of the eigenvalues of random matrices are of big interest. A remarkable fact is that local statistics of eigenvalues of random matrices are the same for large classes of random matrix ensembles, whereas the global statistics depend highly on the probability distribution of the ensemble. This important phenomenon is called *universality*. We will restrict ourselves to the study of Hermitian matrices, although in the literature unitary, real symmetric, and quaternion self-dual matrices have also been studied. A general reference for random matrices is the book of Mehta [83]. An overview of the history of random matrices can also be found in [51]. In [27] random matrices are viewed from a perspective that is very close to ours, using a Riemann-Hilbert approach.

Probably the most extensively studied ensemble of random Hermitian matrices is the Gaussian Unitary Ensemble (GUE). The probability measure on the space of Hermitian $n \times n$ matrices in the GUE is

$$\frac{1}{Z_n} \exp(-\operatorname{Tr} M^2) dM,$$

where Z_n is a normalization constant and dM is the flat Lebesgue measure

$$dM = \prod_{i=1}^{n} d\, M_{ii} \prod_{i<j} d \operatorname{Re} M_{ij} d \operatorname{Im} M_{ij}.$$

In this ensemble the entries of the random matrices are independent (up to the requirement that the matrices are Hermitian) Gaussian distributed random variables and the probability measure is invariant under unitary conjugation. This last property is shared by a large class of random matrix ensembles, which are called *unitary ensembles*. Other unitary invariant distributions are given by

$$\frac{1}{Z_n} \exp(-n \operatorname{Tr} V(M)) dM, \tag{1}$$

where V is a real analytic function satisfying some growth condition at infinity. The factor n turns out to be convenient when the size n of the matrices increases. Due to this factor, the *limiting mean density of eigenvalues* exists when letting $n \to \infty$. In the Gaussian case where $V(x) = x^2$, the density is given by the Wigner semi-circle law. The limiting mean eigenvalue distribution, which is highly dependent on the potential V, is characterized as the unique measure that minimizes the logarithmic energy in external field V,

$$I_V(\mu) = \iint \log \frac{1}{|x-y|} d\mu(x) d\mu(y) + \int V(x) d\mu(x), \tag{2}$$

among all probability measures μ on $\mathbb{R}$. The first term in (2) expresses a repulsion between the eigenvalues, while the second term avoids the eigenvalues to drift away to infinity. Due to the factor n in the exponent of (1), those two effects balance with each other so that the limiting mean eigenvalue density is supported on a compact set.

Useful information about the statistics of the eigenvalues is contained in the so-called *two-point kernel* or *eigenvalue correlation kernel*

$$K_n(x,y) = e^{-\frac{n}{2}V(x)} e^{-\frac{n}{2}V(y)} \sum_{k=0}^{n-1} p_k^{(n)}(x) p_k^{(n)}(y), \tag{3}$$

where $p_k^{(n)}$ denotes the k-th degree orthonormal polynomial with respect to the weight $e^{-nV(x)}$. This connection between random matrices and orthogonal polynomials holds also for more general ensembles than the ones with a measure of the form (1), see [83]. The limiting mean density of eigenvalues ψ_V as $n \to \infty$ can be retrieved from the kernel by the formula

$$\psi_V(x) = \lim_{n\to\infty} \frac{1}{n} K_n(x,x)$$

and the m-point correlation functions $\mathcal{R}_m$ are given by

$$\mathcal{R}_m(x_1,\ldots,x_m) = \det(K_n(x_i,x_j))_{1\le i,j\le m}. \tag{4}$$

Gap probabilities and the distribution of the largest eigenvalue can also be expressed in terms of K_n, and the joint probability density of the eigenvalues is given by

$$\frac{1}{n!} \det(K_n(x_i,x_j))_{1\le i,j\le n}.$$

It turns out that local scaling limits of the two-point kernel near a point x^* are universal: the limiting kernel depends on the scaling regime, but further not on the potential V or on the position of x^*. The two regular scaling regimes are the following,

- bulk scaling, near points where the limiting mean eigenvalue density is positive,
- edge scaling, near edge points of the spectrum (i.e. the support of the limiting mean eigenvalue density).

In the case where x^* lies in the bulk of the spectrum the limiting kernel is the *sine kernel*,

$$\lim_{n\to\infty} \frac{1}{n\psi_V(x^*)} K_n\left(x^* + \frac{u}{n\psi_V(x^*)}, x^* + \frac{v}{n\psi_V(x^*)}\right) = \mathbb{K}^{\text{bulk}}(u,v),$$

where $\mathbb{K}^{\text{bulk}}$ is given by

$$\mathbb{K}^{\text{bulk}}(u,v) = \frac{\sin \pi(u-v)}{\pi(u-v)}. \tag{5}$$

This was proven by Bleher and Its [13] in the case where V is a quartic polynomial and by Deift et al. [27, 32, 33] for general real analytic V, see also [86]. In the edge case where there is square root vanishing of the density at x^*, the *Airy kernel* appears in the large n limit, see e.g. [30],

$$\lim_{n\to\infty} \frac{1}{(cn)^{2/3}} K_n\left(x^* + \frac{u}{(cn)^{2/3}}, x^* + \frac{v}{(cn)^{2/3}}\right) = \mathbb{K}^{\text{edge}}(u,v),$$

where c is a constant depending on V and $\mathbb{K}^{\text{edge}}$ is given by

$$\mathbb{K}^{\text{edge}}(u,v) = \frac{\operatorname{Ai}(x)\operatorname{Ai}'(y) - \operatorname{Ai}(y)\operatorname{Ai}'(x)}{x-y}. \tag{6}$$

The Airy function Ai is a special function characterized by the differential equation $\operatorname{Ai}_{xx} = x \operatorname{Ai}$ and the asymptotics

$$\operatorname{Ai}(x) \sim \frac{e^{-\frac{2}{3}x^{3/2}}}{2\sqrt{\pi}x^{1/4}} \qquad \text{as } x \to +\infty.$$

Note that there is a difference in scaling between the bulk, where the scaling is with $1/n$, and the edge, where the scaling is with $1/n^{2/3}$. The scaling corresponds to the mean distance between consecutive eigenvalues, which is of order $1/n$ in the bulk and of order $1/n^{2/3}$ near the edges.

In critical random matrix ensembles, *singular points* occur. We distinguish three other scaling regimes, depending on the type of the singular point. Singular points are classified as follows, see e.g. [33, 74],

- type I singular points, which are singular points outside the support of the limiting mean eigenvalue density,
- type II singular points, which are points in the interior of the spectrum where the limiting mean eigenvalue density vanishes,
- type III singular points, which are edge points of the spectrum where the limiting mean eigenvalue density vanishes at a higher order than in the regular case.

Near singular points, the universality results stated above are no longer valid and limiting kernels are different from the sine and the Airy kernel. However local scaling limits should remain in some sense universal, they should depend only on the nature of the singular point. Critical random matrix ensembles with singular

points are such that the number of intervals in the support can change when modifying the external field V slightly. For example, the presence of a type II singular points indicates the closing or opening of a gap, or in other words the transition where two intervals merge together to a single interval. Type I and type III singular points also indicate a possible change in the number of intervals where eigenvalues accumulate. To describe these transitions fully, it is useful to deform the weight $w_0 = e^{-nV}$ to a weight w_t depending on a parameter t. In *double scaling limits* where we let $n \to \infty$ and at the same time $t \to 0$ at an appropriate rate, limiting kernels should depend on a deformation parameter.

Unitary random matrix ensembles of the form (1) can be modified in such a way that a so-called spectral singularity at the origin is included. We then have probability distributions (which are also unitary invariant) of the form

$$\frac{1}{Z_n} |\det M|^{2\alpha} \exp(-n \operatorname{Tr} V(M)) dM, \qquad \alpha > -1/2. \tag{7}$$

The factor $|\det M|^{2\alpha}$ leads to an additional repulsion between the eigenvalues close to the origin when $\alpha > 0$ and an attraction when $\alpha < 0$. This factor does not change the limiting mean density of eigenvalues, but it gives rise to a different local behavior of the eigenvalues that are close to 0. The two-point kernels associated with such random matrix ensembles can be expressed as in (3), but in terms of polynomials orthogonal with respect to the weight $|x|^{2\alpha} e^{-nV(x)}$. Due to the special point at the origin, local scaling limits of the two-point kernel are different near 0. For example, if the limiting mean eigenvalue density is positive in 0, this leads (instead of the sine kernel) to the Bessel kernel [77] given by

$$\mathbb{J}_\alpha^o(u,v) = \pi\sqrt{u}\sqrt{v} \frac{J_{\alpha+\frac{1}{2}}(\pi u) J_{\alpha-\frac{1}{2}}(\pi v) - J_{\alpha+\frac{1}{2}}(\pi v) J_{\alpha-\frac{1}{2}}(\pi u)}{2(u-v)}. \tag{8}$$

Here $J_{\alpha\pm\frac{1}{2}}$ are Bessel functions of order $\alpha \pm \frac{1}{2}$.

In the last kind of unitary ensembles we consider, the eigenvalues are restricted to $[0, +\infty)$ by putting a probability distribution on the positive-definite Hermitian matrices,

$$\frac{1}{Z_n} (\det M)^\alpha \exp(-n \operatorname{Tr} V(M)) dM, \qquad \alpha > -1. \tag{9}$$

Here the related orthogonal polynomials are orthogonal on $[0, +\infty)$ with respect to the weight $x^\alpha e^{-nV(x)}$. Local scaling limits of the two-point kernel are different near 0 for two reasons. First there is again the repulsion or attraction because of the factor $(\det M)^\alpha$, and secondly 0 is now a *hard edge* where the eigenvalues are not allowed to pass through. If the limiting mean eigenvalue density is positive at the origin, the limiting kernel is again built out of Bessel functions [99].

Universality is not a phenomenon that occurs only in unitary random matrix ensembles. Recently, universality results have also been obtained for other random matrix ensembles such as orthogonal and symplectic ensembles, see [29, 30].

Painlevé equations

At the beginning of the 20th century, Painlevé wanted to classify second order ordinary differential equations (ODEs) of the form

$$u_{xx} = F(x, u, u_x), \tag{10}$$

where F is a rational function. Solutions of such ODEs can have singularities, some of them fixed (independent of the chosen solution, fixed by the differential equation itself) and some of them movable (depending on the chosen solution). Painlevé aimed to classify all nonlinear ODEs of the form (10) for which the movable singularities are restricted to be poles. It turned out that each of those ODEs can be reduced either to a previously known differential equation or to an equation in a list of six equations, which are now known as the Painlevé equations. Solutions of those equations are called Painlevé transcendents.

The Painlevé equations are integrable, which means that they can be expressed as the compatibility condition of a system of linear differential equations, called the Lax pair. Associated with the Lax pairs, Painlevé equations have Riemann-Hilbert problems in which the so-called monodromy data of a solution of the linear system are contained. The Painlevé equation characterizes the monodromy preserving deformations of the linear system [44, 62]. Generalizations of the Lax pairs (which are obtained by increasing the degree of the polynomial coefficients of the linear system) have higher order differential equations as underlying compatibility conditions. Those higher order ODEs are collected together with the Painlevé equations in Painlevé hierarchies.

Besides the fact that Painlevé equations constitute a rich theory from a purely mathematical point of view, they also have a wide variety of applications, e.g. in

- random matrix theory:
  - gap probabilities and the distribution of the largest eigenvalue can be expressed in terms of Painlevé transcendents [97, 103],
  - in critical ensembles with type II or type III singularities, local behavior of eigenvalues near the singular point is described in terms of functions associated with Painlevé equations [2, 12, 14, 18],
- combinatorics: in analogy to the largest eigenvalue of unitary random matrices, fluctuations of the length of a largest increasing subsequence of large random permutations are described by a special solution of the Painlevé II equation [6],

- Hamiltonian perturbations of hyperbolic equations are in certain cases described by Painlevé equations [39, 40],
- various models in statistical mechanics, like the Ising model, are related to Painlevé equations [11].

An extended survey, including history and applications, on Painlevé equations and associated Riemann-Hilbert problems can be found in the very recent book [45].

Riemann-Hilbert problems

"The Riemann-Hilbert problem" has its origin around 1900, when it referred to Hilbert's 21st problem from his famous list of 23 problems. It was only later that the term "Riemann-Hilbert (RH) problem" began to be used in the context we use it, for a whole class of boundary value problems. For our concerns, a RH problem is typically as follows. We seek for a (scalar or matrix-valued) function satisfying conditions of the following form:

(a) the function is analytic outside a given contour in the complex plane,

(b) the boundary values of the function satisfy prescribed jump conditions across the contour,

(c) some additional conditions are required, e.g. at infinity, in order to have a unique solution.

In the 70's and 80's, RH techniques were used for solving integrable systems. Integrable systems are, as mentioned above, nonlinear differential equations which can be expressed as the compatibility condition of a system of linear differential equations. This linear system of equations is called the Lax pair of the underlying nonlinear differential equation. There is a RH problem which characterizes solutions of the Lax pair, and this RH problem also contains information about the original nonlinear differential equation. An important issue in this context is the solvability of RH problems. In some cases, existence of a solution is equivalent with a *vanishing lemma* [48, 49, 104], which states that a related homogeneous RH problem has only the trivial vanishing solution. This related RH problem has typically the same jump conditions as the original one, but different behavior at infinity. The idea behind this technique is that a RH problem is typically equivalent with an integral equation, to which a certain operator is related. The solvability of the RH problem then follows from the fact that this operator is bijective. If the considered operator is Fredholm with zero index, this is equivalent with triviality of the kernel of the operator, which in turn can be translated to a vanishing lemma.

In the early 90's, Fokas, Its, and Kitaev [47] introduced a 2×2 matrix-valued RH problem for which the unique solution was given in terms of orthogonal polynomials. Given a weight w, the problem is to find a 2×2 matrix-valued function $Y = Y_n$ that satisfies the following conditions:

(a) Y is analytic in $\mathbb{C} \setminus \mathbb{R}$,

(b) $Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix}$ for $x \in \mathbb{R}$,

(c) $Y(z) = \big(I + O(z^{-1})\big) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$ as $z \to \infty$,

where $Y_+(x)$ ($Y_-(x)$) denotes the limit when we approach $x \in \mathbb{R}$ from the upper (lower) half-plane. The solution of this RH problem is given by

$$Y(z) = \begin{pmatrix} \kappa_n^{-1} p_n(z) & \kappa_n^{-1} \dfrac{1}{2\pi i} \displaystyle\int_{\mathbb{R}} \dfrac{p_n(s)w(s)}{s-z} ds \\ -2\pi i \kappa_{n-1} p_{n-1}(z) & -\kappa_{n-1} \displaystyle\int_{\mathbb{R}} \dfrac{p_{n-1}(s)w(s)}{s-z} ds \end{pmatrix}, \tag{11}$$

where $p_n(x) = \kappa_n x^n + \cdots$ denotes the n-th degree orthonormal polynomial with respect to the weight w on $\mathbb{R}$, see also [27]. For the choice of $w = |x|^{2\alpha} e^{-nV(x)}$, the eigenvalue correlation kernel for random matrix ensembles of the form (7) can be expressed explicitly in terms of the RH solution Y,

$$\begin{aligned} &K_{n,N}(x,y) \\ &= |x|^{\alpha} e^{-\frac{1}{2}NV(x)} |y|^{\alpha} e^{-\frac{1}{2}NV(y)} \frac{1}{2\pi i(x-y)} \begin{pmatrix} 0 & 1 \end{pmatrix} Y_{\pm}^{-1}(y) Y_{\pm}(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \end{aligned}$$

which indicates the interest of this RH problem in random matrix theory.

Also in the beginning of the 90's, Deift and Zhou [36] developed a powerful machinery to find asymptotics for solutions of matrix RH problems depending on a parameter tending to infinity. This *Deift/Zhou steepest descent method* consists of a series of transformations of the RH problem in order to obtain a new RH problem which can be solved approximately. It can be seen as an analogue to the classical method of steepest descent used to approximate certain integrals. Instead of approximating an integral, the goal is here to approximate the solution of a RH problem or, equivalently, the solution of a matrix integral equation.

The Deift/Zhou steepest descent method was used by Deift, Kriecherbauer, McLaughlin, Venakides, and Zhou [27, 32, 33] to obtain asymptotics for the Fokas-Its-Kitaev RH problem characterizing orthogonal polynomials on the real line. This leads also to asymptotics for the eigenvalue correlation kernel in unitary random matrix ensembles and to universal local scaling limits of this kernel. The techniques introduced in [27, 32, 33] have led to many asymptotic results for

orthogonal polynomials and random matrices, see [28]. We also use this method in the thesis to obtain universality in critical random matrix ensembles. We give a brief sketch of the ideas used in this analysis.

The starting point is the RH problem for orthogonal polynomials with respect to an exponential weight depending on a parameter n. One would like to compute asymptotics as $n \to \infty$ for the solution of this RH problem, from which asymptotics for the orthogonal polynomials and for the eigenvalue correlation kernel would follow. The strategy is to find an equivalent 'easier' RH problem for which we can approximate the solution asymptotically. To find such a RH problem, we need to perform several transformations. Without going into detail, we describe the main ideas behind those transformations.

The first transformation, the normalization of the RH problem, takes care of the behavior at infinity and leads to a RH problem with a solution tending to the identity matrix at infinity. The appropriate way to do this, is to construct a so-called g-function which is related to an equilibrium problem. Due to this transformation, oscillating jumps are created.

In the second transformation, the jump contour is deformed to a lens-shaped contour in order to turn the oscillating jumps into exponentially decaying jumps. Ignoring those exponentially small jumps and small neighborhoods of some special points, the RH solution can now be constructed explicitly. This is called the construction of the outside parametrix.

The outside parametrix determines the asymptotic behavior of the RH solution outside small neighborhoods of the special points, where the local behavior is determined by local parametrices which have to be constructed. Those can, generically, be constructed using Airy functions.

The construction of parametrices leads to a final RH transformation, which results in a RH problem normalized at infinity and with jumps which are uniformly close to the identity matrix as $n \to \infty$. For such a RH problem it is known that the solution is also close to the identity matrix. Reversing the series of transformations now leads to asymptotics for the RH problem for orthogonal polynomials.

In critical cases, the local parametrices cannot be constructed using Airy functions. Here the parametrices have to be built using other kinds of functions, such as for example functions related to Painlevé equations. This is one of the main issues in this thesis.

We refer to [52, 27] for more information on RH problems and to [59] for historical details.

Outline of the thesis

Outline for Chapter 1 and Chapter 2

In Chapter 1, we consider random matrix ensembles of the form

$$\frac{1}{Z_{n,N}} \exp(-N \operatorname{Tr} V(M)) dM \tag{12}$$

on the space of $n \times n$ Hermitian matrices, with a potential V which is such that there is a type II singular point where the limiting mean density of eigenvalues (as $n, N \to \infty$ such that $n/N \to 1$) vanishes quadratically. We establish universality for a double scaling limit of the eigenvalue correlation kernel near the singular point. This generalizes a result of Bleher and Its [14] who considered critical quartic symmetric potentials V. The limiting kernel $\mathbb{K}^{\mathrm{crit,II}}$ is built out of functions related to the Hastings-McLeod solution of the second Painlevé equation

$$q_{ss}(s) = sq(s) + 2q^3(s),$$

which is characterized by the condition

$$q(s) \sim \mathrm{Ai}(s) \qquad \text{as } s \to +\infty.$$

In Chapter 2, we extend our result to ensembles of the form

$$\frac{1}{Z_{n,N}} |\det M|^{2\alpha} \exp(-N \operatorname{Tr} V(M)) dM, \qquad \alpha > -1/2, \tag{13}$$

where a type II singular point (with quadratic vanishing) lies at the origin. Here the limiting kernel is built out of functions that are related to the general Painlevé II equation with parameter α,

$$q_{ss}(s) = sq(s) + 2q^3(s) - \alpha. \tag{14}$$

The generalized Hastings-McLeod solution q_α is for $\alpha \neq 0$ characterized by the conditions

$$q_\alpha(s) \sim \frac{\alpha}{s} \quad \text{as } s \to +\infty, \qquad q_\alpha(s) \sim \sqrt{\frac{-s}{2}} \quad \text{as } s \to -\infty.$$

The universal limiting kernel appearing in the double scaling limit of the eigenvalue correlation kernel is of the form

$$\begin{aligned} &\mathbb{K}_\alpha^{\mathrm{crit,II}}(u, v; s) \\ &\qquad = -e^{\frac{1}{2}\pi i \alpha [\operatorname{sgn}(u) + \operatorname{sgn}(v)]} \frac{\Phi_{\alpha,1}(u;s)\Phi_{\alpha,2}(v;s) - \Phi_{\alpha,1}(v;s)\Phi_{\alpha,2}(u;s)}{2\pi i(u-v)}, \end{aligned} \tag{15}$$

where $\begin{pmatrix}\Phi_{\alpha,1}\\ \Phi_{\alpha,2}\end{pmatrix}$ is a solution of the Lax pair associated to the Hastings-McLeod solution of the general Painlevé II equation with parameter α. The vector $\begin{pmatrix}\Phi_{\alpha,1}\\ \Phi_{\alpha,2}\end{pmatrix}$ is defined as a special solution of the linear differential equation

$$\Psi_\zeta(\zeta) = \begin{pmatrix} -4i\zeta^2 - i(s+2q^2) & 4\zeta q + 2ir + \alpha/\zeta \\ 4\zeta q - 2ir + \alpha/\zeta & 4i\zeta^2 + i(s+2q^2)\end{pmatrix}\Psi(\zeta), \tag{16}$$

where $q(s) = q_\alpha(s)$ is the Hastings-McLeod solution of the general Painlevé II equation (14), and where $r(s) = q_\alpha'(s)$. The solution $\begin{pmatrix}\Phi_{\alpha,1}\\ \Phi_{\alpha,2}\end{pmatrix}$ is analytic in $\mathbb{C} \setminus (-i\infty, 0]$ and characterized by the condition

$$e^{i(\frac{4}{3}\zeta^3 + s\zeta)}\begin{pmatrix}\Phi_{\alpha,1}(\zeta;s)\\ \Phi_{\alpha,2}(\zeta;s)\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} + \mathcal{O}(\zeta^{-1}), \tag{17}$$

uniformly as $\zeta \to \infty$ in the sector $\varepsilon < \arg\zeta < \pi - \varepsilon$ for any $\varepsilon > 0$. For the limiting kernel (15), we need the values of $\Phi_{\alpha,1}$ and $\Phi_{\alpha,2}$ on the real line.

We can now formulate our main results of Chapter 1 and Chapter 2.

0.1 Theorem. (see Theorem 1.1 in Chapter 1) *Consider random matrix ensembles of the form* (12) *where the limiting mean eigenvalue density ψ_V (as $n, N \to \infty$ in such a way that $n/N \to 1$) vanishes quadratically at an interior point x^* of $S_V = \operatorname{supp}\psi_V$, which is such that*

$$\psi_V(x^*) = \psi_V'(x^*) = 0, \qquad \psi_V''(x^*) > 0.$$

Assume also that there are no other singular points besides x^*. Then we take a double scaling limit where we let $n, N \to \infty$ such that the limit

$$\lim_{n,N\to\infty} n^{2/3}\left(\frac{n}{N} - 1\right) = L$$

exists with $L \in \mathbb{R}$. Then there exist explicit constants $c, c_1 > 0$ such that for $s = c_1 L \in \mathbb{R}$ the following limit holds for the eigenvalue correlation kernel $K_{n,N}$,

$$\lim_{n,N\to\infty} \frac{1}{(cn)^{1/3}} K_{n,N}\left(x^* + \frac{u}{(cn)^{1/3}}, x^* + \frac{v}{(cn)^{1/3}}\right) = \mathbb{K}_0^{\mathrm{crit,II}}(u,v;s). \tag{18}$$

0.2 Theorem. (see Theorem 2.2 in Chapter 2) *Consider random matrix ensembles of the form* (13) *where the limiting mean eigenvalue density ψ_V (as $n, N \to \infty$ in such a way that $n/N \to 1$) vanishes quadratically at the origin,*

$$\psi_V(0) = \psi_V'(0) = 0, \qquad \psi_V''(0) > 0.$$

We take a double scaling limit where we let $n, N \to \infty$ such that the limit

$$\lim_{n,N\to\infty} n^{2/3}\left(\frac{n}{N} - 1\right) = L$$

exists with $L \in \mathbb{R}$. Then there exists constants $c, c_1 > 0$ such that for $s = c_1 L$, the eigenvalue correlation kernel $K_{n,N}$ has the following double scaling limit,

$$\lim_{n,N\to\infty} \frac{1}{(cn)^{1/3}} K_{n,N}\left(\frac{u}{(cn)^{1/3}}, \frac{v}{(cn)^{1/3}}\right) = \mathbb{K}_\alpha^{\mathrm{crit,II}}(u, v; s), \tag{19}$$

uniformly for u, v in compact subsets of $\mathbb{R} \setminus \{0\}$.

We should note that the existence of $\mathbb{K}_\alpha^{\mathrm{crit,II}}(u, v; s)$ for all real s is only guaranteed if the Hastings-McLeod solution q_α has no poles on the real line. This result was only known in the literature for $\alpha = 0$ [57], but we could prove it for general $\alpha > -1/2$ too (see Theorem 2.1) by proving the solvability of an associated RH problem via a vanishing lemma.

The simplest examples for which the conditions of Theorem 0.1 and Theorem 0.2 are satisfied, are the models with quartic potentials V that were also considered by Bleher and Its [14], $V(x) = \frac{g}{4}x^4 - \sqrt{g}x^2$ for $g > 0$. Here the limiting mean eigenvalue density is given by

$$\psi_V(x) = \frac{1}{2\pi} g x^2 \sqrt{\frac{4}{\sqrt{g}} - x^2}, \qquad \text{for } x \in [-2g^{-1/4}, 2g^{-1/4}],$$

and the constants c, c_1 are given by

$$c = \frac{g^{3/4}}{4}, \qquad c_1 = \frac{1}{2^{1/3}}.$$

In the case where supp ψ_V consists of a single interval, we also compute, in addition to the double scaling limit of the kernel, asymptotics for the recurrence coefficients of orthogonal polynomials with respect to the weight $|x|^{2\alpha} e^{-NV(x)}$ on $\mathbb{R}$. The normalized orthogonal polynomials satisfy a three-term recurrence relation of the form

$$x p_k^{(N)}(x) = a_{k+1}^{(N)} p_{k+1}^{(N)}(x) + b_k^{(N)} p_k^{(N)}(x) + a_k^{(N)} p_{k-1}^{(N)}(x).$$

In the double scaling limit, the recurrence coefficients $a_n^{(N)}$ and $b_n^{(N)}$ tend to a limit and the fluctuations around this limit are of order $n^{-1/3}$. The coefficients describing the fluctuations are given in terms of the Hastings-McLeod solution q_α, see Theorem 2.7 for details.

We prove the universality results in Chapter 1 and Chapter 2 by applying the Deift/Zhou steepest descent method to the RH problem for orthogonal polynomials. A key step in the asymptotic analysis is the construction of modified equilibrium measures which can have a negative density near the critical point. Using these modified equilibrium measures, we can construct a local parametrix near the singular point using the model RH problem associated with the Hastings-McLeod solution of the Painlevé II equation. For the proof of Theorem 0.2, there

is an additional difficulty since we have to deal with RH problems with singular behavior at the origin. Chapter 1 corresponds to the paper [22] and is joint work with Arno Kuijlaars, Chapter 2 corresponds to [23] and is joint work with Arno Kuijlaars and Maarten Vanlessen.

Outline for Chapter 3

In Chapter 3, we consider the P_I^2 equation

$$s = ty - \left(\frac{1}{6}y^3 + \frac{1}{24}(y_s^2 + 2yy_{ss}) + \frac{1}{240}y_{ssss}\right), \tag{20}$$

which is a fourth order analogue of the Painlevé I equation. It is the second member of the Painlevé I hierarchy. It was conjectured by Brézin, Marinari, and Parisi for $t = 0$ [20], and by Dubrovin for general t [40], that this equation has a real solution with no poles on the real line. Our main result in Chapter 3 is the following.

0.3 Theorem. (see Theorem 3.1 in Chapter 3) *There exists a solution $y(s,t)$ to the P_I^2 equation* (20) *which is real valued and pole-free for $s, t \in \mathbb{R}$.*

In addition we compute asymptotics for $y(s,t)$ as $s \to \pm\infty$. We find that

$$y(s,t) \sim \mp 6^{1/3}|s|^{1/3} \qquad \text{as } s \to \pm\infty,$$

as it was suggested by Kapaev [66]. We construct the pole-free solution y in terms of the solution of a RH problem. Showing that this RH problem is solvable is then sufficient to prove that the solution of the P_I^2 equation is pole-free on the real line. We do this by using a vanishing lemma. In addition, we apply the Deift/Zhou steepest descent method on the RH problem in order to find asymptotics at $\pm\infty$ for the real pole-free solution. This chapter corresponds to the paper [24] and is joint work with Maarten Vanlessen.

Outline for Chapter 4

In Chapter 4, we study type III singular (edge) points in random matrix ensembles of the form (1), where the limiting mean eigenvalue density vanishes with a power 5/2. We consider potentials $V = V_{s,t} = V_0 + sV_1 + tV_2$, with V_0, V_1, and V_2 real analytic, depending on two parameters. We assume that V_0 is such that the limiting mean eigenvalue density ψ_0 for $s = t = 0$ of the random matrix ensemble is supported on a single interval $[a, b]$. Furthermore b is a singular endpoint where, for some $c > 0$,

$$\psi_0(x) \sim c(b-x)^{5/2} \qquad \text{as } x \nearrow b.$$

V_1 can be arbitrary, but V_2 needs to satisfy the critical condition

$$\int_a^b \sqrt{\frac{u-a}{b-u}} V_2'(u)du = 0. \tag{21}$$

We perform double scaling limits of the eigenvalue correlation kernel where we let $n \to \infty$ and at the same time $s, t \to 0$ at an appropriate rate. We obtain a universal limiting kernel $\mathbb{K}^{\text{crit,III}}$ which is associated with the pole-free solution of the P_I^2 equation considered in Chapter 3. The limiting kernel is of the form

$$\mathbb{K}^{\text{crit,III}}(u, v; s, t) = \frac{-\Phi_1(u; s, t)\Phi_2(v; s, t) + \Phi_2(u; s, t)\Phi_1(v; s, t)}{2\pi i(u - v)},$$

where the vector $\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}$ is a special solution of the Lax pair associated to the pole-free solution of the P_I^2 equation. To be precise,

$$\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}_\zeta = U \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}, \tag{22}$$

with U given by

$$U = \frac{1}{240} \begin{pmatrix} -4y_s\zeta - (12yy_s + y_{sss}) & 8\zeta^2 + 8y\zeta + (12y^2 + 2y_{ss} - 120t) \\ U_{21} & 4y_s\zeta + (12yy_s + y_{sss}) \end{pmatrix},$$

$$\begin{aligned} U_{21} = 8\zeta^3 - 8y\zeta^2 - (4y^2 + 2y_{ss} + 120t)\zeta \\ + (16y^3 - 2y_s^2 + 4yy_{ss} + 240s), \end{aligned}$$

where y is the real pole-free solution of P_I^2 considered in Chapter 3. For this choice of y, the solution of (22) we are looking for, is the unique one for which the following limit holds,

$$\zeta^{\frac{1}{4}\sigma_3} \begin{pmatrix} \Phi_1(\zeta; s, t) \\ \Phi_2(\zeta; s, t) \end{pmatrix} e^{\frac{1}{105}\zeta^{7/2} - \frac{1}{3}t\zeta^{3/2} + s\zeta^{1/2}} \longrightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-\frac{1}{4}\pi i},$$
$$\text{as } \zeta \to \infty \text{ with } 0 < \arg \zeta < 6\pi/7.$$

Here is the main result of Chapter 4.

0.4 Theorem. (see Theorem 4.7 in Chapter 4) *Let $V_{s,t} = V_0 + sV_1 + tV_2$ be such that for $s = t = 0$, $\operatorname{supp} \psi_0 = [a, b]$, with b a singular (edge) point where ψ_0 vanishes with a power $5/2$. Assume also that V_2 satisfies condition (21) and that there are no other singular points besides b. We take a double scaling limit where we let $n \to \infty$ and at the same time $s, t \to 0$, in such a way that, for appropriately chosen constants c_1, c_2,*

$$c_1 \cdot \lim n^{6/7} s = s_0 \in \mathbb{R}, \qquad c_2 \cdot \lim n^{4/7} t = t_0 \in \mathbb{R}. \tag{23}$$

Then, the eigenvalue correlation kernel $K_n^{(s,t)}$ satisfies the following universality result for a constant $c > 0$,

$$\lim \frac{1}{cn^{2/7}} K_n^{(s,t)} \left(b + \frac{u}{cn^{2/7}}, b + \frac{v}{cn^{2/7}} \right) = \mathbb{K}^{\mathrm{crit,III}}(u, v; s_0, t_0), \tag{24}$$

uniformly for u, v in compact subsets of $\mathbb{R}$.

A simple example where the conditions of Theorem 0.4 are satisfied, is the one where

$$V_0(x) = \frac{1}{20}x^4 - \frac{4}{15}x^3 + \frac{1}{5}x^2 + \frac{8}{5}x, \qquad V_1(x) = x, \qquad V_2(x) = x^3 - 6x. \tag{25}$$

Then ψ_0 is supported on the interval $[-2, 2]$ and given by

$$\psi_0(x) = \frac{1}{10\pi}(x+2)^{1/2}(x-2)^{5/2}\chi_{[-2,2]}(x)dx. \tag{26}$$

We also obtain asymptotics for the recurrence coefficients of the related orthogonal polynomials with respect to the weight $e^{-nV_{s,t}}$. In the double scaling limit, the recurrence coefficients show fluctuations of order $n^{-2/7}$. Those fluctuations are described by the real pole-free solution of P_I^2. In particular we have that

$$a_n^{(n,s,t)} = \frac{b-a}{4} + \frac{1}{2c}\, y(s_0, t_0) n^{-2/7}(1 + o(1)), \tag{27}$$

$$b_n^{(n,s,t)} = \frac{b+a}{2} + \frac{1}{c}\, y(s_0, t_0) n^{-2/7}(1 + o(1)), \tag{28}$$

see Theorem 4.11.

As in Chapter 1 and Chapter 2, we prove these results by applying the Deift/Zhou steepest descent method on the RH problem for orthogonal polynomials. The major difference compared with the first two chapters is the construction of the local parametrix near the singular point. Here we use the RH problem associated with the real pole-free solution of the P_I^2 equation. This chapter corresponds with the paper [25] and is joint work with Maarten Vanlessen.

Outline for Chapter 5

In Chapter 5, we consider random matrix models of the form

$$\frac{1}{Z_{n,N}} (\det M)^{\alpha} \exp(-N \operatorname{Tr} V(M)) dM, \qquad \alpha > -1 \tag{29}$$

on the positive-definite Hermitian $n \times n$ matrices. We consider the critical case where the soft edge coincides with the hard edge at the origin in the limit where $n, N \to \infty$ such that $n/N \to 1$, in such a way that the limiting mean eigenvalue density ψ_V vanishes like a square root at the origin,

$$\psi_V(x) \sim c x^{1/2} \qquad \text{as } x \searrow 0.$$

In a double scaling limit, we see the transition where the left endpoint approaches the origin, where we observe the critical case before the density blows up at the origin. The limiting kernel in this case is related to the limiting kernel $\mathbb{K}_\alpha^{\mathrm{crit,II}}$ from Chapter 2, associated with the general Painlevé II equation.

If we let $n, N \to \infty$ such that the limit

$$\lim_{n,N\to\infty} (2n)^{2/3}\left(\frac{n}{N}-1\right) = L$$

exists with $L \in \mathbb{R}$, we obtain the following double scaling limit for the eigenvalue correlation kernel $K_{n,N}$, see Theorem 5.1. There exist constants $c, c_1 > 0$ such that, for $s = c_1 L$, we have that

$$\begin{aligned}\lim_{n,N\to\infty} &\frac{1}{(2cn)^{2/3}} K_{n,N}\left(\frac{u}{(2cn)^{2/3}}, \frac{v}{(2cn)^{2/3}}\right)\\ &= \frac{1}{2}(uv)^{-1/4}\left(\mathbb{K}_{\alpha+\frac{1}{2}}^{\mathrm{crit,II}}(\sqrt{u},\sqrt{v};s) + \mathbb{K}_{\alpha+\frac{1}{2}}^{\mathrm{crit,II}}(\sqrt{u},-\sqrt{v};s)\right), \quad (30)\end{aligned}$$

uniformly for u, v in compact subsets of $(0, +\infty)$. If $\alpha > 0$, this can be rewritten as

$$\begin{aligned}\lim_{n,N\to\infty} &\frac{1}{(2cn)^{2/3}} K_{n,N}\left(\frac{u}{(2cn)^{2/3}}, \frac{v}{(2cn)^{2/3}}\right)\\ &= \frac{1}{2}(uv)^{-1/4}\left(\mathbb{K}_{\alpha-\frac{1}{2}}^{\mathrm{crit,II}}(\sqrt{u},\sqrt{v};s) - \mathbb{K}_{\alpha-\frac{1}{2}}^{\mathrm{crit,II}}(\sqrt{u},-\sqrt{v};s)\right). \quad (31)\end{aligned}$$

Here the kernels $\mathbb{K}_{\alpha\pm\frac{1}{2}}^{\mathrm{crit,II}}$ are the ones that appear also in Chapter 2. The simplest examples for which this critical situation occurs, are the ensembles with a potential of the form $V(x) = \frac{g}{2}x^2 - 2\sqrt{g}x$ for $g > 0$, where ψ_V is given by

$$\psi_V(x) = \frac{g}{2\pi}\sqrt{x}\sqrt{\frac{4}{\sqrt{g}} - x}, \qquad \text{for } x \in [0, 4g^{-1/2}],$$

and where the constants c, c_1 are given by

$$c = \frac{g^{3/4}}{4}, \qquad c_1 = \frac{g^{1/4}}{2c^{1/3}}.$$

To prove our result, it is not necessary to perform a Deift/Zhou steepest descent analysis. In fact, it is a quite straightforward consequence of our results obtained in Chapter 2 (see Theorem 0.2). This chapter is joint work with Arno Kuijlaars.

Outline for Chapter 6

In Chapter 6, we indicate that the different limiting kernels that appeared in this thesis are related to each other. We prove that the Bessel kernel and the Airy

kernel can be obtained as limits of the critical kernel associated with the Hastings-McLeod solution of the Painlevé II equation. Also we prove that the sine kernel and the Airy kernel can be obtained as limits of the critical kernel associated with the real pole-free solution of the P_I^2 equation, see Theorem 6.1 for the formulation of the results. We prove our results by applying the Deift/Zhou steepest descent method to the model RH problems associated with those Painlevé equations.

We conclude the thesis with a summary of the new results obtained and an outlook on possible problems for future research.

Chapter 1

Universality of a double scaling limit near a singular interior point

Summary[1] We study unitary random matrix ensembles in the critical case where the limiting mean eigenvalue density vanishes quadratically at an interior point of the support. We establish universality of the limits of the eigenvalue correlation kernel at such a critical point in a double scaling limit. The limiting kernels are constructed out of functions associated with the second Painlevé equation. This extends a result of Bleher and Its for the special case of a critical quartic potential.

The two main tools we use are equilibrium measures and Riemann-Hilbert problems. In our treatment of equilibrium measures we allow a negative density near the singular point, which enables us to treat all cases simultaneously. The asymptotic analysis of the Riemann-Hilbert problem is done with the Deift/Zhou steepest descent analysis. For the construction of a local parametrix at the singular point we introduce a modification of the approach of Baik, Deift, and Johansson so that we are able to satisfy the required jump properties exactly.

[1] This chapter corresponds to the following paper [22]:

T. Claeys and A.B.J. Kuijlaars, Universality of the double scaling limit in random matrix models, *Communications on Pure and Applied Mathematics* **59** (2006), no. 11, 1573–1603.

1.1 Introduction and statement of result

1.1.1 Unitary random matrix ensembles

We consider the unitary random matrix model

$$Z_{n,N}^{-1} \exp(-N \operatorname{Tr} V(M)) dM \tag{1.1}$$

defined on Hermitian $n \times n$ matrices M in a critical regime where the limiting mean density of eigenvalues vanishes at an interior point. It is a basic fact of random matrix theory [27, 83] that the eigenvalues of the random matrix ensemble (1.1) follow a determinantal point process with correlation kernel

$$K_{n,N}(x,y) = e^{-\frac{N}{2}V(x)} e^{-\frac{N}{2}V(y)} \sum_{k=0}^{n-1} p_k^{(N)}(x) p_k^{(N)}(y), \tag{1.2}$$

where $p_k^{(N)}$ denotes the kth degree orthonormal polynomial with respect to the weight $e^{-NV(x)}$ on $\mathbb{R}$.

We assume in this chapter that the confining potential $V : \mathbb{R} \to \mathbb{R}$ in (1.1) is real analytic and that it satisfies the growth condition

$$\frac{V(x)}{\log(x^2+1)} \to +\infty \quad \text{as } |x| \to +\infty. \tag{1.3}$$

These assumptions ensure that the mean eigenvalue density $\frac{1}{n} K_{n,N}(x,x)$ has a limit as $n, N \to \infty$, $n/N \to 1$, see e.g. [27], which we denote by $\psi_V(x)$. It is known that ψ_V is the density of the measure μ_V which minimizes the weighted energy

$$I_V(\mu) = \iint \log \frac{1}{|x-y|} d\mu(x) d\mu(y) + \int V(x) d\mu(x) \tag{1.4}$$

among all probability measure on $\mathbb{R}$. The measure μ_V is called the equilibrium measure in the external field V, and it satisfies the following variational conditions for some constant ℓ_V:

$$2 \int \log|x-y| \, d\mu_V(y) - V(x) + \ell_V = 0 \qquad \text{for } x \in \operatorname{supp} \mu_V, \tag{1.5}$$

$$2 \int \log|x-y| \, d\mu_V(y) - V(x) + \ell_V \leq 0 \qquad \text{for } x \in \mathbb{R} \setminus \operatorname{supp} \mu_V. \tag{1.6}$$

The fact that V is real analytic ensures that the support $S_V = \operatorname{supp} \mu_V$ consists of a finite union of intervals [31].

It is a remarkable fact that local scaling limits of the kernel (1.2) depend only on the nature of the density ψ_V. This has been proven rigorously in the bulk

of the spectrum for a quartic V in [13] and for general real analytic V in [32]. Indeed, if $\psi_V(x^*) > 0$, then

$$\lim_{n\to\infty} \frac{1}{n\psi_V(x^*)} K_{n,n}\left(x^* + \frac{u}{n\psi_V(x^*)}, x^* + \frac{v}{n\psi_V(x^*)}\right) = \mathbb{K}^{\text{bulk}}(u,v) \tag{1.7}$$

exists, and

$$\mathbb{K}^{\text{bulk}}(u,v) = \frac{\sin \pi(u-v)}{\pi(u-v)}. \tag{1.8}$$

The scaling limits are different at special points of the spectrum. At edge points of the spectrum the density ψ_V typically vanishes like a square root, and then it is known that for some constant $c > 0$,

$$\lim_{n\to\infty} \frac{1}{(cn)^{2/3}} K_{n,n}\left(x^* + \frac{u}{(cn)^{2/3}}, x^* + \frac{v}{(cn)^{2/3}}\right) = \mathbb{K}^{\text{edge}}(u,v) \tag{1.9}$$

where

$$\mathbb{K}^{\text{edge}}(u,v) = \frac{\operatorname{Ai}(u)\operatorname{Ai}'(v) - \operatorname{Ai}'(u)\operatorname{Ai}(v)}{u-v} \tag{1.10}$$

and Ai is the Airy function. The Airy kernel is related to the Tracy-Widom distribution [97]. In (1.9) we have assumed that x^* is a right edge point. For a left edge point we change $u \mapsto -u$, $v \mapsto -v$ in the left-hand side of (1.9).

Other special points in the spectrum include

- Edge points where the density vanishes to a higher order. The possible edge point behaviors (at a right end point x^*) are

$$\psi_V(x) = c(x^* - x)^{2k+\frac{1}{2}}(1 + o(1)) \quad \text{as } x \to x^*+ \tag{1.11}$$

  where $c > 0$ and k is a non-negative integer.

- Interior points where the density vanishes. Then

$$\psi_V(x) = c(x - x^*)^{2k}(1 + o(1)) \quad \text{as } x \to x^* \tag{1.12}$$

  where $c > 0$ and k is a positive integer.

In these critical cases it is believed that the local scaling limit at x^* of the kernel only depends on the order of vanishing of the density at x^* [12].

1.1.2 Singular interior point

The case where ψ_V vanishes quadratically at an interior point of S_V, that is, the case $k = 1$ in (1.12), was considered by Bleher and Its [14] for the case of a critical quartic potential $V(x) = \frac{g}{4}x^4 + \frac{t}{2}x^2$, with $g > 0$ and $t = t_c = -2\sqrt{g}$. Then

$$\psi_V(x) = \frac{1}{2\pi} g x^2 \sqrt{\frac{4}{\sqrt{g}} - x^2}, \qquad \text{for } x \in [-2g^{-1/4}, 2g^{-1/4}],$$

so that ψ_V vanishes quadratically at the origin. Bleher and Its consider the double scaling limit where t changes with n and tends to t_c as $n \to \infty$ in such a way that $n^{2/3}(t - t_c)$ remains constant. For the quartic potential this is equivalent to considering (1.1) where $n, N \to \infty$, $n/N \to 1$, such that

$$\lim_{n,N\to\infty} n^{2/3}\left(\frac{n}{N} - 1\right) \tag{1.13}$$

exists. Bleher and Its gave a one-paremeter family $\mathbb{K}^{\mathrm{crit,II}}(u, v; s)$ of limiting kernels, depending on $s \in \mathbb{R}$, so that for some $c > 0$,

$$\lim_{n,N\to\infty} \frac{1}{(cn)^{1/3}} K_{n,N}\left(\frac{u}{(cn)^{1/3}}, \frac{v}{(cn)^{1/3}}\right) = \mathbb{K}^{\mathrm{crit,II}}(u, v; s) \tag{1.14}$$

where s is proportional to the value of the limit (1.13).

1.1.3 Hastings-McLeod solution of Painlevé II

The critical kernels $\mathbb{K}^{\mathrm{crit,II}}(u, v; s)$ are expressed in terms of so-called ψ-functions associated with the Hastings-McLeod solution of the Painlevé II equation [57]. Consider as in [14] the linear differential equations for a 2-vector (or 2×2 matrix) $\Psi = \Psi(\zeta; s)$,

$$\frac{d}{d\zeta}\Psi = A\Psi, \qquad \frac{\partial}{\partial s}\Psi = B\Psi \tag{1.15}$$

where

$$A = A(\zeta; s) = \begin{pmatrix} 4\zeta q & 4\zeta^2 + s + 2q^2 + 2r \\ -4\zeta^2 - s - 2q^2 + 2r & -4\zeta q \end{pmatrix}, \tag{1.16}$$

and

$$B = B(\zeta; s) = \begin{pmatrix} q & \zeta \\ -\zeta & -q \end{pmatrix}. \tag{1.17}$$

The compatibility condition for the Lax pair (1.15) is that $q = q(s)$ satisfies the Painlevé II equation $q'' = sq + 2q^3$ and that $r = r(s) = q'(s)$. We assume that $q(s)$ is the Hastings-McLeod solution of Painlevé II, which is characterized by the asymptotic condition

$$q(s) = \mathrm{Ai}(s)(1 + o(1)) \qquad \text{as } s \to +\infty.$$

The critical kernels [14] are given by

$$\mathbb{K}^{\mathrm{crit,II}}(u, v; s) = \frac{\Phi^1(u; s)\Phi^2(v; s) - \Phi^2(u; s)\Phi^1(v; s)}{\pi(u - v)} \tag{1.18}$$

where $\left(\begin{smallmatrix}\Phi^1(\zeta;s)\\ \Phi^2(\zeta;s)\end{smallmatrix}\right)$ is the special solution to (1.15) which is real for real ζ, satisfies

$$\Phi^1(-\zeta;s) = \Phi^1(\zeta;s), \qquad \Phi^2(-\zeta;s) = -\Phi^2(\zeta;s),$$

and has asymptotics on the real line

$$\begin{aligned}\Phi^1(\zeta;s) &= \cos\left(\frac{4}{3}\zeta^3 + s\zeta\right) + O(\zeta^{-1}),\\ \Phi^2(\zeta;s) &= -\sin\left(\frac{4}{3}\zeta^3 + s\zeta\right) + O(\zeta^{-1})\end{aligned} \qquad \text{as } \zeta \to \pm\infty.$$

If we put

$$\Phi_1 = \Phi^1 + i\Phi^2, \qquad \Phi_2 = \Phi^1 - i\Phi^2 \tag{1.19}$$

then

$$\mathbb{K}^{\mathrm{crit,II}}(u,v;s) = \frac{-\Phi_1(u;s)\Phi_2(v;s) + \Phi_2(u;s)\Phi_1(v;s)}{2\pi i(u-v)} \tag{1.20}$$

and $\left(\begin{smallmatrix}\Phi_1\\ \Phi_2\end{smallmatrix}\right)$ is a special solution of the differential equations

$$\frac{d}{d\zeta}\Psi(\zeta;s) = \begin{pmatrix} -4i\zeta^2 - i(s+2q^2) & 4\zeta q + 2ir \\ 4\zeta q - 2ir & 4i\zeta^2 + i(s+2q)\end{pmatrix}\Psi(\zeta;s) \tag{1.21}$$

and

$$\frac{\partial}{\partial s}\Psi(\zeta;s) = \begin{pmatrix} -i\zeta & q \\ q & i\zeta\end{pmatrix}\Psi(\zeta;s). \tag{1.22}$$

The equations (1.21)-(1.22) for the ψ-functions correspond to the ones used by Flaschka and Newell [44] and we will also use those in what follows. The vector $\left(\begin{smallmatrix}\Phi_1\\ \Phi_2\end{smallmatrix}\right)$ is the unique solution of (1.21) with asymptotics

$$e^{i(\frac{4}{3}\zeta^3 + s\zeta)}\begin{pmatrix}\Phi_1(\zeta;s)\\ \Phi_2(\zeta;s)\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} + O\left(\zeta^{-1}\right) \tag{1.23}$$

as $\zeta \to \infty$ uniformly in $\varepsilon \le \arg\zeta \le \pi - \varepsilon$ for any $\varepsilon > 0$.

Before discussing our results, we like to point out an integral formula for the kernel $\mathbb{K}^{\mathrm{crit,II}}(u,v;s)$. If we take a derivative of (1.18) with respect to s and use (1.15) and (1.17), we get after some calculations

$$\frac{d}{ds}\mathbb{K}^{\mathrm{crit,II}}(u,v;s) = \frac{1}{\pi}\left[\Phi^1(u;s)\Phi^1(v;s) + \Phi^2(u;s)\Phi^2(v;s)\right].$$

Using the Deift/Zhou steepest descent method for $s \to -\infty$ as done in [37] (see also Section 6.4 in Chapter 6 of this thesis), one can show that $\mathbb{K}^{\mathrm{crit,II}}(u,v;s) \to 0$ as $s \to -\infty$, so that we get

$$\mathbb{K}^{\mathrm{crit,II}}(u,v;s) = \frac{1}{\pi}\int_{-\infty}^{s}\left[\Phi^1(u;\sigma)\Phi^1(v;\sigma) + \Phi^2(u;\sigma)\Phi^2(v;\sigma)\right]d\sigma. \tag{1.24}$$

Since $\Phi^1(\zeta;s)$ and $\Phi^2(\zeta;s)$ are real for real ζ, formula (1.24) clearly shows that $\mathbb{K}^{\mathrm{crit,II}}(u,u;s) > 0$, as it should be.

1.1.4 Statement of result

It is the aim of this chapter to show that the kernel $\mathbb{K}^{\mathrm{crit,II}}(u,v;s)$ is a universal limit. Whenever the limiting mean eigenvalue density ψ_V vanishes quadratically at an interior point x^*, the correlation kernel $K_{n,N}$ has a double scaling limit given by (1.14), assuming that there are no other singular points besides x^*. Here singular point refers to the classification of [32] according to which there are three types of non-regular behavior for a real analytic external field V, see also [74]. The singular points of type I are points in $\mathbb{R}\setminus S_V$ where equality in the variational inequality (1.6) holds. Singular points of type II are interior points of S_V where the density ψ_V vanishes, and singular points of type III are edge points of S_V where ψ_V vanishes to higher order than a square root.

We have not been able to show universality of the double scaling limit in the presence of other singular points besides x^*, although we strongly believe that the universality result should hold in full generality. The problem lies in the existence of suitable parametrices around other singular points which we have been unable to prove in the double scaling regime, see also Remark 1.9.

In our Theorem 1.1 below, we use the equilibrium measure ω_S of a compact set $S\subset\mathbb{R}$. This is the unique probability measure on S that minimizes the logarithmic energy

$$I(\mu)=\iint \log\frac{1}{|s-t|}d\mu(s)d\mu(t) \tag{1.25}$$

among all Borel probability measures μ on S. If S is a single interval $[a,b]$, then ω_S has a density w_S given by

$$w_S(x)=\frac{1}{\pi\sqrt{(b-x)(x-a)}},\qquad x\in(a,b).$$

If S is a finite union of disjoint intervals, say $S=\bigcup_{j=1}^n[a_j,b_j]$ with $a_j<b_j<a_{j+1}$. Then ω_S has density

$$w_S(x)=\frac{|p(x)|}{\pi\sqrt{\prod_{j=1}^n|(b_j-x)(x-a_j)|}},\qquad x\in\bigcup_{j=1}^n(a_j,b_j), \tag{1.26}$$

where $p(x)$ is a monic polynomial of degree $n-1$ with exactly one zero in each of the gaps (b_j,a_{j+1}), $j=1,\dots,n-1$, see e.g. [94, Lemma 4.4.1]. Note that (1.26) has an extension to an analytic function in $\mathbb{C}\setminus(\mathbb{R}\setminus S^o)$, where $S^o=\bigcup_j(a_j,b_j)$, which is a fact that we will use in what follows.

The following is our main result.

1.1 Theorem. *Let V be real analytic on $\mathbb{R}$ such that $\lim_{x\to\pm\infty}\frac{V(x)}{\log(x^2+1)}=+\infty$. Let ψ_V be the density of the equilibrium measure in the external field, and let x^* be an interior singular point of $S_V=\operatorname{supp}\psi_V$ which is such that*

$$\psi_V(x^*)=\psi_V'(x^*)=0,\qquad \psi_V''(x^*)>0.$$

Assume that there are no other singular points besides x^*. Let $n, N \to \infty$ such that the limit

$$\lim_{n,N\to\infty} n^{2/3}\left(\frac{n}{N}-1\right) = L$$

exists with $L \in \mathbb{R}$. Let $K_{n,N}$ be the correlation kernel (1.2) *for the eigenvalues of the random matrix model* (1.1)*. Then there exist constants $c > 0$ and $s \in \mathbb{R}$ such that*

$$\lim_{n,N\to\infty} \frac{1}{(cn)^{1/3}} K_{n,N}\left(x^* + \frac{u}{(cn)^{1/3}}, x^* + \frac{v}{(cn)^{1/3}}\right) = \mathbb{K}^{\mathrm{crit,II}}(u,v;s) \quad (1.27)$$

uniformly for u, v in compact subsets of $\mathbb{R}$.

Explicit formulas for the constants c and s are

$$c = \frac{\pi\psi_V''(x^*)}{8} \quad (1.28)$$

and

$$s = L\frac{\pi}{c^{1/3}} w_{S_V}(x^*), \quad (1.29)$$

where w_{S_V} is the density of the equilibrium measure of S_V.

As noted before, Bleher and Its [14] proved (1.27) for the case of a critical quartic V. See [15] for a rigorous expansion of the free energy in this critical case. Recently Shcherbina generalized Theorem 1.1 for certain non-analytic symmetric potentials [93].

1.2 Remark. The random matrix model (1.1) may be generalized to include a spectral singularity at the origin

$$Z_{n,N}^{-1} |\det M|^{2\alpha} \exp(-N \operatorname{Tr} V(M)) dM, \qquad \alpha > -1/2. \quad (1.30)$$

If $\psi_V(0) > 0$ and $n = N \to \infty$, then the scaled limit of the correlation kernels is a Bessel kernel which involves Bessel functions of order $\alpha \pm \frac{1}{2}$, see [2, 77]. In the multi-critical case where ψ_V vanishes quadratically at 0, an analogue of Theorem 1.1 is valid, see Theorem 2.2. In Chapter 2, we are considering the double scaling limit of (1.30) and we show that the limiting kernels are expressed in terms of the ψ-functions associated with a special solution of the general Painlevé II equation

$$q'' = sq + 2q^3 - \alpha.$$

The main ingredients in the proof of Theorem 1.1 are equilibrium measures and Riemann-Hilbert problems. We give some comments on both.

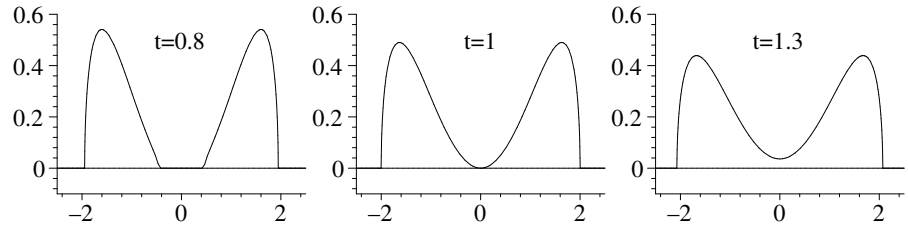

Figure 1.1 *The density of* μ_t *for* $V(x) = \frac{x^4}{4} - x^2$ *with* t *equal to* 0.8, 1 *and* 1.3, *respectively.*

1.1.5 Equilibrium measures

Recall that the equilibrium measure in external field V minimizes (1.4). We need to know how the equilibrium measure μ_V in the external field changes as a result of a change in V. The particular modification we consider here is

$$V_t = \frac{1}{t}V, \qquad t > 0,$$

so that $V_1 = V$. Let us put

$$\mu_t = \mu_{V_t}.$$

Then it is known that $t\mu_t$ and S_t are increasing as a function of t, see e.g. [26, 91, 96]. We also have the Buyarov-Rakhmanov formula [21]

$$\mu_t = \frac{1}{t}\int_0^t \omega_{\text{supp } \mu_\tau} d\tau, \tag{1.31}$$

which expresses the equilibrium measure in the external field as an average of equilibrium measures of sets. A consequence of (1.31) is that

$$\frac{d}{dt}(t\mu_t)\bigg|_{t=1} = \omega_{S_V}, \tag{1.32}$$

which partly explains why the equilibrium measure ω_{S_V} plays a role in the formula (1.29) for s.

For the case of interest in this chapter we have that ψ_V vanishes at $x = x^*$. Then for $t > 1$, there is a positive density at x^*, while for $t < 1$, x^* is out of the support of μ_t. For t slightly less than 1, there is a gap in supp μ_t, see Figure 1.1. An asymptotic analysis based on the equilibrium measure μ_t would require a discussion of the two different situations $t > 1$ and $t < 1$, as is done in [6].

Therefore we found it convenient to introduce a modification of the equilibrium problem in external field, which will enable us to treat both cases simultaneously.

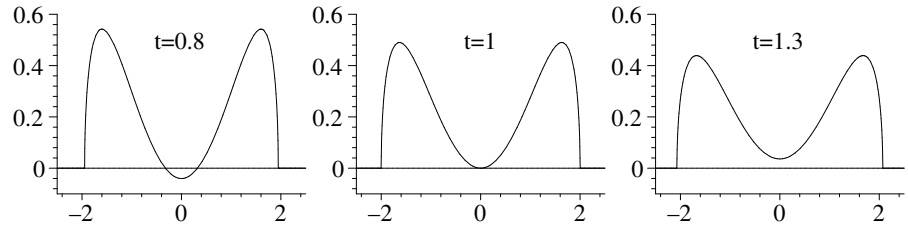

Figure 1.2 *The density of ν_t for $V(x) = \frac{x^4}{4} - x^2$ with t equal to* 0.8, 1 *and* 1.3, *respectively (compare with Figure 1.1).*

The modification we make is that we do not require the measure to be nonnegative in a neighborhood of the point x^*. For a sufficiently small $\delta_0 > 0$, we consider the problem to minimize

$$I_{V_t}(\nu) = \iint \log \frac{1}{|x-y|} d\nu(x)d\nu(y) + \frac{1}{t}\int V(x)d\nu(x) \tag{1.33}$$

among all signed measures $\nu = \nu^+ - \nu^-$ on $\mathbb{R}$, where $\nu^\pm$ are nonnegative measures, such that

$$\int d\nu = 1, \quad \text{and} \quad \operatorname{supp} \nu^- \subset [x^* - \delta_0, x^* + \delta_0]. \tag{1.34}$$

We denote the minimizer by ν_t and we let $S_t = \operatorname{supp} \nu_t$.

Then $\nu_t = \mu_t$ for $t \geq 1$, but for $t < 1$ there is a clear distinction between ν_t and μ_t, see Figure 1.2. However we still have the analogue of (1.32) (as we prove)

$$\frac{d}{dt}(t\nu_t)\Big|_{t=1} = \omega_{S_V}, \tag{1.35}$$

What's more, we also have (1.35) at the level of densities, that is, if ψ_t denotes the density of ν_t, and if x is an interior point of S_V (in particular if $x = x^*$), then

$$\frac{d}{dt}(t\psi_t(x))\Big|_{t=1} = w_{S_V}(x), \tag{1.36}$$

where w_{S_V} is the density of the equilibrium measure of S_V, which is what we need for the proof of Theorem 1.1. Note that we do not have (1.36) for $x = x^*$ if ψ_t is the density of μ_t.

1.1.6 Riemann-Hilbert problem

The second main tool for the proof of Theorem 1.1 is the characterization of orthonormal polynomials by means of a Riemann-Hilbert problem, due to Fokas, Its,

and Kitaev [47], and the subsequent asymptotic analysis of the Riemann-Hilbert problem by means of the Deift/Zhou steepest descent analysis of Riemann-Hilbert problems, introduced in [36], and further developed in [35, 33, 32], and other more recent papers. The Deift/Zhou steepest descent analysis of the Riemann-Hilbert problem consists of a sequence of explicit transformations, which result in a Riemann-Hilbert problem that is explicitly solvable in terms of Neumann series. (In fact, in this chapter, we only need the first term of this series.) The critical point x^* needs special attention.

Of particular interest for us is the paper [6] by Baik, Deift, and Johansson on the length of the longest increasing subsequence of a random permutation of $\{1, 2, \ldots, n\}$. These authors show that the fluctuations of this random variable are distributed according to the Tracy-Widom distribution [97] in the limit as $n \to \infty$. One of the technical tools in this important paper is the asymptotic analysis of a Riemann-Hilbert problem on the unit circle which is related to an equilibrium measure (also on the unit circle) whose density vanishes at the point -1. This situation is comparable to ours. The authors of [6], see also subsequent papers [7, 8, 9, 10], construct a local parametrix near -1 with the aid of the ψ-functions associated with the Hastings-McLeod solution of Painlevé II. These ψ-functions satisfy a model Riemann-Hilbert problem and the local parametrix is constructed by appropriately mapping the model Riemann-Hilbert problem onto a neighborhood of -1 so that it satisfies certain desired jump properties approximately.

We follow the approach of [6] but we introduce a modification in the construction of the local parametrix so that it has the desired jump properties *exactly*, in contrast to [6] where the desired jump properties only hold *approximately*. The fact that we have the exact jump properties simplifies the arguments considerably and we feel that this is also a main contribution of the present work.

1.1.7 Outline of the rest of the chapter.

In Section 1.2 we collect the necessary facts about equilibrium measures. In particular we study the modified equilibrium problem with external field in some detail. In Section 1.3 we discuss the Riemann-Hilbert problem satisfied by the ψ-functions. Here we follow [44]. Then in Section 1.4 we state the Riemann-Hilbert problem for orthogonal polynomials, discuss the relation with the correlation kernel $K_{n,N}$, and perform the transformations in the steepest descent analysis. Finally in Section 1.5 we give the proof of Theorem 1.1.

1.2 Equilibrium Measures

As explained in the previous section, we consider a modification of the equilibrium problem where we drop the non-negativity condition in a small neighborhood of

x^*. We take $\delta_0 > 0$ sufficiently small so that

$$\psi_V(x) > 0 \qquad \text{for all } x \in [x^* - \delta_0, x^* + \delta_0] \setminus \{x^*\}. \tag{1.37}$$

and we use ν_t to denote the signed measure that minimizes (1.33) under the conditions (1.34). We define

$$S_t = \text{supp } \nu_t. \tag{1.38}$$

The existence and uniqueness of ν_t follows as in [91].

Let

$$U^\nu(x) = \int \log \frac{1}{|x-y|} d\nu(y)$$

be the logarithmic potential of ν. Then standard arguments of potential theory [27, 91] show that ν_t is the unique signed measure satisfying (1.34) with the property that

$$2U^{\nu_t}(x) + \frac{1}{t}V(x) = \ell_t, \qquad x \in \text{supp } \nu_t \cup [x^* - \delta_0, x^* + \delta_0], \tag{1.39}$$

$$2U^{\nu_t}(x) + \frac{1}{t}V(x) \geq \ell_t, \qquad x \in \mathbb{R}. \tag{1.40}$$

for some constant ℓ_t.

As before we use μ_t to denote the equilibrium measure in the external field V_t. This is a probability measure that satisfies for some constant $\tilde{\ell}_t$,

$$2U^{\mu_t}(x) + \frac{1}{t}V(x) = \tilde{\ell}_t, \qquad x \in \text{supp } \mu_t, \tag{1.41}$$

$$2U^{\mu_t}(x) + \frac{1}{t}V(x) \geq \tilde{\ell}_t, \qquad x \in \mathbb{R}. \tag{1.42}$$

It is known that $t\mu_t$ and supp μ_t are increasing with $t > 0$, see [21, 91]. For $t \geq 1$, we have supp $\mu_V \subset$ supp μ_t which implies that $\nu_t = \mu_t$ in view of (1.37) and the variational conditions (1.39)–(1.42). For $t < 1$, we have that x^* is outside the support of μ_t, and in fact the strict inequality in (1.42) holds for $x = x^*$ if $t < 1$.

In general there is the following inequality between μ_t and ν_t.

1.3 Lemma. *For every $t > 0$ we have*

$$\mu_t \leq \nu_t^+. \tag{1.43}$$

Proof. Let $\lambda = \nu_t - \mu_t$. From (1.39), (1.40), (1.41), and (1.42) it follows that

$$2U^\lambda(x) \leq \ell_t - \tilde{\ell}_t, \qquad \text{for } x \in S_t, \tag{1.44}$$

$$2U^\lambda(x) \geq \ell_t - \tilde{\ell}_t, \qquad \text{for } x \in \text{supp } \mu_t. \tag{1.45}$$

The potential U^λ is subharmonic on $\mathbb{C}\setminus\text{supp }\lambda^+$ and since $\int d\lambda = 0$, it is subharmonic at infinity as well. By the maximum principle for subharmonic functions [88, 91], the maximum of U^λ is attained in supp λ^+ only. Since supp $\lambda^+ \subset S_t$ we then have by (1.44) that equality in (1.45) holds for every $x \in \text{supp }\mu_t$, and so supp $\mu_t \subset$ supp λ^+. This implies (1.43). □

It follows from (1.43) that supp $\nu_t^- \cap$ supp μ_t is empty. Since for t slightly less than 1 a gap opens in supp μ_t, which depends continuously on t, see [74], it follows that for any given $\delta \in (0, \delta_0)$, there is $t_0 < 1$ such that

$$\text{supp }\nu_t^- \subset [x^* - \delta, x^* + \delta], \qquad \text{for } t > t_0.$$

This shows that for $t < 1$ sufficiently close to 1, the definition of ν_t is independent of the choice of δ_0.

A very useful fact is that for real analytic V, say V is analytic in a neighborhood $\mathcal{V}$ of the real line, the measures μ_t have densities $\tilde{\psi}_t$ which can be expressed in terms of the negative part of an analytic function in $\mathcal{V}$. Indeed, if

$$\tilde{Q}_t(z) = \left(\frac{V'(z)}{2t}\right)^2 - \frac{1}{t}\int \frac{V'(z) - V'(y)}{z - y} d\mu_t(y), \tag{1.46}$$

then it was shown in [31] that

$$\tilde{\psi}_t(x) = \frac{1}{\pi}\sqrt{\tilde{Q}_t^-(x)}, \tag{1.47}$$

where $\tilde{Q}_t^- = \max(0, -\tilde{Q}_t)$ denotes the negative part of $\tilde{Q}_t$. A consequence of (1.47) is that supp μ_t is the closure of the set where $\tilde{Q}_t$ is negative.

The arguments of [31] can be readily extended to the signed measures ν_t, provided that supp $\nu_t^- \subset [x^* - \delta, x^* + \delta]$ for some $\delta < \delta_0$. We define

$$Q_t(z) = \left(\frac{V'(z)}{2t}\right)^2 - \frac{1}{t}\int \frac{V'(z) - V'(y)}{z - y} d\nu_t(y), \qquad z \in \mathcal{V}. \tag{1.48}$$

and then the following holds.

1.4 Proposition. *There exists $t_0 \in (0,1)$ such that for every $t > t_0$, the signed measure ν_t has a density ψ_t and*

$$Q_t(x) = -\left[\pi\psi_t(x)\right]^2, \qquad \textit{for } x \in S_t. \tag{1.49}$$

In addition we have

$$Q_t(x) = \left[\int_{S_t} \frac{\psi_t(y)}{y - x} dy + \frac{V'(x)}{2}\right]^2 \qquad \textit{for } x \in \mathbb{R}\setminus S_t. \tag{1.50}$$

Proof. This follows as in [31, Proposition 2.51]. □

Obviously, for $t \geq 1$ we have $Q_t = \tilde{Q}_t$. Since $\psi_1 = \psi_V$ has a double zero at x^*, we see from (1.49) that Q_1 has a zero at x^* of order four. For t slightly bigger than 1, this fourth order zero splits into two double zeros in the complex plane away from the real axis.

For t slightly less than 1, there is a difference in the behavior of the zeros of Q_t and $\tilde{Q}_t$. Indeed, $\tilde{Q}_t$ has two simple real zeros near x^* which are endpoints of the support of μ_t, and in addition there is a double real zero in between them. On the other hand, we have that Q_t has two double real zeros near x^*, which are endpoints of the support of ν_t^-.

The fact that $-Q_t$ has only double zeros near x^* allows us to take an analytic square root, and in view of (1.49) we choose it so that for $z \in S_t$,

$$\psi_t(z) = \frac{1}{\pi}(-Q_t(z))^{1/2}, \tag{1.51}$$

where the sign of the square root at $x = x^*$ is taken negative if $t < 1$ and positive if $t > 1$. The right-hand side of (1.51) has an analytic extension to a neighborhood of x^*, which is independent of the value of $t > t_0$. Thus ψ_t has an analytic extension to a fixed neighborhood of x^*, which will also be denoted by ψ_t.

Recall from the discussion before the statement of Theorem 2.1, that the density w_{S_V} of the equilibrium measure of S_V also has an analytic extension to a neighborhood of x^*, which we also denote by w_{S_V}. The remaining part of this section is devoted to the proof of the following proposition.

1.5 Proposition. *We have*

$$\lim_{t \to 1} \frac{t\psi_t(z) - \psi_1(z)}{t-1} = w_{S_V}(z) \tag{1.52}$$

uniformly for z in a neighborhood of x^*.

We start with a lemma which contains a weaker form of (1.52).

1.6 Lemma. *We have*

$$\frac{d}{dt}(t\nu_t)\bigg|_{t=1} = \omega_{S_V} \tag{1.53}$$

where ω_{S_V} is the equilibrium measure of S_V.

Proof. Buyarov and Rakhmanov [21] proved (for a very general class of V) that

$$\lim_{t \to 1-} \frac{t\mu_t - \mu_V}{t-1} = \omega_{S_V} \tag{1.54}$$

and

$$\lim_{t \to 1+} \frac{t\mu_t - \mu_V}{t-1} = \omega_{S_V^*} \tag{1.55}$$

where $\omega_{S_V^*}$ is the equilibrium measure of the set

$$S_V^* = \{x \in \mathbb{R} \mid 2U^{\mu_V}(x) + V(x) = \tilde{\ell}_1\},$$

see (1.41) with $t = 1$. For real analytic V, the sets S_V and S_V^* differ by an at most finite number of points, so that $\omega_{S_V} = \omega_{S_V^*}$. Thus by (1.54) and (1.55) we have

$$\lim_{t\to 1} \frac{t\mu_t - \mu_V}{t-1} = \omega_{S_V}. \tag{1.56}$$

Now write

$$\frac{t\nu_t - \nu_1}{t-1} = \frac{t\mu_t - \mu_V}{t-1} + \frac{t}{t-1}(\nu_t - \mu_t).$$

In view of (1.56) it suffices to prove that

$$\|\nu_t - \mu_t\| = o(t-1) \qquad \text{as } t \to 1, \tag{1.57}$$

where the norm denotes the total variation of a signed measure. We may assume that $t < 1$. Because of (1.43) we have

$$\|\nu_t - \mu_t\| = \|\nu_t^+ - \mu_t - \nu_t^-\| = \int d(\nu_t^+ - \mu_t) + \int d\nu_t^- = 2\int d\nu_t^-. \tag{1.58}$$

For $t < 1$, the support of ν_t^- is contained in an interval $[x^* - \delta(t), x^* + \delta(t)]$, with

$$\delta(t) = O\left((1-t)^{1/2}\right) \qquad \text{as } t \to 1-, \tag{1.59}$$

see [74, Lemma 8.1(iii)]. From (1.51) and the fact that Q_t has two double zeros in $[x^* - \delta(t), x^* + \delta(t)]$ which are the end-points of the support of ν_t^-, we then easily get that

$$\frac{d}{dx}\nu_t^-(x) = O(\delta(t)^2) = O(1-t) \qquad \text{as } t \to 1-. \tag{1.60}$$

Combining (1.59) and (1.60), we find $\int d\nu_t^- = O\left((1-t)^{3/2}\right)$ as $t \to 1-$, which by (1.58) implies (1.57). This completes the proof of the lemma. □

We now give a characterization of ψ_V, which will be of use in the proof of Proposition 1.5.

1.7 Lemma. *For $x \in S_V$ we have*

$$\psi_V(x) = \frac{1}{2\pi^2}\frac{1}{w_{S_V}(x)}\int \frac{V'(x) - V'(y)}{x-y} d\omega_{S_V}(y), \tag{1.61}$$

where w_{S_V} is the density of ω_{S_V}.

Proof. By the Sokhotski-Plemelj formulas, see e.g. [52], we have that

$$F(z) = \int_{S_V} \frac{\psi_V(s)}{z-s} ds, \qquad \text{for } z \in \mathbb{C} \setminus S_V,$$

satisfies

$$\begin{aligned} F_+(x) + F_-(x) &= V'(x), & x \in S_V, \quad (1.62)\\ F_+(x) - F_-(x) &= -2\pi i \psi_V(x), & x \in S_V. \quad (1.63) \end{aligned}$$

Let $S_V = \bigcup_{j=1}^n [a_j, b_j]$ and set

$$R(z) = \left(\prod_{j=1}^n (z-b_j)(z-a_j) \right)^{1/2}, \qquad z \in \mathbb{C} \setminus S_V,$$

where the square root is positive for $z > b_n$. Using (1.62) and the fact that $R_+(x) = -R_-(x)$ as $x \in S_V$, we see that

$$\frac{p(x)F_+(x)}{R_+(x)} - \frac{p(x)F_-(x)}{R_-(x)} = \frac{p(x)V'(x)}{R_+(x)} \tag{1.64}$$

for any polynomial p. Suppose p has degree at most $n-1$. Then (1.64) and the fact that $\frac{p(z)F(z)}{R(z)} \to 0$ as $z \to \infty$ imply that

$$\frac{p(z)F(z)}{R(z)} = \frac{1}{2\pi i} \int_{S_V} \frac{p(s)V'(s)}{R_+(s)} \frac{1}{s-z} ds, \qquad z \in \mathbb{C} \setminus S_V$$

which we can rewrite for $z \in \mathcal{V}$ as

$$\begin{aligned} \frac{p(z)F(z)}{R(z)} &= \frac{1}{2\pi i} \int_{S_V} \frac{V'(s)-V'(z)}{s-z} \frac{p(s)}{R_+(s)} ds + \frac{V'(z)}{2\pi i} \int_{S_V} \frac{p(s)}{R_+(s)} \frac{1}{s-z} ds \\ &= \frac{1}{2\pi i} \int_{S_V} \frac{V'(z)-V'(s)}{z-s} \frac{p(s)}{R_+(s)} ds + \frac{V'(z)}{2} \frac{p(z)}{R(z)} \end{aligned} \tag{1.65}$$

by contour integration. Now we use (1.65) and (1.63) to obtain

$$\psi_V(x) = \frac{F_+(x) - F_-(x)}{-2\pi i} = \frac{1}{2\pi^2} \frac{R_+(x)}{p(x)} \int_{S_V} \frac{V'(x)-V'(y)}{x-y} \frac{p(y)}{R_+(y)} dy,$$

and this holds for any polynomial p of degree at most $n-1$. Since we know by (1.26) that

$$w_{S_V}(x) = \frac{i}{\pi} \frac{p(x)}{R_+(x)}, \qquad x \in S_V,$$

for some monic polynomial p of degree $n-1$, we get (1.61). □

Now we can give the proof of Proposition 1.5.

Proof of Proposition 1.5. By (1.48) we have for $z \in \mathcal{V}$,

$$t^2 Q_t(z) = \left(\frac{V'(z)}{2}\right)^2 - \int \frac{V'(z) - V'(y)}{z - y} d(t\nu_t)(y)$$

so that in view of (1.53) we find,

$$\frac{d}{dt}(t^2 Q_t(z))\Big|_{t=1} = -\int \frac{V'(z) - V'(y)}{z - y} d\omega_{S_V}(y).$$

Thus by (1.51) for $z \neq x^*$ in a neighborhood of x^*,

$$\begin{aligned}\frac{d}{dt}(t\psi_t(z))\Big|_{t=1} &= \frac{d}{dt}\frac{1}{\pi}(-t^2 Q_t(z))^{1/2}\Big|_{t=1} \\ &= \frac{1}{2\pi}(-Q_V(z))^{-1/2} \int \frac{V'(z) - V'(y)}{z - y} d\omega_{S_V}(y) \\ &= \frac{1}{2\pi^2}\frac{1}{\psi_V(z)} \int \frac{V'(z) - V'(y)}{z - y} d\omega_{S_V}(y). \end{aligned} \tag{1.66}$$

By (1.61), the right-hand side of (1.66) is $\omega_{S_V}(z)$ in case $z \neq x^*$ is real, and by analytic continuation it continues to be $\omega_{S_V}(z)$ in a punctured neighborhood of x^*. Thus (1.52) holds for $z \neq x^*$ in a neighborhood of x^*, and then it easily follows for $z = x^*$ as well. □

1.3 Riemann-Hilbert problem for Painlevé II

In this section we recall the RH problem associated with the second Painlevé equation, see [44, 62, 37, 6, 60]. We transform this RH problem into a RH problem that will be used in the next section to construct a parametrix around x^*.

1.3.1 ψ-functions associated with Painlevé II

Consider the differential equation of (1.21),

$$\frac{d}{d\zeta}\Psi(\zeta) = \begin{pmatrix} -4i\zeta^2 - i(s + 2q^2) & 4\zeta q + 2ir \\ 4\zeta q - 2ir & 4i\zeta^2 + i(s + 2q^2) \end{pmatrix} \Psi(\zeta), \tag{1.67}$$

where Ψ is a 2×2 complex matrix-valued function and s, q and r are considered as parameters. All solutions of (1.67) are entire functions of ζ.

For $j = 1, \ldots, 6$, let S_j be the sector

$$S_j = \{\zeta \in \mathbb{C} \mid \frac{2j-3}{6}\pi < \arg \zeta < \frac{2j-1}{6}\pi\}. \tag{1.68}$$

There exists a unique solution Ψ_j of equation (1.67) so that

$$\Psi_j(\zeta)e^{i(\frac{4}{3}\zeta^3+s\zeta)\sigma_3} = I + O(\zeta^{-1}) \tag{1.69}$$

as $\zeta \to \infty$ in the sector S_j. Here we use $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ to denote the third Pauli matrix. There exist complex values s_j, $j = 1, \ldots, 6$ (called Stokes multipliers) so that

$$\Psi_{j+1}(\zeta) = \Psi_j(\zeta)A_j, \quad \text{for } j = 1, \ldots, 5, \qquad \Psi_1(\zeta) = \Psi_6(\zeta)A_6. \tag{1.70}$$

with

$$A_j = \begin{pmatrix} 1 & 0 \\ s_j & 1 \end{pmatrix} \quad \text{if } j \text{ is odd}, \tag{1.71}$$

and

$$A_j = \begin{pmatrix} 1 & s_j \\ 0 & 1 \end{pmatrix} \quad \text{if } j \text{ is even}. \tag{1.72}$$

Furthermore, we have

$$s_{j+3} = s_j \qquad \text{and} \qquad s_1 s_2 s_3 + s_1 + s_2 + s_3 = 0. \tag{1.73}$$

Now define the rays

$$\Gamma_j = \{\zeta \mid \arg\zeta = \frac{2j-1}{6}\pi\} \quad \text{for } j = 1, \ldots, 6, \tag{1.74}$$

oriented away from the origin, and the matrix-valued function

$$\Psi(\zeta) := \Psi_j(\zeta), \qquad \text{for } \zeta \in S_j. \tag{1.75}$$

Then Ψ satisfies the following RH problem

(a) Ψ is analytic on $\mathbb{C} \setminus \bigcup_{j=1}^{6} \Gamma_j$.

(b) $\Psi_+ = \Psi_- A_j$ on Γ_j,

(c) $\Psi(\zeta)e^{i(\frac{4}{3}\zeta^3+s\zeta)\sigma_3} = I + O(\zeta^{-1})$ as $\zeta \to \infty$,

(d) Ψ is bounded near 0.

The Stokes multipliers s_j depend on s, q and r. An isomonodromy deformation is a variation of these parameters such that the Stokes multipliers remain constant. Flaschka and Newell [44] showed that the isomonodromy deformations are given by the Painlevé II equation $q''(s) = sq(s) + 2q^3(s)$ and $r(s) = q'(s)$.

Any solution of the Painlevé II equation is a meromorphic function with an infinite number of poles. We write $\Psi(\zeta; s)$ for the Ψ function (1.75) with parameters s, $q = q(s)$, $r = r(s)$, where q is the Hastings-McLeod solution and

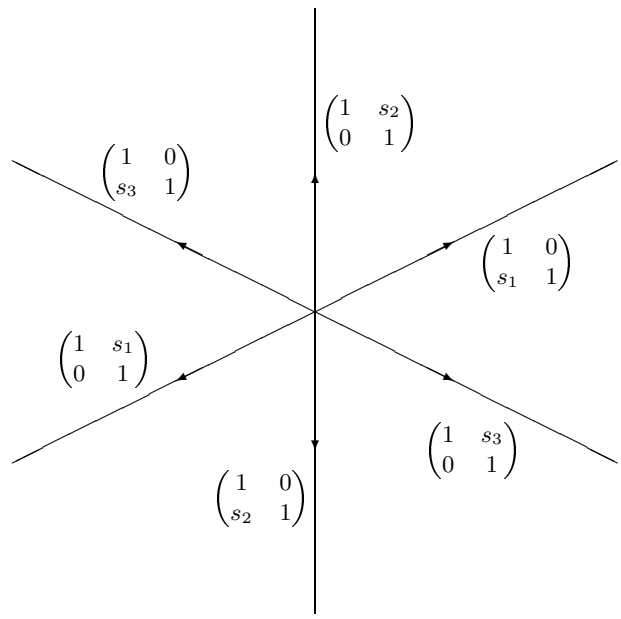

Figure 1.3 *Jumps for $\Psi(\zeta; s)$ for the Ψ-functions associated with the Painlevé II equation (general case).*

$r(s) = q'(s)$. Let $\mathcal{P}$ be the set of poles of q. Then $\Psi(\zeta; s)$ is defined and analytic for $\zeta \in \mathbb{C} \setminus \bigcup_{j=1}^{6} \Gamma_j$ and for $s \in \mathbb{C} \setminus \mathcal{P}$. It is known that there are no poles on the real line [57]. The Stokes multipliers corresponding to the Hastings-McLeod solution are

$$s_1 = 1, \quad s_2 = 0, \quad s_3 = -1.$$

Thus $\Psi(\zeta; s)$ is analytic across the imaginary axis. We reverse the orientation of Γ_3 and Γ_4, and we define

$$\Sigma_1 = \Gamma_1 \cup \Gamma_3, \quad \text{and} \quad \Sigma_2 = \Gamma_4 \cup \Gamma_6.$$

Then $\Psi(\zeta; s)$ solves the following RH problem, see also Figure 1.4.

(a) Ψ is analytic on $\mathbb{C} \setminus (\Sigma_1 \cup \Sigma_2)$,

(b) Ψ satisfies the following jump conditions on Σ_1 and Σ_2,

$$\Psi_+ = \Psi_- \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad \text{on } \Sigma_1,$$

$$\Psi_+ = \Psi_- \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \qquad \text{on } \Sigma_2,$$

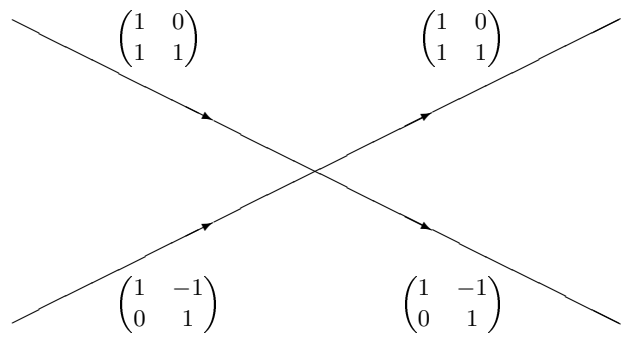

Figure 1.4 *Jumps for $\Psi(\zeta; s)$ for the special Ψ-function associated with the Hastings-McLeod solution of the Painlevé II equation.*

(c) $\Psi(\zeta; s)e^{i(\frac{4}{3}\zeta^3+s\zeta)\sigma_3} = I + O(\zeta^{-1})$ as $\zeta \to \infty$,

(d) Ψ is bounded near 0.

The RH problem has a solution if and only if $s \in \mathbb{C} \setminus \mathcal{P}$. The properties (c) and (d) are valid uniformly for s in compact subsets of $\mathbb{C} \setminus \mathcal{P}$.

1.8 Remark. We also note that $\Psi(\zeta; s)$ is jointly analytic in $\zeta \in \mathbb{C}$ and $s \in \mathbb{C} \setminus \mathcal{P}$. This follows from basic results on the analyticity of solutions of differential equations with analytic parameters, since Ψ satisfies both (1.21) and (1.22) where $q = q(s)$ and $r = q'(s)$ depend analytically on s.

1.3.2 Transformation of the RH problem

We modify the RH problem for Ψ so that it resembles the RH problem that we will need locally near x^*.

We introduce an additional parameter $\theta \in \mathbb{R}$ and define

$$M(\zeta; s, \theta) = \begin{cases} e^{i\theta\sigma_3}\Psi(\zeta; s)e^{i(\frac{4}{3}\zeta^3+s\zeta)\sigma_3}e^{-i\theta\sigma_3} & \text{for } \operatorname{Im}\zeta > 0, \\ e^{i\theta\sigma_3}\Psi(\zeta; s)e^{i(\frac{4}{3}\zeta^3+s\zeta)\sigma_3}e^{-i\theta\sigma_3}\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) & \text{for } \operatorname{Im}\zeta < 0. \end{cases} \tag{1.76}$$

For any given $s \in \mathbb{C} \setminus \mathcal{P}$ and $\theta \in \mathbb{R}$, we then have that M is defined for $\zeta \in \mathbb{C} \setminus (\mathbb{R} \cup \Sigma_1 \cup \Sigma_2)$, see Figure 1.5, and satisfies the following RH problem

(a) $M(\zeta; s, \theta)$ is analytic (both in ζ and s) for $\zeta \in \mathbb{C}\setminus(\mathbb{R}\cup\Sigma_1\cup\Sigma_2)$ and $s \in \mathbb{C}\setminus\mathcal{P}$, see also Remark 1.8.

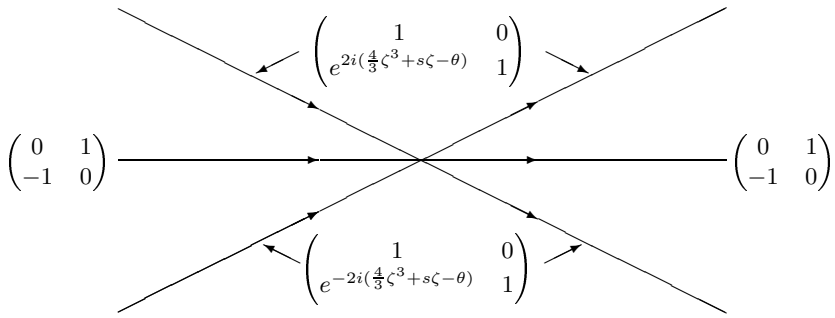

Figure 1.5 *Jumps for* $M(\zeta; s, \theta)$.

(b) M satisfies the following jump conditions,

$$M_+(\zeta; s, \theta) = M_-(\zeta; s, \theta) \begin{pmatrix} 1 & 0 \\ e^{2i(\frac{4}{3}\zeta^3+s\zeta-\theta)} & 1 \end{pmatrix} \qquad \text{for } \zeta \in \Sigma_1, \tag{1.77}$$

$$M_+(\zeta; s, \theta) = M_-(\zeta; s, \theta) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{for } \zeta \in \mathbb{R}, \tag{1.78}$$

$$M_+(\zeta; s, \theta) = M_-(\zeta; s, \theta) \begin{pmatrix} 1 & 0 \\ e^{-2i(\frac{4}{3}\zeta^3+s\zeta-\theta)} & 1 \end{pmatrix} \qquad \text{for } \zeta \in \Sigma_2. \tag{1.79}$$

(c) $M(\zeta; s, \theta) = I + O(\zeta^{-1})$ as $\zeta \to \infty$ in the upper half plane,
$M(\zeta; s, \theta) = (I + O(\zeta^{-1})) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ as $\zeta \to \infty$ in the lower half plane,

(d) $M(\zeta; s, \theta)$ is bounded for ζ near 0.

The properties (c) and (d) hold uniformly for s in compact subsets of $\mathbb{C} \setminus \mathcal{P}$ and for $\theta \in \mathbb{R}$.

1.4 Steepest descent analysis of Riemann-Hilbert problem

The proof of Theorem 1.1 is based on the steepest descent analysis of the Riemann-Hilbert problem for orthogonal polynomials.

Since the main point of the present discussion is the treatment of the critical point x^*, we will restrict ourselves to the one-interval case. We also assume that there are no other singular points besides x^*. Thus $S_V = [a, b]$ and ψ_V vanishes like a square root at a and b, and quadratically at x^* but at no other points of S_V. In addition, we assume that the inequality in the variational condition (1.42)

is strict for $x \in \mathbb{R} \setminus [a,b]$. We then have that

$$S_t = \operatorname{supp} \nu_t = [a_t, b_t] \tag{1.80}$$

for certain $a_t < b_t$ for every t close to $t = 1$. Note that a_t is an increasing and b_t a decreasing function of t.

We will comment below on the modifications that have to be made in the multi-interval case, see Remark 1.9.

1.4.1 RH problem for orthogonal polynomials

For each n and N, we consider a Riemann-Hilbert problem, introduced by Fokas, Its, and Kitaev [46, 47]. We will look for a 2×2 matrix-valued function $Y = Y_{n,N}$ (we drop the subscripts for simplicity) that satisfies the following conditions:

(a) Y is analytic in $\mathbb{C} \setminus \mathbb{R}$,

(b) $Y_+(x) = Y_-(x) \begin{pmatrix} 1 & e^{-NV(x)} \\ 0 & 1 \end{pmatrix}$ for $x \in \mathbb{R}$,

(c) $Y(z) = \big(I + O(z^{-1})\big) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$ as $z \to \infty$.

Here $Y_+(x)$ (resp. $Y_-(x)$) denotes the limit as we approach $x \in \mathbb{R}$ from the upper (resp. lower) half-plane. The RH problem possesses a unique solution which is given by

$$Y(z) = \begin{pmatrix} \kappa_{n,N}^{-1} p_n^{(N)}(z) & \kappa_{n,N}^{-1} \dfrac{1}{2\pi i} \displaystyle\int_{\mathbb{R}} \dfrac{p_n^{(N)}(s) e^{-NV(s)}}{s - z} ds \\ -2\pi i \kappa_{n-1,N} p_{n-1}^{(N)}(z) & -\kappa_{n-1,N} \displaystyle\int_{\mathbb{R}} \dfrac{p_{n-1}^{(N)}(s) e^{-NV(s)}}{s - z} ds \end{pmatrix} \tag{1.81}$$

where $p_n^{(N)}(x) = \kappa_{n,N} x^n + \cdots$ denotes the orthonormal polynomial as before, see also [27, 73].

The correlation kernel (1.2) can be expressed directly in terms of the solution of the RH problem. Indeed, by the Christoffel-Darboux formula for orthogonal polynomials

$$K_{n,N}(x,y) = e^{-\frac{N}{2}V(x)} e^{-\frac{N}{2}V(y)} \frac{\kappa_{n-1,N}}{\kappa_{n,N}} \frac{p_n^{(N)}(x) p_{n-1}^{(N)}(y) - p_{n-1}^{(N)}(x) p_n^{(N)}(y)}{x - y}$$

which involves the orthogonal polynomials of degrees n and $n-1$ only. Using (1.81) and the fact that $\det Y(z) = 1$ for every $z \in \mathbb{C} \setminus \mathbb{R}$, we then get

$$K_{n,N}(x,y) = e^{-\frac{N}{2}V(x)} e^{-\frac{N}{2}V(y)} \frac{1}{2\pi i (x-y)} \begin{pmatrix} 0 & 1 \end{pmatrix} Y_+^{-1}(y) Y_+(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{1.82}$$

Applying the Deift/Zhou steepest descent method to the RH problem will allow us to find the asymptotics of $K_{n,N}$ as given in Theorem 1.1.

We note that an expression like (1.82) was first given in the context of random matrices with external source [3, 16, 17] where the eigenvalue correlation kernel is given in terms of the solution of a 3×3 matrix-valued RH problem, and the expression similar to (1.82) was found to be very convenient for asymptotic analysis. Also in the present 2×2-case we find it helpful to work with (1.82).

1.4.2 Normalization of the RH problem at infinity: $Y \mapsto T$

In the first transformation we normalize the RH problem at infinity. A standard approach would be to use the equilibrium measure μ_t in external field V_t where $t = n/N$. If we would do this for the case where $t < 1$, we would have an equilibrium measure with a gap in the support around x^*, and an annoying consequence is that the equality in the variational conditions is not valid near x^*. For this reason we have introduced the signed measures ν_t in Section 1.2 and we will use these measures now to normalize the RH problem.

We let $t = n/N$ and assume t is sufficiently close to 1 so that (1.80) holds. As before we use

$$\psi_t := \frac{d\nu_t}{dx} \tag{1.83}$$

to denote the density of ν_t. Then we define the g-function

$$g_t(z) = \int \log(z-s)d\nu_t(s) = \int \log(z-s)\psi_t(s)ds, \tag{1.84}$$

where we take the branch cut of the logarithm along the negative real axis. Then the following properties of g_t are easy to check.

(i) $e^{g_t(z)}$ is analytic in $\mathbb{C} \setminus [a_t, b_t]$,

(ii) $e^{g_t(z)} = z + O(\frac{1}{z})$ as $z \to \infty$.

(iii) $g_{t+}(x) - g_{t-}(x) = 2\pi i \int_x^{b_t} \psi_t(s)ds$ for $x \in \mathbb{R}$,

(iv) $g_{t+}(x) + g_{t-}(x) - \frac{1}{t}V(x) + \ell_t \leq 0$ for $x \in \mathbb{R} \setminus [a_t, b_t]$,

(v) $g_{t+}(x) + g_{t-}(x) - \frac{1}{t}V(x) + \ell_t = 0$ for $x \in [a_t, b_t]$.

In terms of the analytic function Q_t, see (1.48) and Proposition 1.4, we have

$$g_{t+}(x) - g_{t-}(x) = -2\int_{b_t}^{x} (Q_t(s))_+^{1/2} ds, \qquad \text{for } x \in [a_t, b_t], \tag{1.85}$$

and

$$g_{t+}(x) + g_{t-}(x) - \frac{1}{t}V(x) + \ell_t = -2\int_{b_t}^{x} (Q_t(s))^{1/2} ds \qquad \text{for } x > b_t, \tag{1.86}$$

$$g_{t+}(x) + g_{t-}(x) - \frac{1}{t}V(x) + \ell_t = -2\int_{a_t}^{x} (Q_t(s))^{1/2} ds \qquad \text{for } x < a_t \tag{1.87}$$

where $(Q_t(s))^{1/2}$ is analytic for $s \in \mathcal{V} \setminus [a_t, b_t]$ and that square root is taken which is positive for large real s. We define

$$\varphi_t(z) = \int_{b_t}^{z} (Q_t(s))^{1/2} ds \tag{1.88}$$

and

$$\tilde{\varphi}_t(z) = \int_{a_t}^{z} (Q_t(s))^{1/2} ds \tag{1.89}$$

which are defined and analytic in the neighborhood $\mathcal{V}$ of the real line where V is analytic with cuts along $(-\infty, b_t)$ and $(a_t, +\infty)$, respectively. Note that Q_t has simple zeros in a_t and b_t and only double zeros in $\mathcal{V} \setminus [a_t, b_t]$, so that $Q_t^{1/2}$ is indeed analytic there. It is possible that Q_t has double real zeros, and that $Q_t^{1/2}$ has sign changes in $(-\infty, a_t)$ or (b_t, ∞). However, by (1.86) and (1.87) we have that $\varphi_t(x) > 0$ for $x > b_t$ and $\tilde{\varphi}_t(x) > 0$ for $x < a_t$, since we are in a situation with strict inequality in (1.40) for $x > b_t$ and $x < a_t$.

We will now perform the first transformation of the RH problem: we take $t = n/N$ and define

$$T(z) = e^{\frac{n}{2}\ell_t \sigma_3} Y(z) e^{-n g_t(z)\sigma_3} e^{-\frac{n}{2}\ell_t \sigma_3} \qquad \text{for } z \in \mathbb{C} \setminus \mathbb{R}. \tag{1.90}$$

Using the jump condition (b) in the RH problem of Y and (1.90), we easily check that $T_+(x) = T_-(x) v_T(x)$ for $x \in \mathbb{R}$, where

$$\begin{aligned} v_T(x) &= e^{\frac{n}{2}\ell_t \sigma_3} e^{n g_{t-}(x)\sigma_3} \begin{pmatrix} 1 & e^{-NV(x)} \\ 0 & 1 \end{pmatrix} e^{-n g_{t+}(x)\sigma_3} e^{-\frac{n}{2}\ell_t \sigma_3} \\ &= \begin{pmatrix} e^{-n(g_{t+}(x) - g_{t-}(x))} & e^{n(g_{t+}(x) + g_{t-}(x) - \frac{1}{t}V(x) + \ell_t)} \\ 0 & e^{n(g_{t+}(x) - g_{t-}(x))}. \end{pmatrix} \end{aligned} \tag{1.91}$$

Because of the properties (1.85)–(1.87) and the definitions (1.88)–(1.89), we see that the jump matrix v_T has the following forms on the respective intervals (a_t, b_t), (b_t, ∞), and $(-\infty, a_t)$,

$$v_T = \begin{pmatrix} e^{2n\varphi_{t+}} & 1 \\ 0 & e^{2n\varphi_{t-}} \end{pmatrix} \qquad \text{on } (a_t, b_t)$$

$$v_T = \begin{pmatrix} 1 & e^{-2n\varphi_t} \\ 0 & 1 \end{pmatrix} \qquad \text{on } (b_t, \infty)$$

$$v_T = \begin{pmatrix} 1 & e^{-2n\tilde{\varphi}_t} \\ 0 & 1 \end{pmatrix} \qquad \text{on } (-\infty, a_t).$$

Thus T is the unique solution of the RH problem

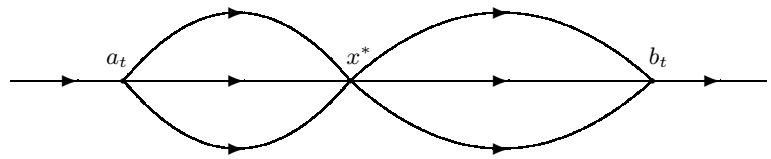

Figure 1.6 *The contour after the opening of the lens.*

(a) T is analytic in $\mathbb{C} \setminus \mathbb{R}$,

(b) $T_+(x) = T_-(x) v_T(x) \quad$ for $x \in \mathbb{R}$,

(c) $T(z) = I + O(z^{-1}) \quad$ as $z \to \infty$.

1.4.3 Opening of the lens: $T \mapsto S$

The jump matrix v_T on the interval (a_t, b_t) has the factorization

$$\begin{aligned} v_T &= \begin{pmatrix} e^{2n\varphi_{t+}} & 1 \\ 0 & e^{2n\varphi_{t-}} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ e^{2n\varphi_{t-}} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{2n\varphi_{t+}} & 1 \end{pmatrix}. \end{aligned} \tag{1.92}$$

Now we open lenses around the intervals (a_t, x^*) and (x^*, b_t) as shown in Figure 1.6. Let C_1 be the upper lips of the lenses and C_2 the lower lips, with orientation as in Figure 1.6. We open the lenses in such a way that they are fully contained in $\mathcal{V}$, the region of analyticity of V and Q_t. In addition, we can take C_1 and C_2 in such a way that $\operatorname{Re} \varphi_t < 0$ on C_1 and C_2, with the exception of a neighborhood of x^* if $t < 1$. This effect is due to the fact that ν_t has a negative density near x^* if $t < 1$. However, if $t < 1$ increases to one, the exceptional neighborhood shrinks to a point. It follows that for any given $\delta > 0$, there is a constant $\gamma > 0$ such that for t sufficiently close to 1 we have

$$\operatorname{Re} \varphi_t(z) < -\gamma < 0$$

for all $z \in C_1 \cup C_2$ with $\min(|z - x^*|, |z - a|, |z - b|) > \delta$.

We define

$$S = \begin{cases} T & \text{outside the lenses,} \\ T \begin{pmatrix} 1 & 0 \\ -e^{2n\varphi_t} & 1 \end{pmatrix} & \text{in upper parts of the lenses,} \\ T \begin{pmatrix} 1 & 0 \\ e^{2n\varphi_t} & 1 \end{pmatrix} & \text{in lower parts of the lenses.} \end{cases} \tag{1.93}$$

The RH problem for T and the factorization (1.92) imply that S solves the following RH problem:

(a) S is analytic in $\mathbb{C} \setminus (\mathbb{R} \cup C_1 \cup C_2)$,

(b) $S_+(z) = S_-(z) v_S(z) \quad$ for $z \in \mathbb{R} \cup C_1 \cup C_2$,

(c) $S(z) = I + O(z^{-1}) \quad$ as $z \to \infty$,

(d) S remains bounded near a_t, b_t, and x^*,

where v_S is given by

$$v_S = \begin{cases} \begin{pmatrix} 1 & e^{-2n\varphi_t} \\ 0 & 1 \end{pmatrix} & \text{on } (b_t, \infty), \\ \begin{pmatrix} 1 & e^{-2n\tilde{\varphi}_t} \\ 0 & 1 \end{pmatrix} & \text{on } (-\infty, a_t), \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{on } (a_t, b_t), \\ \begin{pmatrix} 1 & 0 \\ e^{2n\varphi_t(z)} & 1 \end{pmatrix} & \text{on } C_1 \cup C_2, \end{cases} \tag{1.94}$$

The jump matrices on $\mathbb{R} \setminus (a_t, b_t)$ and on $C_1 \cup C_2$ tend to the identity matrix as $n \to \infty$ and $t \to 1$. Ignoring these jumps, we find the parametrix for the outside region. Uniform convergence breaks down in neighborhoods of a_t, b_t, and x^*, so that we will also need to construct local parametrices near those points.

1.4.4 Parametrix $P^{(\infty)}$ for the outside region

The outside parametrix $P^{(\infty)} = P_t^{(\infty)}$ solves the following RH problem

(a) $P^{(\infty)}$ is analytic in $\mathbb{C} \setminus [a_t, b_t]$,

(b) $P_+^{(\infty)} = P_-^{(\infty)} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on (a_t, b_t),

(c) $P^{(\infty)}(z) = I + O(z^{-1})$ as $z \to \infty$.

As in [32] it has the solution

$$P_t^{(\infty)}(z) = \begin{pmatrix} \frac{\beta(z)+\beta(z)^{-1}}{2} & \frac{\beta(z)-\beta(z)^{-1}}{2i} \\ -\frac{\beta(z)-\beta(z)^{-1}}{2i} & \frac{\beta(z)+\beta(z)^{-1}}{2} \end{pmatrix}, \qquad z \in \mathbb{C} \setminus [a_t, b_t], \tag{1.95}$$

where

$$\beta(z) = \beta_t(z) = \left(\frac{z - b_t}{z - a_t}\right)^{1/4}, \qquad z \in \mathbb{C} \setminus [a_t, b_t]. \tag{1.96}$$

Note that $P_t^{(\infty)}$ depends on t.

1.4.5 Parametrices near edge points

At edge points a_t and b_t the density ψ_t vanishes like a square root. This allows the construction of local parametrices near a_t and b_t with the use of Airy functions. We will not give the details, see [27, 33, 32].

1.4.6 Parametrix near critical point

In a neighborhood $U_\delta = \{z \in \mathbb{C} \mid |x^* - z| < \delta\}$ of x^*, we want to construct a parametrix P with the following properties

(a) P is analytic in $U_\delta \setminus (\mathbb{R} \cup C_1 \cup C_2)$,

(b) $P_+ = P_- v_S(z)$ on $(\mathbb{R} \cup C_1 \cup C_2) \cap U_\delta$,

(c) $P(z) = (I + O(n^{-1/3}))P_t^{(\infty)}(z)$ as $n \to \infty$, $t \to 1$, uniformly for $z \in \partial U_\delta$,

(d) $P(z)$ remains bounded for z near x^*.

We seek a parametrix P near x^* in the form

$$P(z) = E_t(z) M\left(n^{1/3} f(z); n^{2/3} s_t(z), n\theta_t\right) \tag{1.97}$$

where E_t is analytic in U_δ, f is a conformal map from U_δ to a neighborhood of 0, s_t is analytic in U_δ, and θ_t is a real constant. Recall that M is given by (1.76), and since M is analytic as a function of both variables ζ and s, see part (a) in the RH problem for M in Section 1.3.2, we have that (1.97) is well-defined and piecewise analytic in U_δ with jumps on the contours where $f(z) \in \mathbb{R} \cup \Sigma_1 \cup \Sigma_2$, for any choice of analytic E_t, f, and s_t, provided that $n^{2/3} s_t(z) \in \mathbb{C} \setminus \mathcal{P}$ for every $z \in U_\delta$.

In view of the jump properties of M, we seek f, s_t, and θ_t so that for $z \in U_\delta$,

$$i\left(\frac{4}{3} f(z)^3 + s_t(z) f(z) - \theta_t\right) = \begin{cases} \varphi_t(z) & \text{if } \operatorname{Im} z > 0, \\ -\varphi_t(z) & \text{if } \operatorname{Im} z < 0. \end{cases} \tag{1.98}$$

Since $\varphi_{t+} = -\varphi_{t-}$, the right-hand side of (1.98) does indeed define an analytic function in U_δ.

The conformal map f will not depend on t. Recall that Q_V has a zero of order four at x^*, and that it is negative on $(a, b) \setminus \{x^*\}$. Thus $\int_{x^*}^z (-Q_V(y))^{1/2} dy$ is

analytic in U_δ and has a zero of order three at x^*. We may assume there are no other zeros in U_δ. Then we can take a third root and define

$$f(z) = \left[\frac{3}{4}\int_{x^*}^{z}(-Q_V(y))^{1/2}dy\right]^{1/3}, \qquad z \in U_\delta. \tag{1.99}$$

Then f is analytic in U_δ with

$$\begin{aligned} f(z) &= \left(\frac{\pi\psi_V''(x^*)}{8}\right)^{1/3}(z-x^*) + O\left((z-x^*)^2\right) \\ &= c^{1/3}(z-x^*) + O\left((z-x^*)^2\right), \qquad \text{as } z \to x^* \end{aligned} \tag{1.100}$$

where c is given by (1.28). Taking smaller δ if necessary, we then have that f is indeed a conformal map on U_δ. Note that $f(x^*) = 0$ and that f is real and positive on $(x^*, x^*+\delta)$ and real and negative on $(x^*-\delta, x^*)$. We still have some freedom in opening the lenses. We take C_1 and C_2 in δ so that f maps them to the rays where M has its jumps. That is, $C_1 \cap U_\delta$ is mapped into $\arg\zeta = \pi/6$ and $\arg\zeta = 5\pi/6$, and $C_2 \cap U_\delta$ is mapped into $\arg\zeta = -\pi/6$ and $\arg\zeta = -5\pi/6$.

Having f with $f(x^*) = 0$, we take $z = x^*$ in (1.98) and we see that we should take

$$\theta_t = i\varphi_{t+}(x^*) = -i\varphi_{t-}(x^*). \tag{1.101}$$

Then θ_t is real and it is also given by

$$\theta_t = -\int_{b_t}^{x^*}(-Q_t(y))^{1/2}dy$$

and so

$$\theta_t \mp i\varphi_t(z) = \int_{x^*}^{z}(-Q_t(y))^{1/2}dy, \qquad \text{for } \pm\operatorname{Im} z > 0. \tag{1.102}$$

Having f and θ_t we finally take $s_t(z)$ so that (1.98) holds, that is,

$$\begin{aligned} s_t(z)f(z) &= -\frac{4}{3}f(z)^3 + \theta_t \mp i\varphi_t(z) \\ &= \int_{x^*}^{z}\left((-Q_t(y))^{1/2} - (-Q_V(y))^{1/2}\right)dy \end{aligned} \tag{1.103}$$

where for the last line we used (1.99) and (1.102). Since the right-hand side of (1.103) is analytic in U_δ and vanishes for $z = x^*$ we can divide by $f(z)$ (which has a simple zero at $z = x^*$) and obtain an analytic function s_t in U_δ. Then $s_t(z)$ is real for real z, and $s_1(z) \equiv 0$.

So the above construction yields analytic functions f, s_t, and a constant θ_t so that (1.98) holds. Then we can define the parametrix P by (1.97), provided that

$n^{2/3}s_t(z)$ for $z \in U_\delta$ stays away from the poles of the Hastings-McLeod solution of Painlevé II, since for $s \in \mathcal{P}$ we have that $M(\zeta; s, \theta)$ is not defined. For given values of n and t, we can take δ small enough so that this is indeed the case. However, we want to let $n \to \infty$, $t \to 1$ so that $n^{2/3}(t-1) \to L$, and work with a neighborhood U_δ that is independent of n and t, although it may depend on L.

Note that by (1.103) and (1.51)

$$s_t(z)f(z) = \pi \int_{x^*}^{z} (\psi_t(y) - \psi_V(y))dy. \tag{1.104}$$

Because of (1.52) we have

$$\psi_t(y) - \psi_V(y) = (t-1)(w_{S_V}(y) - \psi_1(y)) + o(t-1),$$

as $t \to 1$, uniformly for y in a neighborhood of x^*. Using this in (1.104) we get that

$$\begin{aligned} s_t(z)\frac{f(z)}{z-x^*} &= \pi(t-1)\frac{1}{z-x^*}\int_{x^*}^{z}(w_{S_V}(y) - \psi_1(y))dy + o(t-1) \\ &= \pi(t-1)w_{S_V}(x^*) + (t-1)O(z-x^*) + o(t-1), \end{aligned} \tag{1.105}$$

where $o(t-1)$ is uniformly in z as $t \to 1$, and $O(z-x^*)$ is uniformly in t as $z \to x^*$. By (1.100) and (1.105) we then also have

$$s_t(z) = \pi c^{1/3}(t-1)w_{S_V}(x^*) + (t-1)O(z-x^*) + o(t-1), \tag{1.106}$$

where again $o(t-1)$ is uniformly in z as $t \to 1$, and $O(z-x^*)$ is uniformly in t as $z \to x^*$. Then if $n^{2/3}(t-1) \to L$ we get

$$\begin{aligned} n^{2/3}s_t(z) &= \pi c^{-1/3}Lw_{S_V}(x^*) + O(z-x^*) + o(1) \\ &= s + O(z-x^*) + o(1), \end{aligned} \tag{1.107}$$

where we used the definition (1.29) of the constant s. Since there are no poles on the real line, we can find a neighborhood U_δ of x^* such that $n^{2/3}s_t(z) \notin \mathcal{P}$ for all $z \in U_\delta$ if n is large enough and $n^{2/3}(t-1) \to L$. Note that δ depends on L, but not on n and t. Then

$$M(n^{1/3}f(z); n^{2/3}s_t(z), n\theta_t)$$

is well-defined for z in a fixed neighborhood U_δ of x^*.

Finally, we define E_t in such a way that the matching condition at ∂U_δ is satisfied. We do this by defining

$$E_t(z) = \begin{cases} P_t^{(\infty)}(z) & \text{for } \operatorname{Im} z > 0, \\ P_t^{(\infty)}(z)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{for } \operatorname{Im} z \le 0, \end{cases} \tag{1.108}$$

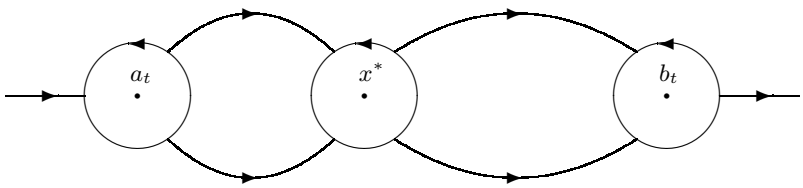

Figure 1.7 *The contour* Γ *after the third and final transformation.*

so that $E_t(z)$ is analytic near x^* by the jump of $P_t^{(\infty)}$ on (a_t, b_t), see Section 1.4.4. Because of the asymptotic behavior of $M(\zeta; s, \theta)$ given in Section 1.3.2 (which is valid uniformly for s away from the poles and for $\theta \in \mathbb{R}$) we have the matching condition

$$P(z) = P_t^{(\infty)}(z)(I + O(n^{-1/3})) = (I + O(n^{-1/3}))P_t^{(\infty)}(z),$$

uniformly for $z \in \partial U_\delta$. This completes the construction of the parametrix P in the neighborhood of x^*.

1.4.7 Final transformation: $S \mapsto R$

We set

$$R(z) = \begin{cases} S(z)P^{-1}(z) & \text{for } z \text{ in disks around } x^*, a, \text{ and } b, \\ S(z)(P_t^{(\infty)})^{-1}(z) & \text{for } z \text{ outside the disks.} \end{cases} \tag{1.109}$$

Since S and P have the same jumps inside each of the disks, and S and $P_t^{(\infty)}$ have the same jumps on $[a_t, b_t]$, R has jumps on a contour Γ as shown in Figure 1.7.

R solves the following RH problem:

(a) R is analytic in $\mathbb{C} \setminus \Gamma$,

(b) $R_+(z) = R_-(z)v_R(z)$ as $z \in \Gamma$,

(c) $R(z) = I + O(z^{-1})$ as $z \to \infty$,

(d) R is bounded,

for certain jump matrices v_R, which as $n \to \infty$, $t \to 1$ so that $n^{2/3}(t-1) \to L$, satisfy

$$v_R(z) = \begin{cases} I + O(n^{-1}) & \text{for } z \text{ in circles around } a \text{ and } b \\ I + O(n^{-1/3}) & \text{for } z \in \partial U_\delta, \\ I + O(e^{-\gamma n}) & \text{for some fixed } \gamma > 0 \text{ elsewhere on } \Gamma. \end{cases}$$

As in [27, 33] it now follows that

$$\|R(z) - I\|_\infty = O(n^{-1/3}) \tag{1.110}$$

uniformly for $z \in \mathbb{C} \setminus \Gamma$.

1.5 Proof of Theorem 1.1

Now we are ready for the proof of Theorem 1.1. We start with the expression (1.82) for the kernel $K_{n,N}$. After the transformation (1.90), we find that for $x, y \in (a_t, b_t)$,

$$K_{n,N}(x,y) = \frac{1}{2\pi i(x-y)} \begin{pmatrix} 0 & e^{-n\varphi_{t+}(y)} \end{pmatrix} T_+^{-1}(y) T_+(x) \begin{pmatrix} e^{-n\varphi_{t+}(x)} \\ 0 \end{pmatrix}. \tag{1.111}$$

Using formula (1.93) for S in the upper parts of the lenses, we get for $x, y \in (a_t, b_t)$,

$$K_{n,N}(x,y) = \frac{1}{2\pi i(x-y)} \begin{pmatrix} -e^{n\varphi_{t+}(y)} & e^{n\varphi_{t+}(y)} \end{pmatrix} \times S_+^{-1}(y) S_+(x) \begin{pmatrix} e^{-n\varphi_{t+}(x)} \\ e^{n\varphi_{t+}(x)} \end{pmatrix} \tag{1.112}$$

Assume that x and y are inside the disk U_δ around x^*. Then by (1.109), (1.97), (1.76), and (1.98) we get

$$S_+(x) = R(x) E_t(x) e^{in\theta_t \sigma_3} \Psi_+(n^{1/3} f(x); n^{2/3} s_t(x)) e^{n\varphi_{t+}(x)\sigma_3},$$

and so

$$S_+(x) \begin{pmatrix} e^{-n\varphi_{t+}(x)} \\ e^{n\varphi_{t+}(x)} \end{pmatrix} = R(x) E_t(x) e^{in\theta_t \sigma_3} \Psi(n^{1/3} f(x); n^{2/3} s_t(x)) \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{1.113}$$

Similarly

$$\begin{aligned} &\begin{pmatrix} -e^{n\varphi_{t+}(y)} & e^{n\varphi_{t+}(y)} \end{pmatrix} S_+^{-1}(y) \\ &\quad = \begin{pmatrix} -1 & 1 \end{pmatrix} \Psi^{-1}(n^{1/3} f(y); n^{2/3} s_t(y)) e^{-in\theta_t \sigma_3} E_t^{-1}(y) R^{-1}(y). \end{aligned} \tag{1.114}$$

Now we fix u and v and take

$$x = x^* + \frac{u}{(cn)^{1/3}} \quad \text{and} \quad y = x^* + \frac{v}{(cn)^{1/3}}, \tag{1.115}$$

so that for n large enough, x and y are inside the disk around x^*, so that (1.112), (1.113), and (1.114) hold. Then it follows from (1.100) and (1.115) that

$$n^{1/3} f(x) \to u, \quad \text{and} \quad n^{1/3} f(y) \to v \qquad \text{as } n \to \infty. \tag{1.116}$$

From (1.107) and (1.115) we get

$$n^{2/3}s_t(x) \to s, \quad \text{and} \quad n^{2/3}s_t(y) \to s \tag{1.117}$$

as $n \to \infty$, $t \to 1$ such that $n^{2/3}(t-1) \to L$. Furthermore, from (1.110), (1.115), and the fact that R is analytic near x^*, we get

$$R^{-1}(y)R(x) = I + O\left(\frac{x-y}{n^{1/3}}\right) = I + O\left(\frac{u-v}{n^{2/3}}\right). \tag{1.118}$$

From (1.95), (1.108), and (1.115) we easily get

$$E_t^{-1}(y)E_t(x) = I + O\left(\frac{u-v}{n^{1/3}}\right) \tag{1.119}$$

as $n \to \infty$. The constants implied by the O-symbols in (1.118) and (1.119) are independent of u and v, when u and v are restricted to a compact subset of $\mathbb{R}$. Combining (1.118) and (1.119) we get that

$$e^{-in\theta_t\sigma_3}E_t^{-1}(y)R^{-1}(y)R(x)E_t(x)e^{in\theta_t\sigma_3} = I + O\left(\frac{u-v}{n^{1/3}}\right), \tag{1.120}$$

since θ_t is real.

Then multiplying (1.114) and (1.113) and letting $n \to \infty$, $t \to 1$ such that $n^{2/3}(t-1) \to L$, we get by using (1.112), (1.115), (1.116), (1.117), and (1.120) that

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{(cn)^{1/3}} K_{n,N}\left(x^* + \frac{u}{(cn)^{1/3}}, x^* + \frac{v}{(cn)^{1/3}}\right) \\
&\quad = \frac{1}{2\pi i(u-v)} \lim_{n\to\infty} \begin{pmatrix} -e^{n\varphi_{t+}(y)} & e^{n\varphi_{t+}(y)} \end{pmatrix} S_+^{-1}(y)S_+(x) \begin{pmatrix} e^{-n\varphi_{t+}(x)} \\ e^{n\varphi_{t+}(x)} \end{pmatrix} \\
&\quad = \frac{1}{2\pi i(u-v)} \begin{pmatrix} -1 & 1 \end{pmatrix} \Psi^{-1}(v;s)\Psi(u;s) \begin{pmatrix} 1 \\ 1 \end{pmatrix},
\end{aligned} \tag{1.121}$$

uniformly for u and v in compact subsets of $\mathbb{R}$.

By (1.23), (1.69) and (1.75) we have that

$$\begin{pmatrix} \Phi_1(\zeta;s) \\ \Phi_2(\zeta;s) \end{pmatrix} = \Psi(\zeta;s) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{for } \zeta \in S_2 \cup S_3.$$

In view of the jump $\Psi_+ = \Psi_- \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ satisfied by $\Psi(\zeta;s)$ on Σ_1 we have that

$$\begin{pmatrix} \Phi_1(\zeta;s) \\ \Phi_2(\zeta;s) \end{pmatrix} = \Psi(\zeta;s) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{for } \zeta \in S_1 \cup S_4, \tag{1.122}$$

Since $\det \Psi \equiv 1$, we also get (after simple calculation)

$$\begin{pmatrix} -1 & 1 \end{pmatrix} \Psi^{-1}(\zeta;s) = \begin{pmatrix} -\Phi_2(\zeta;s) & \Phi_1(\zeta;s) \end{pmatrix} \quad \text{for } \zeta \in S_1 \cup S_4. \tag{1.123}$$

Note that $S_1 \cup S_4$ includes the full real line, so that we can take $\zeta = u$ in (1.122) and $\zeta = v$ in (1.123) which we use in (1.121) to obtain (1.27). This completes the proof of Theorem 1.1.

1.9 Remark. The above proof of Theorem 1.1 was given under the assumption that S_V consists of one interval and that there are no singular points except for x^*. Here we indicate briefly the modifications that have to be made if these assumptions are not satisfied.

The main complication in the multi-interval case is that the construction of the outside parametrix $P_t^{(\infty)}$ is more complicated, since it uses Θ-functions as in [32, Lemma 4.3]. It should be noted that $P_t^{(\infty)}$ will also depend on n. As a result it will follow that E_t as defined in (1.108) also depends on n. However this will not effect the asymptotic behavior (1.119) as $n \to \infty$, so that the above proof goes through.

If other singular points occur, we need local parametrices near those points. Since our focus is on a small neighborhood of x^*, it suffices to know the existence of an appropriate parametrix. In the case $n = N$, the existence of local parametrices is proven in [32, Section 5]. For $n \neq N$, we also believe that local parametrices exist in all singular cases, but we have not been able to find a rigorous proof of this fact.

Chapter 2

Universality of a double scaling limit near a multi-critical singularity and the general Painlevé II equation

Summary [1] We study unitary random matrix ensembles which are of the form $Z_{n,N}^{-1}|\det M|^{2\alpha}\exp(-N\operatorname{Tr}V(M))dM$, where $\alpha > -1/2$ and V is such that the limiting mean eigenvalue density for $n, N \to \infty$ and $n/N \to 1$ vanishes quadratically at the origin. In order to compute the double scaling limits of the eigenvalue correlation kernel near the origin, we use the Deift/Zhou steepest descent method applied to the Riemann-Hilbert problem for orthogonal polynomials on the real line with respect to the weight $|x|^{2\alpha}e^{-NV(x)}$. Here the main focus is on the construction of a local parametrix near the origin with ψ-functions associated with a special solution q_α of the Painlevé II equation $q'' = sq + 2q^3 - \alpha$. We show that q_α has no real poles for $\alpha > -1/2$, by proving the solvability of the corresponding Riemann-Hilbert problem. We also show that the asymptotics of the recurrence coefficients of the orthogonal polynomials can be expressed in terms of q_α in the double scaling limit.

[1] This chapter corresponds to the following paper [23]:

T. Claeys, A.B.J. Kuijlaars, and M. Vanlessen, Multi-critical unitary random matrix ensembles and the general Painlevé equation, arxiv:math-ph/0508062, to appear in *Annals of Mathematics*

2.1 Introduction and statement of results

2.1.1 Unitary random matrix ensembles

For $n \in \mathbb{N}$, $N > 0$, and $\alpha > -1/2$, we consider the unitary random matrix ensemble

$$Z_{n,N}^{-1} |\det M|^{2\alpha} \exp(-N \operatorname{Tr} V(M))\, dM, \tag{2.1}$$

on the space of $n \times n$ Hermitian matrices M, where $V : \mathbb{R} \to \mathbb{R}$ is a real analytic function satisfying

$$\lim_{x \to \pm\infty} \frac{V(x)}{\log(x^2+1)} = +\infty. \tag{2.2}$$

Because of (2.2) and $\alpha > -1/2$, the integral

$$Z_{n,N} = \int |\det M|^{2\alpha} \exp(-N \operatorname{Tr} V(M))\, dM \tag{2.3}$$

converges and the matrix ensemble (2.1) is well-defined. It is well known, see for example [27, 83], that the eigenvalues of M are distributed according to a determinantal point process with a correlation kernel given by

$$K_{n,N}(x,y) = |x|^\alpha e^{-\frac{N}{2}V(x)} |y|^\alpha e^{-\frac{N}{2}V(y)} \sum_{k=0}^{n-1} p_k^{(N)}(x) p_k^{(N)}(y), \tag{2.4}$$

where $p_k^{(N)} = \kappa_{k,N} x^k + \cdots$, $\kappa_{k,N} > 0$, denotes the k-th degree orthonormal polynomial with respect to the weight $|x|^{2\alpha} e^{-NV(x)}$ on $\mathbb{R}$.

Scaling limits of the kernel (2.4) as $n, N \to \infty$, $n/N \to 1$, show a remarkable universal behavior which is determined to a large extent by the limiting mean density of eigenvalues

$$\psi_V(x) = \lim_{n\to\infty} \frac{1}{n} K_{n,n}(x,x). \tag{2.5}$$

Indeed, for the case $\alpha = 0$, Bleher and Its [13] (for quartic V) and Deift et al. [32] (for general real analytic V) showed that the sine kernel is universal in the bulk of the spectrum, i.e.,

$$\lim_{n\to\infty} \frac{1}{n\psi_V(x_0)} K_{n,n}\left(x_0 + \frac{u}{n\psi_V(x_0)}, x_0 + \frac{v}{n\psi_V(x_0)}\right) = \frac{\sin \pi(u-v)}{\pi(u-v)}$$

whenever $\psi_V(x_0) > 0$. In addition, the Airy kernel appears generically at endpoints of the spectrum. If x_0 is a right endpoint and $\psi_V(x) \sim (x_0 - x)^{1/2}$ as $x \to x_0-$, then there exists a constant $c > 0$ such that

$$\lim_{n\to\infty} \frac{1}{cn^{2/3}} K_{n,n}\left(x_0 + \frac{u}{cn^{2/3}}, x_0 + \frac{v}{cn^{2/3}}\right) = \frac{\operatorname{Ai}(u)\operatorname{Ai}'(v) - \operatorname{Ai}'(u)\operatorname{Ai}(v)}{u-v},$$

where Ai denotes the Airy function, see also [30].

The extra factor $|\det M|^{2\alpha}$ in (2.1) introduces singular behavior at 0 if $\alpha \neq 0$. The pointwise limit (2.5) does not hold if $\psi_V(0) > 0$, since $K_{n,n}(0,0) = 0$ if $\alpha > 0$ and $K_{n,n}(0,0) = +\infty$ if $\alpha < 0$, due to the factor $|x|^\alpha |y|^\alpha$ in (2.4). However (2.5) continues to hold for $x \neq 0$ and also in the sense of weak* convergence of probability measures

$$\frac{1}{n} K_{n,n}(x,x)dx \xrightarrow{*} \psi_V(x)dx, \qquad \text{as } n \to \infty.$$

Therefore we can still call ψ_V the limiting mean density of eigenvalues. Observe that ψ_V does not depend on α.

However, at a microscopic level the introduction of the factor $|\det M|^{2\alpha}$ changes the eigenvalue correlations near the origin. Indeed, for the case of a non-critical V for which $\psi_V(0) > 0$, it was shown in [77] that

$$\lim_{n\to\infty} \frac{1}{n\psi_V(0)} K_{n,n}\left(\frac{u}{n\psi_V(0)}, \frac{v}{n\psi_V(0)}\right) = \pi\sqrt{u}\sqrt{v}\frac{J_{\alpha+\frac{1}{2}}(\pi u)J_{\alpha-\frac{1}{2}}(\pi v) - J_{\alpha-\frac{1}{2}}(\pi u)J_{\alpha+\frac{1}{2}}(\pi v)}{2(u-v)}, \tag{2.6}$$

where J_ν denotes the usual Bessel function of order ν.

2.1.2 Multi-critical case

It is the goal of this chapter to study (2.1) in a critical case where ψ_V vanishes quadratically at 0, i.e.,

$$\psi_V(0) = \psi_V'(0) = 0, \qquad \text{and} \qquad \psi_V''(0) > 0. \tag{2.7}$$

The behavior (2.7) is among the possible singular behaviors that were classified in [31]. The classification depends on the characterization of the measure $\psi_V(x)dx$ as the unique minimizer of the logarithmic energy

$$I_V(\mu) = \iint \log\frac{1}{|x-y|} d\mu(x)d\mu(y) + \int V(x)d\mu(x) \tag{2.8}$$

among all probability measures μ on $\mathbb{R}$. The corresponding Euler-Lagrange variational conditions give that for some constant $\ell \in \mathbb{R}$,

$$2\int \log|x-y|\psi_V(y)dy - V(x) + \ell = 0, \qquad \text{for } x \in \operatorname{supp}\psi_V, \tag{2.9}$$

$$2\int \log|x-y|\psi_V(y)dy - V(x) + \ell \leq 0, \qquad \text{for } x \in \mathbb{R}. \tag{2.10}$$

In addition one has that ψ_V is supported on a finite union of disjoint intervals, and

$$\psi_V(x) = \frac{1}{\pi}\sqrt{Q_V^-(x)}, \tag{2.11}$$

where Q_V is a real analytic function, and Q_V^- denotes its negative part. Note that the endpoints of the support correspond to zeros of Q_V with odd multiplicity.

The possible singular behaviors are as follows, see [31, 74].

Singular case I Equality holds in the variational inequality (2.10) for some $x \in \mathbb{R} \setminus \operatorname{supp} \psi_V$.

Singular case II ψ_V vanishes at an interior point of supp ψ_V, which corresponds to a zero of Q_V in the interior of the support. The multiplicity of such a zero is necessarily a multiple of 4.

Singular case III ψ_V vanishes at an endpoint to higher order than a square root. This corresponds to a zero of the function Q_V in (2.11) of odd multiplicity $4k+1$ with $k \geq 1$. (The multipicity $4k+3$ cannot occur in these matrix models.)

In each of the above cases, V is called singular, otherwise regular. The above conditions correspond to a singular exterior point, a singular interior point, and a singular endpoint, respectively.

In each of the singular cases one expects a family of possible limiting kernels in a double scaling limit as $n, N \to \infty$ and $n/N \to 1$ at some critical rate [12]. As said before we consider the case (2.7) which corresponds to the singular case II with $k = 1$ at the singular point $x = 0$. For technical reasons we assume that there are no other singular points besides 0. Setting $t = n/N$, and letting $n, N \to \infty$ such that $t \to 1$, we have that the parameter t describes the transition from the case where $\psi_V(0) > 0$ (for $t > 1$) through the multi-critical case ($t = 1$) to the case where 0 lies in a gap between two intervals of the spectrum ($t < 1$). The appropriate double scaling limit will be such that the limit $\lim_{n,N\to\infty} n^{2/3}(t-1)$ exists.

The double scaling limit for $\alpha = 0$ was considered in [6, 14, 15] for certain special cases, and in [22] (which corresponds with Chapter 1) in general. The limiting kernel is built out of ψ-functions associated with the Hastings-McLeod solution [57] of the Painlevé II equation $q'' = sq + 2q^3$.

For general $\alpha > -1/2$, we are led to the general Painlevé II equation

$$q'' = sq + 2q^3 - \alpha. \tag{2.12}$$

The Painlevé II equation for general α has been suggested by the physical papers [2, 92]. The limiting kernels in the double scaling limit are associated with a special distinguished solution of (2.12), which we describe first. We assume from now on that $\alpha \neq 0$.

2.1.3 General Painlevé II equation

Balancing sq and α in the differential equation (2.12), we find that there exist solutions such that

$$q(s) \sim \frac{\alpha}{s}, \qquad \text{as } s \to +\infty, \tag{2.13}$$

and balancing sq and $2q^3$, we see that there also exist solutions of (2.12) such that

$$q(s) \sim \sqrt{\frac{-s}{2}}, \qquad \text{as } s \to -\infty. \tag{2.14}$$

There is exactly one solution of (2.12) that satisfies both (2.13) and (2.14), see [60, 61, 68], and we denote it by q_α. This is the special solution that we need. It corresponds to the choice of Stokes multipliers

$$s_1 = e^{-\pi i\alpha}, \qquad s_2 = 0, \qquad s_3 = -e^{\pi i\alpha},$$

see Section 2 below. We call q_α the Hastings-McLeod solution of the general Painlevé II equation (2.12), since it seems to be the natural analogue of the Hastings-McLeod solution for $\alpha = 0$.

The Hastings-McLeod solution is meromorphic in s (as are all solutions of (2.12)) with an infinite number of poles. We need that it has no poles on the real line. From the asymptotic behavior (2.13) and (2.14) we know that there are no real poles for $|s|$ large enough, but that does not exclude the possibility of a finite number of real poles. While there is a substantial literature on Painlevé equations and Painlevé transcendents, we have not been able to find the following result.

2.1 Theorem. *Let q_α be the Hastings-McLeod solution of the general Painlevé II equation* (2.12) *with $\alpha > -1/2$. Then q_α is a meromorphic function with no poles on the real line.*

2.1.4 Main result

To describe our main result, we recall the notion of ψ-functions associated with the Painlevé II equation, see [44]. The Painlevé II equation (2.12) is the compatibility condition for the following system of linear differential equations for $\Psi = \Psi_\alpha(\zeta; s)$, which is called the Lax pair for the Painlevé II equation.

$$\frac{\partial \Psi}{\partial \zeta} = A\Psi, \qquad \frac{\partial \Psi}{\partial s} = B\Psi, \tag{2.15}$$

where

$$A = \begin{pmatrix} -4i\zeta^2 - i(s+2q^2) & 4\zeta q + 2ir + \alpha/\zeta \\ 4\zeta q - 2ir + \alpha/\zeta & 4i\zeta^2 + i(s+2q^2) \end{pmatrix}, \qquad B = \begin{pmatrix} -i\zeta & q \\ q & i\zeta \end{pmatrix}. \tag{2.16}$$

That is, (2.15) has a solution where $q = q(s)$ and $r = r(s)$ depend on s but not on ζ, if and only if q satisfies Painlevé II and $r = q'$.

Given s, q and r, the solutions of

$$\frac{\partial}{\partial \zeta}\begin{pmatrix}\Phi_1(\zeta)\\ \Phi_2(\zeta)\end{pmatrix} = A\begin{pmatrix}\Phi_1(\zeta)\\ \Phi_2(\zeta)\end{pmatrix} \tag{2.17}$$

are analytic with branch point at $\zeta = 0$. For $\alpha > -1/2$ and $s \in \mathbb{R}$, we take $q = q_\alpha(s)$ and $r = q_\alpha'(s)$ where q_α is the Hastings-McLeod solution of the Painlevé II equation, and we define $\begin{pmatrix}\Phi_{\alpha,1}(\zeta;s)\\ \Phi_{\alpha,2}(\zeta;s)\end{pmatrix}$ as the unique solution of (2.17) with asymptotics

$$e^{i(\frac{4}{3}\zeta^3+s\zeta)}\begin{pmatrix}\Phi_{\alpha,1}(\zeta;s)\\ \Phi_{\alpha,2}(\zeta;s)\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix} + \mathcal{O}(\zeta^{-1}), \tag{2.18}$$

uniformly as $\zeta \to \infty$ in the sector $\varepsilon < \arg\zeta < \pi - \varepsilon$ for any $\varepsilon > 0$. Note that this is well-defined for every $s \in \mathbb{R}$ because of Theorem 2.1.

The functions $\Phi_{\alpha,1}$ and $\Phi_{\alpha,2}$ extend to analytic functions on $\mathbb{C} \setminus (-i\infty, 0]$, which we also denote by $\Phi_{\alpha,1}$ and $\Phi_{\alpha,2}$, see also Remark 2.62 below. Their values on the real line appear in the limiting kernel. The following is the main result of this chapter.

2.2 Theorem. *Let V be real analytic on $\mathbb{R}$ such that* (2.2) *holds. Suppose that ψ_V vanishes quadratically in the origin, i.e., $\psi_V(0) = \psi_V'(0) = 0$, and $\psi_V''(0) > 0$, and that there are no other singular points besides* 0*. Let $n, N \to \infty$ such that*

$$\lim_{n,N\to\infty} n^{2/3}(n/N - 1) = L \in \mathbb{R}$$

exists. Define constants

$$c = \left(\frac{\pi\psi_V''(0)}{8}\right)^{1/3}, \tag{2.19}$$

and

$$s = 2\pi^{2/3} L\,[\psi_V''(0)]^{-1/3}\, w_{S_V}(0), \tag{2.20}$$

where w_{S_V} is the equilibrium density of the support of ψ_V (see Remark 2.3 *below). Then*

$$\lim_{n,N\to\infty} \frac{1}{cn^{1/3}} K_{n,N}\left(\frac{u}{cn^{1/3}}, \frac{v}{cn^{1/3}}\right) = \mathbb{K}_\alpha^{\mathrm{crit,II}}(u,v;s), \tag{2.21}$$

uniformly for u, v in compact subsets of $\mathbb{R} \setminus \{0\}$, where

$$\mathbb{K}_\alpha^{\mathrm{crit,II}}(u,v;s) = -e^{\frac{1}{2}\pi i\alpha[\mathrm{sgn}(u)+\mathrm{sgn}(v)]} \times \frac{\Phi_{\alpha,1}(u;s)\Phi_{\alpha,2}(v;s) - \Phi_{\alpha,1}(v;s)\Phi_{\alpha,2}(u;s)}{2\pi i(u-v)}. \tag{2.22}$$

2.3 Remark. The equilibrium measure of $S_V = \operatorname{supp} \psi_V$ is the unique probability measure ω_{S_V} on S_V that minimizes the logarithmic energy

$$I(\mu) = \iint \log \frac{1}{|x-y|} d\mu(x) d\mu(y)$$

among all probability measures on S_V. Since S_V consists of a finite union of intervals, and since 0 is an interior point of one of these intervals, ω_{S_V} has a density w_{S_V} with respect to Lebesgue measure, and $w_{S_V}(0) > 0$. This number is used in (2.20).

2.4 Remark. One can refine the calculations of Section 2.4 to obtain the following stronger result:

$$\frac{1}{cn^{1/3}} K_{n,N}\left(\frac{u}{cn^{1/3}}, \frac{v}{cn^{1/3}}\right) = \mathbb{K}_\alpha^{\mathrm{crit,II}}(u,v;s) + \mathcal{O}\left(\frac{|u|^\alpha |v|^\alpha}{n^{1/3}}\right), \tag{2.23}$$

uniformly for u, v in bounded subsets of $\mathbb{R} \setminus \{0\}$.

2.5 Remark. It is not immediate from the expression (2.22) that $\mathbb{K}_\alpha^{\mathrm{crit,II}}$ is real. This property follows from the symmetry

$$e^{\frac{1}{2}\pi i \alpha \operatorname{sgn}(u)} \Phi_{\alpha,2}(u;s) = \overline{e^{\frac{1}{2}\pi i \alpha \operatorname{sgn}(u)} \Phi_{\alpha,1}(u;s)}, \qquad \text{for } u \in \mathbb{R} \setminus \{0\},$$

which leads to the "real formula"

$$\mathbb{K}_\alpha^{\mathrm{crit,II}}(u,v;s) = -\frac{1}{\pi(u-v)} \operatorname{Im}\left(e^{\frac{1}{2}\pi i \alpha (\operatorname{sgn}(u) - \operatorname{sgn}(v))} \Phi_{\alpha,1}(u;s) \overline{\Phi_{\alpha,1}(v;s)}\right),$$

see Remark 2.21 below.

2.6 Remark. For $\alpha = 0$, we proved the theorem already in Chapter 1. The proof for the general case follows along similar lines, but we need the information about the existence of $q_\alpha(s)$ for real s, as guaranteed by Theorem 2.1.

2.1.5 Recurrence coefficients for orthogonal polynomials

In order to prove Theorem 2.2, we will study the Riemann-Hilbert (RH) problem for orthogonal polynomials with respect to the weight $|x|^{2\alpha} e^{-NV(x)}$. This analysis leads to asymptotics for the kernel $K_{n,N}$, but also provides the ingredients to derive asymptotics for the orthogonal polynomials and for the coefficients in the recurrence relation that is satisfied by them.

To state these results we introduce measures ν_t in the following way, see also Chapter 1 and Section 2.3.2. Take $\delta_0 > 0$ sufficiently small and let ν_t be the minimizer of $I_{V/t}(\nu)$ (see (2.8) for the definition of I_V) among all measures $\nu = \nu^+ - \nu^-$, where $\nu^\pm$ are nonnegative measures on $\mathbb{R}$ such that $\nu(\mathbb{R}) = 1$ and $\operatorname{supp} \nu^- \subset [-\delta_0, \delta_0]$. We use ψ_t to denote the density of ν_t.

We restrict ourselves to the one-interval case without singular points except for 0. Then supp $\psi_V = [a,b]$ and supp $\psi_t = [a_t, b_t]$ for t close to 1, where a_t and b_t are real analytic functions of t.

We write $p_k^{(N)}$ for the orthonormal polynomial of degree n with respect to the weight $|x|^{2\alpha}e^{-NV(x)}$. Those polynomials satisfy a three-term recurrence relation

$$xp_k^{(N)}(x) = a_{k+1}^{(N)}p_{k+1}^{(N)}(x) + b_k^{(N)}p_k^{(N)}(x) + a_k^{(N)}p_{k-1}^{(N)}(x), \tag{2.24}$$

with recurrence coefficients $a_k^{(N)}$ and $b_k^{(N)}$. In the large n expansion of $a_n^{(N)}$ and $b_n^{(N)}$, we observe oscillations in the $\mathcal{O}(n^{-1/3})$-term. The amplitude of the oscillations is proportional to $q_\alpha(s)$, while in general the frequency of the oscillations slowly varies with $t = n/N$.

2.7 Theorem. *Let the conditions of Theorem* 2.2 *be satisfied and suppose in addition that* supp $\psi_V = [a,b]$ *consists of one single interval. Consider the three-term recurrence relation* (2.24) *for the orthonormal polynomials* $p_k^{(N)}$ *with respect to the weight* $|x|^{2\alpha}e^{-NV(x)}$. *Then as* $n, N \to \infty$ *such that* $n/N - 1 = \mathcal{O}(n^{-2/3})$, *we have*

$$a_n^{(N)} = \frac{b-a}{4} - \frac{q_\alpha(s_{t,n})\cos(2\pi n\omega_t + 2\alpha\theta)}{2c}n^{-1/3} + \mathcal{O}(n^{-2/3}), \tag{2.25}$$

$$b_n^{(N)} = \frac{b+a}{2} + \frac{q_\alpha(s_{t,n})\sin(2\pi n\omega_t + (2\alpha+1)\theta)}{c}n^{-1/3} + \mathcal{O}(n^{-2/3}), \tag{2.26}$$

where $t = n/N$, c *is given by* (2.19),

$$s_{t,n} = n^{2/3}\frac{\pi}{c}\psi_t(0), \tag{2.27}$$

$$\theta = \arcsin\frac{b+a}{b-a}, \tag{2.28}$$

and

$$\omega_t = \int_0^{b_t} \psi_t(x)dx. \tag{2.29}$$

2.8 Remark. We have also shown in Chapter 1 that $\frac{d}{dt}\psi_t(0)\big|_{t=1} = w_{S_V}(0)$, which in the situation of Theorem 2.7 implies that (since $S_V = [a,b]$ and $\psi_t(0)$ is real analytic as a function of t near $t = 1$),

$$\psi_t(0) = (t-1)\frac{1}{\pi\sqrt{-ab}} + \mathcal{O}((t-1)^2), \qquad \text{as } t \to 1.$$

Then it follows from (2.27) that $s_{t,n} = n^{2/3}(t-1)\frac{1}{c\sqrt{-ab}} + \mathcal{O}(n^{-2/3})$ and we could in fact replace $s_{t,n}$ in (2.25) and (2.26) by

$$s_{t,n}^* = n^{2/3}(t-1)\frac{1}{c\sqrt{-ab}}.$$

We prefer to use $s_{t,n}$ since it appears more naturally from our analysis.

2.9 Remark. In [14], Bleher and Its derived (2.25) in the case where $\alpha = 0$ and where V is a critical even quartic polynomial. They also computed the $\mathcal{O}(n^{-2/3})$-term in the large n expansion for $a_n^{(N)}$. For even V we have that $a = -b$, $\theta = 0$, $\omega_t = 1/2$ and thus $\cos(2\pi n\omega_t + 2\alpha\theta) = (-1)^n$, so that (2.25) reduces to

$$a_n^{(N)} = \frac{b}{2} - \frac{q_\alpha(s_{t,n})(-1)^n}{2c} n^{-1/3} + \mathcal{O}(n^{-2/3}),$$

which is in agreement with the result of [14]. Also for even V the recurrence coefficient $b_n^{(N)}$ vanishes which is in agreement with (2.26).

2.10 Remark. In [12] an ansatz was made about the recurrence coefficients associated with a general (not necessarily even) critical quartic polynomial V in the case $\alpha = 0$. For fixed large N, the ansatz agrees with (2.25) and (2.26) up to an N-dependent phase shift in the trigonometric functions.

2.1.6 Outline of the rest of the chapter

In Section 2.2, we comment on the RH problem associated with the Painlevé II equation. We also prove the existence of a solution to this RH problem for real values of the parameter s, and this existence provides the proof of Theorem 2.1. In Section 2.3, we state the RH problem for orthogonal polynomials and apply the Deift/Zhou steepest descent method. Our main focus will be the construction of a local parametrix near the origin. For this construction, we will use the RH problem from Section 2.2. In Section 2.4 and Section 2.5 finally, we use the results obtained in Section 2.3 to prove Theorem 2.2 and Theorem 2.7.

2.2 RH problem for Painlevé II and the proof of Theorem 2.1

As before, we assume $\alpha > -1/2$.

2.2.1 Statement of RH problem

Let $\Sigma = \bigcup_j \Gamma_j$ be the contour consisting of four straight rays oriented to infinity,

$$\Gamma_1 : \arg\zeta = \frac{\pi}{6}, \qquad \Gamma_2 : \arg\zeta = \frac{5\pi}{6}, \qquad \Gamma_3 : \arg\zeta = -\frac{5\pi}{6}, \qquad \Gamma_4 : \arg\zeta = -\frac{\pi}{6}.$$

The contour Σ divides the complex plane into four regions $S_1, \ldots, S_4$ as shown in Figure 2.1. For $\alpha > -1/2$ and $s \in \mathbb{C}$, we seek a 2×2 matrix valued function $\Psi_\alpha(\zeta; s) = \Psi_\alpha(\zeta)$ (we suppress notation of s for brevity) satisfying the following.

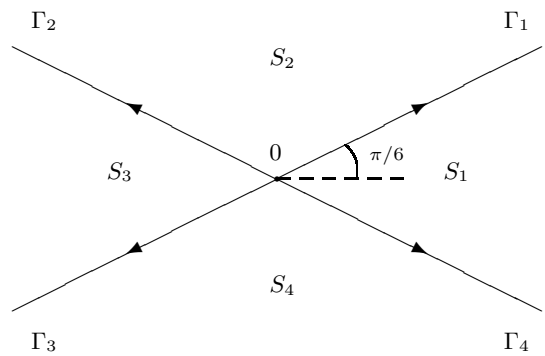

Figure 2.1 *The contour Σ consisting of four straight rays oriented to infinity.*

RH problem for Ψ_α:

(a) Ψ_α is analytic in $\mathbb{C} \setminus \Sigma$.

(b) Ψ_α satisfies the following jump relations on $\Sigma \setminus \{0\}$,

$$\Psi_{\alpha,+}(\zeta) = \Psi_{\alpha,-}(\zeta) \begin{pmatrix} 1 & 0 \\ e^{-\pi i\alpha} & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_1, \tag{2.30}$$

$$\Psi_{\alpha,+}(\zeta) = \Psi_{\alpha,-}(\zeta) \begin{pmatrix} 1 & 0 \\ -e^{\pi i\alpha} & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_2, \tag{2.31}$$

$$\Psi_{\alpha,+}(\zeta) = \Psi_{\alpha,-}(\zeta) \begin{pmatrix} 1 & e^{-\pi i\alpha} \\ 0 & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_3, \tag{2.32}$$

$$\Psi_{\alpha,+}(\zeta) = \Psi_{\alpha,-}(\zeta) \begin{pmatrix} 1 & -e^{\pi i\alpha} \\ 0 & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_4. \tag{2.33}$$

(c) Ψ_α has the following behavior at infinity,

$$\Psi_\alpha(\zeta) = (I + \mathcal{O}(1/\zeta))e^{-i(\frac{4}{3}\zeta^3 + s\zeta)\sigma_3}, \qquad \text{as } \zeta \to \infty. \tag{2.34}$$

Here $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ denotes the third Pauli matrix.

(d) Ψ_α has the following behavior near the origin. If $\alpha < 0$,

$$\Psi_\alpha(\zeta) = \mathcal{O}\begin{pmatrix} |\zeta|^\alpha & |\zeta|^\alpha \\ |\zeta|^\alpha & |\zeta|^\alpha \end{pmatrix}, \qquad \text{as } \zeta \to 0, \tag{2.35}$$

and if $\alpha \geq 0$,

$$\Psi_\alpha(\zeta) = \begin{cases} \mathcal{O}\begin{pmatrix} |\zeta|^{-\alpha} & |\zeta|^{-\alpha} \\ |\zeta|^{-\alpha} & |\zeta|^{-\alpha} \end{pmatrix}, & \text{as } \zeta \to 0,\ \zeta \in S_1 \cup S_3, \\ \mathcal{O}\begin{pmatrix} |\zeta|^{\alpha} & |\zeta|^{-\alpha} \\ |\zeta|^{\alpha} & |\zeta|^{-\alpha} \end{pmatrix}, & \text{as } \zeta \to 0,\ \zeta \in S_2, \\ \mathcal{O}\begin{pmatrix} |\zeta|^{-\alpha} & |\zeta|^{\alpha} \\ |\zeta|^{-\alpha} & |\zeta|^{\alpha} \end{pmatrix}, & \text{as } \zeta \to 0,\ \zeta \in S_4. \end{cases} \tag{2.36}$$

Note that Ψ_α depends on s only through the asymptotic condition (2.34).

2.11 Remark. This RH problem is a generalization of the RH problem for the case where $\alpha = 0$, used in [6, 22].

2.12 Remark. By standard arguments based on Liouville's theorem, see e.g. [27, 75], it can be verified that the solution of this RH problem, if it exists, is unique. Here it is important that $\alpha > -1/2$.

In the following we need more information on the behavior of solutions of the RH problem near 0. To this end, we make use of the following proposition, cf. [61]. We use G_j to denote the jump matrix of Ψ_α on Γ_j as given by (2.30)–(2.33).

2.13 Proposition. *Let Ψ satisfy conditions* (a), (b), *and* (d) *of the RH problem for Ψ_α.*

(1) *If $\alpha - \frac{1}{2} \notin \mathbb{N}_0$, then there exists an analytic matrix-valued function E and constant matrices A_j such that*

$$\Psi(\zeta) = E(\zeta) \begin{pmatrix} \zeta^{-\alpha} & 0 \\ 0 & \zeta^{\alpha} \end{pmatrix} A_j, \qquad \text{for } \zeta \in S_j, \tag{2.37}$$

where the branch cut of ζ^α is chosen along Γ_4. The matrices A_j satisfy

$$A_{j+1} = A_j G_j, \qquad \text{for } j = 1, 2, 3, \tag{2.38}$$

and

$$A_2 = \begin{pmatrix} 0 & -p^{-1} \\ p & \frac{p}{2\cos(\pi\alpha)} \end{pmatrix}, \qquad \text{for some } p \in \mathbb{C} \setminus \{0\}. \tag{2.39}$$

(2) *If $\alpha - \frac{1}{2} \in \mathbb{N}_0$, then there is logarithmic behavior of Ψ at the origin. There exists an analytic matrix-valued function E and constant matrices A_j such that*

$$\Psi(\zeta) = E(\zeta) \begin{pmatrix} \zeta^{-\alpha} & 0 \\ \frac{1}{\pi}\zeta^{\alpha}\ln\zeta & \zeta^{\alpha} \end{pmatrix} A_j, \qquad \text{for } \zeta \in S_j, \tag{2.40}$$

where again the branch cuts of ζ^α *and* $\ln\zeta$ *are chosen along* Γ_4*. The matrices* A_j *satisfy*

$$A_{j+1} = A_j G_j, \qquad \text{for } j = 1, 2, 3, \tag{2.41}$$

and for some $p \in \mathbb{C}$*,*

$$A_2 = \begin{cases} \begin{pmatrix} 0 & -1 \\ 1 & p \end{pmatrix}, & \text{if } \alpha - \frac{1}{2} \text{ is even,} \\ \begin{pmatrix} 0 & i \\ i & p \end{pmatrix}, & \text{if } \alpha - \frac{1}{2} \text{ is odd.} \end{cases} \tag{2.42}$$

Proof. (1) Define E by equation (2.37) with matrices A_j satisfying (2.38) and (2.39). Then E is analytic across Γ_1, Γ_2, and Γ_3 because of (2.38). For $\zeta \in \Gamma_4$ there is a jump

$$E_+(\zeta) = E_-(\zeta) \begin{pmatrix} \zeta^{-\alpha} & 0 \\ 0 & \zeta^\alpha \end{pmatrix}_- A_4 G_4 A_1^{-1} \begin{pmatrix} \zeta^\alpha & 0 \\ 0 & \zeta^{-\alpha} \end{pmatrix}_+ . \tag{2.43}$$

Using $\zeta_-^\alpha = e^{2\pi i\alpha}\zeta_+^\alpha$ and the explicit expressions for the matrices G_j and A_j, we get from (2.43) that E is analytic across Γ_4 as well.

What remains to be shown is that the possible isolated singularity of E at the origin is removable. If $\alpha < 0$ it follows from (2.35) and (2.37) that

$$E(\zeta) = \mathcal{O}\begin{pmatrix} |\zeta|^{2\alpha} & 1 \\ |\zeta|^{2\alpha} & 1 \end{pmatrix}, \qquad \text{as } \zeta \to 0,$$

so that (since $2\alpha > -1$) the isolated singularity at the origin is indeed removable. If $\alpha > 0$ we have in sector S_2 by (2.36), (2.37), and (2.39) that

$$\begin{aligned} E(\zeta) &= \Psi(\zeta) A_2^{-1} \begin{pmatrix} \zeta^\alpha & 0 \\ 0 & \zeta^{-\alpha} \end{pmatrix} \\ &= \mathcal{O}\begin{pmatrix} |\zeta|^\alpha & |\zeta|^{-\alpha} \\ |\zeta|^\alpha & |\zeta|^{-\alpha} \end{pmatrix} \begin{pmatrix} * & * \\ * & 0 \end{pmatrix} \begin{pmatrix} \zeta^\alpha & 0 \\ 0 & \zeta^{-\alpha} \end{pmatrix} \\ &= \mathcal{O}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \end{aligned}$$

as $\zeta \to 0$, $\zeta \in S_2$, where $*$ denotes an unimportant constant. Hence the singularity at the origin is not a pole. Moreover, from (2.36) and (2.37) it is also easy to check that E does not have an essential singularity at the origin either. Therefore the singularity is removable for the case $\alpha > 0$ as well, and the proof of part (1) is complete.

(2) The proof of part (2) is similar. □

2.14 Remark. The matrix A_2 in Proposition 2.13 is called the connection matrix, cf. [44, 49]. In all cases we have $\det A_2 = 1$ and the $(1,1)$-entry of A_2 is zero.

2.2.2 Solvability of the RH problem for Ψ_α

We are going to prove that the RH problem for Ψ_α is solvable for every $s \in \mathbb{R}$. We do that, as in [32, 49, 104], by showing that every solution of the homogeneous RH problem is identically zero. Such a result is known as a vanishing lemma [48, 49].

We briefly indicate why the vanishing lemma is enough to establish the solvability of the RH problem for Ψ_α. The RH problem is equivalent to a singular integral equation on the contour Σ. The singular integral equation can be stated in operator theoretic terms, and the operator is a Fredholm operator of zero index. The vanishing lemma yields that the kernel is trivial, and so the operator is onto which implies that the singular integral equation is solvable, and therefore the RH problem is solvable. For more details and other examples of this procedure see [32, 49, 104] and [65, Appendix A].

2.15 Proposition. *(**vanishing lemma**) Let $\alpha > -1/2$ and $s \in \mathbb{R}$. Suppose that $\widehat{\Psi}$ satisfies conditions* (a), (b), *and* (d) *of the RH problem for Ψ_α with the following asymptotic condition (instead of condition* (c)*)*

$$\widehat{\Psi}(\zeta)e^{i(\frac{4}{3}\zeta^3+s\zeta)\sigma_3} = \mathcal{O}(1/\zeta), \qquad \text{as } \zeta \to \infty. \tag{2.44}$$

Then $\widehat{\Psi} \equiv 0$.

Proof. As before, we use G_j to denote the jump matrix of Γ_j, given by (2.30)–(2.33). Introduce an auxiliary matrix-valued function H with jumps only on $\mathbb{R}$, as follows.

$$H(\zeta) = \begin{cases} \widehat{\Psi}(\zeta)e^{i(\frac{4}{3}\zeta^3+s\zeta)\sigma_3}, & \text{for } \zeta \in S_2 \cup S_4, \\ \widehat{\Psi}(\zeta)G_1 e^{i(\frac{4}{3}\zeta^3+s\zeta)\sigma_3}, & \text{for } \zeta \in S_1 \cap \mathbb{C}_+, \\ \widehat{\Psi}(\zeta)G_2^{-1} e^{i(\frac{4}{3}\zeta^3+s\zeta)\sigma_3}, & \text{for } \zeta \in S_3 \cap \mathbb{C}_+, \\ \widehat{\Psi}(\zeta)G_3 e^{i(\frac{4}{3}\zeta^3+s\zeta)\sigma_3}, & \text{for } \zeta \in S_3 \cap \mathbb{C}_-, \\ \widehat{\Psi}(\zeta)G_4^{-1} e^{i(\frac{4}{3}\zeta^3+s\zeta)\sigma_3}, & \text{for } \zeta \in S_1 \cap \mathbb{C}_-. \end{cases} \tag{2.45}$$

Then H satisfies the following RH problem.

RH problem for H:

(a) $H : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{2\times 2}$ is analytic and satisfies the following jump relations on $\mathbb{R} \setminus \{0\}$,

$$H_+(\zeta) = H_-(\zeta) e^{-i(\frac{4}{3}\zeta^3 + s\zeta)\sigma_3} \begin{pmatrix} 0 & -e^{-\pi i\alpha} \\ e^{\pi i\alpha} & 1 \end{pmatrix} e^{i(\frac{4}{3}\zeta^3 + s\zeta)\sigma_3}, \quad \text{for } \zeta \in (-\infty, 0), \tag{2.46}$$

$$H_+(\zeta) = H_-(\zeta) e^{-i(\frac{4}{3}\zeta^3 + s\zeta)\sigma_3} \begin{pmatrix} 0 & -e^{\pi i\alpha} \\ e^{-\pi i\alpha} & 1 \end{pmatrix} e^{i(\frac{4}{3}\zeta^3 + s\zeta)\sigma_3}, \quad \text{for } \zeta \in (0, \infty). \tag{2.47}$$

(b) $H(\zeta) = \mathcal{O}(1/\zeta), \qquad \text{as } \zeta \to \infty.$

(c) H has the following behavior near the origin: If $\alpha < 0$,

$$H(\zeta) = \mathcal{O}\begin{pmatrix} |\zeta|^\alpha & |\zeta|^\alpha \\ |\zeta|^\alpha & |\zeta|^\alpha \end{pmatrix}, \qquad \text{as } \zeta \to 0, \tag{2.48}$$

and if $\alpha > 0$,

$$H(\zeta) = \begin{cases} \mathcal{O}\begin{pmatrix} |\zeta|^\alpha & |\zeta|^{-\alpha} \\ |\zeta|^\alpha & |\zeta|^{-\alpha} \end{pmatrix}, & \text{as } \zeta \to 0, \operatorname{Im}\zeta > 0, \\ \mathcal{O}\begin{pmatrix} |\zeta|^{-\alpha} & |\zeta|^\alpha \\ |\zeta|^{-\alpha} & |\zeta|^\alpha \end{pmatrix}, & \text{as } \zeta \to 0, \operatorname{Im}\zeta < 0. \end{cases} \tag{2.49}$$

The jumps in (a) follow from straightforward calculation. The vanishing behavior (b) of H at infinity (in all sectors) follows from the triangular shape of the jump matrices G_j, see (2.30)–(2.33). For example, for $\zeta \in S_1 \cap \mathbb{C}_+$ we have $\operatorname{Re} i(\frac{4}{3}\zeta^3 + s\zeta) < 0$ so that by (2.44) and (2.45)

$$H(\zeta) = \mathcal{O}(1/\zeta) \begin{pmatrix} 1 & 0 \\ e^{-\pi i\alpha} e^{2i(\frac{4}{3}\zeta^3 + s\zeta)} & 1 \end{pmatrix} = \mathcal{O}(1/\zeta), \qquad \text{as } \zeta \to \infty.$$

The behavior near the origin in (c) follows from Proposition 2.13. This is immediate for (2.48), while for $\alpha > 0$, $\alpha - \frac{1}{2} \notin \mathbb{N}_0$, we have by (2.37), (2.38), (2.39), and (2.45),

$$H(\zeta) e^{-i(\frac{4}{3}\zeta^3 + s\zeta)\sigma_3} = \begin{cases} E(\zeta)\zeta^{-\alpha\sigma_3} A_2 = E(\zeta)\zeta^{-\alpha\sigma_3} \begin{pmatrix} 0 & * \\ * & * \end{pmatrix}, & \text{if } \operatorname{Im}\zeta > 0, \\ E(\zeta)\zeta^{-\alpha\sigma_3} A_4 = E(\zeta)\zeta^{-\alpha\sigma_3} \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}, & \text{if } \operatorname{Im}\zeta < 0, \end{cases}$$

which yields (2.49) in case $\alpha - \frac{1}{2} \notin \mathbb{N}_0$, since E is analytic. Using (2.42) instead of (2.39), we will see that the same argument works if $\alpha - \frac{1}{2} \in \mathbb{N}_0$.

Next we define, cf. [32, 49, 104],

$$M(\zeta) = H(\zeta)H(\bar{\zeta})^*, \qquad \text{for } \zeta \in \mathbb{C} \setminus \mathbb{R}, \tag{2.50}$$

where H^* denotes the Hermitian conjugate of H. From condition (c) of the RH problem for H it follows that M has the following behavior near the origin

$$M(\zeta) = \begin{cases} \mathcal{O}\begin{pmatrix} |\zeta|^{2\alpha} & |\zeta|^{2\alpha} \\ |\zeta|^{2\alpha} & |\zeta|^{2\alpha} \end{pmatrix}, & \text{as } \zeta \to 0, \text{ in case } \alpha < 0, \\ \mathcal{O}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \text{as } \zeta \to 0, \text{ in case } \alpha > 0. \end{cases}$$

Since $\alpha > -1/2$, it follows that each entry of M has an integrable singularity at the origin. Because $M(\zeta) = \mathcal{O}(1/\zeta^2)$ as $\zeta \to \infty$, and M is analytic in the upper half plane, it then follows by Cauchy's theorem that $\int_{\mathbb{R}} M_+(\zeta)d\zeta = 0$, and hence by (2.50)

$$\int_{\mathbb{R}} H_+(\zeta)H_-(\zeta)^* d\zeta = 0.$$

Adding this equation to its Hermitian conjugate, we find

$$\int_{\mathbb{R}} \left[H_+(\zeta)H_-(\zeta)^* + H_-(\zeta)H_+(\zeta)^*\right] d\zeta = 0. \tag{2.51}$$

Using (2.46), (2.47) and the fact that $(e^{i(\frac{4}{3}\zeta^3+s\zeta)\sigma_3})^* = e^{-i(\frac{4}{3}\zeta^3+s\zeta)\sigma_3}$ for $\zeta, s \in \mathbb{R}$, (here we use the fact that s is real!), we obtain from (2.51),

$$0 = \int_{\mathbb{R}} H_-(\zeta) \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} H_-(\zeta)^* d\zeta = 2\int_{\mathbb{R}} \left[|(H_-)_{12}(\zeta)|^2 + |(H_-)_{22}(\zeta)|^2\right] d\zeta.$$

This implies that the second column of H_- is identically zero. The jump relations (2.46) and (2.47) of H then imply that the first column of H_+ is identically zero as well.

To show that the second column of H_+ and the first column of H_- are also identically zero, we use an idea of Deift et al. [32, Proof of Theorem 5.3, Step 3]. Since the second column of H_- is identically zero, the jump relations (2.46) and (2.47) for H yield for $j = 1, 2$,

$$(H_{j2})_+(\zeta) = -e^{\operatorname{sgn}(\zeta)\pi i\alpha} e^{-2i(\frac{4}{3}\zeta^3+s\zeta)} (H_{j1})_-(\zeta), \qquad \text{for } \zeta \in \mathbb{R} \setminus \{0\}.$$

Thus if we define for $j = 1, 2$,

$$h_j(\zeta) = \begin{cases} H_{j2}(\zeta), & \text{if } \operatorname{Im}\zeta > 0, \\ H_{j1}(\zeta), & \text{if } \operatorname{Im}\zeta < 0, \end{cases} \tag{2.52}$$

then both h_1 and h_2 satisfy the following RH problem for a scalar function h.

RH problem for h:

(a) h is analytic on $\mathbb{C} \setminus \mathbb{R}$ and satisfies the following jump relation

$$h_+(\zeta) = -e^{\operatorname{sgn}(\zeta)\pi i\alpha} e^{-2i(\frac{4}{3}\zeta^3+s\zeta)} h_-(\zeta), \qquad \text{for } \zeta \in \mathbb{R} \setminus \{0\},$$

(b) $h(\zeta) = \mathcal{O}(1/\zeta)$ as $\zeta \to \infty$.

(c) $h(\zeta) = \begin{cases} \mathcal{O}(|\zeta|^{\alpha}), & \text{as } \zeta \to 0, \text{ in case } \alpha < 0, \\ \mathcal{O}(|\zeta|^{-\alpha}), & \text{as } \zeta \to 0, \text{ in case } \alpha > 0. \end{cases}$

Take ζ_0 with $\operatorname{Im}\zeta_0 < -1$ and define

$$\widehat{h}(\zeta) = \begin{cases} \frac{\zeta^{\alpha}}{(\zeta-\zeta_0)^{\alpha}} h(\zeta), & \text{if } \operatorname{Im}\zeta > 0, \\ \frac{\zeta^{\alpha}}{(\zeta-\zeta_0)^{\alpha}} \left(-e^{\pi i\alpha} e^{-2i(\frac{4}{3}\zeta^3+s\zeta)}\right) h(\zeta), & \text{if } -1 < \operatorname{Im}\zeta < 0, \end{cases} \tag{2.53}$$

where we use principal branches of the powers, so that ζ^{α} is defined with a branch cut along the negative real axis. Then it is easy to check that $\widehat{h}$ is analytic in $\operatorname{Im}\zeta > -1$, continuous and uniformly bounded in $\operatorname{Im}\zeta \geq -1$, and

$$\widehat{h}(\zeta) = \mathcal{O}(e^{-3|\operatorname{Re}\zeta|^2}), \qquad \text{as } \zeta \to \infty \text{ on the horizontal line } \operatorname{Im}\zeta = -1.$$

By Carlson's theorem, see e.g. [89], this implies that $\widehat{h} \equiv 0$, so that $h \equiv 0$, as well. This in turn implies that $h_1 \equiv 0$ and $h_2 \equiv 0$, so that $H \equiv 0$. Then also $\widehat{\Psi} \equiv 0$ and the proposition is proven. □

As noted before, Proposition 2.15 has the following consequence.

2.16 Corollary. *The RH problem for Ψ_{α}, see Section 2.1, has a unique solution for every $s \in \mathbb{R}$ and $\alpha > -1/2$.*

2.2.3 Proof of Theorem 2.1

Theorem 2.1 follows from the connection of the RH problem for Ψ_{α} of Section 2.1 with the RH problem associated with the general Painlevé II equation (2.12) as first described by Flaschka and Newell [44, Section 3D].

Proof of Theorem 2.1. Consider the matrix differential equation

$$\frac{\partial \Psi}{\partial \zeta} = A\Psi, \tag{2.54}$$

where A is as in (2.16) and s, q, and r are constants. For every $k = 0, 1, \ldots, 5$, there is a unique solution Ψ_k of (2.54) such that $\Psi_k(\zeta) = (I + \mathcal{O}(1/\zeta)) e^{-i(\frac{4}{3}\zeta^3+s\zeta)\sigma_3}$ as $\zeta \to \infty$ in the sector $(2k-1)\frac{\pi}{6} < \arg\zeta < (2k+1)\frac{\pi}{6}$. The function

$$\Psi(\zeta) = \Psi_k(\zeta), \qquad \text{for } (2k-1)\tfrac{\pi}{6} < \arg\zeta < (2k+1)\tfrac{\pi}{6}, \tag{2.55}$$

is then defined on $\mathbb{C} \setminus (\Sigma \cup i\mathbb{R})$ and satisfies the following conditions.

(a) Ψ is analytic in $\mathbb{C} \setminus (\Sigma \cup i\mathbb{R})$.

(b) There exist constants $s_1, s_2, s_3 \in \mathbb{C}$ (Stokes multipliers) satisfying

$$s_1 + s_2 + s_3 + s_1 s_2 s_3 = -2i \sin \pi\alpha \tag{2.56}$$

such that the following jump conditions hold, where all rays are oriented to infinity,

$$\Psi_+ = \begin{cases} \Psi_- \begin{pmatrix} 1 & 0 \\ s_1 & 1 \end{pmatrix}, & \text{on } \Gamma_1, \\ \Psi_- \begin{pmatrix} 1 & s_2 \\ 0 & 1 \end{pmatrix}, & \text{on } i\mathbb{R}^+, \\ \Psi_- \begin{pmatrix} 1 & 0 \\ s_3 & 1 \end{pmatrix}, & \text{on } \Gamma_2, \end{cases} \qquad \Psi_+ = \begin{cases} \Psi_- \begin{pmatrix} 1 & s_1 \\ 0 & 1 \end{pmatrix}, & \text{on } \Gamma_3, \\ \Psi_- \begin{pmatrix} 1 & 0 \\ s_2 & 1 \end{pmatrix}, & \text{on } i\mathbb{R}^-, \\ \Psi_- \begin{pmatrix} 1 & s_3 \\ 0 & 1 \end{pmatrix}, & \text{on } \Gamma_4. \end{cases}$$

(c) $\Psi(\zeta) = (I + \mathcal{O}(1/\zeta)) e^{-i(\frac{4}{3}\zeta^3 + s\zeta)\sigma_3}, \qquad \text{as } \zeta \to \infty.$

The Stokes multipliers s_1, s_2, s_3 depend on s, q and r. However, if $q = q(s)$ satisfies the second Painlevé equation $q'' = sq + 2q^3 - \alpha$, and if $r = q'(s)$, then the Stokes multipliers are constant. In this way there is a one-to-one correspondence between solutions of the Painlevé II equation and Stokes multipliers s_1, s_2, s_3 satisfying (2.56). This also means that there exists a solution of the above RH problem which is built out of solutions of (2.54) if and only if s is not a pole of the Painlevé II function that corresponds to the Stokes multipliers s_1, s_2, s_3. The Painlevé II function itself may then be recovered from the RH problem by the formula [44]

$$q(s) = \lim_{\zeta \to \infty} 2i\zeta \Psi_{12}(\zeta) e^{-i(\frac{4}{3}\zeta^3 + s\zeta)},$$

with Ψ_{12} the $(1,2)$-entry of Ψ. In particular, condition (c) of the RH problem can be strengthened to the following formula as $\zeta \to \infty$

$$\Psi(\zeta) = \left(I + \frac{1}{2i\zeta} \begin{pmatrix} u(s) & q(s) \\ -q(s) & -u(s) \end{pmatrix} + \mathcal{O}(1/\zeta^2) \right) e^{-i(\frac{4}{3}\zeta^3 + s\zeta)\sigma_3}, \tag{2.57}$$

where $u = (q')^2 - sq^2 - q^4 + 2\alpha q$.

The RH problem for Ψ_α in Section 2.1 corresponds to

$$s_1 = e^{-\pi i\alpha}, \qquad s_2 = 0, \qquad s_3 = -e^{\pi i\alpha}. \tag{2.58}$$

These Stokes multipliers are very special in two respects [60, 68]. First, since $s_2 = 0$, the corresponding solution of the Painlevé II equation decays as $s \to +\infty$, i.e.,

$$q(s) \sim \frac{\alpha}{s}, \qquad \text{as } s \to +\infty. \tag{2.59}$$

Secondly, since $s_1 s_3 = -1$ the Painlevé II solution increases as $s \to -\infty$, i.e.,

$$q(s) \sim \pm\sqrt{-\frac{s}{2}}, \qquad \text{as } s \to -\infty. \tag{2.60}$$

where the choice $s_1 = e^{-\pi i\alpha}$, $s_3 = -e^{\pi i\alpha}$ corresponds to the +sign, while the interchange of s_1 and s_3 corresponds to the $-$ sign in (2.60). Thus the special choice (2.58) corresponds to q_α, the Hastings-McLeod solution of the general Painlevé II equation, see (2.13) and (2.14).

Then as a consequence of the fact that the RH problem for Ψ_α stated in Section 2.1 is solvable for every real s by Corollary 2.16, we conclude that q_α has no poles on the real line, which proves Theorem 2.1. □

2.17 Remark. Its and Kapaev [60] use a slightly modified, but equivalent, version of the RH problem for Ψ_α. The solutions are connected by the transformation

$$\Psi_\alpha \leftrightarrow e^{\frac{\pi i}{4}\sigma_3}\Psi_\alpha e^{-\frac{\pi i}{4}\sigma_3}, \tag{2.61}$$

which results in a transformation of the Stokes multipliers $s_j \leftrightarrow (-1)^j i s_j$.

For later use, we record the following corollary.

2.18 Corollary. *For every fixed $s_0 \in \mathbb{R}$, there exists an open neighborhood U of s_0 such that the RH problem for Ψ_α is solvable for every $s \in U$.*

Proof. Since q_α is meromorphic in $\mathbb{C}$, there is an open neighborhood of s_0 without poles. This implies [44] that the RH problem for Ψ_α is solvable for every s in that open neighborhood of s_0, as well. □

2.19 Remark. The function $\Psi_\alpha(\zeta; s)$ is analytic as a function of both $\zeta \in \mathbb{C} \setminus \Sigma$ and $s \in \mathbb{C}\setminus\mathcal{P}_\alpha$, where $\mathcal{P}_\alpha$ denotes the set of poles of q_α, see [44]. As a consequence, one can check that (2.34), (2.35) and (2.36) hold uniformly for s in compact subsets of $\mathbb{C} \setminus \mathcal{P}_\alpha$.

2.20 Remark. The functions $\Phi_{\alpha,1}$ and $\Phi_{\alpha,2}$ defined by (2.15) and (2.18) are connected with Ψ_α as follows. Define

$$\Phi_\alpha(\zeta; s) = \begin{cases} \Psi_\alpha(\zeta; s)\begin{pmatrix} 1 & 0 \\ e^{-\pi i\alpha} & 1 \end{pmatrix}, & \text{for } \zeta \in S_1, \\ \Psi_\alpha(\zeta; s), & \text{for } \zeta \in S_2, \\ \Psi_\alpha(\zeta; s)\begin{pmatrix} 1 & 0 \\ e^{\pi i\alpha} & 1 \end{pmatrix}, & \text{for } \zeta \in S_3, \\ \Psi_\alpha(\zeta; s)\begin{pmatrix} 1 & -e^{\pi i\alpha} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ e^{-\pi i\alpha} & 1 \end{pmatrix}, & \text{for } \zeta \in S_4,\ \operatorname{Re}\zeta > 0, \\ \Psi_\alpha(\zeta; s)\begin{pmatrix} 1 & -e^{-\pi i\alpha} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ e^{\pi i\alpha} & 1 \end{pmatrix}, & \text{for } \zeta \in S_4,\ \operatorname{Re}\zeta < 0. \end{cases} \tag{2.62}$$

Then it follows from the RH problem for Ψ_α that Φ_α is analytic on $\mathbb{C} \setminus (-i\infty, 0]$. Moreover, we also see from (2.15) and (2.18) that

$$\Phi_\alpha = \begin{pmatrix} \Phi_{\alpha,1} & * \\ \Phi_{\alpha,2} & * \end{pmatrix}, \tag{2.63}$$

where $*$ denotes an unspecified unimportant entry. It also follows that $\Phi_{\alpha,1}$ and $\Phi_{\alpha,2}$ have analytic continuations to $\mathbb{C} \setminus (-i\infty, 0]$.

2.21 Remark. We show that the kern $\mathbb{K}_\alpha^{\mathrm{crit,II}}(u, v; s)$ is real. This will follow from the identity

$$e^{\frac{1}{2}\pi i\alpha \mathrm{sgn}(u)}\Phi_{\alpha,2}(u; s) = \overline{e^{\frac{1}{2}\pi i\alpha \mathrm{sgn}(u)}\Phi_{\alpha,1}(u; s)}, \qquad \text{for } u, s \in \mathbb{R}, u \neq 0, \tag{2.64}$$

since obviously this implies that

$$\mathbb{K}_\alpha^{\mathrm{crit,II}}(u, v; s) = -\frac{1}{\pi(u-v)} \mathrm{Im}\left(e^{\frac{1}{2}\pi i\alpha(\mathrm{sgn}(u)-\mathrm{sgn}(v))}\Phi_{\alpha,1}(u; s)\overline{\Phi_{\alpha,1}(v; s)}\right).$$

The identity (2.64) will follow from the RH problem. It is easy to check that $\sigma_1\overline{\Psi_\alpha(\overline{\zeta}; s)}\sigma_1$, with $\sigma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, also satisfies the RH conditions for Ψ_α. Because of the uniqueness of the solution of the RH problem, this implies

$$\Psi_\alpha(\zeta; s) = \sigma_1\overline{\Psi_\alpha(\overline{\zeta}; s)}\sigma_1. \tag{2.65}$$

For $\zeta \in S_4$, the equality of the $(2, 1)$ entries of (2.65) yields by (2.62) and (2.63)

$$e^{\pi i\alpha}\Phi_{\alpha,2}(\zeta; s) = \overline{\Phi_{\alpha,1}(\zeta; s)}, \qquad \text{for } \zeta \in S_4,\ \mathrm{Re}\,\zeta > 0, \tag{2.66}$$

and

$$e^{-\pi i\alpha}\Phi_{\alpha,2}(\zeta; s) = \overline{\Phi_{\alpha,1}(\zeta; s)}, \qquad \text{for } \zeta \in S_4,\ \mathrm{Re}\,\zeta < 0. \tag{2.67}$$

Since both sides of (2.66) are analytic in the right half-plane we find the identity (2.64) for $u > 0$, and similarly since both sides of (2.67) are analytic in the left half-plane, we obtain (2.64) for $u < 0$.

2.3 Steepest descent analysis of Riemann-Hilbert problem

In this section we write the kernel $K_{n,N}$ in terms of the solution Y of the RH problem for orthogonal polynomials (due to Fokas, Its and Kitaev [47]) and apply the Deift/Zhou steepest descent method [36] to the RH problem for Y to get the asymptotics for Y. These asymptotics will be used in the next sections to prove Theorems 2.2 and 2.7.

We will restrict ourselves to the one-interval case, which means that ψ_V is supported on one interval, although the RH analysis can be done in general. We comment below in Remark 2.22 (see the end of this section) on the modifications that have to be made in the multi-interval case.

As in Theorems 2.2 and 2.7 we also assume that besides 0 there are no other singular points.

2.3.1 RH problem for orthogonal polynomials

The starting point is the RH problem that characterizes the orthogonal polynomials associated with the weight $|x|^{2\alpha}e^{-NV(x)}$. The 2×2 matrix-valued function $Y = Y_{n,N}$ satisfies the following conditions.

RH problem for Y

(a) $Y : \mathbb{C}\setminus\mathbb{R} \to \mathbb{C}^{2\times 2}$ is analytic.

(b) $Y_+(x) = Y_-(x)\begin{pmatrix} 1 & |x|^{2\alpha}e^{-NV(x)} \\ 0 & 1 \end{pmatrix}, \qquad \text{for } x\in\mathbb{R}.$

(c) $Y(z) = (I + \mathcal{O}(1/z))\begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}, \qquad \text{as } z\to\infty.$

(d) Y has the following behavior near the origin,

$$Y(z) = \begin{cases} \mathcal{O}\begin{pmatrix} 1 & |z|^{2\alpha} \\ 1 & |z|^{2\alpha} \end{pmatrix}, & \text{as } z\to 0, \text{ if } \alpha<0, \\ \mathcal{O}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \text{as } z\to 0, \text{ if } \alpha\geq 0. \end{cases} \tag{2.68}$$

Here we have oriented the real axis from the left to the right and $Y_+(x)$ ($Y_-(x)$) in part (b) denotes the limit as we approach $x\in\mathbb{R}$ from the upper (lower) half-plane. This RH problem possesses a unique solution given by [47] (see [75, 77] for the condition (d)),

$$Y(z) = \begin{pmatrix} \dfrac{1}{\kappa_{n,N}}p_n^{(N)}(z) & \dfrac{1}{2\pi i\kappa_{n,N}}\displaystyle\int_{\mathbb{R}} \dfrac{p_n^{(N)}(y)|y|^{2\alpha}e^{-NV(y)}}{y-z}dy \\ -2\pi i\kappa_{n-1,N}p_{n-1}^{(N)}(z) & -\kappa_{n-1,N}\displaystyle\int_{\mathbb{R}} \dfrac{p_{n-1}^{(N)}(y)|y|^{2\alpha}e^{-NV(y)}}{y-z}dy \end{pmatrix}, \tag{2.69}$$

for $z\in\mathbb{C}\setminus\mathbb{R}$, where $p_n^{(N)}(z) = \kappa_{n,N}z^n + \cdots$, is the n-th degree orthonormal polynomial with respect to the weight $|x|^{2\alpha}e^{-NV(x)}$, and $\kappa_{n,N}$ is the leading coefficient of $p_n^{(N)}$.

The correlation kernel $K_{n,N}$ can be expressed in terms of the solution of this RH problem. Indeed, using the Christoffel-Darboux formula for orthogonal polynomials and the fact that $\det Y = 1$, we get from (2.4) and (2.69),

$$\begin{aligned} &K_{n,N}(x,y) \\ &= |x|^\alpha e^{-\frac{1}{2}NV(x)} |y|^\alpha e^{-\frac{1}{2}NV(y)} \frac{\kappa_{n-1,N}}{\kappa_{n,N}} \frac{p_n^{(N)}(x)p_{n-1}^{(N)}(y) - p_{n-1}^{(N)}(x)p_n^{(N)}(y)}{x-y} \\ &= |x|^\alpha e^{-\frac{1}{2}NV(x)} |y|^\alpha e^{-\frac{1}{2}NV(y)} \frac{1}{2\pi i(x-y)} \begin{pmatrix} 0 & 1 \end{pmatrix} Y_\pm^{-1}(y) Y_\pm(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{aligned} \tag{2.70}$$

The asymptotics of $K_{n,N}$ follow from a steepest descent analysis of the RH problem for Y, see [22, 32, 33, 76, 77, 99]. The Deift/Zhou steepest descent analysis consists of a series of explicit transformations $Y \mapsto T \mapsto S \mapsto R$ so that it leads to a RH problem for R which is normalized at infinity and which has jumps uniformly close to the identity matrix I. Then R itself is uniformly close to I. By going back in the series of transformations we then have the asymptotics for Y from which the asymptotics of $K_{n,N}$ in different scaling regimes can be deduced.

The main issue of the present situation is the construction of a local parametrix near 0 with the aid of the RH problem for Ψ_α introduced in Section 2.2. For the case $\alpha = 0$ this was done in Chapter 1 and we use some ideas introduced there.

Throughout the rest of this chapter we use the notation

$$t = n/N, \qquad \text{and} \qquad V_t = \frac{1}{t} V. \tag{2.71}$$

2.3.2 Normalization of the RH problem at infinity: $Y \mapsto T$

In the first transformation we normalize the RH problem at infinity. The standard approach would be to use the equilibrium measure in the external field V_t, see [27, 91]. This is the probability measure that minimizes

$$I_{V_t}(\mu) = \iint \log \frac{1}{|x-y|} d\mu(x) d\mu(y) + \int V_t(x) d\mu(x)$$

among all Borel probability measures μ on $\mathbb{R}$. The minimizer for $t = 1$ has density ψ_V which by assumption vanishes at the origin. For $t < 1$, the origin is outside of the support and for t slightly less than 1, there is a gap in the support around 0. An annoying consequence is that the equality in the variational conditions is not valid near the origin. To overcome this problem, we introduced a modified measure ν_t in Chapter 1.

Here, we follow Section 1.3. We take a small $\delta_0 > 0$ so that $\psi_V(x) > 0$ for $x \in [-\delta_0, \delta_0] \setminus \{0\}$, and we consider the problem to minimize $I_{V_t}(\nu)$ among all signed measures $\nu = \nu^+ - \nu^-$ where $\nu^\pm$ are nonnegative measures such that

$\int d\nu = 1$ and supp $\nu^- \subset [-\delta_0, \delta_0]$. There is a unique minimizer which we denote by ν_t. This signed measure is absolutely continuous with density ψ_t and its support $S_t = [a_t, b_t]$ is an interval if t is sufficiently close to 1. The following variational conditions are satisfied: there exists a constant $\ell_t \in \mathbb{R}$ such that

$$2\int \log|x-y|\psi_t(y)dy - V_t(x) + \ell_t = 0, \qquad \text{for } x \in [a_t, b_t], \tag{2.72}$$

$$2\int \log|x-y|\psi_t(y)dy - V_t(x) + \ell_t \leq 0, \qquad \text{for } x \in \mathbb{R}. \tag{2.73}$$

In addition, we have shown in Chapter 1 that for t sufficiently close to 1,

$$\psi_t(x) = \frac{1}{\pi}(-Q_t(x))^{1/2}, \qquad \text{for } x \in [a_t, b_t], \tag{2.74}$$

where

$$Q_t(z) = \left(\frac{V'(z)}{2t}\right)^2 - \frac{1}{t}\int \frac{V'(z)-V'(y)}{z-y}\psi_t(y)dy. \tag{2.75}$$

For $t > 1$, we take the square root in (2.74) which is positive for $x = 0$, while for $t < 1$ we take the square root which is negative for $x = 0$.

For the first transformation, we introduce the following 'g-function' associated with ν_t,

$$g_t(z) = \int \log(z-y)d\nu_t(y) = \int \log(z-y)\psi_t(y)dy, \quad \text{for } z \in \mathbb{C}\setminus\mathbb{R}, \tag{2.76}$$

where we take the branch cut of the logarithm along the negative real axis. We define

$$T(z) = e^{\frac{1}{2}n\ell_t\sigma_3}Y(z)e^{-ng_t(z)\sigma_3}e^{-\frac{1}{2}n\ell_t\sigma_3}, \qquad \text{for } z \in \mathbb{C}\setminus\mathbb{R}. \tag{2.77}$$

We also use the functions

$$\varphi_t(z) = \int_{b_t}^{z} (Q_t(s))^{1/2}ds, \tag{2.78}$$

$$\tilde{\varphi}_t(z) = \int_{a_t}^{z} (Q_t(s))^{1/2}ds, \tag{2.79}$$

where the path of integration does not cross the real axis. The relations that exist between g_t, φ_t and $\tilde{\varphi}_t$ are described in Section 1.5.2]. Using these, we find that T is the unique solution of the following RH problem.

RH problem for T:

(a) $T : \mathbb{C}\setminus\mathbb{R} \to \mathbb{C}^{2\times 2}$ is analytic.

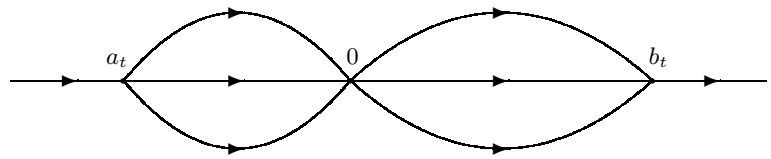

Figure 2.2 *The lens shaped contour Σ_S going through the origin.*

(b) $T_+(x) = T_-(x)v_T(x)$ for $x \in \mathbb{R}$, with

$$v_T(x) = \begin{cases} \begin{pmatrix} e^{2n\varphi_{t,+}(x)} & |x|^{2\alpha} \\ 0 & e^{2n\varphi_{t,-}(x)} \end{pmatrix}, & \text{for } x \in (a_t, b_t), \\ \begin{pmatrix} 1 & |x|^{2\alpha} e^{-2n\varphi_t(x)} \\ 0 & 1 \end{pmatrix}, & \text{for } x \in (b_t, \infty), \\ \begin{pmatrix} 1 & |x|^{2\alpha} e^{-2n\tilde{\varphi}_t(x)} \\ 0 & 1 \end{pmatrix}, & \text{for } x \in (-\infty, a_t). \end{cases}$$

(c) $T(z) = I + \mathcal{O}(1/z), \qquad \text{as } z \to \infty.$

(d) T has the same behavior as Y near the origin, given by (2.68).

2.3.3 Opening of the lens: $T \mapsto S$

In this subsection, we open the lens as in Figure 2.2. The opening of the lens is based on the factorization of the jump matrix v_T for $x \in (a_t, b_t)$, which is

$$\begin{aligned} v_T(x) &= \begin{pmatrix} e^{2n\varphi_{t,+}(x)} & |x|^{2\alpha} \\ 0 & e^{2n\varphi_{t,-}(x)} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ |x|^{-2\alpha} e^{2n\varphi_{t,-}(x)} & 1 \end{pmatrix} \begin{pmatrix} 0 & |x|^{2\alpha} \\ -|x|^{-2\alpha} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ |x|^{-2\alpha} e^{2n\varphi_{t,+}(x)} & 1 \end{pmatrix}. \end{aligned} \tag{2.80}$$

We deform the RH problem for T into a RH problem for S by opening a lens around $[a_t, b_t]$ going through the origin, as shown in Figure 2.2. The precise form of the lens is not yet specified but for now we choose the lens to be contained in the region of analyticity of V and we can do it in such a way that for any given $\delta > 0$, there exists a constant $\gamma > 0$ so that, for every t sufficiently close to 1, we have that

$$\operatorname{Re} \varphi_t(z) < -\gamma, \tag{2.81}$$

for all z on the upper and lower lips of the lens with $\min\{|z|, |z-a|, |z-b|\} > \delta$, see also Section 1.4.3].

Let ω be the analytic continuation of $x \mapsto |x|^{2\alpha}$ to $\mathbb{C} \setminus (i\mathbb{R})$, i.e.,

$$\omega(z) = \begin{cases} z^{2\alpha}, & \text{if } \operatorname{Re} z > 0, \\ (-z)^{2\alpha}, & \text{if } \operatorname{Re} z < 0. \end{cases} \tag{2.82}$$

The second transformation is then defined by

$$S(z) = \begin{cases} T(z), & \text{outside the lens.} \\ T(z) \begin{pmatrix} 1 & 0 \\ -\omega(z)^{-1} e^{2n\varphi_t(z)} & 1 \end{pmatrix}, & \text{in the upper parts of the lens,} \\ T(z) \begin{pmatrix} 1 & 0 \\ \omega(z)^{-1} e^{2n\varphi_t(z)} & 1 \end{pmatrix}, & \text{in the lower parts of the lens.} \end{cases} \tag{2.83}$$

Then S is the unique solution of the following RH problem posed on the contour Σ_S which is the union of $\mathbb{R}$ with the upper and lower lips of the lens.

RH problem for S:

(a) $S : \mathbb{C} \setminus \Sigma_S \to \mathbb{C}^{2\times 2}$ is analytic.

(b) $S_+ = S_- v_S$ on Σ_S, where

$$v_S(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ \omega(z)^{-1} e^{2n\varphi_t(z)} & 1 \end{pmatrix}, & \text{for } z \in \Sigma_S \setminus \mathbb{R}, \\ \begin{pmatrix} 0 & |z|^{2\alpha} \\ -|z|^{-2\alpha} & 0 \end{pmatrix}, & \text{for } z \in (a_t, b_t), \\ \begin{pmatrix} 1 & |z|^{2\alpha} e^{-2n\varphi_t(z)} \\ 0 & 1 \end{pmatrix}, & \text{for } z \in (b_t, \infty), \\ \begin{pmatrix} 1 & |z|^{2\alpha} e^{-2n\tilde{\varphi}_t(z)} \\ 0 & 1 \end{pmatrix}, & \text{for } z \in (-\infty, a_t). \end{cases}$$

(c) $S(z) = I + \mathcal{O}(1/z), \qquad \text{as } z \to \infty.$

(d) S has the following behavior near the origin. If $\alpha < 0$,

$$S(z) = \mathcal{O} \begin{pmatrix} 1 & |z|^{2\alpha} \\ 1 & |z|^{2\alpha} \end{pmatrix}, \qquad \text{as } z \to 0, z \in \mathbb{C} \setminus \Sigma_S, \tag{2.84}$$

and if $\alpha \geq 0$,

$$S(z) = \begin{cases} \mathcal{O}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \text{as } z \to 0 \text{ from outside the lens,} \\ \mathcal{O}\begin{pmatrix} |z|^{-2\alpha} & 1 \\ |z|^{-2\alpha} & 1 \end{pmatrix}, & \text{as } z \to 0 \text{ from inside the lens.} \end{cases} \tag{2.85}$$

2.3.4 Parametrix $P^{(\infty)}$ for the outside region

On the lips of the lens and on $(-\infty, a_t) \cup (b_t, \infty)$, the jump matrix for S is close to the identity matrix if n is large and t is close to 1. This follows from the inequality (2.81) and the fact that $\varphi_t(x) > 0$ for $x > b_t$ and $\tilde{\varphi}_t(x) > 0$ for $x < a_t$. Ignoring these jumps we are led to the following RH problem.

RH problem for $P^{(\infty)}$:

(a) $P^{(\infty)} : \mathbb{C} \setminus [a_t, b_t] \to \mathbb{C}^{2\times 2}$ is analytic.

(b) $P_+^{(\infty)}(x) = P_-^{(\infty)}(x) \begin{pmatrix} 0 & |x|^{2\alpha} \\ -|x|^{-2\alpha} & 0 \end{pmatrix}, \qquad \text{for } x \in (a_t, b_t) \setminus \{0\}.$

(c) $P^{(\infty)}(z) = I + \mathcal{O}(1/z), \qquad \text{as } z \to \infty.$

Note that $P^{(\infty)}$ depends on n and N through the parameter t. As in [73, 75, 77] we construct $P^{(\infty)}$ in terms of the Szegő function D associated with $|x|^{2\alpha}$ on (a_t, b_t). This is an analytic function in $\mathbb{C} \setminus [a_t, b_t]$ satisfying $D_+(x)D_-(x) = |x|^{2\alpha}$ for $x \in (a_t, b_t) \setminus \{0\}$ and which does not vanish anywhere in $\overline{\mathbb{C}} \setminus [a_t, b_t]$. It is easy to check that D is given by

$$D(z) = z^\alpha \phi\left(\frac{2z - a_t - b_t}{b_t - a_t}\right)^{-\alpha}, \qquad \text{for } z \in \mathbb{C} \setminus [a_t, b_t], \tag{2.86}$$

where $\phi(z) = z + (z-1)^{1/2}(z+1)^{1/2}$ is the conformal map from $\mathbb{C} \setminus [-1, 1]$ onto the exterior of the unit circle. Since $\phi(z) = 2z + \mathcal{O}(1/z)$ as $z \to \infty$ we have

$$\lim_{z\to\infty} D(z) = \left(\frac{4}{b_t - a_t}\right)^{-\alpha} \equiv D_\infty.$$

Now the transformed matrix-valued function

$$\widehat{P}^{(\infty)} = D_\infty^{-\sigma_3} P^{(\infty)} D^{\sigma_3} \tag{2.87}$$

satisfies conditions (a) and (c) of the RH problem and it has the jump matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ on (a_t, b_t). The construction of $\widehat{P}^{(\infty)}$ has been done in [27, 32, 33], and

leads us to the solution of the RH problem for $P^{(\infty)}$: for $z \in \mathbb{C} \setminus [a_t, b_t]$,

$$P^{(\infty)}(z) = D_\infty^{\sigma_3} \begin{pmatrix} \frac{\beta(z)+\beta(z)^{-1}}{2} & \frac{\beta(z)-\beta(z)^{-1}}{2i} \\ \frac{\beta(z)-\beta(z)^{-1}}{-2i} & \frac{\beta(z)+\beta(z)^{-1}}{2} \end{pmatrix} D(z)^{-\sigma_3}, \tag{2.88}$$

where

$$\beta(z) = \frac{(z-b_t)^{1/4}}{(z-a_t)^{1/4}}, \qquad \text{for } z \in \mathbb{C} \setminus [a_t, b_t]. \tag{2.89}$$

2.3.5 Parametrices near endpoints

The jump matrices of S and $P^{(\infty)}$ are not uniformly close to each other near the origin and near the endpoints of $[a_t, b_t]$. We surround a_1 and b_1 (the endpoints of S_V) with small disks $U_\delta(a)$ and $U_\delta(b)$ of radius δ. For t sufficiently close to 1, the endpoints a_t and b_t are in these disks, and then local parametrices $P^{(a_t)}$ and $P^{(b_t)}$ can be constructed with Airy functions as in [27, 32, 33].

2.3.6 Parametrix near the origin

Near the origin a local parametrix will be constructed with the aid of the RH problem for Ψ_α of Section 2.2. Let U_δ be a small disk with center at 0 and radius $\delta > 0$. We seek a 2×2 matrix-valued function P in U_δ with the same jumps as S, with the same behavior as S near the origin, and which matches with $P^{(\infty)}$ on the boundary ∂U_δ of the disk. We thus seek a 2×2 matrix-valued function that satisfies the following RH problem.

RH problem for P:

(a) P is defined and analytic in $U_{\delta'} \setminus \Sigma_S$ for some $\delta' > \delta$.

(b) On $\Sigma_S \cap U_\delta$, P satisfies the jump relations

$$P_+(z) = P_-(z) \begin{pmatrix} 1 & 0 \\ \omega(z)^{-1} e^{2n\varphi_t(z)} & 1 \end{pmatrix}, \quad \text{for } z \in (\Sigma_S \setminus \mathbb{R}) \cap U_\delta, \tag{2.90}$$

$$P_+(x) = P_-(x) \begin{pmatrix} 0 & |x|^{2\alpha} \\ -|x|^{-2\alpha} & 0 \end{pmatrix}, \qquad \text{for } x \in (-\delta, \delta) \setminus \{0\}. \tag{2.91}$$

(c) P satisfies the matching condition

$$P(z) = (I + \mathcal{O}(n^{-1/3}))P^{(\infty)}(z), \tag{2.92}$$

as $n, N \to \infty$ such that $n^{2/3}(n/N - 1) \to L$, uniformly for $z \in \partial U_\delta \setminus \Sigma_S$.

(d) P has the same behavior near the origin as S, given by (2.84) and (2.85).

In order to solve the RH problem for P we work as follows. First, we seek P such that it satisfies conditions (a), (b), and (d). To do this, we transform (in the first step) the RH problem into a RH problem for $\widehat{P}$ with constant jump matrices. In the second step we solve the RH problem for $\widehat{P}$ explicitly by using the RH problem for Ψ_α. In the final step we take also the matching condition (c) into account.

Step 1: Transformation to constant jump matrices

In the first step we transform the RH problem for P into a RH problem for $\widehat{P}$ with constant jump matrices. We seek P in the form

$$P(z) = E(z)\widehat{P}(z)e^{n\varphi_t(z)\sigma_3}e^{\frac{1}{2}\pi i\alpha\sigma_3}z^{-\alpha\sigma_3}, \qquad \text{if } \operatorname{Im} z > 0, \tag{2.93}$$

$$P(z) = E(z)\widehat{P}(z)e^{-n\varphi_t(z)\sigma_3}e^{\frac{1}{2}\pi i\alpha\sigma_3}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} z^{-\alpha\sigma_3}, \qquad \text{if } \operatorname{Im} z < 0, \tag{2.94}$$

where the invertible matrix-valued function $E = E_{n,N}$ (we suppress notation of the indices) is analytic in $U_{\delta'}$ and where the branch cut of z^α is chosen along the negative real axis.

Using (2.91), (2.93) and (2.94), and keeping track of the branches of z^α, we can easily check that $\widehat{P}$ has no jumps on $(-\delta,\delta)\setminus\{0\}$. What remains are jumps on the contour $(\Sigma_S\setminus\mathbb{R})\cap U_\delta = \bigcup_{j=1}^4 \Sigma_j$, which is shown in Figure 2.3. We have reversed the orientation of Σ_2 and Σ_3 towards infinity, so that now the orientation of the Σ_j's corresponds to the orientation of the Γ_j's in Figure 2.1. The contour divides U_δ into four regions I, II, III and IV, also shown in Figure 2.3.

We will now determine the jump relations for $\widehat{P}$. By (2.82), (2.90), and (2.93), $\widehat{P}$ should have the following jump matrix on Σ_1,

$$\begin{aligned}
&\widehat{P}_-(z)^{-1}\widehat{P}_+(z) \\
&= e^{n\varphi_t(z)\sigma_3}e^{\frac{1}{2}\pi i\alpha\sigma_3}z^{-\alpha\sigma_3}\begin{pmatrix} 1 & 0 \\ \omega(z)^{-1}e^{2n\varphi_t(z)} & 1 \end{pmatrix} z^{\alpha\sigma_3}e^{-\frac{1}{2}\pi i\alpha\sigma_3}e^{-n\varphi_t(z)\sigma_3} \\
&= \begin{pmatrix} 1 & 0 \\ \omega(z)^{-1}z^{2\alpha}e^{-\pi i\alpha} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ e^{-\pi i\alpha} & 1 \end{pmatrix}.
\end{aligned} \tag{2.95}$$

For $z\in\Sigma_2$ we have, because of the reversal of the orientation, an extra minus sign in the $(2,1)$-entry of the jump matrix. The result is

$$\widehat{P}_-(z)^{-1}\widehat{P}_+(z) = \begin{pmatrix} 1 & 0 \\ -\omega(z)^{-1}z^{2\alpha}e^{-\pi i\alpha} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -e^{\pi i\alpha} & 1 \end{pmatrix}, \tag{2.96}$$

where the last equality follows from the fact that $\omega(z)^{-1}z^{2\alpha} = e^{2\pi i\alpha}$ by (2.82), since $\operatorname{Re} z < 0$ in this case. Using equations (2.91) and (2.94), the jump matrices

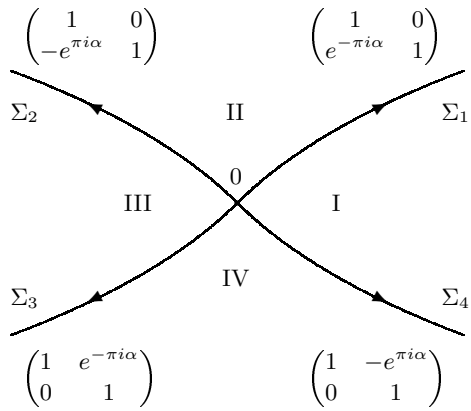

Figure 2.3 *Contour and jumps for the RH problem for $\widehat{P}$.*

for $\widehat{P}$ on Σ_3 and Σ_4 can be determined similarly. The result is that

$$\widehat{P}_+(z) = \begin{cases} \widehat{P}_-(z) \begin{pmatrix} 1 & e^{-\pi i\alpha} \\ 0 & 1 \end{pmatrix}, & \text{for } z \in \Sigma_3, \\ \widehat{P}_-(z) \begin{pmatrix} 1 & -e^{\pi i\alpha} \\ 0 & 1 \end{pmatrix}, & \text{for } z \in \Sigma_4. \end{cases} \tag{2.97}$$

We arrive at the following RH problem for $\widehat{P}$. If it is satisfied by $\widehat{P}$ then P defined by (2.93)-(2.94) satisfies the parts (a), (b), and (d) of the RH problem for P.

RH problem for $\widehat{P}$:

(a) $\widehat{P}$ is defined and analytic in $U_{\delta'} \setminus \bigcup_j \Sigma_j$ for some $\delta' > \delta$.

(b) $\widehat{P}$ satisfies the following jump relations

$$\widehat{P}_+(z) = \begin{cases} \widehat{P}_-(z)\begin{pmatrix} 1 & 0 \\ e^{-\pi i\alpha} & 1 \end{pmatrix}, & \text{for } z \in \Sigma_1, \\ \widehat{P}_-(z)\begin{pmatrix} 1 & 0 \\ -e^{\pi i\alpha} & 1 \end{pmatrix}, & \text{for } z \in \Sigma_2, \\ \widehat{P}_-(z)\begin{pmatrix} 1 & e^{-\pi i\alpha} \\ 0 & 1 \end{pmatrix}, & \text{for } z \in \Sigma_3, \\ \widehat{P}_-(z)\begin{pmatrix} 1 & -e^{\pi i\alpha} \\ 0 & 1 \end{pmatrix}, & \text{for } z \in \Sigma_4. \end{cases} \tag{2.98}$$

(c) $\widehat{P}$ has the following behavior near the origin. If $\alpha < 0$,

$$\widehat{P}(z) = \mathcal{O}\begin{pmatrix} |z|^\alpha & |z|^\alpha \\ |z|^\alpha & |z|^\alpha \end{pmatrix}, \qquad \text{as } z \to 0, \tag{2.99}$$

and if $\alpha \geq 0$,

$$\widehat{P}(z) = \begin{cases} \mathcal{O}\begin{pmatrix} |z|^{-\alpha} & |z|^{-\alpha} \\ |z|^{-\alpha} & |z|^{-\alpha} \end{pmatrix}, & \text{as } z \to 0,\ z \in \mathrm{I} \cup \mathrm{III}, \\ \mathcal{O}\begin{pmatrix} |z|^{\alpha} & |z|^{-\alpha} \\ |z|^{\alpha} & |z|^{-\alpha} \end{pmatrix}, & \text{as } z \to 0,\ z \in \mathrm{II}, \\ \mathcal{O}\begin{pmatrix} |z|^{-\alpha} & |z|^{\alpha} \\ |z|^{-\alpha} & |z|^{\alpha} \end{pmatrix}, & \text{as } z \to 0,\ z \in \mathrm{IV}. \end{cases} \tag{2.100}$$

Note that if $\widehat{P}$ has the behavior near the origin as described in part (c) of the RH problem, then P defined by (2.93) and (2.94) has the same behavior near the origin as S, as required by part (d) of the RH problem for P.

Step 2: Construction of $\widehat{P}$

Observe that the jump matrices and the behavior near the origin of the RH problem for $\widehat{P}$ correspond exactly to the jump matrices and the behavior near the origin of the RH problem for Ψ_α. We use the solution of the latter RH problem to solve the RH problem for $\widehat{P}$.

We seek $\widehat{P}$ in the form

$$\widehat{P}(z) = \Psi_\alpha\left(n^{1/3}f(z); n^{2/3}s_t(z)\right), \tag{2.101}$$

where f and s_t are analytic functions on U_δ which are real on $(-\delta,\delta)$, and s_t is such that

$$n^{2/3}s_t(z) \in \mathbb{C} \setminus \mathcal{P}_\alpha, \qquad \text{for } z \in U_\delta, \tag{2.102}$$

where $\mathcal{P}_\alpha$ is the set of poles of q_α. In addition, f is a conformal map from U_δ onto a convex neighborhood $f(U_\delta)$ of 0 such that $f(0)=0$ and $f'(0)>0$. Depending on f we open the lens around $[a_t,b_t]$ such that that $f(\Sigma_i)=\Gamma_i$ for $i=1,2,3,4$, where the Γ_i's are the jump contours for the RH problem for Ψ_α, see Figure 2.1. Recall that the lens was not fully specified and we still have the freedom to make this choice.

It remains to determine f and s_t so that the matching condition for P is also satisfied. As in Section 1.5.6] we take

$$\begin{aligned} f(z) &= \left[\frac{3}{4}\int_0^z (-Q_1(y))^{1/2}dy\right]^{1/3} && (2.103)\\ &= z\left(\frac{\pi\psi_V''(0)}{8}\right)^{1/3} + \mathcal{O}\left(z^2\right), \qquad \text{as } z\to 0, && (2.104)\end{aligned}$$

and

$$s_t(z)f(z) = \int_0^z \left((-Q_t(y))^{1/2} - (-Q_1(y))^{1/2}\right)dy. \tag{2.105}$$

Then f is analytic with $f(0)=0$ and $f'(0)>0$, it does not depend on t, and it is a conformal mapping on U_δ provided δ is small enough. Since the right-hand side of (2.105) is analytic and vanishes for $z=0$, we can divide by $f(z)$ and obtain an analytic function s_t. From (1.106), we get that there exists a constant $K>0$ such that

$$|s_t(z) - \pi c^{1/3}(t-1)w_{S_V}(0)| \le K(t-1)|z| + o(t-1) \quad \text{as } t\to 1, \tag{2.106}$$

uniformly for z in a neighborhood of 0. Now assume that $|n^{2/3}(t-1)|\le M$ and n large enough. Then it easily follows from (2.106) and the fact that q_α has no real poles, that there exists a $\delta>0$, depending only on M, such that

$$|\operatorname{Im} n^{2/3}s_t(z)| < \min\{|\operatorname{Im} s| \mid s \text{ is a pole of } q_\alpha\} \qquad \text{for } |z|\le\delta. \tag{2.107}$$

Then (2.102) holds and (2.101) is well-defined and analytic since $\Psi_\alpha(\zeta;s)$ is jointly analytic in its two arguments, see Remark 2.19.

It follows from (2.103) and (2.105) that

$$-i\left[\frac{4}{3}f(z)^3 + s_t(z)f(z)\right] = \begin{cases}\varphi_{t,+}(0)-\varphi_t(z), & \text{if } \operatorname{Im} z>0,\\ \varphi_{t,+}(0)+\varphi_t(z), & \text{if } \operatorname{Im} z<0,\end{cases} \tag{2.108}$$

see also Section 1.4.6. Hence by (2.34), which by Remark 2.19 holds uniformly for s in compact subsets of $\mathbb{C} \setminus \mathcal{P}_\alpha$, we have

$$\widehat{P}(z) = \Psi_\alpha\left(n^{1/3}f(z); n^{2/3}s_t(z)\right)\left[I + \mathcal{O}(1/n^{1/3})\right] e^{n\varphi_{t,+}(0)\sigma_3} \times \begin{cases} e^{-n\varphi_t(z)\sigma_3}, & \text{if } \operatorname{Im} z > 0, \\ e^{n\varphi_t(z)\sigma_3}, & \text{if } \operatorname{Im} z < 0, \end{cases} \tag{2.109}$$

as $n, N \to \infty$, uniformly for $z \in \partial U_\delta$.

Step 3: Matching condition

In the final step we determine E such that the matching condition (c) of the RH problem for P is satisfied. By (2.93), (2.94), and (2.109) we have for $z \in \partial U_\delta$,

$$P(z) = \begin{cases} E(z)\left[I + \mathcal{O}(1/n^{1/3})\right] e^{n\varphi_{t,+}(0)\sigma_3} e^{\frac{1}{2}\pi i\alpha\sigma_3} z^{-\alpha\sigma_3}, & \text{if } \operatorname{Im} z > 0, \\ E(z)\left[I + \mathcal{O}(1/n^{1/3})\right] e^{n\varphi_{t,+}(0)\sigma_3} e^{\frac{1}{2}\pi i\alpha\sigma_3} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} z^{-\alpha\sigma_3}, & \text{if } \operatorname{Im} z < 0, \end{cases}$$

as $n, N \to \infty$. This has to match the outside parametrix $P^{(\infty)}$, so that we are led to the following definition for the prefactor $E(z)$, for $z \in U_\delta$,

$$E(z) = \begin{cases} P^{(\infty)}(z) z^{\alpha\sigma_3} e^{-\frac{1}{2}\pi i\alpha\sigma_3} e^{-n\varphi_{t,+}(0)\sigma_3}, & \text{if } \operatorname{Im} z > 0, \\ P^{(\infty)}(z) z^{\alpha\sigma_3} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} e^{-\frac{1}{2}\pi i\alpha\sigma_3} e^{-n\varphi_{t,+}(0)\sigma_3}, & \text{if } \operatorname{Im} z < 0. \end{cases} \tag{2.110}$$

One can check as in [77, 98] that E is invertible and analytic in a full neighborhood of U_δ. In addition we have the matching condition (2.92). This completes the construction of the parametrix near the origin.

2.3.7 Final transformation: $S \mapsto R$

Having the parametrices $P^{(\infty)}$, $P^{(a_t)}$, $P^{(b_t)}$, and P, we now define

$$R(z) = \begin{cases} S(z)P^{-1}(z), & \text{for } z \in U_\delta, \\ S(z)\left(P^{(a_t)}\right)^{-1}(z), & \text{for } z \in U_\delta(a), \\ S(z)\left(P^{(b_t)}\right)^{-1}(z), & \text{for } z \in U_\delta(b), \\ S(z)\left(P^{(\infty)}\right)^{-1}(z), & \text{for } z \in \mathbb{C} \setminus (U_\delta \cup U_\delta(a) \cup U_\delta(b) \cup \Sigma_S). \end{cases} \tag{2.111}$$

Then R has only jumps on the reduced system of contours Σ_R shown in Figure 2.4, and R satisfies the following RH problem, cf. Chapter 1. The circles around 0, a_t and b_t are oriented counterclockwise.

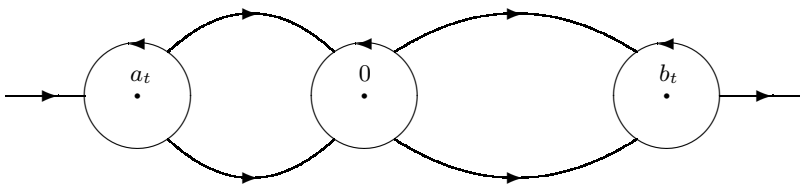

Figure 2.4 *The contour Σ_R after the third and final transformation.*

RH problem for R:

(a) $R : \mathbb{C} \setminus \Sigma_R \to \mathbb{C}^{2\times 2}$ is analytic.

(b) $R_+(z) = R_-(z)v_R(z)$ for $z \in \Sigma_R$, with

$$v_R = \begin{cases} P^{(\infty)}(P^{(a_t)})^{-1}, & \text{on } \partial U_\delta(a), \\ P^{(\infty)}(P^{(b_t)})^{-1}, & \text{on } \partial U_\delta(b), \\ P^{(\infty)}P^{-1}, & \text{on } \partial U_\delta, \\ P^{(\infty)}v_S(P^{(\infty)})^{-1}, & \text{on the rest of } \Sigma_R. \end{cases} \tag{2.112}$$

(c) $R(z) = I + \mathcal{O}(1/z), \qquad \text{as } z \to \infty,$

(d) R remains bounded near the intersection points of Σ_R.

Now we let $n, N \to \infty$ such that $|n^{2/3}(n/N - 1)| \leq M$, so that δ does not depend on n. Then it follows from the construction of the parametrices that

$$v_R = \begin{cases} I + \mathcal{O}(1/n), & \text{on } \partial U_\delta(a) \cup \partial U_\delta(b), \\ I + \mathcal{O}(n^{-1/3}), & \text{on } \partial U_\delta, \\ I + \mathcal{O}(e^{-\gamma n}), & \text{on the rest of } \Sigma_R, \end{cases} \tag{2.113}$$

where $\gamma > 0$ is some fixed constant. All $\mathcal{O}$-terms hold uniformly on their respective contours.

For large n, the jump matrix of R is close to the identity matrix, both in L^∞ and in L^2-sense on Σ_R. Then arguments as in [27, 32, 33] (which are based on estimates on Cauchy operators as well as on contour deformations), guarantee that

$$R(z) = I + \mathcal{O}(n^{-1/3}), \qquad \text{uniformly for } z \in \mathbb{C} \setminus \Sigma_R, \tag{2.114}$$

as $n, N \to \infty$ such that $|n^{2/3}(n/N - 1)| \leq M$.

This completes the steepest descent analysis. Following the effect of the transformation on the correlation kernel $K_{n,N}$ and using (2.114) we will prove the main Theorem 2.2. This will be done in the next section. For the proof of Theorem 2.7 we need to expand $v_R(z)$ in (2.113) up to order $n^{-1/3}$, from which it follows that

$$R(z) = I + \frac{R^{(1)}(z)}{n^{1/3}} + \mathcal{O}(n^{-2/3}), \qquad \text{uniformly for } z \in \mathbb{C} \setminus \Sigma_R,$$

with an explicitly computable $R^{(1)}(z)$. The asymptotic behavior of the recurrence coefficients is expressed in terms of $R^{(1)}$ and this leads to the proof of Theorem 2.7. This will be done in Section 2.5.

2.22 Remark. The steepest descent analysis was done under the assumption that supp ψ_V consists of one interval. In the multi-interval case, the construction of the outside parametrix $P^{(\infty)}$ is more complicated, since it uses Θ-functions as in [32, Lemma 4.3] and the Szegő function for multiple intervals as in [77, Section 4]. With these modifications the asymptotic analysis can be carried through in the multi-interval case without any additional difficulty.

2.4 Proof of Theorem 2.2

As in the statement of Theorem 2.2, we assume that $n, N \to \infty$ with $n^{2/3}(t-1) \to L$, where $t = n/N$. Let $M > |L|$ and take n sufficiently large so that $|n^{2/3}(t-1)| \leq M$. Let $\delta > 0$ be such that (2.107) holds. We start by writing the kernel $K_{n,N}$ explicitly in terms of the matrix-valued function Φ_α defined in (2.62). For notational convenience we introduce

$$B(z) = R(z)E(z), \tag{2.115}$$

where E and R are given by (2.110) and (2.111), respectively.

2.23 Proposition. *Let $x, y \in (-\delta, \delta) \setminus \{0\}$. Then*

$$K_{n,N}(x,y) = \frac{1}{2\pi i(x-y)} e^{\frac{1}{2}\pi i\alpha(\operatorname{sgn}(x)+\operatorname{sgn}(y))} \begin{pmatrix} 0 & 1 \end{pmatrix} \Phi_\alpha^{-1}\left(n^{1/3}f(y); n^{2/3}s_t(y)\right)$$
$$\times B^{-1}(y)B(x)\Phi_\alpha\left(n^{1/3}f(x); n^{2/3}s_t(x)\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{2.116}$$

where Φ_α is given by (2.62).

Proof. From (2.70), (2.77), and the fact that $NV = nV_t$, the kernel $K_{n,N}$ can be written as

$$K_{n,N}(x,y) = |x|^\alpha e^{\frac{1}{2}n(2g_{t,+}(x)-V_t(x)+\ell_t)}|y|^\alpha e^{\frac{1}{2}n(2g_{t,+}(y)-V_t(y)+\ell_t)}$$
$$\times \frac{1}{2\pi i(x-y)} \begin{pmatrix} 0 & 1 \end{pmatrix} T_+^{-1}(y)T_+(x)\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Using the relation

$$2g_{t,+} - V_t + \ell_t = -2\varphi_{t,+} \qquad \text{on } [a_t, b_t],$$

see Chapter 1, and (2.83) to express T in terms of S, we find for x and y in $(a_t, b_t) \setminus \{0\}$,

$$\begin{aligned} K_{n,N}(x,y) &= \frac{|x|^\alpha e^{-n\varphi_{t,+}(x)} |y|^\alpha e^{-n\varphi_{t,+}(y)}}{2\pi i(x-y)} \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -|y|^{-2\alpha} e^{2n\varphi_{t,+}(y)} & 1 \end{pmatrix} \\ &\qquad \times S_+^{-1}(y) S_+(x) \begin{pmatrix} 1 & 0 \\ |x|^{-2\alpha} e^{2n\varphi_{t,+}(x)} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{2\pi i(x-y)} \begin{pmatrix} -1 & 1 \end{pmatrix} |y|^{-\alpha\sigma_3} e^{n\varphi_{t,+}(y)\sigma_3} S_+^{-1}(y) \\ &\qquad \times S_+(x) |x|^{\alpha\sigma_3} e^{-n\varphi_{t,+}(x)\sigma_3} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \end{aligned} \tag{2.117}$$

We further simplify this expression by writing S in terms of R and the parametrix P near the origin. Consider the case that $x \in (0, \delta)$. Then, since $S_+(x) = R(x)P_+(x)$ by (2.111), we have by (2.93),

$$S_+(x) = B(x)\widehat{P}(x) e^{\frac{1}{2}\pi i\alpha\sigma_3} e^{n\varphi_{t,+}(x)\sigma_3} |x|^{-\alpha\sigma_3}, \qquad \text{for } x \in (0, \delta), \tag{2.118}$$

where B is given by (2.115). By (2.118), (2.101), and (2.62) we then find for $x \in (0, \delta)$,

$$\begin{aligned} &S_+(x)|x|^{\alpha\sigma_3} e^{-n\varphi_{t,+}(x)\sigma_3} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &\qquad = B(x)\Phi_\alpha\left(n^{1/3} f(x); n^{2/3} s_t(x)\right) \begin{pmatrix} 1 & 0 \\ -e^{-\pi i\alpha} & 1 \end{pmatrix} e^{\frac{1}{2}\pi i\alpha\sigma_3} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &\qquad = e^{\frac{1}{2}\pi i\alpha \operatorname{sgn}(x)} B(x)\Phi_\alpha\left(n^{1/3} f(x); n^{2/3} s_t(x)\right) \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{aligned} \tag{2.119}$$

A similar calculation shows that (2.119) also holds for $x \in (-\delta, 0)$. Similarly, we have

$$\begin{aligned} &\begin{pmatrix} -1 & 1 \end{pmatrix} |y|^{-\alpha\sigma_3} e^{n\varphi_{t,+}(y)\sigma_3} S_+^{-1}(y) \\ &\qquad = e^{\frac{1}{2}\pi i\alpha \operatorname{sgn}(y)} \begin{pmatrix} 0 & 1 \end{pmatrix} \Phi_\alpha^{-1}\left(n^{1/3} f(y); n^{2/3} s_t(y)\right) B^{-1}(y), \end{aligned} \tag{2.120}$$

for $y \in (-\delta, \delta) \setminus \{0\}$. Inserting (2.119) and (2.120) into (2.117), we arrive at (2.116), which proves the proposition. □

Proof of Theorem 2.2. Let $u, v \in \mathbb{R} \setminus \{0\}$, and put $u_n = u/(cn^{1/3})$ and $v_n = v/(cn^{1/3})$ with c given by (2.19). Note that, by (2.104),

$$\lim_{n\to\infty} n^{1/3} f(u_n) = u, \qquad \lim_{n\to\infty} n^{1/3} f(v_n) = v. \tag{2.121}$$

Furthermore, by (2.106), (2.19), and (2.20),

$$|n^{2/3} s_t(z) - s| \leq K n^{2/3}(t-1)|z| + n^{2/3} o(t-1) + |n^{2/3}(t-1) - L|\pi c^{1/3} w_{S_V}(0)$$

uniformly for z in a neighborhood of 0. Then, since $n^{2/3}(t-1) \to L$, it easily follows that

$$\lim_{n,N\to\infty} n^{2/3} s_t(u_n) = \lim_{n,N\to\infty} n^{2/3} s_t(v_n) = s. \tag{2.122}$$

Now, similar as in [77], we use the fact that the entries of B are analytic and uniformly bounded in U_δ, to obtain

$$\lim_{n,N\to\infty} B^{-1}(v_n) B(u_n) = I. \tag{2.123}$$

Inserting (2.121), (2.122), and (2.123) into (2.116), we find that

$$\begin{aligned}
&\lim_{n,N\to\infty} \frac{1}{cn^{1/3}} K_{n,N}(u_n, v_n) \\
&\qquad = \frac{1}{2\pi i(u-v)} e^{\frac{1}{2}\pi i\alpha(\operatorname{sgn}(u)+\operatorname{sgn}(v))} \begin{pmatrix} 0 & 1 \end{pmatrix} \Phi_\alpha^{-1}(v;s)\Phi_\alpha(u;s) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&\qquad = -e^{\frac{1}{2}\pi i\alpha(\operatorname{sgn}(u)+\operatorname{sgn}(v))} \frac{\Phi_{\alpha,1}(u;s)\Phi_{\alpha,2}(v;s) - \Phi_{\alpha,1}(v;s)\Phi_{\alpha,2}(u;s)}{2\pi i(u-v)}.
\end{aligned}$$

This completes the proof of Theorem 2.2. □

2.5 Proof of Theorem 2.7

In this section we will determine the asymptotic behavior of the recurrence coefficients $a_n^{(N)}$ and $b_n^{(N)}$ as $n, N \to \infty$ such that $|n^{2/3}(n/N - 1)| \leq M$ for some $M > 0$. As in Theorem 2.7 we assume that $S_V = [a, b]$ is an interval, and that there are no other singular points besides 0. Then it follows that supp ψ_t consists of one interval $[a_t, b_t]$ if t is sufficiently close to 1. In addition we have that the endpoints a_t and b_t are real analytic functions in t, see [74, Theorem 1.3], so that

$$a_t = a + \mathcal{O}(n^{-2/3}), \qquad b_t = b + \mathcal{O}(n^{-2/3}), \tag{2.124}$$

since $t = n/N = 1 + \mathcal{O}(n^{-2/3})$.

We make use of the following result, see for example [27, 33]. Let Y be the unique solution of the RH problem for Y. There exist 2×2 constant (independent of z, but depending on n, N) matrices Y_1, Y_2 such that

$$Y(z)\begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} = I + \frac{Y_1}{z} + \frac{Y_2}{z^2} + \mathcal{O}(1/z^3), \qquad \text{as } z \to \infty,$$

and

$$a_n^{(N)} = \sqrt{(Y_1)_{12}(Y_1)_{21}}, \qquad b_n^{(N)} = (Y_1)_{11} + \frac{(Y_2)_{12}}{(Y_1)_{12}}. \tag{2.125}$$

We need to determine the constant matrices Y_1 and Y_2. For large $|z|$ we have by (2.77), (2.83) and (2.111) that

$$Y(z) = e^{-\frac{1}{2}n\ell_t\sigma_3} R(z) P^{(\infty)}(z) e^{ng_t(z)\sigma_3} e^{\frac{1}{2}n\ell_t\sigma_3}. \tag{2.126}$$

So in order to compute Y_1 and Y_2 we need the asymptotic behavior of $P^{(\infty)}(z)$, $e^{ng_t(z)\sigma_3}$ and $R(z)$ as $z \to \infty$.

Asymptotic behavior of $P^{(\infty)}(z)$ as $z \to \infty$:

From (2.86) and (2.89) it is straightforward to determine the asymptotic behavior of the scalar functions $D(z)$ and $\beta(z)$ as $z \to \infty$. Indeed, as $z \to \infty$,

$$\begin{pmatrix} \frac{\beta(z)+\beta(z)^{-1}}{2} & \frac{\beta(z)-\beta(z)^{-1}}{2i} \\ \frac{\beta(z)-\beta(z)^{-1}}{-2i} & \frac{\beta(z)+\beta(z)^{-1}}{2} \end{pmatrix}$$

$$= I - \frac{1}{4}(b_t - a_t)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\frac{1}{z} + \frac{i}{8}(b_t^2 - a_t^2)\begin{pmatrix} * & 1 \\ -1 & * \end{pmatrix}\frac{1}{z^2} + \mathcal{O}(1/z^3),$$

and

$$D(z)^{-\sigma_3} = \left[I - \frac{\alpha}{2}(b_t + a_t)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{z} + \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}\frac{1}{z^2} + \mathcal{O}(1/z^3) \right] D_\infty^{-\sigma_3},$$

where $*$ denotes an unspecified unimportant entry. Inserting these equations into (2.88) and using (2.124) gives us the asymptotic behavior of $P^{(\infty)}$ at infinity,

$$P^{(\infty)}(z) = I + \frac{P_1^{(\infty)}}{z} + \frac{P_2^{(\infty)}}{z^2} + \mathcal{O}(1/z^3), \qquad \text{as } z \to \infty, \tag{2.127}$$

with

$$P_1^{(\infty)} = D_\infty^{\sigma_3}\begin{pmatrix} -\frac{\alpha}{2}(b_t + a_t) & \frac{i}{4}(b_t - a_t) \\ -\frac{i}{4}(b_t - a_t) & \frac{\alpha}{2}(b_t + a_t) \end{pmatrix} D_\infty^{-\sigma_3}$$

$$= D_\infty^{\sigma_3}\begin{pmatrix} -\frac{\alpha}{2}(b + a) & \frac{i}{4}(b - a) \\ -\frac{i}{4}(b - a) & \frac{\alpha}{2}(b + a) \end{pmatrix} D_\infty^{-\sigma_3} + \mathcal{O}(n^{-2/3}), \tag{2.128}$$

and

$$P_2^{(\infty)} = D_\infty^{\sigma_3} \begin{pmatrix} * & \frac{i}{8}(\alpha+1)(b_t^2 - a_t^2) \\ \frac{i}{8}(\alpha-1)(b_t^2 - a_t^2) & * \end{pmatrix} D_\infty^{-\sigma_3}$$
$$= D_\infty^{\sigma_3} \begin{pmatrix} * & \frac{i}{8}(\alpha+1)(b^2 - a^2) \\ \frac{i}{8}(\alpha-1)(b^2 - a^2) & * \end{pmatrix} D_\infty^{-\sigma_3} + \mathcal{O}(n^{-2/3}). \quad (2.129)$$

Asymptotic behavior of $e^{ng_t(z)\sigma_3}$ as $z \to \infty$:

By (2.76) we have

$$e^{ng_t(z)\sigma_3} \begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} = I + \frac{G_1}{z} + \frac{G_2}{z^2} + \mathcal{O}(1/z^3), \qquad \text{as } z \to \infty, \quad (2.130)$$

with

$$G_1 = -n \int_{a_t}^{b_t} y\psi_t(y)dy \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad G_2 = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}. \quad (2.131)$$

Asymptotic behavior of $R(z)$ as $z \to \infty$:

The computation of R_1 and R_2 is more involved. For $z \in \partial U_\delta \cap \mathbb{C}_+$, we have by (2.112), (2.93), (2.101), and (2.110),

$$\begin{aligned} v_R(z) &= P^{(\infty)}(z)P^{-1}(z) \\ &= P^{(\infty)}(z) z^{\alpha\sigma_3} e^{-\frac{1}{2}\pi i\alpha\sigma_3} e^{-n\varphi_t(z)\sigma_3} \Psi_\alpha^{-1}(n^{1/3}f(z); n^{2/3}s_t(z)) \\ &\qquad \times e^{n\varphi_{t,+}(0)\sigma_3} e^{\frac{1}{2}\pi i\alpha\sigma_3} z^{-\alpha\sigma_3} (P^{(\infty)})^{-1}(z). \end{aligned} \quad (2.132)$$

Using (2.57) and (2.108), we then find

$$v_R(z) = I + \frac{\Delta^{(1)}(z)}{n^{1/3}} + \mathcal{O}(n^{-2/3}), \quad (2.133)$$

where

$$\Delta^{(1)}(z) = -\frac{1}{2if(z)} P^{(\infty)}(z) z^{\alpha\sigma_3} e^{-\frac{1}{2}\pi i\alpha\sigma_3} e^{-n\varphi_{t,+}(0)\sigma_3}$$
$$\times \begin{pmatrix} u_\alpha(n^{2/3}s_t(z)) & q_\alpha(n^{2/3}s_t(z)) \\ -q_\alpha(n^{2/3}s_t(z)) & -u_\alpha(n^{2/3}s_t(z)) \end{pmatrix}$$
$$\times \quad e^{n\varphi_{t,+}(0)\sigma_3} e^{\frac{1}{2}\pi i\alpha\sigma_3} z^{-\alpha\sigma_3} (P^{(\infty)})^{-1}(z), \quad (2.134)$$

for $z \in \partial U_\delta \cap \mathbb{C}_+$. A similar calculation leads to an analogous formula for $z \in \partial U_\delta \cap \mathbb{C}_-$, which together with (2.134) shows that $\Delta^{(1)}$ has an extension to an analytic function in a punctured neighborhood of 0 with a simple pole at 0.

To calculate the residue at 0, we use (2.86) together with the fact $\phi_+(x) = \exp(i \arccos x)$ for $x \in [-1,1]$ to find

$$\lim_{z\to 0+i0} \frac{D(z)}{z^\alpha} = \exp\left(-i\alpha \arccos\left(-\frac{b_t+a_t}{b_t-a_t}\right)\right),$$

so that by (2.87)

$$\lim_{z\to 0+i0} P^{(\infty)}(z) z^{\alpha\sigma_3} e^{-\frac{1}{2}\pi i\alpha\sigma_3} = D_\infty^{\sigma_3} \widehat{P}_+^{(\infty)}(0) e^{i\alpha\theta_t\sigma_3}, \tag{2.135}$$

with $\theta_t = \arcsin \frac{b_t+a_t}{b_t-a_t}$. Also note that by (2.74), (2.78), (2.29) and

$$-\varphi_{t,+}(0) = \pi i \int_0^{b_t} \psi_t(x)dx = \pi i \omega_t. \tag{2.136}$$

Now use (2.103), (2.19), (2.135), and (2.136) in (2.134) to find

$$\begin{aligned}\mathrm{Res}(\Delta^{(1)};0) = -\frac{1}{2ic} D_\infty^{\sigma_3} \widehat{P}_+^{(\infty)}(0) e^{i(\pi n\omega_t+\alpha\theta_t)\sigma_3} \\ \times \quad \begin{pmatrix} u_\alpha(n^{2/3}s_t(0)) & q_\alpha(n^{2/3}s_t(0)) \\ -q_\alpha(n^{2/3}s_t(0)) & -u_\alpha(n^{2/3}s_t(0)) \end{pmatrix} \\ \times \quad e^{-i(\pi n\omega_t+\alpha\theta)\sigma_3} (\widehat{P}_+^{(\infty)})^{-1}(0) D_\infty^{-\sigma_3}.\end{aligned} \tag{2.137}$$

Combining (2.105), (2.104), and (2.74) we see that $n^{2/3}s_t(0) = s_{t,n}$ as defined in (2.27). From (2.87), (2.88), and (2.89) it follows that

$$\widehat{P}_+^{(\infty)}(0) = \begin{pmatrix} \frac{\beta_+(0)+\beta_+(0)^{-1}}{2} & \frac{\beta_+(0)-\beta_+(0)^{-1}}{2i} \\ \frac{\beta_+(0)-\beta_+(0)^{-1}}{2i} & \frac{\beta_+(0)+\beta_+(0)^{-1}}{2} \end{pmatrix},$$

where $\beta_+(0) = e^{i\pi/4}\left(-b_t/a_t\right)^{1/4}$.

We insert this into (2.137) and after some straightforward calculations we find

$$-\mathrm{Res}(\Delta^{(1)};0) = D_\infty^{\sigma_3}\left(r_1\sigma_1 + r_2\sigma_2 + r_3\sigma_3\right) D_\infty^{-\sigma_3}, \tag{2.138}$$

where the Pauli matrices are $\sigma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\sigma_2 = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$, and $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, and

$$\begin{aligned} r_1 &= -\frac{1}{2ic}\left(u_\alpha(s_{t,n})\frac{b_t-a_t}{2\sqrt{-a_tb_t}} + q_\alpha(s_{t,n})\frac{b_t+a_t}{2\sqrt{-a_tb_t}}\sin(2\pi n\omega_t + 2\alpha\theta_t)\right) \\ &= -\frac{1}{2ic}\left(u_\alpha(s_{t,n})\frac{b-a}{2\sqrt{-ab}} + q_\alpha(s_{t,n})\frac{b+a}{2\sqrt{-ab}}\sin(2\pi n\omega_t + 2\alpha\theta)\right) \\ &\qquad + \mathcal{O}(n^{-2/3}), \end{aligned} \tag{2.139}$$

$$\begin{aligned} r_2 &= \frac{q_\alpha(s_{t,n})}{2c}\cos(2\pi n\omega_t + 2\alpha\theta_t) \\ &= \frac{q_\alpha(s_{t,n})}{2c}\cos(2\pi n\omega_t + 2\alpha\theta) + \mathcal{O}(n^{-2/3}), \end{aligned} \tag{2.140}$$

$$\begin{aligned} r_3 &= \frac{1}{2c}\left(q_\alpha(s_{t,n})\frac{b_t - a_t}{2\sqrt{-a_t b_t}}\sin(2\pi n\omega_t + 2\alpha\theta_t) + u_\alpha(s_{t,n})\frac{b_t + a_t}{2\sqrt{-a_t b_t}}\right) \\ &= \frac{1}{2c}\left(q_\alpha(s_{t,n})\frac{b - a}{2\sqrt{-ab}}\sin(2\pi n\omega_t + 2\alpha\theta) + u_\alpha(s_{t,n})\frac{b + a}{2\sqrt{-ab}}\right) \\ &\qquad + \mathcal{O}(n^{-2/3}), \end{aligned} \tag{2.141}$$

where we used (2.124).

From (2.133) it follows that

$$R(z) = I + \frac{R^{(1)}(z)}{n^{1/3}} + \mathcal{O}(n^{-2/3}), \tag{2.142}$$

where $R_+^{(1)} = R_-^{(1)} + \Delta^{(1)}$ on ∂U_δ and $R^{(1)}(z) \to 0$ as $z \to \infty$. Since $\Delta^{(1)}$ is analytic with a simple pole at $z = 0$, we can find explicitly

$$R^{(1)}(z) = \begin{cases} -\frac{1}{z}\mathrm{Res}(\Delta^{(1)}; 0) + \Delta^{(1)}(z), & \text{for } z \in U_\delta, \\ -\frac{1}{z}\mathrm{Res}(\Delta^{(1)}; 0), & \text{for } z \in \mathbb{C} \setminus \overline{U}_\delta. \end{cases} \tag{2.143}$$

As in [33] the matrix-valued function R has the following asymptotic behavior at infinity,

$$R(z) = I + \frac{R_1}{z} + \frac{R_2}{z^2} + \mathcal{O}(1/z^3), \qquad \text{as } z \to \infty. \tag{2.144}$$

The compatibility with (2.142) and (2.143) yields that

$$R_1 = -\mathrm{Res}(\Delta^{(1)}; 0)n^{-1/3} + \mathcal{O}(n^{-2/3}), \qquad R_2 = \mathcal{O}(n^{-2/3}). \tag{2.145}$$

Now, we are ready to determine the asymptotics of the recurrence coefficients.

Proof of Theorem 2.7. Note that by (2.126), (2.127), (2.130) and (2.144),

$$Y_1 = e^{-\frac{1}{2}n\ell_t\sigma_3}\left[P_1^{(\infty)} + G_1 + R_1\right]e^{\frac{1}{2}n\ell_t\sigma_3} \tag{2.146}$$

and

$$Y_2 = e^{-\frac{1}{2}n\ell_t\sigma_3}\left[P_2^{(\infty)} + G_2 + R_2 + R_1P_1^{(\infty)} + \left(P_1^{(\infty)} + R_1\right)G_1\right]e^{\frac{1}{2}n\ell_t\sigma_3} \tag{2.147}$$

We start with the recurrence coefficient $a_n^{(N)}$. Inserting (2.146) into (2.125) and using (2.128) and the facts that $(G_1)_{12} = (G_1)_{21} = 0$ (by (2.131)), and $(R_1)_{12}(R_1)_{21} = \mathcal{O}(n^{-2/3})$ (by (2.145)), we obtain

$$\begin{aligned} a_n^{(N)} &= \Big[(P_1^{(\infty)})_{12}(P_1^{(\infty)})_{21} + (P_1^{(\infty)})_{12}(R_1)_{21} + \\ &\qquad\qquad (P_1^{(\infty)})_{21}(R_1)_{12} + \mathcal{O}(n^{-2/3})\Big]^{1/2} \\ &= \left[\left(\frac{b-a}{4}\right)^2 + i\frac{b-a}{4}\left(D_\infty^2(R_1)_{21} - D_\infty^{-2}(R_1)_{12}\right) + \mathcal{O}(n^{-2/3})\right]^{1/2} \\ &= \frac{b-a}{4} + \frac{i}{2}\left(D_\infty^2(R_1)_{21} - D_\infty^{-2}(R_1)_{12}\right) + \mathcal{O}(n^{-2/3}). \end{aligned}$$

From (2.145) and (2.138) we then arrive at,

$$\begin{aligned} a_n^{(N)} &= \frac{b-a}{4} - r_2 n^{-1/3} + \mathcal{O}(n^{-2/3}) \\ &= \frac{b-a}{4} - \frac{q_\alpha(s_{t,n})\cos(2\pi n\omega_t + 2\alpha\theta)}{2c} n^{-1/3} + \mathcal{O}(n^{-2/3}). \end{aligned} \tag{2.148}$$

Next, we consider the recurrence coefficient $b_n^{(N)}$. Inserting (2.146) and (2.147) into (2.125), and using the facts that $(G_1)_{11} + (G_1)_{22} = 0$ (by (2.131)), and $(R_2)_{12} = \mathcal{O}(n^{-2/3})$ (by (2.145)) we obtain

$$\begin{aligned} b_n^{(N)} &= (P_1^{(\infty)})_{11} + (R_1)_{11} + \frac{(P_2^{(\infty)})_{12} + (R_1 P_1^{(\infty)})_{12} + \mathcal{O}(n^{2/3})}{(P_1^{(\infty)} + R_1)_{12}} \\ &= (P_1^{(\infty)})_{11} + (R_1)_{11} + \left(1 - \frac{(R_1)_{12}}{(P_1^{(\infty)})_{12}} + \mathcal{O}(n^{-2/3})\right) \\ &\qquad \times \left(\frac{(P_2^{(\infty)})_{12}}{(P_1^{(\infty)})_{12}} + (R_1)_{11} + \frac{(P_1^{(\infty)})_{22}}{(P_1^{(\infty)})_{12}}(R_1)_{12} + \mathcal{O}(n^{-2/3})\right). \end{aligned}$$

From equations (2.128), (2.129), (2.145), and (2.138), we then arrive at

$$\begin{aligned} b_n^{(N)} &= \frac{b+a}{2} + 2(R_1)_{11} + 2i\frac{b+a}{b-a}D_\infty^{-2}(R_1)_{12} + \mathcal{O}(n^{-2/3}) \\ &= \frac{b+a}{2} + 2\left(r_3 + i\frac{b+a}{b-a}(r_1 - ir_2)\right) + \mathcal{O}(n^{-2/3}). \end{aligned} \tag{2.149}$$

Using (2.139), (2.140), and (2.141) in (2.149) we see that the terms containing

u_α cancel against each other. The remaining terms are the ones containing q_α:

$$b_n^{(N)} = \frac{b+a}{2} + \frac{q_\alpha(s_{t,n})}{c}\left[\frac{b+a}{b-a}\cos(2\pi n\omega_t + 2\alpha\theta) + \frac{2\sqrt{-ab}}{b-a}\sin(2\pi n\omega_t + 2\alpha\theta)\right] n^{-1/3} + \mathcal{O}(n^{-2/3}). \quad (2.150)$$

Since $\frac{b+a}{b-a} = \sin\theta$ and $\frac{2\sqrt{-ab}}{b-a} = \cos\theta$, we can combine the two terms within square brackets and the result is

$$b_n^{(N)} = \frac{b+a}{2} + \frac{q_\alpha(s_{t,n})\sin(2\pi n\omega_t + (2\alpha+1)\theta)}{c} n^{-1/3} + \mathcal{O}(n^{-2/3}). \quad (2.151)$$

Theorem 2.7 is proven by (2.148) and (2.151). □

Chapter 3

The existence of a real pole-free solution of the fourth order analogue of the Painlevé I equation

Summary[1] We establish the existence of a real solution $y(s,t)$ with no poles on the real line of the following fourth order analogue of the Painlevé I equation,

$$s = ty - \left(\frac{1}{6}y^3 + \frac{1}{24}(y_s^2 + 2yy_{ss}) + \frac{1}{240}y_{ssss}\right).$$

This proves the existence part of a conjecture posed by Dubrovin. We obtain our result by proving the solvability of an associated Riemann-Hilbert problem through the approach of a vanishing lemma. In addition, by applying the Deift/Zhou steepest descent method to this Riemann-Hilbert problem, we obtain the asymptotics for $y(s,t)$ as $s \to \pm\infty$.

3.1 Introduction and statement of results

3.1.1 The P_I^2 equation

The first Painlevé equation is the second order differential equation

$$y_{ss} = 6y^2 + s. \tag{3.1}$$

[1] This chapter corresponds to the following paper [24]:

T. Claeys and M. Vanlessen, The existence of a real pole-free solution of the fourth order analogue of the Painlevé I equation, arxiv:math-ph/0604046.

This equation has higher order analogues of even order $2m$ for $m \geq 1$, which are collected, together with the first Painlevé equation itself, in the Painlevé I hierarchy, see e.g. [69, 72]. The second member of the hierarchy is the fourth order differential equation

$$s = -\left(\frac{1}{6}y^3 + \frac{1}{24}(y_s^2 + 2yy_{ss}) + \frac{1}{240}y_{ssss}\right), \tag{3.2}$$

and has solutions that are meromorphic in the complex plane. In 1990, Brézin, Marinari, and Parisi [20] argued numerically that there exists a solution y to (3.2) with no poles on the real line, and with asymptotic behavior

$$y(s) \sim \mp|6s|^{1/3}, \qquad \text{as } s \to \pm\infty. \tag{3.3}$$

Moore [85] proved the existence of a unique real solution to (3.2) with asymptotic behavior given by (3.3), and he gave a line of argument why this solution is probably pole-free on the real line.

A generalization of (3.2) can be obtained by introducing an additional variable t, as done by Dubrovin in [40], so that we get the following differential equation for $y = y(s,t)$, which we denote as the P_I^2 equation (cf. [66] for $t = 0$),

$$s = ty - \left(\frac{1}{6}y^3 + \frac{1}{24}(y_s^2 + 2yy_{ss}) + \frac{1}{240}y_{ssss}\right). \tag{3.4}$$

In recent work [40], Dubrovin conjectured (see Section 3.1.3 below for more details) the existence of a unique real solution to (3.4) with no poles on the real line. We prove the existence part of this conjecture.

3.1.2 Statement of results

Our results are the following.

3.1 Theorem. *There exists a solution $y(s,t)$ to the P_I^2 equation* (3.4) *with the following properties:*

(i) $y(s,t)$ is real valued and pole-free for $s, t \in \mathbb{R}$.

(ii) For fixed $t \in \mathbb{R}$, $y(s,t)$ has the following asymptotic behavior,

$$y(s,t) = \frac{1}{2}z_0|s|^{1/3} + \mathcal{O}(|s|^{-2}), \qquad \text{as } s \to \pm\infty, \tag{3.5}$$

where $z_0 = z_0(s,t)$ is the real solution of

$$z_0^3 = -48\mathrm{sgn}(s) + 24z_0|s|^{-2/3}t. \tag{3.6}$$

3.2 Remark. Observe that z_0 is negative (positive) for $s > 0$ ($s < 0$) with the following asymptotic behavior as $s \to \pm\infty$,

$$z_0 = \hat{z}_0 - \operatorname{sgn}(s)\,\frac{2}{3}\,6^{2/3} t|s|^{-2/3} + \mathcal{O}(|s|^{-4/3}), \qquad \hat{z}_0 = -\operatorname{sgn}(s)\,2\cdot 6^{1/3}, \tag{3.7}$$

so that the asymptotics (3.5) for y can be rewritten as, cf. (3.3)

$$y(s,t) = \mp(6|s|)^{1/3} \mp \frac{1}{3} 6^{2/3} t|s|^{-1/3} + \mathcal{O}(|s|^{-1}), \qquad \text{as } s \to \pm\infty. \tag{3.8}$$

Power expansions for solutions of (3.2) were found in [71].

3.3 Remark. One expects, see [85, Appendix A] for $t = 0$, that the solution y considered in Theorem 3.1 is uniquely determined by realness and the asymptotics (3.5).

3.1.3 Motivation

Hamiltonian perturbations of hyperbolic equations

Hyperbolic equations of the form

$$u_t + a(u)u_s = 0 \tag{3.9}$$

can be perturbed to a Hamiltonian equation of the form

$$\begin{aligned} u_t + a(u)u_s + \epsilon\left[b_1(u)u_{ss} + b_2(u)u_s^2\right] \\ + \epsilon^2\left[b_3(u)u_{sss} + b_4(u)u_s u_{ss} + b_5(u)u_s^3\right] + \cdots = 0, \end{aligned} \tag{3.10}$$

where ϵ is small and $b_1, b_2, \ldots$ are smooth functions. These equations have been studied by Dubrovin in [40], see also [39], where he formulated the universality conjecture about the behavior of a generic solution to a general perturbed Hamiltonian equation (3.10) near the point (s_0, t_0) of *gradient catastrophe* of the unperturbed solution (3.9). He argued that this behavior is described by a special solution to the P_I^2 equation (3.4). To be more precise, his conjecture is the following.

3.4 Conjecture. (Dubrovin, [40])

(i) Let $u_0 = u_0(s,t)$ be a smooth solution to the unperturbed equation (3.9)*, defined for all $s \in \mathbb{R}$ and $0 \le t < t_0$, and monotone in s for any t. Then there exists a solution $u = u(s,t;\epsilon)$ to the perturbed equation* (3.10) *defined on the same domain in the (s,t)-plane with the asymptotics as $\epsilon \to 0$ of the form*

$$u(s,t;\epsilon) = u_0(s,t) + \epsilon^2 u_1(s,t) + \epsilon^4 u_2(s,t) + o(\epsilon^4), \tag{3.11}$$

where u_1 and u_2 can be written down explicitly.

(ii) The ODE (3.4) *has a unique solution $y = y(s,t)$ smooth for all real $s \in \mathbb{R}$ for all values of the parameter t.*

(iii) The generic solution u described in part (i) of the conjecture can be extended up to $t = t_0 + \delta$ for sufficiently small positive $\delta = \delta(\epsilon)$; near the point (s_0, t_0) it behaves in the following way

$$u(s,t;\epsilon) = u_0(s,t) + a\epsilon^{2/7} y\left(b\epsilon^{-6/7}(s - c(t-t_0) - s_0), d\epsilon^{-4/7}(t-t_0)\right) + \mathcal{O}(\epsilon^{4/7}),$$

for some constants a, b, c, d which depend on the hyperbolic equation, the solution u, and on the choice of perturbation. Here y is the unique smooth solution described in part (ii) of the conjecture.

So Theorem 3.1 in fact proves the existence part of part (ii) of Dubrovin's conjecture.

In [55], numerical calculations were done for the particular example (of a perturbed Hamiltonian equation) of the small dispersion limit of the KdV equation, see also [78, 79, 80, 100],

$$u_t + 6uu_s + \epsilon^2 u_{sss} = 0, \qquad \text{with initial condition} \quad u(s,0) = u_0(s).$$

Before the time of gradient catastrophe t_0, solutions turn out to behave nicely. Around the critical time t_0, fast oscillations near s_0 set in. The behavior of the solution near the critical point (s_0, t_0) should be described in terms of the real pole-free solution to (3.4) we consider in this chapter.

Random matrix theory

The local eigenvalue correlations of unitary random matrix ensembles on the space of $n \times n$ Hermitian matrices have universal behavior (when the size n of the matrices is going to infinity) in different regimes of the spectrum. In the bulk of the spectrum it is known, see e.g. [13, 27, 32, 86], that the local correlations can be expressed in terms of the sine kernel, while at the soft edge of the spectrum they generically (i.e. when the limiting mean eigenvalue density vanishes like a square root) can be expressed in terms of the Airy kernel, see e.g. [13, 32, 50, 97].

In the presence of singular points, one observes different types of limiting kernels in double scaling limits, see e.g. [14] and Chapters 1 and 2. Near singular edge points, where the limiting mean eigenvalue density vanishes at a higher order than a square root (the regular case) the local eigenvalue correlations are expected [18] to be described in terms of functions associated with real pole-free solutions of the even members of the Painlevé I hierarchy. The particular case where the limiting mean eigenvalue density vanishes like a power 5/2, which is the lowest non-regular order of vanishing, should correspond with the real pole-free solution of P_I^2 considered in Theorem 3.1. We come back to this in Chapter 4.

3.1.4 Riemann-Hilbert problem and Lax pair for P_I^2

Consider the following Riemann-Hilbert (RH) problem for given complex parameters s and t, on a contour $\Sigma = \left(\cup_{j=0}^{6}\Sigma_j\right)\cup\mathbb{R}^-$, with $\Sigma_j = e^{j\frac{2\pi i}{7}}\mathbb{R}^+$, where each of the eight rays are orientated from 0 to infinity.

RH problem for Ψ:

(a) Ψ is analytic in $\mathbb{C}\setminus\Sigma$.

(b) Ψ satisfies the following jump relations on Σ, for some complex numbers $s_0,\dots,s_6$ which do not depend on ζ, s, and t,

$$\Psi_+(\zeta) = \Psi_-(\zeta)\begin{pmatrix}1 & s_j\\ 0 & 1\end{pmatrix}, \qquad \text{for } \zeta\in\Sigma_j \text{ for even } j, \tag{3.12}$$

$$\Psi_+(\zeta) = \Psi_-(\zeta)\begin{pmatrix}1 & 0\\ s_j & 1\end{pmatrix}, \qquad \text{for } \zeta\in\Sigma_j \text{ for odd } j, \tag{3.13}$$

$$\Psi_+(\zeta) = \Psi_-(\zeta)\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \qquad \text{for } \zeta\in\mathbb{R}^-. \tag{3.14}$$

(c) There exist complex numbers y and h, which depend on s and t but not on ζ, such that Ψ has the following asymptotic behavior as $\zeta\to\infty$,

$$\Psi(\zeta) = \zeta^{-\frac{1}{4}\sigma_3}N\left(I - h\sigma_3\zeta^{-1/2} + \frac{1}{2}\begin{pmatrix}h^2 & iy\\ -iy & h^2\end{pmatrix}\zeta^{-1} + \mathcal{O}(\zeta^{-2})\right) \times \quad e^{-\theta(\zeta;s,t)\sigma_3}, \tag{3.15}$$

where

$$N = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}e^{-\frac{1}{4}\pi i\sigma_3}, \quad \theta(\zeta;s,t) = \frac{1}{105}\zeta^{7/2} - \frac{t}{3}\zeta^{3/2} + s\zeta^{1/2}. \tag{3.16}$$

3.5 Remark. In [66], Kapaev uses a slightly modified RH problem for the P_I^2 equation with parameter $t=0$. However a transformation shows that both RH problems are equivalent.

3.6 Remark. The RH problem for P_I^2 is similar to the RH problem for the Painlevé I equation, see [67]. The only differences are that, for Painlevé I, there are only six rays in the jump contour, and that the highest exponent of ζ in θ is $5/2$. For the m-th member of the Painlevé I hierarchy, there are $4+2m$ rays in the jump contour, and the highest exponent of ζ in θ is $m+3/2$.

The complex numbers $s_0,\dots,s_6$ are the Stokes multipliers and do not depend on s and t, so that varying the parameters s and t leads to a monodromy preserving

deformation [44, 54, 62, 63]. The RH problem can only be solvable if the Stokes multipliers satisfy the relation

$$\begin{pmatrix} 1 & 0 \\ s_4 & 1 \end{pmatrix}\begin{pmatrix} 1 & s_5 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ s_6 & 1 \end{pmatrix}\begin{pmatrix} 1 & s_0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ s_1 & 1 \end{pmatrix}\begin{pmatrix} 1 & s_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ s_3 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.17}$$

As we will show in Section 3.2.3 (in fact we only treat one particular choice of Stokes multipliers, but the proof holds in general), a solution Ψ of the RH problem for Ψ also satisfies the following system of differential equations, which is the Lax pair for the P_I^2 equation,

$$\frac{\partial \Psi}{\partial \zeta} = U\Psi, \qquad \frac{\partial \Psi}{\partial s} = W\Psi, \tag{3.18}$$

where

$$U = \frac{1}{240}\begin{pmatrix} -4y_s\zeta - (12yy_s + y_{sss}) & 8\zeta^2 + 8y\zeta + (12y^2 + 2y_{ss} - 120t) \\ U_{21} & 4y_s\zeta + (12yy_s + y_{sss}) \end{pmatrix}, \tag{3.19}$$

$$\begin{aligned} U_{21} = 8\zeta^3 - 8y\zeta^2 - (4y^2 + 2y_{ss} + 120t)\zeta \\ +(16y^3 - 2y_s^2 + 4yy_{ss} + 240s), \end{aligned} \tag{3.20}$$

and

$$W = \begin{pmatrix} 0 & 1 \\ \zeta - 2y & 0 \end{pmatrix}. \tag{3.21}$$

This Lax pair appeared first in work of Moore [85] for $T = 0$ and was derived in [69] for general T. The compatibility condition of the Lax pair (4.15)–(4.18) is exactly the P_I^2 equation (4.14), see e.g. [69]. Different choices of Stokes multipliers $s_0, \ldots, s_6$ correspond to different solutions of the P_I^2 equation. The particular solution we are interested in, is the unique solution with Stokes multipliers $s_1 = s_2 = s_5 = s_6 = 0$. It then follows by (3.17) that $s_0 = 1$ and $s_3 = s_4 = -1$. This choice of Stokes multipliers was suggested by Kapaev in [66], where he proved that the solution of (3.2) with asymptotics given by (4.19), if it exists, is indeed the one corresponding to $s_1 = s_2 = s_5 = s_6 = 0$, $s_0 = 1$, and $s_3 = s_4 = -1$. This proves the uniqueness part of Dubrovin's conjecture for the case $T = 0$. One can expect that similar arguments, based on the asymptotic solution of the direct monodromy problem, hold for $T \neq 0$ as well.

3.1.5 Outline of the rest of the chapter

In the next section, we prove the first part (the existence part) of Theorem 3.1. In order to do this, we introduce in Section 3.2.1 a RH problem for Φ, which is equivalent to the RH problem for Ψ (the RH problem for P_I^2) with Stokes

multipliers $s_1 = s_2 = s_5 = s_6 = 0$, $s_0 = 1$, and $s_3 = s_4 = -1$. Afterwards, we prove in Section 3.2.2 the solvability of the RH problem for Φ for real s and t by proving that the associated homogeneous RH problem has only the trivial solution. This approach is often referred to in the literature as a vanishing lemma, see e.g. [23, 32, 48, 49, 104]. We are only able to prove the vanishing lemma for real s and t due to symmetries in the RH problem. In Section 3.2.3 we show that Ψ satisfies a Lax pair of the form (3.18)–(3.21), with y given in terms of Φ. By compatibility of the Lax pair, it follows that y solves the P_I^2 equation, and by the solvability of the RH problem, y has no real poles.

In Section 3 we prove the second part (the asymptotics part) of Theorem 3.1. We do this by applying the Deift/Zhou steepest descent method [36, 37] to the RH problem for Φ. In this method, we perform a series of transformations to reduce the RH problem for Φ to a RH problem that we can solve approximately for large $|s|$. By unfolding the series of the transformations, we obtain the asymptotics for y.

3.2 The existence of a real pole-free solution to P_I^2

3.2.1 Statement of an associated RH problem to P_I^2

Let $\Gamma = \bigcup_{j=1}^4 \Gamma_j$ be the contour consisting of four straight rays,

$$\Gamma_1 : \arg\zeta = 0, \qquad \Gamma_2 : \arg\zeta = \frac{6\pi}{7}, \qquad \Gamma_3 : \arg\zeta = \pi, \qquad \Gamma_4 : \arg\zeta = -\frac{6\pi}{7},$$

oriented as shown in Figure 3.1. We seek (for $s, t \in \mathbb{C}$) a 2×2 matrix-valued function $\Phi(\zeta; s, t) = \Phi(\zeta)$ (we suppress notation of s and t for brevity) satisfying the following RH problem.

RH problem for Φ:

(a) Φ is analytic in $\mathbb{C} \setminus \Gamma$.

(b) Φ satisfies the following constant jump relations on Γ,

$$\Phi_+(\zeta) = \Phi_-(\zeta) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_1, \tag{3.22}$$

$$\Phi_+(\zeta) = \Phi_-(\zeta) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_2 \cup \Gamma_4, \tag{3.23}$$

$$\Phi_+(\zeta) = \Phi_-(\zeta) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_3. \tag{3.24}$$

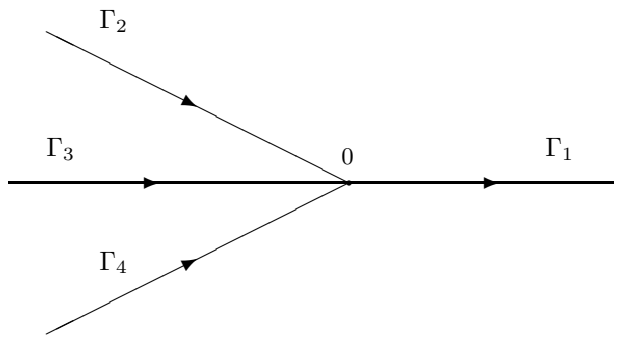

Figure 3.1 *The oriented contour* Γ *consisting of the four straight rays* Γ_1, Γ_2, Γ_3, *and* Γ_4.

(c) Φ has the following behavior at infinity,

$$\Phi(\zeta) = (I + \mathcal{O}(1/\zeta))\zeta^{-\frac{1}{4}\sigma_3} N e^{-\theta(\zeta;s,t)\sigma_3}, \qquad \text{as } \zeta \to \infty, \tag{3.25}$$

where N and θ are given by (3.16).

3.7 Remark. By multiplying Φ to the left with an appropriate matrix independent of ζ, see (3.53) below, we obtain by Proposition 3.11 the RH problem for Ψ, as stated in Section 3.1.4, for the particular choice of Stokes multipliers $s_1 = s_2 = s_5 = s_6 = 0$, $s_0 = 1$, and $s_3 = s_4 = -1$.

3.8 Remark. Let Φ be a solution of the RH problem. By using the jump relations (3.22)–(3.24) one has that $\det \Phi_+ = \det \Phi_-$ on Γ. This yields that $\det \Phi$ is entire. From (3.25) we have that $\det \Phi(\zeta) \to 1$ as $\zeta \to \infty$, and thus, by Liouville's theorem, we have that $\det \Phi \equiv 1$.

Now, suppose that $\tilde{\Phi}$ is a second solution of the RH problem. Then, since $\tilde{\Phi}$ and Φ satisfy the same jump relations on Γ, one has that $\tilde{\Phi}\,\Phi^{-1}$ is entire (observe that Φ^{-1} exists since $\det \Phi \equiv 1$). From (3.25) we have that $\tilde{\Phi}(\zeta)\Phi(\zeta)^{-1} \to I$ as $\zeta \to \infty$, and thus, by Liouville's theorem, we have that $\tilde{\Phi}\,\Phi^{-1} \equiv I$. We now have shown that if the RH problem for Φ has a solution, then this solution is unique.

3.2.2 Solvability of the RH problem for Φ

Here, our goal is to prove that the RH problem for Φ is solvable for $s, t \in \mathbb{R}$. Moreover, we will also strengthen the asymptotic condition (c) of the RH problem and prove analyticity properties in the variables s and t. In case $s = t = 0$, the solvability of the RH problem for Φ has been proven by Deift et al. in [32, Section 5.3]. The general case is analogous but for the convenience of the reader we will

recall the different steps in the proof and indicate where we need the restriction to $s, t \in \mathbb{R}$. The result of this subsection is the following lemma.

3.9 Lemma. *For every $s_0, t_0 \in \mathbb{R}$, there exist complex neighborhoods $\mathcal{V}$ of s_0 and $\mathcal{W}$ of t_0 such that for all $s \in \mathcal{V}$ and $t \in \mathcal{W}$ the following holds.*

(i) The RH problem for Φ is solvable.

(ii) The solution Φ of the RH problem for Φ has a full asymptotic expansion in powers of ζ^{-1} as follows,

$$\Phi(\zeta; s, t) \sim \Big(I + \sum_{k=1}^{\infty} A_k \zeta^{-k}\Big) \zeta^{-\frac{1}{4}\sigma_3} N e^{-\theta(\zeta;s,t)\sigma_3}, \tag{3.26}$$

as $\zeta \to \infty$, uniformly in $\mathbb{C} \setminus \Gamma$. Here, the $A_k = A_k(s,t)$ are real-valued for $s, t \in \mathbb{R}$.

(iii) The solution Φ of the RH problem for Φ, as well as the A_k in (3.26), are analytic both as functions of s and t.

3.10 Remark. The important feature of this lemma is the following. In the next subsection we will show that $y = 2A_{1,11} - A_{1,12}^2$, where $A_{1,ij}$ is the (i,j)-th entry of A_1, is a solution to the P_I^2 equation. From the above lemma we then have that this y is real-valued and pole-free on the real axis, so that the first part of Theorem 3.1 is proven.

In order to prove Lemma 3.9, we transform, as in [32, Section 5.3], the RH problem for Φ into an equivalent RH problem for $\widehat{\Phi}$ such that the jump matrix for $\widehat{\Phi}$ is continuous on Γ and converges exponentially to the identity matrix as $\zeta \to \infty$ on Γ, and such that the RH problem for $\widehat{\Phi}$ is normalized at infinity. To do this, we introduce an auxiliary 2×2 matrix-valued function M satisfying the following RH problem on a contour $\Gamma^\sigma = \bigcup_{j=1}^4 \Gamma_j^\sigma$ consisting of four straight rays

$$\Gamma_1^\sigma : \arg\zeta = 0, \quad \Gamma_2^\sigma : \arg\zeta = \sigma, \quad \Gamma_3^\sigma : \arg\zeta = \pi, \quad \Gamma_4^\sigma : \arg\zeta = -\sigma, \tag{3.27}$$

where $\sigma \in (\frac{\pi}{3}, \pi)$. We orientate the straight rays from the left to the right, as shown in Figure 3.1 for the contour Γ. The dependence on the parameter σ is needed in Section 3.3. In this section, we take $\sigma = 6\pi/7$ fixed, so that $\Gamma^\sigma = \Gamma$.

RH problem for M:

(a) M is analytic in $\mathbb{C} \setminus \Gamma^\sigma$.

(b) M satisfies the following jump relations on Γ^σ,

$$M_+(\zeta) = M_-(\zeta)\begin{pmatrix} 1 & e^{-\frac{4}{3}\zeta^{3/2}} \\ 0 & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_1^\sigma, \tag{3.28}$$

$$M_+(\zeta) = M_-(\zeta)\begin{pmatrix} 1 & 0 \\ e^{\frac{4}{3}\zeta^{3/2}} & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_2^\sigma \cup \Gamma_4^\sigma, \tag{3.29}$$

$$M_+(\zeta) = M_-(\zeta)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_3^\sigma. \tag{3.30}$$

(c) M has the following behavior at infinity,

$$M(\zeta) \sim \Big(I + \sum_{k=1}^{\infty} B_k \zeta^{-k}\Big)\zeta^{-\frac{1}{4}\sigma_3} N, \qquad \text{as } \zeta \to \infty, \tag{3.31}$$

uniformly for $\zeta \in \mathbb{C} \setminus \Gamma^\sigma$ and σ in compact subsets of $(\frac{\pi}{3}, \pi)$. Here, N is given by equation (3.16), and for $k \geq 1$,

$$B_{3k-2} = \begin{pmatrix} 0 & 0 \\ t_{2k-1} & 0 \end{pmatrix}, \quad B_{3k-1} = \begin{pmatrix} 0 & \widehat{t}_{2k-1} \\ 0 & 0 \end{pmatrix}, \quad B_{3k} = \begin{pmatrix} \widehat{t}_{2k} & 0 \\ 0 & t_{2k} \end{pmatrix}, \tag{3.32}$$

with

$$\widehat{t}_k = \frac{\Gamma(3k+1/2)}{36^k k! \Gamma(k+1/2)}, \qquad t_k = -\frac{6k+1}{6k-1}\widehat{t}_k. \tag{3.33}$$

It is well-known, see e.g. [27, 33], that there exists a unique solution M to the above RH problem given in terms of Airy functions Ai. The matrix-valued function M is the so-called Airy parametrix and for the purpose of this chapter we will not need its exact expression but refer the reader to [27, 33] for this.

We now define $\widehat{\Phi}(\zeta; s, t) = \widehat{\Phi}(\zeta)$ by

$$\widehat{\Phi}(\zeta) = \Phi(\zeta) e^{\theta(\zeta)\sigma_3} M(\zeta)^{-1}, \qquad \text{for } \zeta \in \mathbb{C} \setminus \Gamma. \tag{3.34}$$

A straightforward calculation, using (3.22)–(3.25), (3.28)–(3.31), and $\theta_+(\zeta) + \theta_-(\zeta) = 0$ for $\zeta \in \mathbb{R}_-$, shows that $\widehat{\Phi}$ satisfies the following RH problem.

RH problem for $\widehat{\Phi}$:

(a) $\widehat{\Phi}$ is analytic in $\mathbb{C} \setminus \Gamma$.

(b) $\widehat{\Phi}_+(\zeta) = \widehat{\Phi}_-(\zeta)\hat{v}(\zeta)$ for $\zeta \in \Gamma$, where $v(\zeta) = v(\zeta; s, t)$ is given by

$$v(\zeta) = \begin{cases} M_-(\zeta)\begin{pmatrix} 1 & e^{-2\theta(\zeta)} - e^{-\frac{4}{3}\zeta^{3/2}} \\ 0 & 1 \end{pmatrix} M_-(\zeta)^{-1}, & \text{on } \Gamma_1, \\ M_-(\zeta)\begin{pmatrix} 1 & 0 \\ e^{2\theta(\zeta)} - e^{\frac{4}{3}\zeta^{3/2}} & 1 \end{pmatrix} M_-(\zeta)^{-1}, & \text{on } \Gamma_2 \cup \Gamma_4, \\ I, & \text{on } \Gamma_3. \end{cases} \tag{3.35}$$

(c) $\widehat{\Phi}(\zeta) = I + \mathcal{O}(1/\zeta), \qquad \text{as } \zeta \to \infty.$

Observe that the jump matrix v is indeed continuous on Γ and that it converges exponentially to the identity matrix as $\zeta \to \infty$ on Γ. This RH problem corresponds to the RH problem [32, (5.108)–(5.110)], and the only difference is that we now have a factor $e^{\pm 2\theta}$ (containing the s, t dependence) instead of $e^{\pm\zeta^{(4\nu+3)/2}}$ in the jump matrices.

Proof of Lemma 3.9 (i). From (3.34) it follows that proving the solvability of the RH problem for Φ is equivalent to proving the solvability of the RH problem for $\widehat{\Phi}$. By general theory of the construction of solutions of RH problems, this is reduced to the study of the singular integral operator,

$$C_v : L^2(\Gamma) \to L^2(\Gamma) : f \mapsto C_+\left[f\left(I - v^{-1}\right)\right], \tag{3.36}$$

where v is the jump matrix (3.35) of the RH problem for $\widehat{\Phi}$, and where C_+ is the $+$boundary value of the Cauchy operator

$$Cf(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(s)}{s - z}ds, \qquad \text{for } z \in \mathbb{C} \setminus \Gamma.$$

Indeed, suppose that $I - C_v$ is invertible in $L^2(\Gamma)$. Then, there exists $\mu \in L^2(\Gamma)$ such that $(I - C_v)\mu = C_+(I - v^{-1})$, and it is immediate that

$$\widehat{\Phi}(\zeta) \equiv I + \frac{1}{2\pi i}\int_\Gamma \frac{(I + \mu(s))(I - v(s)^{-1})}{s - \zeta}ds, \qquad \text{for } \zeta \in \mathbb{C} \setminus \Gamma, \tag{3.37}$$

is analytic in $\mathbb{C} \setminus \Gamma$ and satisfies (since $C_+ - C_- = I$) condition (b) of the RH problem for $\widehat{\Phi}$ in the so-called L^2-sense. However, as in [32, Step 3 of Sections 5.2 and 5.3], one can use the analyticity of v to show that $\widehat{\Phi}$ satisfies jump condition (b) in the sense of continuous boundary values, as well. Further, as in [32, Proposition 5.4], it follows from the exponential decaying of $I - v^{-1}$ as $\zeta \to \infty$ on Γ that the asymptotic condition (c) of the RH problem for $\widehat{\Phi}$ is also satisfied. We summarize that the RH problem for $\widehat{\Phi}$ is solvable, with solution given by (3.37), provided the singular integral operator $I - C_v$ is invertible in $L^2(\Gamma)$.

First, we consider the case $s, t \in \mathbb{R}$. For this case, we show that $I - C_v$ is invertible by showing that it is a Fredholm operator with zero index and kernel $\{0\}$. Exactly as in [32, Steps 1 and 2 of Section 5.3] one has that $I - C_v$ is a Fredholm operator with zero index. In this step, one does not need the restriction to real s and t. It remains to prove that the kernel of $I - C_v$ is $\{0\}$, and it is in this step that we will need the restriction that $s, t \in \mathbb{R}$. This is (again) as in [32, Section 5.3] but for the convenience of the reader we will indicate were we need s and t to be real.

Suppose there exists $\mu_0 \in L^2(\Gamma)$ such that $(I - C_v)\mu_0 = 0$. One can then show that the matrix-valued function $\widehat{\Phi}_0$ defined by

$$\widehat{\Phi}_0(\zeta) \equiv \frac{1}{2\pi i} \int_\Gamma \frac{\mu_0(s)(I - v(s)^{-1})}{s - \zeta} ds, \qquad \text{for } \zeta \in \mathbb{C} \setminus \Gamma, \tag{3.38}$$

is a solution to the RH problem for $\widehat{\Phi}$, but with the asymptotic condition (c) replaced by the *homogeneous* condition

$$\widehat{\Phi}_0(\zeta) = \mathcal{O}(1/\zeta), \qquad \text{as } \zeta \to \infty, \text{ uniformly for } \zeta \in \mathbb{C} \setminus \Gamma. \tag{3.39}$$

Since $\mu_0 = \widehat{\Phi}_{0,+}$ (which follows from (3.38) together with $(I - C_v)\mu_0 = 0$), we need to show that $\widehat{\Phi}_0 \equiv 0$. Showing that a solution of the homogeneous RH problem is identically zero, is known in the literature as a vanishing lemma, see [32, 48, 49].

Now, let

$$\Phi_0(\zeta) = \widehat{\Phi}_0(\zeta) M(\zeta), \qquad \text{for } \zeta \in \mathbb{C} \setminus \Gamma,$$

then it is straightforward to check, using (3.28)–(3.31), (3.35), and (3.39), that Φ_0 solves the following RH problem.

RH problem for Φ_0:

(a) Φ_0 is analytic in $\mathbb{C} \setminus \Gamma$.

(b) Φ_0 satisfies the following jump relations on Γ,

$$\Phi_{0,+}(\zeta) = \Phi_{0,-}(\zeta) \begin{pmatrix} 1 & e^{-2\theta(\zeta)} \\ 0 & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_1, \tag{3.40}$$

$$\Phi_{0,+}(\zeta) = \Phi_{0,-}(\zeta) \begin{pmatrix} 1 & 0 \\ e^{2\theta(\zeta)} & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_2 \cup \Gamma_4, \tag{3.41}$$

$$\Phi_{0,+}(\zeta) = \Phi_{0,-}(\zeta) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_3. \tag{3.42}$$

(c) $\Phi_0(\zeta) = \mathcal{O}(1/\zeta)\zeta^{-\frac{1}{4}\sigma_3} N, \qquad \text{as } \zeta \to \infty, \text{ uniformly for } \zeta \in \mathbb{C} \setminus \Gamma.$

Further, we introduce an auxiliary matrix-valued function A with jumps only on $\mathbb{R}$, as follows, cf. [32, Equations (5.135)–(5.138)]

$$A(\zeta) = \begin{cases} \Phi_0(\zeta)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & \text{for } 0 < \arg\zeta < \frac{6\pi}{7}, \\ \Phi_0(\zeta)\begin{pmatrix} 1 & 0 \\ e^{2\theta(\zeta)} & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & \text{for } \frac{6\pi}{7} < \arg\zeta < \pi, \\ \Phi_0(\zeta)\begin{pmatrix} 1 & 0 \\ -e^{2\theta(\zeta)} & 1 \end{pmatrix}, & \text{for } -\pi < \arg\zeta < -\frac{6\pi}{7}, \\ \Phi_0(\zeta), & \text{for } -\frac{6\pi}{7} < \arg\zeta < 0. \end{cases} \tag{3.43}$$

Using (3.40)–(3.42) and condition (c) of the RH problem for Φ_0 one can then check that A is a solution to the following RH problem.

RH problem for A:

(a) A is analytic in $\mathbb{C} \setminus \mathbb{R}$

(b) A satisfies the following jump relation on $\mathbb{R}$,

$$A_+(\zeta) = A_-(\zeta)\begin{pmatrix} 1 & -e^{2\theta_+(\zeta)} \\ e^{2\theta_-(\zeta)} & 0 \end{pmatrix}, \qquad \text{for } \zeta \in \mathbb{R}_-, \tag{3.44}$$

$$A_+(\zeta) = A_-(\zeta)\begin{pmatrix} e^{-2\theta(\zeta)} & -1 \\ 1 & 0 \end{pmatrix}, \qquad \text{for } \zeta \in \mathbb{R}_+, \tag{3.45}$$

(c) $A(\zeta) = \mathcal{O}(\zeta^{-3/4})$, as $\zeta \to \infty$, uniformly for $\zeta \in \mathbb{C} \setminus \mathbb{R}$.

Now, we define $Q(\zeta) = A(\zeta)A^*(\bar{\zeta})$, where A^* denotes the Hermitian conjugate of A. The matrix-valued function Q is analytic in the upper half plane, continuous up to $\mathbb{R}$, and decays like $\zeta^{-3/2}$ as $\zeta \to \infty$. By Cauchy's theorem this implies, $\int_{\mathbb{R}} Q_+(s)ds = 0$. Using the jump relations (3.44) and (3.45) we then have,

$$\int_{\mathbb{R}_-} A_-(s)\begin{pmatrix} 1 & -e^{2\theta_+(s)} \\ e^{2\theta_-(s)} & 0 \end{pmatrix} A_-^*(s)ds + \int_{\mathbb{R}_+} A_-(s)\begin{pmatrix} e^{-2\theta(s)} & -1 \\ 1 & 0 \end{pmatrix} A_-^*(s)ds = 0.$$

Adding this to its Hermitian conjugate, and using the fact $\overline{\theta_+(s)} = \theta_-(s)$ for $s \in \mathbb{R}_-$ (which is true since $s, t \in \mathbb{R}$), we arrive at, cf. [32, Equation (5.146)]

$$\int_{\mathbb{R}_-} A_-(s)\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} A_-^*(s)ds + \int_{\mathbb{R}_+} A_-(s)\begin{pmatrix} 2e^{-2\theta(s)} & 0 \\ 0 & 0 \end{pmatrix} A_-^*(s)ds = 0. \tag{3.46}$$

This is the crucial step where we need s and t to be real. The latter relation implies that the first column of A_- is identically zero, and the jump relations (3.44) and (3.45) then imply that the second column of A_+ is identically zero, as well.

By writing out the RH conditions for each entry of A and using the vanishing of the first column of A_- and the second column of A_+, the matrix RH problem reduces to two scalar RH problems. The proof that the solutions of those scalar RH problems (and thus also the second column of A_- and the first column of A_+) are identically zero, is exactly as in [32, Step 3 of Section 5.3] using Carlson's theorem, see [89], and we will not go into detail about this. We then have shown that $A \equiv 0$, so that also $\widehat{\Phi}_0 \equiv 0$ and thus $\mu_0 \equiv 0$. We now have proven that $I - C_v$ is invertible for $s, t \in \mathbb{R}$, which implies that the RH problem for $\widehat{\Phi}$ (and thus also the RH problem for Φ) is solvable for $s, t \in \mathbb{R}$.

Next, fix $s_0, t_0 \in \mathbb{R}$. Above, we have shown that the singular integral operator $I - C_{v(\cdot\,;\,s_0,t_0)}$ is invertible. Since

$$I - C_{v(\cdot\,;\,s,t)} = \big(I - C_{v(\cdot\,;\,s_0,t_0)}\big)\Big[I + \big(I - C_{v(\cdot\,;\,s_0,t_0)}\big)^{-1}\big(C_{v(\cdot\,;\,s_0,t_0)} - C_{v(\cdot\,;\,s,t)}\big)\Big],$$

it then follows that $I - C_{v(\cdot\,;\,s,t)}$ is invertible provided

$$\left\|\big(I - C_{v(\cdot\,;\,s_0,t_0)}\big)^{-1}\big(C_{v(\cdot\,;\,s_0,t_0)} - C_{v(\cdot\,;\,s,t)}\big)\right\| < 1, .$$

where $\|\cdot\|$ denotes the operator norm. It is straightforward to check that there exist neighborhoods $\mathcal{V}$ of s_0 and $\mathcal{W}$ of t_0 such that for all $s \in \mathcal{V}$ and $t \in \mathcal{W}$,

$$\begin{aligned}\left\|C_{v(\cdot\,;\,s_0,t_0)} - C_{v(\cdot\,;\,s,t)}\right\| &\le \left\|C_+\right\| \left\|v(\cdot\,;\,s_0,t_0) - v(\cdot\,;\,s,t)\right\|_{L^{(\infty)}(\Gamma)} \\ &< \left\|\big(I - C_{v(\cdot\,;\,s_0,t_0)}\big)^{-1}\right\|^{-1},\end{aligned}$$

which implies that the operator $I - C_{v(\cdot\,;\,s,t)}$ is invertible. Hence the RH problem for $\widehat{\Phi}$, and thus also the RH problem for Φ, is solvable for $s \in \mathcal{V}$ and $t \in \mathcal{W}$. This finishes the proof of the first part of the lemma. □

Proof of Lemma 3.9 (ii). It follows from the asymptotic expansion (3.31) of M together with $\Phi = \widehat{\Phi} M e^{-\theta\sigma_3}$, see (3.34), that we need to show that $\widehat{\Phi}$ has a full asymptotic expansion in powers of ζ^{-1}. Insert the relation

$$\frac{1}{s-\zeta} = -\sum_{k=1}^{n} s^{k-1}\zeta^{-k} + \frac{s^n}{\zeta^n(s-\zeta)}, \qquad \text{for } n \in \mathbb{N},$$

into the solution $\widehat{\Phi}$ of the RH problem for $\widehat{\Phi}$, which is given by (3.37). We then obtain for any $n \in \mathbb{N}$,

$$\widehat{\Phi} = I + \sum_{k=1}^{n} \widehat{B}_k \zeta^{-k} + \frac{1}{2\pi i}\int_\Gamma \frac{s^n(I+\mu(s))(I - v(\zeta)^{-1})}{\zeta^n(s-\zeta)} ds, \tag{3.47}$$

where

$$\widehat{B}_k = -\frac{1}{2\pi i}\int_\Gamma s^{k-1}(I+\mu(s))(I-v(s)^{-1})ds. \tag{3.48}$$

As in [32, Proposition 5.4] one can check that the integral in (3.47) is of order $\mathcal{O}(\zeta^{-(n+1)})$ as $\zeta \to \infty$ uniformly for $\zeta \in \mathbb{C}\setminus\Gamma$. We then have shown that $\widehat{\Phi}$ has the following asymptotic expansion in powers of ζ^{-1},

$$\widehat{\Phi}(\zeta) \sim I + \sum_{k=1}^{\infty} \widehat{B}_k \zeta^{-k}, \qquad \text{as } \zeta \to \infty, \text{ uniformly for } \zeta \in \mathbb{C}\setminus\Gamma. \tag{3.49}$$

From (3.31), (3.49), and the fact that $\Phi = \widehat{\Phi} P e^{-\theta\sigma_3}$ it now follows that Φ has a full asymptotic expansion in the form (3.26), where (with $B_0 = \widehat{B}_0 = I$)

$$A_k = \sum_{j=0}^{k} B_j \widehat{B}_{k-j}. \tag{3.50}$$

It remains to show that the A_k are real-valued for $s,t \in \mathbb{R}$. It is straightforward to verify that for $s,t \in \mathbb{R}$ the matrix-valued function $-i\overline{\Phi(\overline{\zeta};s,t)}\sigma_3$ is a solution to the RH problem for Φ. By uniqueness we then have

$$\Phi(\zeta;s,t) = -i\overline{\Phi(\overline{\zeta};s,t)}\sigma_3, \qquad \text{for } s,t \in \mathbb{R},$$

which yields

$$\Big(I + \sum_{k=1}^{\infty} A_k \zeta^{-k}\Big)\zeta^{-\frac{1}{4}\sigma_3} N e^{-\theta(\zeta;s,t)\sigma_3} = \Big(I + \sum_{k=1}^{\infty} \overline{A_k} \zeta^{-k}\Big)\zeta^{-\frac{1}{4}\sigma_3} N e^{-\theta(\zeta;s,t)\sigma_3},$$

and hence $A_k = \overline{A_k}$ for $s,t \in \mathbb{R}$. This proves the second part of the lemma. □

Proof of Lemma 3.9 (iii). We show that Φ and A_k are analytic in s, for $s \in \mathcal{V}$. The analyticity in t follows in a similar fashion. In order to show that $\widehat{\Phi}$ (and thus also Φ) is analytic for $s \in \mathcal{V}$ we need to show that, letting $h \to 0$ in the complex plane,

$$\lim_{h\to 0} \frac{1}{h}\big(\widehat{\Phi}(\zeta;s+h,t) - \widehat{\Phi}(\zeta;s,t)\big)$$

exists. Consider the 2×2 auxiliary matrix-valued function $H(\zeta;s,t;h) = H(\zeta)$ defined as follows,

$$H(\zeta) = \widehat{\Phi}(\zeta;s+h,t)\widehat{\Phi}(\zeta;s,t)^{-1}, \qquad \text{for } \zeta \in \mathbb{C}\setminus\Gamma. \tag{3.51}$$

Here we take h sufficiently small, so that $\Phi(\zeta;s+h,t)$ exists by part (i) of the lemma. It is straightforward to check that H satisfies the following RH problem.

RH problem for H:

(a) H is analytic in $\mathbb{C} \setminus \Gamma$.

(b) H satisfies the jump relation $H_+(\zeta) = H_-(\zeta) v_H(\zeta)$ for $\zeta \in \Gamma$, where

$$\begin{aligned} v_H(\zeta) &= I + e^{-2\theta(\zeta;s,t)}(e^{-2h\zeta^{1/2}} - 1)\widehat{\Phi}_-(\zeta;s,t) \\ &\qquad \times \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \widehat{\Phi}_-(\zeta;s,t)^{-1}, && \zeta \in \Gamma_1, \\ v_H(\zeta) &= I + e^{2\theta(\zeta;s,t)}(e^{2h\zeta^{1/2}} - 1)\widehat{\Phi}_-(\zeta;s,t) \\ &\qquad \times \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \widehat{\Phi}_-(\zeta;s,t)^{-1}, && \zeta \in \Gamma_2 \cup \Gamma_4, \\ v_H(\zeta) &= I, && \zeta \in \Gamma_3. \end{aligned}$$

(c) $H(\zeta) = I + \mathcal{O}(1/\zeta)$, as $\zeta \to \infty$, uniformly for $\zeta \in \mathbb{C} \setminus \Gamma$.

Since $v_H(\zeta) = I + \mathcal{O}(h)$ as $h \to 0$ uniformly for $\zeta \in \Gamma$, where the $\mathcal{O}(h)$-term can be expanded into a full asymptotic expansion in powers of h, it follows as in [27, 33, 32] that, as $h \to 0$,

$$\widehat{\Phi}(\zeta; s+h, t)\widehat{\Phi}(\zeta; s, t)^{-1} = H(\zeta) = I + hH_1(\zeta; s, t) + \mathcal{O}(h^2), \tag{3.52}$$

where H_1 is a 2×2 matrix-valued function independent of h. This yields,

$$\lim_{h \to 0} \frac{1}{h}\big(\widehat{\Phi}(\zeta; s+h, t) - \widehat{\Phi}(\zeta; s, t)\big) = H_1(\zeta; s, t)\widehat{\Phi}(\zeta; s, t),$$

which implies that $\widehat{\Phi}$ (and thus also Φ) is analytic for $s \in \mathcal{V}$.

It remains to show that the matrix-valued functions A_k are analytic for $s \in \mathcal{V}$. By (3.37), it is immediate that

$$\widehat{\Phi}_+(\zeta) = I + \mu(\zeta), \qquad \text{for } \zeta \in \Sigma,$$

so that μ is analytic for $s \in \mathcal{V}$. By (3.48) it then follows that $\widehat{B}_k$ is also analytic for $s \in \mathcal{V}$. This yields, by (3.50) and (3.32), the analyticity of A_k, and hence the last part of the lemma is proven. □

3.2.3 Proof of Theorem 3.1 (i)

In order to prove the existence part of Theorem 3.1 we proceed as follows. Introduce, for $s, t \in \mathbb{R}$, a 2×2 matrix-valued function $\Psi(\zeta; s, t) = \Psi(\zeta)$ by multiplying

the solution Φ of the RH problem for Φ to the left with an appropriate matrix independent of ζ,

$$\Psi(\zeta) = \begin{pmatrix} 1 & 0 \\ A_{1,12} & 1 \end{pmatrix} \Phi(\zeta), \qquad \text{for } \zeta \in \mathbb{C} \setminus \Gamma. \tag{3.53}$$

Here $A_{1,12}$ is the $(1,2)$-entry of the 2×2 matrix $A_1 = A_1(s,t)$ appearing in the asymptotic expansion (3.26) of Φ at infinity. The important feature of this transformation is that Ψ satisfies the RH problem for P_I^2, see Section 3.1.4, as we will show in the following proposition.

3.11 Proposition. *The matrix-valued function Ψ, defined by* (3.53)*, is a solution to the RH problem for Ψ, see Section* 3.1.4*, with Stokes multipliers $s_1 = s_2 = s_5 = s_6 = 0$, $s_0 = 1$, and $s_3 = s_4 = -1$, and with the asymptotic condition (c) replaced by the stronger condition*

$$\Psi(\zeta) = \zeta^{-\frac{1}{4}\sigma_3} N \widehat{\Psi}(\zeta) e^{-\theta(\zeta)\sigma_3}, \tag{3.54}$$

where $\widehat{\Psi}$ has a full asymptotic expansion in powers of $\zeta^{-1/2}$ as follows,

$$\begin{aligned} \widehat{\Psi}(\zeta; s,t) \sim I - h\sigma_3 \zeta^{-1/2} + \frac{1}{2} \begin{pmatrix} h^2 & iy \\ -iy & h^2 \end{pmatrix} \zeta^{-1} \\ + \frac{1}{2} \sum_{k=1}^{\infty} \left[\begin{pmatrix} q_k & ir_k \\ ir_k & -q_k \end{pmatrix} \zeta^{-k-\frac{1}{2}} + \begin{pmatrix} v_k & iw_k \\ -iw_k & v_k \end{pmatrix} \zeta^{-k-1} \right], \end{aligned} \tag{3.55}$$

as $\zeta \to \infty$ uniformly for $\zeta \in \mathbb{C} \setminus \Gamma$. Here, $y = y(s,t)$ is given by

$$y = 2A_{1,11} - A_{1,12}^2. \tag{3.56}$$

Further, $h = A_{1,12}$ and the q_k, r_k, v_k and w_k are some unimportant functions of s and t (independent of ζ).

Proof. The fact that Ψ satisfies conditions (a) and (b) of the RH problem for Ψ follows trivially from (3.53) together with conditions (a) and (b) of the RH problem for Φ. So, it remains to show that $\widehat{\Psi}$ given by

$$\widehat{\Psi} = N^{-1} \zeta^{\frac{\sigma_3}{4}} \Psi(\zeta) e^{\theta(\zeta)\sigma_3}, \tag{3.57}$$

satisfies an asymptotic expansion of the form (3.55) with y given by (3.56). It follows from (3.57), (3.53) and (3.26) that

$$\begin{aligned} \widehat{\Psi}(\zeta) &\sim N^{-1} \zeta^{\frac{\sigma_3}{4}} \begin{pmatrix} 1 & 0 \\ A_{1,12} & 1 \end{pmatrix} \left[I + \sum_{k=1}^{\infty} A_k \zeta^{-k} \right] \zeta^{-\frac{\sigma_3}{4}} N \\ &\sim N^{-1} \left(\sum_{k=0}^{\infty} \zeta^{\frac{\sigma_3}{4}} \tilde{A}_k \zeta^{-\frac{\sigma_3}{4}} \zeta^{-k} \right) N, \end{aligned} \tag{3.58}$$

where

$$\tilde{A}_0 = \begin{pmatrix} 1 & 0 \\ A_{1,12} & 1 \end{pmatrix}, \qquad \text{and} \quad \tilde{A}_k = \begin{pmatrix} 1 & 0 \\ A_{1,12} & 1 \end{pmatrix} A_k, \qquad \text{for } k \geq 1.$$

Now, using the facts that $\tilde{A}_{0,11} = \tilde{A}_{0,22} = 1$, that $\tilde{A}_{0,12} = 0$ and that $\tilde{A}_{0,21} = \tilde{A}_{1,12} = A_{1,12}$ we find,

$$\begin{aligned}
&\sum_{k=0}^{\infty} \zeta^{\frac{\sigma_3}{4}} \tilde{A}_k \zeta^{-\frac{\sigma_3}{4}} \zeta^{-k} \\
&= \sum_{k=0}^{\infty} \left[\begin{pmatrix} 0 & 0 \\ \tilde{A}_{k,21} & 0 \end{pmatrix} \zeta^{-k-\frac{1}{2}} + \begin{pmatrix} 0 & \tilde{A}_{k,12} \\ 0 & 0 \end{pmatrix} \zeta^{-k+\frac{1}{2}} + \begin{pmatrix} \tilde{A}_{k,11} & 0 \\ 0 & \tilde{A}_{k,22} \end{pmatrix} \zeta^{-k} \right] \\
&= I + A_{1,12} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \zeta^{-1/2} + \begin{pmatrix} \tilde{A}_{1,11} & 0 \\ 0 & \tilde{A}_{1,22} \end{pmatrix} \zeta^{-1} \\
&\qquad + \sum_{k=1}^{\infty} \left[\begin{pmatrix} 0 & \tilde{A}_{k+1,12} \\ \tilde{A}_{k,21} & 0 \end{pmatrix} \zeta^{-k-\frac{1}{2}} + \begin{pmatrix} \tilde{A}_{k+1,11} & 0 \\ 0 & \tilde{A}_{k+1,22} \end{pmatrix} \zeta^{-k-1} \right].
\end{aligned}$$

Inserting this into (3.58) and using (3.16) we arrive at,

$$\begin{aligned}
\widehat{\Psi}(\zeta; s, t) \sim I - h\sigma_3 \zeta^{-1/2} + \frac{1}{2} \begin{pmatrix} \tilde{A}_{1,11} + \tilde{A}_{1,22} & i(\tilde{A}_{1,11} - \tilde{A}_{1,22}) \\ -i(\tilde{A}_{1,11} - \tilde{A}_{1,22}) & \tilde{A}_{1,11} + \tilde{A}_{1,22} \end{pmatrix} \zeta^{-1} \\
+ \frac{1}{2} \sum_{k=1}^{\infty} \left[\begin{pmatrix} q_k & ir_k \\ ir_k & -q_k \end{pmatrix} \zeta^{-k-\frac{1}{2}} + \begin{pmatrix} v_k & iw_k \\ -iw_k & v_k \end{pmatrix} \zeta^{-k-1} \right], \qquad (3.59)
\end{aligned}$$

where $h = A_{1,12}$ and where the q_k, r_k, v_k, and w_k can be written down explicitly in terms of $\tilde{A}_k$ and $\tilde{A}_{k+1}$. Now, note that since $\det \Phi \equiv 1$ (see Remark 3.8) and since, by (3.26),

$$\det \Phi = 1 + (A_{1,11} + A_{1,22})\zeta^{-1} + \mathcal{O}(\zeta^{-2}), \qquad \text{as } \zeta \to \infty,$$

we have that $A_{1,22} = -A_{1,11}$. This together with the facts that $\tilde{A}_{1,11} = A_{1,11}$ and $\tilde{A}_{1,22} = A_{1,12}^2 + A_{1,22}$ yields,

$$\tilde{A}_{1,11} + \tilde{A}_{1,22} = A_{1,12}^2 = h^2, \qquad \tilde{A}_{1,11} - \tilde{A}_{1,22} = 2A_{1,11} - A_{1,12}^2 = y.$$

Inserting this into (3.59) the proposition is proven. □

The idea is now to show that Ψ satisfies the linear system of differential equations (3.18)–(3.21) with y given by (3.56), so that by compatibility of the Lax pair this y is a solution to the P_I^2 equation (3.4). Since by Lemma 3.9 the functions $A_{1,11}$ and $A_{1,12}$ are real-valued and pole-free for $s, t \in \mathbb{R}$ we have that y itself is real-valued and pole-free for $s, t \in \mathbb{R}$, so that the first part of Theorem 3.1 is proven.

Proof of Theorem 3.1 (i). Recall from the above discussion that we need to show that the matrix-valued functions (note that, by Lemma 3.9 (iii) and (3.53), Ψ is differentiable with respect to s)

$$U = \frac{\partial \Psi}{\partial \zeta}\,\Psi^{-1} \qquad \text{and} \qquad W = \frac{\partial \Psi}{\partial s}\,\Psi^{-1}, \tag{3.60}$$

are of the form (3.19) and (3.21), respectively, with y given by (3.56). Observe that, since Ψ has constant jump matrices, the derivatives $\frac{\partial \Psi}{\partial \zeta}$ and $\frac{\partial \Psi}{\partial s}$ have the same jumps as Ψ, and hence U and W are entire.

First, we focus on U. By (3.54), we have that for $\zeta \to \infty$,

$$U = -\frac{\partial \theta}{\partial \zeta}\,\zeta^{-\frac{\sigma_3}{4}}\left(N\widehat{\Psi}\sigma_3\widehat{\Psi}^{-1}N^{-1}\right)\zeta^{\frac{\sigma_3}{4}} + \begin{pmatrix} \mathcal{O}(\zeta^{-1}) & \mathcal{O}(\zeta^{-2}) \\ \mathcal{O}(\zeta^{-1}) & \mathcal{O}(\zeta^{-1}) \end{pmatrix}. \tag{3.61}$$

Since $\det \Phi \equiv 1$ we obtain from (3.53) and (3.54) that $\det \widehat{\Psi} \equiv 1$, as well. Then, it is easy to verify that

$$\widehat{\Psi}\sigma_3\widehat{\Psi}^{-1} = \begin{pmatrix} 1 + 2\widehat{\Psi}_{12}\widehat{\Psi}_{21} & -2\widehat{\Psi}_{11}\widehat{\Psi}_{12} \\ 2\widehat{\Psi}_{21}\widehat{\Psi}_{22} & -1 - 2\widehat{\Psi}_{12}\widehat{\Psi}_{21} \end{pmatrix} \equiv \begin{pmatrix} Q_{11} & -iQ_{12} \\ -iQ_{21} & -Q_{11} \end{pmatrix},$$

and hence, by (3.16),

$$N\widehat{\Psi}\sigma_3\widehat{\Psi}^{-1}N^{-1} = \begin{pmatrix} \frac{1}{2}(Q_{21} - Q_{12}) & -\frac{1}{2}(Q_{21} + Q_{12}) - Q_{11} \\ \frac{1}{2}(Q_{21} + Q_{12}) - Q_{11} & \frac{1}{2}(Q_{12} - Q_{21}) \end{pmatrix}. \tag{3.62}$$

The asymptotics of the functions Q_{11}, Q_{12} and Q_{21} at infinity follow from the asymptotic behavior (3.55) of $\widehat{\Psi}$. We find, as $\zeta \to \infty$,

$$Q_{11} = 1 + \frac{1}{2}y^2\zeta^{-2} + \big(yw_1 - \frac{1}{2}r_1^2\big)\zeta^{-3} + \mathcal{O}(\zeta^{-4}), \tag{3.63}$$

$$\begin{aligned} Q_{12} = y\zeta^{-1} + (r_1 - yh)\zeta^{-3/2} + \big(\frac{1}{2}yh^2 - hr_1 + w_1\big)\zeta^{-2} \\ + \frac{1}{8}t\zeta^{-5/2} + u\zeta^{-3} + v\zeta^{-7/2} + \mathcal{O}(\zeta^{-4}), \end{aligned} \tag{3.64}$$

$$\begin{aligned} Q_{21} = y\zeta^{-1} - (r_1 - yh)\zeta^{-3/2} + \big(\frac{1}{2}yh^2 - hr_1 + w_1\big)\zeta^{-2} \\ - \frac{1}{8}t\zeta^{-5/2} + u\zeta^{-3} - v\zeta^{-7/2} + \mathcal{O}(\zeta^{-4}), \end{aligned} \tag{3.65}$$

where t, u and v are some functions of s and t. Inserting (3.62)–(3.65) into (3.61) and using the fact that,

$$\frac{\partial \theta}{\partial \zeta} = \frac{1}{30}\zeta^{5/2} - \frac{1}{2}t\zeta^{1/2} + \frac{1}{2}s\zeta^{-1/2},$$

it is straightforward to check that,

$$U = \frac{1}{240}\begin{pmatrix} a\zeta + t & 8\zeta^2 + 8y\zeta + b + e\zeta^{-1} \\ 8\zeta^3 - 8y\zeta^2 + c\zeta + d & -a\zeta - t \end{pmatrix} + \begin{pmatrix} \mathcal{O}(\zeta^{-1}) & \mathcal{O}(\zeta^{-2}) \\ \mathcal{O}(\zeta^{-1}) & \mathcal{O}(\zeta^{-1}) \end{pmatrix},$$

with

$$a = 8r_1 - 8yh, \tag{3.66}$$

$$b = 4y^2 - 120t + 4yh^2 - 8hr_1 + 8w_1, \tag{3.67}$$

$$c = 4y^2 - 120t - 4yh^2 + 8hr_1 - 8w_1, \tag{3.68}$$

$$d = 8yw_1 - 4r_1^2 + 120s + 120yt - 8u, \tag{3.69}$$

$$e = 8yw_1 - 4r_1^2 + 120s - 120yt + 8u. \tag{3.70}$$

Since U is entire, it contains no negative powers of ζ. In particular $e = 0$, so that

$$d = d + e = 16yw_1 - 8r_1^2 + 240s. \tag{3.71}$$

We now have shown that,

$$U = \frac{1}{240}\begin{pmatrix} a\zeta + t & 8\zeta^2 + 8y\zeta + b \\ 8\zeta^3 - 8y\zeta^2 + c\zeta + d & -a\zeta - t \end{pmatrix}, \tag{3.72}$$

where a, b, and c are given by (3.66) and (3.67), and (3.68), and where d is given by (3.71).

Next, we consider W. Observe that by (3.54),

$$W = \zeta^{-\frac{\sigma_3}{4}} N \frac{\partial \widehat{\Psi}}{\partial s} \widehat{\Psi}^{-1} N^{-1} \zeta^{\frac{\sigma_3}{4}} - \frac{\partial \theta}{\partial s} \zeta^{-\frac{\sigma_3}{4}} \left(N \widehat{\Psi} \sigma_3 \widehat{\Psi}^{-1} N^{-1} \right) \zeta^{\frac{\sigma_3}{4}}. \tag{3.73}$$

From (3.55) we obtain

$$\begin{aligned} \zeta^{-\frac{\sigma_3}{4}} N \frac{\partial \widehat{\Psi}}{\partial s} \widehat{\Psi}^{-1} N^{-1} \zeta^{\frac{\sigma_3}{4}} &= \zeta^{-\frac{\sigma_3}{4}} N \left(-h_s \sigma_3 \zeta^{-1/2} + \mathcal{O}(\zeta^{-1}) \right) N^{-1} \zeta^{\frac{\sigma_3}{4}} \\ &= \begin{pmatrix} 0 & 0 \\ h_s & 0 \end{pmatrix} + \mathcal{O}(\zeta^{-1/2}), \end{aligned} \tag{3.74}$$

where h_s denotes the derivative of h with respect to s. Further, using (3.62)–(3.65) together with the fact that $\frac{\partial \theta}{\partial s} = \zeta^{1/2}$, we have

$$-\frac{\partial \theta}{\partial s} \zeta^{-\frac{\sigma_3}{4}} \left(N \widehat{\Psi} \sigma_3 \widehat{\Psi}^{-1} N^{-1} \right) \zeta^{\frac{\sigma_3}{4}} = \begin{pmatrix} 0 & 1 \\ \zeta - y & 0 \end{pmatrix} + \mathcal{O}(\zeta^{-1}). \tag{3.75}$$

Inserting (3.74) and (3.75) into (3.73), and using the fact that W is entire (so that W contains no negative powers of ζ) we arrive at,

$$W = \begin{pmatrix} 0 & 1 \\ \zeta + (h_s - y) & 0 \end{pmatrix}. \tag{3.76}$$

We will now complete the proof by determining the functions a, b, c, d, t and h_s exclusively in terms of y, y_s, y_{ss}, and y_{sss}, using the compatibility condition

$$\frac{\partial^2 \Psi}{\partial \zeta \partial s} = \frac{\partial^2 \Psi}{\partial s \partial \zeta}.$$

This condition is equivalent to $\dfrac{\partial U}{\partial s} - \dfrac{\partial W}{\partial \zeta} + UW - WU = 0$ and leads, after a straightforward calculation, to

$$C_0 \zeta^2 + C_1 \zeta + C_2 = 0,$$

where

$$C_0 = \begin{pmatrix} 8(h_s + y) & 0 \\ -8y_s - 2a & 8(h_s + y) \end{pmatrix}, \tag{3.77}$$

$$C_1 = \begin{pmatrix} a_s + 8y(h_s - y) + b - c & 8y_s + 2a \\ c_s - 2a(h_s - y) - 2t & -a_s - 8y(h_s - y) - b + c \end{pmatrix}, \tag{3.78}$$

$$C_2 = \begin{pmatrix} t_s + b(h_s - y) - d & b_s + 2t \\ d_s - 2t(h_s - y) - 240 & -t_s - b(h_s - y) + d \end{pmatrix}. \tag{3.79}$$

Since $C_0 = 0$ we deduce that $h_s = -y$, and hence by (3.76) W is of the form (3.21), and that $a = -4y_s$. By (3.66) we then have,

$$r_1 = -\frac{1}{2} y_s + yh. \tag{3.80}$$

Further, since $C_{1,11} = 0$ we then obtain from (3.67) and (3.68) that

$$w_1 = \frac{1}{4} y_{ss} + y^2 + \frac{1}{2} yh^2 - \frac{1}{2} y_s h. \tag{3.81}$$

Inserting the expressions (3.80) and (3.81) for r_1 and w_1 into the expressions (3.66), (3.67), (3.68), and (3.71) for a, b, c, and d, and using the fact that $t = -\frac{1}{2} b_s$ (since $C_{2,12} = 0$) we arrive at

$$a = -4y_s, \qquad b = 12y^2 + 2y_{ss} - 120t, \tag{3.82}$$

$$c = -4y^2 - 2y_{ss} - 120t, \qquad d = 16y^3 - 2y_s^2 + 4yy_{ss} + 240s, \tag{3.83}$$

$$t = -12yy_s - y_{sss}. \tag{3.84}$$

Inserting the latter equations into (3.72) we have that U is of the form (3.19). Note that the fact that y satisfies the P_I^2 equation now follows from $C_{2,11} = 0$. This proves the first part of the theorem. □

3.12 Remark. Note that, by Lemma 3.9 (iii), we can safely differentiate y and h with respect to s, as we did in the above proof.

3.3 Asymptotic behavior of $y(s,t)$ as $s \to \pm\infty$

In this section we will determine for fixed $t \in \mathbb{R}$ the asymptotics (as $s \to \pm\infty$) of the particular solution $y(s,t)$ of the P_I^2 equation with no poles on the real line as constructed in the previous section and given by, cf. (3.56),

$$y = 2A_{1,11} - A_{1,12}^2. \tag{3.85}$$

Here, A_1 is the matrix-valued function appearing in the asymptotic expansion (3.26) for Φ. So, it suffices to determine the asymptotics (as $s \to \pm\infty$) of the first row of A_1 which we will do by applying the Deift/Zhou steepest descent method [27, 32, 33, 36, 37] to the RH problem for Φ.

3.3.1 Rescaling of the RH problem and deformation of the jump contour: $\Phi \mapsto Y$

Let $z_0 = z_0(s,t) \in \mathbb{R}$ (to be determined in Section 3.3.2) and let $\hat{\Gamma} = \bigcup_{j=1}^4 \hat{\Gamma}_j$ be the oriented contour through z_0 as shown in Figure 3.2. Here, the dotted lines are in fact Γ_2 and Γ_4, see Figure 3.1, and are not part of the contour. The precise form of the contour $\hat{\Gamma}$ (in particular of $\hat{\Gamma}_2$ and $\hat{\Gamma}_4$) will be determined below. Now, introduce the 2×2 matrix-valued function $Y(\zeta; s,t) = Y(\zeta)$ as follows,

$$Y(\zeta) \equiv \begin{cases} \Phi(|s|^{1/3}\zeta), & \text{for } \zeta \in \mathrm{I} \cup \mathrm{II} \cup \mathrm{III} \cup \mathrm{IV}, \\ \Phi(|s|^{1/3}\zeta) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & \text{for } \zeta \in \mathrm{V}, \\ \Phi(|s|^{1/3}\zeta) \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, & \text{for } \zeta \in \mathrm{VI}, \end{cases} \tag{3.86}$$

where Φ is the solution of the RH problem for Φ, see Section 3.2.1, and where the sets I,II,...,VI are defined by Figure 3.2. Then, it is straightforward to check, using (3.22)–(3.24), (3.26) and (3.16), that Y satisfies the following conditions.

RH problem for Y:

(a) Y is analytic in $\mathbb{C} \setminus \hat{\Gamma}$.

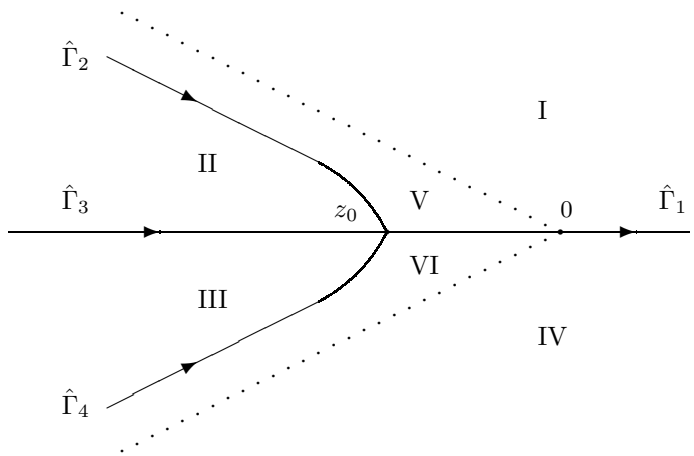

Figure 3.2 *The contour* $\hat{\Gamma} = \bigcup_{j=1}^{4} \hat{\Gamma}_j$. *Note that the dotted lines are not part of the contour.*

(b) Y satisfies the same jump relations on $\hat{\Gamma}$ as Φ does on Γ. Namely,

$$Y_+(\zeta) = Y_-(\zeta) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \hat{\Gamma}_1, \tag{3.87}$$

$$Y_+(\zeta) = Y_-(\zeta) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \hat{\Gamma}_2 \cup \hat{\Gamma}_4, \tag{3.88}$$

$$Y_+(\zeta) = Y_-(\zeta) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \text{for } \zeta \in \hat{\Gamma}_3. \tag{3.89}$$

(c) Y has the following behavior as $\zeta \to \infty$,

$$Y(\zeta) \sim \left(I + \sum_{k=1}^{\infty} A_k |s|^{-k/3} \zeta^{-k} \right) \zeta^{-\frac{\sigma_3}{4}} |s|^{-\frac{\sigma_3}{12}} N e^{-|s|^{7/6} \hat{\theta}(\zeta;s,t)\sigma_3}, \tag{3.90}$$

where

$$\hat{\theta}(\zeta; s, t) = \frac{1}{105}\zeta^{7/2} - \frac{1}{3}|s|^{-2/3} t \zeta^{3/2} + \operatorname{sgn}(s)\zeta^{1/2}. \tag{3.91}$$

3.3.2 Normalization of the RH problem at infinity: $Y \mapsto S$

In order to normalize the RH problem for Y at infinity we proceed as Kapaev in [67]. Introduce a function $g(\zeta; s, t) = g(\zeta)$ of the following form,

$$g(\zeta) = c_1(\zeta - z_0)^{7/2} + c_2(\zeta - z_0)^{5/2} + c_3(\zeta - z_0)^{3/2}. \tag{3.92}$$

where z_0 and the coefficients c_1, c_2, and c_3 are to be chosen independent of ζ (but possibly depending on s and t) in such a way that

$$g(\zeta) = \hat{\theta}(\zeta) + \mathcal{O}(\zeta^{-1/2}), \qquad \text{as } \zeta \to \infty. \tag{3.93}$$

If we let $z_0 = z_0(s,t)$ be the real solution of the following third degree equation (which has one real and two complex conjugate solutions),

$$z_0^3 = -\mathrm{sgn}(s)48 + 24 z_0 |s|^{-2/3} t, \qquad \text{for } s \neq 0, \tag{3.94}$$

and if we set

$$c_1 = \frac{1}{105}, \qquad c_2 = \frac{1}{30} z_0, \qquad c_3 = \frac{1}{36} z_0^2 - \mathrm{sgn}(s) \frac{2}{3 z_0}, \tag{3.95}$$

then it is straightforward to verify, using (3.91) and (3.92), that for ζ sufficiently large,

$$g(\zeta) = \hat{\theta}(\zeta) + \sum_{k=0}^{\infty} b_k \zeta^{-k-\frac{1}{2}}, \tag{3.96}$$

for some unimportant b_k's which depend only on s and t and which can be calculated explicitly. The latter equation yields that for ζ large enough,

$$e^{|s|^{7/6}(g(\zeta)-\hat{\theta}(\zeta))\sigma_3} = I + \sum_{k=1}^{\infty} d_k \sigma_3^k \zeta^{-k/2}, \tag{3.97}$$

where the coefficients d_k can also be calculated explicitly. Further, observe that by (3.97) we have $\det(I + \sum_{k=1}^{\infty} d_k \sigma_3^k \zeta^{-k/2}) = 1$, which yields

$$d_2 = \frac{1}{2} d_1^2. \tag{3.98}$$

Another crucial feature of the g-function is stated in the following proposition, which is important for the choice of the contour $\hat{\Gamma}$, and which is illustrated by Figure 3.3.

3.13 Proposition. *There exist constants $c > 0$, $\varepsilon_0 > 0$ and $s_0 > 0$ such that for $s \geq s_0$,*

$$\mathrm{Re}\, g(\zeta) > c|\zeta - z_0|^{7/2} > 0, \qquad \textit{as } \arg(\zeta - z_0) = 0, \tag{3.99}$$

$$\mathrm{Re}\, g(\zeta) < -c|\zeta - z_0|^{7/2} < 0, \ \textit{as } \tfrac{6\pi}{7} - \varepsilon_0 \leq |\arg(\zeta - z_0)| \leq \tfrac{6\pi}{7} + \varepsilon_0. \tag{3.100}$$

Proof. With $\zeta = z_0 + re^{i\phi}$ we have

$$r^{-7/2} \mathrm{Re}\, g(\zeta) = c_1 \cos(7\phi/2) + c_2 \cos(5\phi/2) r^{-1} + c_3 \cos(3\phi/2) r^{-2}, \tag{3.101}$$

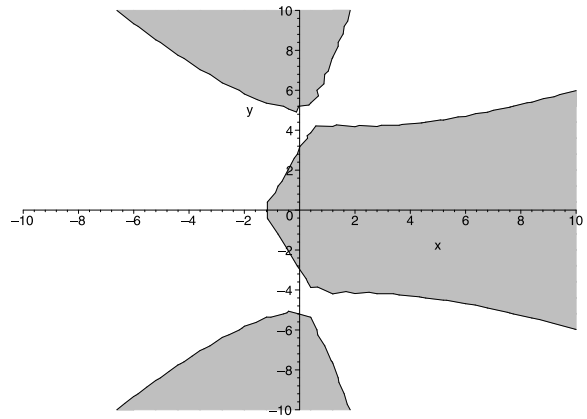

Figure 3.3 *Contour plot of* $\operatorname{Re} g$ *for* $t = 0$ *and* $s > 0$. *The shaded areas indicate where* $\operatorname{Re} g > 0$.

where by using (3.94) and (3.95)

$$c_1 = \frac{1}{105}, \quad c_2 = -\frac{1}{15}\operatorname{sgn}(s)6^{1/3} + \mathcal{O}(s^{-2/3}), \quad c_3 = 6^{-1/3} + \mathcal{O}(s^{-2/3}), \tag{3.102}$$

as $|s| \to \infty$. Observe that the right hand side of (3.101) is a second degree equation in r^{-1}, so that it is straightforward to check that,

$$\min(r^{-7/6}\operatorname{Re} g(\zeta)) = c_1 - \frac{c_2^2}{4c_3} = \frac{1}{350} + \mathcal{O}(s^{-2/3}), \qquad \text{as } \phi = 0, \tag{3.103}$$

which yields already (3.99), and that for $\pi/3 < |\phi| < \pi$,

$$\max(r^{-7/6}\operatorname{Re} g(\zeta)) = c_1 \cos(7\phi/2) - \frac{c_2^2}{4c_3}\frac{\cos^2(5\phi/2)}{\cos(3\phi/2)}. \tag{3.104}$$

Further, since

$$\cos(7\phi/2) = -1, \qquad -\frac{\cos^2(5\phi/2)}{\cos(3\phi/2)} < 1.31, \qquad \text{as } \phi = \tfrac{6\pi}{7},$$

there exists, by continuity in ϕ, a constant $\varepsilon_0 > 0$ sufficiently small such that the following estimates hold,

$$\cos(7\phi/2) < -0.99, \qquad -\frac{\cos^2(5\phi/2)}{\cos(3\phi/2)} < 1.31, \qquad \text{as } \tfrac{6\pi}{7} - \varepsilon_0 \le |\phi| \le \tfrac{6\pi}{7} + \varepsilon_0.$$

This implies by (3.104) and (3.102) that

$$\max(r^{-7/6}\operatorname{Re} g(\zeta)) < -0.99c_1 + 1.31\frac{c_2^2}{4c_3} < -0.00069 + \mathcal{O}(s^{-2/3}),$$
$$\text{as } \tfrac{6\pi}{7} - \varepsilon_0 \le |\phi| \le \tfrac{6\pi}{7} + \varepsilon_0, \tag{3.105}$$

which proves (3.100). □

3.14 Remark. Recall that the contour $\hat{\Gamma}$ (in particular $\hat{\Gamma}_2$ and $\hat{\Gamma}_4$) is not yet explicitly defined. For now, we choose $\hat{\Gamma}_2$ and $\hat{\Gamma}_4$ to lie in the sectors where (3.100) holds.

We are now ready to normalize the RH problem for Y at infinity. We let $S(\zeta; s, t) = S(\zeta)$ be the following 2×2 matrix-valued function,

$$S(\zeta) = \begin{pmatrix} 1 & 0 \\ d_1|s|^{1/6} & 1 \end{pmatrix} Y(\zeta) e^{|s|^{7/6} g(\zeta)\sigma_3}, \qquad \text{for } \zeta \in \mathbb{C} \setminus \hat{\Gamma}, \tag{3.106}$$

where Y, g and d_1 are given by (3.86), (3.92), and (3.97), respectively. It is then straightforward to check, using (3.87)–(3.89), using the fact that $g_+(\zeta)+g_-(\zeta) = 0$ for $\zeta \in (-\infty, z_0)$, and using (3.90), (3.97), (3.16) and (3.98), that S satisfies the following conditions.

RH problem for S:

(a) S is analytic in $\mathbb{C} \setminus \hat{\Gamma}$.

(b) $S_+(\zeta) = S_-(\zeta) v_S(\zeta)$ for $\zeta \in \hat{\Gamma}$, where v_S is given by,

$$v_S(\zeta) = \begin{cases} \begin{pmatrix} 1 & e^{-2|s|^{7/6} g(\zeta)} \\ 0 & 1 \end{pmatrix}, & \text{for } \zeta \in \hat{\Gamma}_1, \\ \begin{pmatrix} 1 & 0 \\ e^{2|s|^{7/6} g(\zeta)} & 1 \end{pmatrix}, & \text{for } \zeta \in \hat{\Gamma}_2 \cup \hat{\Gamma}_4, \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \text{for } \zeta \in \hat{\Gamma}_3. \end{cases} \tag{3.107}$$

(c) S has the following behavior as $\zeta \to \infty$,

$$\begin{aligned} S(\zeta) = \Bigg[I &+ \begin{pmatrix} 1 & 0 \\ d_1|s|^{1/6} & 1 \end{pmatrix} A_1 \begin{pmatrix} 1 & 0 \\ -d_1|s|^{1/6} & 1 \end{pmatrix} |s|^{-1/3} \zeta^{-1} \\ &+ \begin{pmatrix} \frac{1}{2} d_1^2 & -d_1|s|^{-1/6} \\ * & * \end{pmatrix} \zeta^{-1} + \mathcal{O}(\zeta^{-2}) \Bigg] \zeta^{-\frac{\sigma_3}{4}} |s|^{-\frac{\sigma_3}{12}} N, \end{aligned} \tag{3.108}$$

where the *'s denote unimportant functions depending only on s and t.

3.15 Remark. Note that by Proposition 3.13 the jump matrix v_S on $\hat{\Gamma}_1, \hat{\Gamma}_2$ and $\hat{\Gamma}_4$ converges exponentially fast (as $s \to \pm\infty$) to the identity matrix.

3.3.3 Parametrix $P^{(\infty)}$ for the outside region

From Remark 3.15 we expect that the leading order asymptotics of Φ will be determined by a matrix valued function $P^{(\infty)}$ (which will be referred to as the parametrix for the outside region) with jumps only on $(-\infty, z_0)$ satisfying there the same jump relation as S does. Let

$$P^{(\infty)}(\zeta) = |s|^{-\frac{\sigma_3}{12}}(\zeta - z_0)^{-\frac{\sigma_3}{4}} N, \qquad \text{for } \zeta \in \mathbb{C} \setminus (-\infty, z_0]. \tag{3.109}$$

Then, using (3.16) and the fact that $(\zeta - z_0)_-^{\frac{\sigma_3}{4}}(\zeta - z_0)_+^{-\frac{\sigma_3}{4}} = e^{-\frac{\pi i}{2}\sigma_3}$ for $\zeta \in (-\infty, z_0)$, we obtain that

$$\begin{aligned} P_+^{(\infty)}(\zeta) &= P_-^{(\infty)}(\zeta) N^{-1} (\zeta - z_0)_-^{\frac{\sigma_3}{4}} (\zeta - z_0)_+^{-\frac{\sigma_3}{4}} N \\ &= P_-^{(\infty)}(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \text{for } \zeta \in (-\infty, z_0). \end{aligned} \tag{3.110}$$

Before we can do the final transformation $S \mapsto R$ we need to do a local analysis near z_0 since the jump matrices for S and $P^{(\infty)}$ are not uniformly close to each other in the neighborhood of z_0.

3.3.4 Parametrix near z_0

In this subsection, we construct the parametrix near z_0. We surround the fixed point $\hat{z}_0$, see (3.2), by a disk $U_\delta = \{z \in \mathbb{C} : |z - \hat{z}_0| < \delta\}$ with radius $\delta > 0$ (sufficiently small and which will be determined in Proposition 3.16 below as part of the problem) and we seek a 2×2 matrix-valued function $P(\zeta; s, t) = P(\zeta)$ satisfying the following conditions.

RH problem for P:

(a) P is analytic in $U_\delta \setminus \hat{\Gamma}$.

(b) $P_+(\zeta) = P_-(\zeta) v_S(\zeta)$ for $\zeta \in \hat{\Gamma} \cap U_\delta$, where v_S is the jump matrix for S given by (3.107).

(c) $P(\zeta) P^{(\infty)}(\zeta)^{-1} = I + \mathcal{O}(s^{-1}), \qquad$ as $s \to \pm\infty$, uniformly for $\zeta \in \partial U_\delta$.

We start with constructing a matrix-valued function satisfying conditions (a) and (b) of the RH problem. This is based upon the auxiliary RH problem for M with jumps on the contour Γ^σ, see Section 3.2.2. The idea is that, by (3.28)–(3.30), the matrix-valued function $M(|s|^{7/9} f(z))$ will satisfy conditions (a) and (b) of the RH problem for P if we have appropriate biholomorphic maps f on U_δ which satisfy the following proposition.

3.16 Proposition. *There exists $s_1 \geq s_0 > 0$ and $\delta > 0$ such that for all $|s| \geq s_1$ there are biholomorphic maps $f = f(\cdot\,; s, t)$ on U_δ satisfying the following conditions.*

1. *There exists a constant c_0 such that for all $\zeta \in U_\delta$ and $|s| \geq s_1$ the derivative of f can be estimated by: $c_0 < |f'(\zeta)| < 1/c_0$ and $|\arg f'(\zeta)| < \varepsilon_0$ with ε_0 defined in Proposition* 3.13.
2. $f(U_\delta \cap \mathbb{R}) = f(U_\delta) \cap \mathbb{R}$ and $f(U_\delta \cap \mathbb{C}_\pm) = f(U_\delta) \cap \mathbb{C}_\pm$.
3. $\frac{2}{3} f(\zeta)^{3/2} = g(\zeta)$ for $\zeta \in U_\delta \setminus (-\infty, z_0]$.

Proof. One can verify, using (3.102), that there exists $s_1 \geq s_0 > 0$ sufficiently large and $\delta > 0$ sufficiently small, such that for all $|s| \geq s_1$ the function $f(\zeta; s, t) = f(\zeta)$ defined by

$$\begin{aligned} f(\zeta) &= \left(\frac{3}{2}c_3 + \frac{3}{2}c_1(\zeta - z_0)^2 + \frac{3}{2}c_2(\zeta - z_0)\right)^{2/3} (\zeta - z_0) \\ &= \left(\frac{3}{2}\frac{g(\zeta)}{(\zeta - z_0)^{3/2}}\right)^{2/3} (\zeta - z_0), \end{aligned} \tag{3.111}$$

is analytic for $\zeta \in U_\delta$, and that f is uniformly (in s and ζ) bounded in U_δ. By Cauchy's theorem for derivatives we then also have that f'' is uniformly (in s and ζ) bounded in U_δ for a smaller δ. Then, there exists a constant $C > 0$ such that

$$|f'(\zeta) - f'(z_0)| = \left|\int_{z_0}^{\zeta} f''(s)ds\right| \leq C|\zeta - z_0|, \qquad \text{for all } |s| \geq s_1 \text{ and } \zeta \in U_\delta.$$

Since, by (3.102),

$$f'(z_0) = (\frac{3}{2}c_3)^{2/3} \geq const > 0 \qquad \text{for } |s| \text{ large enough,} \tag{3.112}$$

this yields that for all $|s| \geq s_1$ (for a possible larger s_1) the functions f are injective and hence biholomorphic in U_δ (for a possible smaller δ) and that they satisy part 1 of the proposition.

The second part follows from the first part (for a possible smaller δ). The last part follows from the second part and from (3.111). □

Now, let $|s| \geq s_1$ and $\sigma \in (\frac{\pi}{3}, \pi)$ (we will specify our choice of σ below), and recall that the contour $\hat{\Gamma}$ is not yet explicitly defined. We suppose that $\hat{\Gamma}$ is defined in U_δ as the pre-image of $\Gamma^\sigma \cap f(U_\delta)$ under the map f (so $\hat{\Gamma}$ depends on the parameters s and σ), where $\Gamma^\sigma = \cup_{j=1}^4 \Gamma_j^\sigma$ is the jump contour for M, as defined by (3.27). Then, we immediately have, by (3.28)-(3.30) and part 3 of Proposition 3.16, that $M(|s|^{7/9} f(\zeta))$ satisfies conditions (a) and (b) of the RH problem for

P. Moreover, for any invertible analytic matrix-valued function E in U_δ, one has that

$$P(\zeta) = E(\zeta)M(|s|^{7/9}f(\zeta)), \qquad \text{for } \zeta \in U_\delta \setminus \hat{\Gamma}, \tag{3.113}$$

satisfies also conditions (a) and (b) of the RH problem for P. We need E to be such that the matching condition (c) is satisfied as well. Let

$$E(\zeta) = |s|^{-\frac{\sigma_3}{12}}(\zeta - z_0)^{-\frac{\sigma_3}{4}}(|s|^{7/9}f(\zeta))^{\frac{\sigma_3}{4}}, \tag{3.114}$$

which of course is an invertible analytic matrix-valued function in U_δ. Then, using (3.31), (3.32) and (3.109) we have,

$$P(\zeta)P^{(\infty)}(\zeta)^{-1} = I + \Delta_1|s|^{-1} + \Delta_2|s|^{-4/3} + \mathcal{O}\left(|s|^{-7/3}\right), \tag{3.115}$$

as $s \to \pm\infty$ uniformly for $\zeta \in \partial U_\delta$ and σ in compact subsets of $(\frac{\pi}{3}, \pi)$, where Δ_1 and Δ_2 are given by

$$\Delta_1 = \frac{1}{f(\zeta)}\left(\frac{\zeta - z_0}{f(\zeta)}\right)^{1/2}\begin{pmatrix} 0 & 0 \\ t_1 & 0 \end{pmatrix}, \tag{3.116}$$

$$\Delta_2 = \frac{1}{f(\zeta)^2}\left(\frac{\zeta - z_0}{f(\zeta)}\right)^{-1/2}\begin{pmatrix} 0 & \\ widehatt_1 & \\ 0 & 0 \end{pmatrix}, \tag{3.117}$$

and where t_1 and $\widehat{t_1}$ are unimportant constants given by (3.33). We then have shown that P defined by (3.113) satisfies the conditions of the RH problem for P. This ends the construction of the parametrix near z_0.

3.3.5 Final transformation: $S \mapsto R$

We will now perform the final transformation. Recall that the contour $\hat{\Gamma}$ is still not yet explicitly defined. We will now define it in terms of the (sufficiently large) parameter s.

Consider the fixed point $\hat{z}_0 + \delta e^{\frac{6\pi i}{7}}$ (which depends only on $\operatorname{sgn}(s)$) on ∂U_δ. Since $z_0 \to \hat{z}_0$ as $s \to \pm\infty$, see Remark 3.2, there exists $s_2 \geq s_1$ sufficiently large such that for all $|s| \geq s_2$,

$$\frac{6\pi}{7} - \varepsilon_0 < \arg(\hat{z}_0 + \delta e^{\frac{6\pi i}{7}} - z_0) < \frac{6\pi}{7} + \varepsilon_0,$$

where ε_0 is defined in Proposition 3.13. From Proposition 3.16 we then know that for $|s| \geq s_2$ there exists $\sigma = \sigma(s) \in (\frac{6\pi}{7} - 2\varepsilon_0, \frac{6\pi}{7} + 2\varepsilon_0)$ such that $f^{-1}(\Gamma_2^\sigma) \cap \partial U_\delta = \{\hat{z}_0 + \delta e^{\frac{6\pi i}{7}}\}$. By the symmetry $\overline{f(\zeta)} = f(\overline{\zeta})$ we then also have $f^{-1}(\Gamma_4^\sigma) \cap \partial U_\delta = \{\hat{z}_0 + \delta e^{-\frac{6\pi i}{7}}\}$. We now define $\hat{\Gamma}$ in U_δ (for $|s| \geq s_2$) as the inverse f-image of the

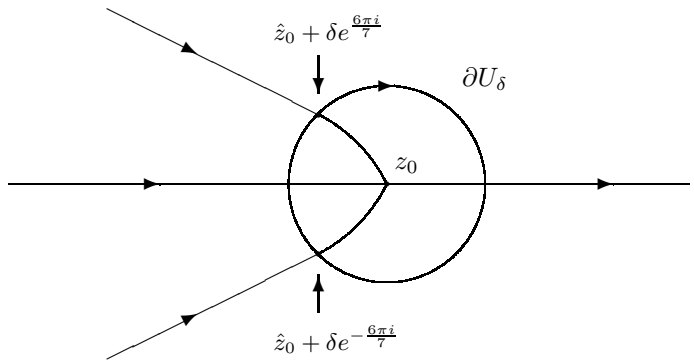

Figure 3.4 *The contour $\Gamma_R = \hat{\Gamma}_R \cup \partial U_\delta$. The part of Γ_R inside U_δ depends on s. The rest of Γ_R is independent of s.*

contour Γ^σ. Outside U_δ, we take $\hat{\Gamma}_1 \cup \hat{\Gamma}_3 = \mathbb{R}$, $\hat{\Gamma}_2 = \{\hat{z}_0 + te^{6\pi i/7} : t \geq \delta\}$, and $\hat{\Gamma}_4 = \{\hat{z}_0 + te^{-6\pi i/7} : t \geq \delta\}$. Note that by Proposition 3.13,

$$\operatorname{Re} g(\zeta) > c|\zeta - z_0|^{7/2} \qquad \text{for } \zeta \in \hat{\Gamma}_1 \setminus U_\delta, \tag{3.118}$$

$$\operatorname{Re} g(\zeta) < -c|\zeta - z_0|^{7/2} \qquad \text{for } \zeta \in (\hat{\Gamma}_2 \cup \hat{\Gamma}_4) \setminus U_\delta. \tag{3.119}$$

Further define a contour Γ_R as $\Gamma_R = \hat{\Gamma} \cup \partial U_\delta$. This leads to Figure 3.4. Note that $\Gamma_R \cap U_\delta$ depends on s. However, the part of Γ_R outside U_δ is independent of s.

Now, we are ready to do the final transformation $S \mapsto R$. Define a 2×2 matrix-valued function $R(\zeta; s, t) = R(\zeta)$ for $\zeta \in \mathbb{C} \setminus \Gamma_R$ as

$$R(\zeta) = \begin{cases} S(\zeta)P(\zeta)^{-1}, & \text{for } \zeta \in U_\delta \setminus \Gamma_R, \\ S(\zeta)P^{(\infty)}(\zeta)^{-1}, & \text{for } \zeta \text{ elsewhere}, \end{cases} \tag{3.120}$$

where P is the parametrix near z_0 given by (3.113), $P^{(\infty)}$ is the parametrix for the outside region given by (3.109), and S is the solution of the RH problem for S.

By definition, R has jumps on the contour Γ_R. However, S and P have the same jumps on $\Gamma_R \cap U_\delta$. Further, S and $P^{(\infty)}$ satisfy the same jump relation on $(-\infty, \hat{z}_0 - \delta)$. This yields that R has only jumps on the reduced system of contours $\hat{\Gamma}_R$ (which is independent of s), shown in Figure 3.5.

Using (3.120), (3.108) and (3.109) one can now show that R is a solution of the following RH problem on the contour $\hat{\Gamma}_R$.

RH problem for R:

(a) R is analytic in $\mathbb{C} \setminus \hat{\Gamma}_R$.

(b) $R_+(\zeta) = R_-(\zeta) v_R(\zeta)$ for $\zeta \in \hat{\Gamma}_R$, with v_R given by

$$v_R(\zeta) = P^{(\infty)}(\zeta) v_S(\zeta) P^{(\infty)}(\zeta)^{-1}, \qquad \text{for } \zeta \in \hat{\Gamma}_R \setminus \partial U_\delta. \tag{3.121}$$

$$v_R(\zeta) = P(\zeta) P^{(\infty)}(\zeta)^{-1}, \qquad \text{for } \zeta \in \partial U_\delta. \tag{3.122}$$

(c) $R(\zeta) = I + \mathcal{O}(\zeta^{-1})$ as $\zeta \to \infty$.

3.17 Remark. Observe that by (3.115), (3.118) and (3.119) we have as $s \to \pm\infty$,

$$v_R(\zeta) = \begin{cases} I + \Delta_1 |s|^{-1} + \Delta_2 |s|^{-4/3} + \mathcal{O}(|s|^{-7/3}), & \text{uniformly for } \zeta \in \partial U_\delta, \\ I + \mathcal{O}(e^{-c|s|^{7/6}|\zeta - z_0|^{7/2}}) & \text{uniformly for } \zeta \in \hat{\Gamma}_R \setminus \partial U_\delta, \end{cases} \tag{3.123}$$

for some constant $\gamma > 0$, and where Δ_1 and Δ_2 are given by (3.116). As in [27, 32, 33], this yields that R itself is uniformly close to the identity matrix,

$$R(\zeta) = I + \mathcal{O}(s^{-1}), \qquad \text{as } s \to \pm\infty, \text{ uniformly for } \zeta \in \mathbb{C} \setminus \hat{\Gamma}_R. \tag{3.124}$$

3.18 Remark. Since $R(\zeta) = S(\zeta) P^{(\infty)}(\zeta)^{-1}$ for ζ large one can use (3.108), (3.109), and the fact that $(\zeta - z_0)^{\frac{\sigma_3}{4}} = \zeta^{\frac{\sigma_3}{4}} \left[I - \frac{1}{4} z_0 \sigma_3 \zeta^{-1} + \mathcal{O}(\zeta^{-2}) \right]$ as $\zeta \to \infty$, to strengthen condition (c) of the RH problem for R to

$$R(\zeta) = I + \frac{R_1}{\zeta} + \mathcal{O}(\zeta^{-2}), \qquad \text{as } \zeta \to \infty, \tag{3.125}$$

where R_1 is a 2×2 matrix-valued function depending on s and t with $(1,1)$ and $(1,2)$ entries given by,

$$R_{1,11} = -\frac{z_0}{4} + \frac{1}{2} d_1^2 + |s|^{-1/3} A_{1,11} - d_1 |s|^{-1/6} A_{1,12}, \tag{3.126}$$

$$R_{1,12} = -d_1 |s|^{-1/6} + |s|^{-1/3} A_{1,12}. \tag{3.127}$$

From (3.123) it follows, as in [32], that

$$R_1 = -\operatorname{Res}(\Delta_1, z_0)|s|^{-1} - \operatorname{Res}(\Delta_2, z_0)|s|^{-4/3} + \mathcal{O}(|s|^{-7/3}), \quad \text{as } s \to \pm\infty,$$

so that by (3.116) and (3.117),

$$R_{1,11} = \mathcal{O}(|s|^{-7/3}), \qquad R_{1,12} = \mathcal{O}(|s|^{-4/3}), \qquad \text{as } s \to \pm\infty. \tag{3.128}$$

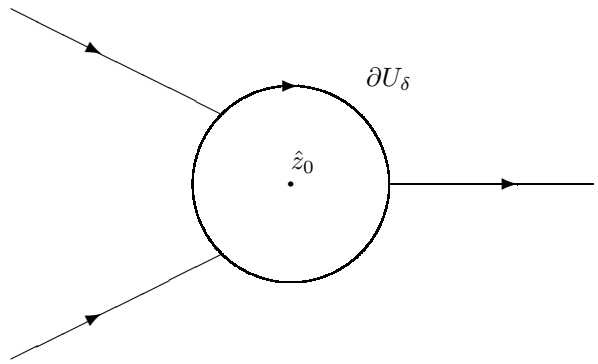

Figure 3.5 *The reduced system of contours $\hat{\Gamma}_R$ independent of s.*

3.3.6 Proof of Theorem 3.1 (ii)

We now have all the necessary ingredients to prove the second part of the main theorem.

Proof of Theorem 3.1 (ii). Recall that $y = 2A_{1,11} - A_{1,12}^2$. Using (3.126) and (3.127) one can then write y in terms of the $(1,1)$ and $(1,2)$ entries of R_1,

$$2A_{1,11} = \frac{1}{2}z_0|s|^{1/3} + 2|s|^{1/3}R_{1,11} - d_1^2|s|^{1/3} + 2d_1|s|^{1/6}A_{1,12},$$

$$A_{1,12}^2 = |s|^{2/3}R_{1,12}^2 - d_1^2|s|^{1/3} + 2d_1|s|^{1/6}A_{1,12},$$

so that

$$y = \frac{1}{2}z_0|s|^{1/3} + 2|s|^{1/3}R_{1,11} - |s|^{2/3}R_{1,12}^2. \tag{3.129}$$

Inserting (3.128) into the latter equation we obtain precisely (3.5). This finishes the proof of Theorem 3.1. □

Chapter 4

Universality of a double scaling limit near a singular edge point

Summary[1] We consider unitary random matrix ensembles $Z_{n,s,t}^{-1} e^{-n \operatorname{tr} V_{s,t}(M)} dM$ on the space of Hermitian $n \times n$ matrices M, where the confining potential $V_{s,t}$ is such that the limiting mean density of eigenvalues (as $n \to \infty$ and $s, t \to 0$) vanishes like a power $5/2$ at a (singular) edge point of its support. The main purpose of this chapter is to prove universality of the eigenvalue correlation kernel in a double scaling limit. The limiting kernel is built out of functions associated with a special solution of the P_I^2 equation, which is a fourth order analogue of the Painlevé I equation. In order to prove our result, we use the well-known connection between the eigenvalue correlation kernel and the Riemann-Hilbert (RH) problem for orthogonal polynomials, together with the Deift/Zhou steepest descent method to analyze the RH problem asymptotically. The key step in the asymptotic analysis will be the construction of a parametrix near the singular endpoint, for which we use the model RH problem for the special solution of the P_I^2 equation.

In addition, the RH method allows us to determine the asymptotics (in a double scaling limit) for the recurrence coefficients of the orthogonal polynomials with respect to the varying weights $e^{-nV_{s,t}}$ on $\mathbb{R}$. The special solution of the P_I^2 equation pops up in the $n^{-2/7}$ term in the asymptotics of the recurrence coefficients.

[1] This chapter corresponds to the following paper [25]:

T. Claeys and M. Vanlessen, Universality of a double scaling limit near singular edge points in random matrix models, arxiv:math-ph/0607043, to appear in *Communications in Mathematical Physics*.

4.1 Introduction and statement of results

4.1.1 Unitary random matrix ensembles

On the space $\mathcal{H}_n$ of Hermitian $n \times n$ matrices M, we consider for $n \in \mathbb{N}$ and $s, t \in \mathbb{R}$ the unitary random matrix ensemble,

$$\frac{1}{Z_{n,s,t}} \exp(-n \operatorname{Tr} V_{s,t}(M)) dM. \tag{4.1}$$

Here, the confining potential $V_{s,t}$ is a real analytic function, depending on two parameters $s, t \in \mathbb{R}$, satisfying the asymptotic condition for some $\delta_0 > 0$,

$$\lim_{x \to \pm\infty} \frac{V_{s,t}(x)}{\log(x^2+1)} = +\infty, \qquad \text{uniformly for } s, t \in [-\delta_0, \delta_0]. \tag{4.2}$$

Then,

$$Z_{n,s,t} = \int_{\mathcal{H}_n} \exp(-n \operatorname{Tr} V_{s,t}(M)) dM$$

is convergent as $n \to \infty$ so that the random matrix model is well-defined.

It is well-known, see e.g. [83], that an important role in the study of the unitary random matrix ensemble (4.1) is played by the following scalar 2-point (correlation) kernel,

$$K_n^{(s,t)}(x,y) = e^{-\frac{n}{2}V_{s,t}(x)} e^{-\frac{n}{2}V_{s,t}(y)} \sum_{k=0}^{n-1} p_k^{(n,s,t)}(x) p_k^{(n,s,t)}(y), \tag{4.3}$$

constructed out of the orthonormal polynomials

$$p_k^{(n,s,t)}(x) = \kappa_k^{(n,s,t)} x^k + \cdots, \qquad \kappa_k^{(n,s,t)} > 0,$$

with respect to the varying weights $e^{-nV_{s,t}(x)}$. Indeed, the correlations between the eigenvalues of M can be written in terms of the correlation kernel. More precisely, the m-point correlation function $\mathcal{R}_{n,m}^{(s,t)}$ satisfies [83],

$$\mathcal{R}_{n,m}^{(s,t)}(x_1, \ldots, x_m) = \det \left(K_n^{(s,t)}(x_i, x_j) \right)_{1 \le i,j \le m}. \tag{4.4}$$

Further, the limiting mean eigenvalue distribution $\mu_{s,t}$ has a density $\rho_{s,t}$ which can be retrieved from the correlation kernel as follows,

$$\rho_{s,t}(x) = \lim_{n \to \infty} \frac{1}{n} K_n^{(s,t)}(x,x). \tag{4.5}$$

The limiting mean eigenvalue distribution $\mu_{s,t}$ equals [32] the equilibrium measure in external field $V_{s,t}$. This is the unique measure minimizing the logarithmic energy [91]

$$I_{V_{s,t}}(\mu) = \iint \log \frac{1}{|x-y|} d\mu(x) d\mu(y) + \int V_{s,t}(y) d\mu(y), \tag{4.6}$$

among all probability measures μ on $\mathbb{R}$. Furthermore, there exists a real analytic function $Q_{s,t}$, such that [31],

$$\rho_{s,t}(x) = \frac{1}{\pi}\sqrt{Q_{s,t}^{-}(x)}, \tag{4.7}$$

where $Q_{s,t}^{-}$ denotes the negative part of $Q_{s,t}$, i.e. $Q_{s,t} = Q_{s,t}^{+} - Q_{s,t}^{-}$, with $Q_{s,t}^{\pm} \geq 0$ and $Q_{s,t}^{+}Q_{s,t}^{-} = 0$. Due to condition (4.2) we have that $Q_{s,t}(x) \to +\infty$ as $x \to \pm\infty$, so that $\mu_{s,t}$ is supported on a finite union of intervals, which we denote by $\mathbb{S}_{s,t}$. It is known [91] that the equilibrium measure $\mu_{s,t}$ satisfies the following Euler-Lagrange variational conditions: there exists a constant $\kappa_{s,t} \in \mathbb{R}$ such that

$$2\int \log|x-u|d\mu_{s,t}(u) - V_{s,t}(x) = \kappa_{s,t}, \qquad \text{for } x \in \mathbb{S}_{s,t}, \tag{4.8}$$

$$2\int \log|x-u|d\mu_{s,t}(u) - V_{s,t}(x) \leq \kappa_{s,t}, \qquad \text{for } x \in \mathbb{R}\setminus\mathbb{S}_{s,t}. \tag{4.9}$$

The external field $V_{s,t}$ is called regular if strict inequality in (4.9) holds, if the density $\rho_{s,t}$ does not vanish in the interior of the support $\mathbb{S}_{s,t}$, and if $Q_{s,t}$ has a simple zero at each of the endpoints of the support $\mathbb{S}_{s,t}$. If one of these conditions is not valid, $V_{s,t}$ is called singular. The singular points x^* are classified as follows, see [32, 74]:

(i) $x^* \in \mathbb{R}\setminus\mathbb{S}_{s,t}$ is a type I singular point if equality in (4.9) holds. Then, x^* is a zero of $Q_{s,t}^{+}$ of multiplicity $4m$ with $m \in \mathbb{N}$.

(ii) $x^* \in \mathbb{S}_{s,t}$ is a type II singular point if it is an interior point of $\mathbb{S}_{s,t}$ where the equilibrium density $\rho_{s,t}$ vanishes. Then, x^* is a zero of $Q_{s,t}^{-}$ of multiplicity $4m$.

(iii) x^* is a type III singular point if it is an edge point of the support $\mathbb{S}_{s,t}$ and a zero of $Q_{s,t}$ of multiplicity larger than one. Then, x^* is a zero of $Q_{s,t}$ of multiplicity $4m+1$, which means that $\rho_{s,t}(x) \sim c|x-x^*|^{(4m+1)/2}$.

In this chapter, we consider external fields $V_{s,t}$ which are such that in the critical case $s = t = 0$, $V_0 = V_{0,0}$ has a type III singular (edge) point x^* with $m = 1$, i.e.

$$\rho_{0,0}(x) \sim c|x-x^*|^{5/2}, \qquad \text{as } x \to x^*. \tag{4.10}$$

Further, we take $V_{s,t}$ of the special form,

$$V_{s,t} = V_0 + sV_1 + tV_2, \tag{4.11}$$

where V_1 is an arbitrary real analytic function, while V_2 is real analytic and in addition satisfies some critical condition which we will specify in Section 4.1.4 below.

4.1.2 Universality in random matrix theory

Consider for now unitary random matrix ensembles $Z_n^{-1}\exp(-n\operatorname{Tr}V(M))dM$ on the space of Hermitian $n\times n$ Hermitian matrices. Scaling limits of the associated correlation kernel K_n show universal behavior.

Near regular points, universality results have been established in [13, 27, 32, 33, 86]. For example, if x^* lies in the bulk of the spectrum (i.e. x^* is such that it lies in the interior of the support $\mathbb{S}$ of the equilibrium measure in external field V, and such that the equilibrium density ρ does not vanish at x^*) there is a constant c such that

$$\lim_{n\to\infty}\frac{1}{cn}K_n\left(x^*+\frac{u}{cn},x^*+\frac{v}{cn}\right)=\frac{\sin\pi(u-v)}{\pi(u-v)}. \tag{4.12}$$

On the other hand, if x^* is a regular edge point of the spectrum (i.e. x^* is an endpoint of $\mathbb{S}$ and $\rho(x)$ vanishes like a square root at $x=x^*$), there is a constant c such that

$$\lim_{n\to\infty}\frac{1}{cn^{2/3}}K_n\left(x^*+\frac{u}{cn^{2/3}},x^*+\frac{v}{cn^{2/3}}\right)=\frac{\operatorname{Ai}(u)\operatorname{Ai}'(v)-\operatorname{Ai}(v)\operatorname{Ai}'(u)}{u-v}, \tag{4.13}$$

where Ai is the Airy function.

Near singular points, similar results hold. In those singular cases it is interesting to consider double scaling limits where the external field V depends on additional parameters. In [14, 22, 23, 93], an external field V was considered such that there is a type II singular point x^* with $m=1$, i.e.

$$\rho(x)\sim c(x-x^*)^2,\qquad \text{as } x\to x^*.$$

If an additional parameter is included in the external field, $V_t=V/t$, one observes for t close to 1 the transition where two intervals in the support of the limiting mean density of eigenvalues merge to one interval through the critical case of a type II singular point. In the double scaling limit where $n\to\infty$ and $t\to 1$ in such a way that $c_0n^{2/3}(t-1)\to s\in\mathbb{R}$ for some appropriately chosen constant c_0, there exists a constant c such that (for the associated correlation kernel $K_{n,t}$),

$$\lim\frac{1}{cn^{1/3}}K_{n,t}\left(x^*+\frac{u}{cn^{1/3}},x^*+\frac{v}{cn^{1/3}}\right)=\mathbb{K}^{\mathrm{crit,II}}(u,v;s).$$

Here, $\mathbb{K}^{\mathrm{crit,II}}(u,v;s)$ is built out of functions associated with the Hastings-McLeod solution of the second Painlevé equation.

The main purpose of this chapter is to obtain, for the random matrix models in Section 4.1.1 above, a similar result near the type III singular (edge) point of V_0 with $m=1$. We take a double scaling limit ($n\to\infty$ and $s,t\to 0$), and the limiting kernel $\mathbb{K}^{\mathrm{crit,III}}$ will be built out of functions which are associated with a special solution of the fourth order analogue of the Painlevé I equation. The case of a type III singular (edge) point was also studied in the Physics literature [18].

In addition, the techniques that we use to prove this allow us to determine the asymptotics (in a double scaling limit) of the recurrence coefficients in the three-term recurrence relation satisfied by the orthogonal polynomials $p_k^{(n,s,t)}$ with respect to the varying weight $e^{-nV_{s,t}}$ on $\mathbb{R}$.

4.1.3 ψ-functions associated with a special solution of the P_I^2 equation

We consider the following differential equation for $y = y(s,t)$, which we denote as the P_I^2 equation,

$$s = ty - \left(\frac{1}{6}y^3 + \frac{1}{24}(y_s^2 + 2yy_{ss}) + \frac{1}{240}y_{ssss}\right). \tag{4.14}$$

For $t = 0$, this equation is the second member in the Painlevé I hierarchy [69, 72]. The P_I^2 equation has been studied for example in [20, 66, 85] (for $t = 0$) and [24, 40] (for general t). The Lax pair for the P_I^2 equation is the linear system of differential equations

$$\frac{\partial \Psi}{\partial \zeta} = U\Psi, \qquad \frac{\partial \Psi}{\partial s} = W\Psi, \tag{4.15}$$

where

$$U = \frac{1}{240}\begin{pmatrix} -4y_s\zeta - (12yy_s + y_{sss}) & 8\zeta^2 + 8y\zeta + (12y^2 + 2y_{ss} - 120t) \\ U_{21} & 4y_s\zeta + (12yy_s + y_{sss}) \end{pmatrix}, \tag{4.16}$$

$$\begin{aligned} U_{21} = 8\zeta^3 - 8y\zeta^2 - (4y^2 + 2y_{ss} + 120t)\zeta \\ + (16y^3 - 2y_s^2 + 4yy_{ss} + 240s), \end{aligned} \tag{4.17}$$

and

$$W = \begin{pmatrix} 0 & 1 \\ \zeta - 2y & 0 \end{pmatrix}. \tag{4.18}$$

The system of differential equations (4.15)–(4.18) can only be solvable if $y = y(s,t)$ is a solution to the P_I^2 equation (4.14). For different solutions y, we have different Lax pairs.

We are interested in the special solution y which was studied in [20, 40] and in Chapter 3. This solution $y = y(s,t)$ is characterized by the vanishing of its Stokes multipliers s_1, s_2, s_5, and s_6, see [66]. We have shown in Chapter 3 that y has no poles for real s and t, and that it has, for fixed $t \in \mathbb{R}$, the following asymptotic behavior,

$$y(s,t) = \mp(6|s|)^{1/3} \mp \frac{1}{3}6^{2/3}t|s|^{-1/3} + \mathcal{O}(|s|^{-1}), \qquad \text{as } s \to \pm\infty. \tag{4.19}$$

It has been shown in [85, Appendix A] that for $t = 0$, y is uniquely determined by realness and asymptotic condition (4.19). For general t we are not aware of a similar result although it is supported by a conjecture of Dubrovin [40] that it should hold for general t. For $s, t \in \mathbb{R}$, the Lax pair (4.15)–(4.18) associated with this special choice of y has a unique solution $\left(\begin{smallmatrix}\Phi_1\\ \Phi_2\end{smallmatrix}\right)$ for which the following limit holds, see [24, 66],

$$\zeta^{\frac{1}{4}\sigma_3}\begin{pmatrix}\Phi_1(\zeta;s,t)\\ \Phi_2(\zeta;s,t)\end{pmatrix} e^{\theta(\zeta;s,t)} \longrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix} e^{-\frac{1}{4}\pi i}, \qquad \text{as } \zeta \to \infty \text{ with } 0 < \arg\zeta < 6\pi/7, \tag{4.20}$$

where $\sigma_3 = \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ denotes the third Pauli-matrix, and where θ is given by

$$\theta(\zeta;s,t) = \frac{1}{105}\zeta^{7/2} - \frac{1}{3}t\zeta^{3/2} + s\zeta^{1/2}. \tag{4.21}$$

The functions Φ_1 and Φ_2 will appear below in the universal limiting correlation kernel near type III singular (edge) points with $m = 1$.

4.1.4 Statement of results

We work under the following assumptions.

4.1 Assumptions.

(i) We consider external fields $V_{s,t}$ of the form

$$V_{s,t} = V_0 + sV_1 + tV_2, \tag{4.22}$$

where V_0, V_1, and V_2 are real analytic and are such that there exists a $\delta_0 > 0$ such that the following holds

$$\lim_{|x|\to\infty} \frac{V_{s,t}(x)}{\log(x^2+1)} = +\infty, \qquad \text{uniformly for } s,t \in [-\delta_0, \delta_0]. \tag{4.23}$$

(ii) V_0 is such that the equilibrium measure ν_0 in external field V_0 is supported on one single interval $[a, b] \subset \mathbb{R}$, and b is a type III singular (edge) point with $m = 1$. Then, ν_0 is of the form [31],

$$d\nu_0(x) = \frac{1}{2\pi}h_0(x)\sqrt{(b-x)(x-a)}\,\chi_{[a,b]}(x)dx, \tag{4.24}$$

with $\chi_{[a,b]}$ the indicator function of the set $[a, b]$, and with h_0 real analytic and satisfying,

$$h_0(b) = h_0'(b) = 0, \qquad \text{and} \qquad h_0''(b) > 0. \tag{4.25}$$

Furthermore, we assume that there are no other singular points besides b. In particular, a is a regular (edge) point and we then have that

$$h_0(a) > 0. \tag{4.26}$$

(iii) V_2 is such that it satisfies the critical condition

$$\int_a^b \sqrt{\frac{u-a}{b-u}} V_2'(u)du = 0. \tag{4.27}$$

Throughout the rest of this chapter we let $\mathcal{V}$ be the neighborhood of the real line where V_0, V_1, V_2, and h_0 are analytic.

4.2 Example. The assumptions above are valid for the particular example where V_0, V_1, and V_2 are given by,

$$V_0(x) = \frac{1}{20}x^4 - \frac{4}{15}x^3 + \frac{1}{5}x^2 + \frac{8}{5}x, \qquad V_1(x) = x, \qquad V_2(x) = x^3 - 6x. \tag{4.28}$$

Then, the equilibrium measure ν_0 is supported on the interval $[-2, 2]$ and given by

$$d\nu_0(x) = \frac{1}{10\pi}(x+2)^{1/2}(x-2)^{5/2}\chi_{[-2,2]}(x)dx. \tag{4.29}$$

It should be noted that a type III singular (edge) point cannot occur when V_0 is a polynomial of degree lower than 4.

4.3 Example. In the continuum limit of the Toda lattice [34], an external field of the form

$$V_{t_1,t_2}(x) = (1+t_1)(V_0(x) + t_2 x)$$

was studied. This deformation of V_0 can be written in the form (4.22). Indeed, if we let $V_1(x) = x$ and $V_2(x) = V_0(x) + cx$, with c chosen such that the critical condition (4.27) holds, we have that

$$V_{t_1,t_2} = V_0 + sV_1 + tV_2,$$

with $s = t_2 + t_1 t_2 - ct_1$ and $t = t_1$.

4.4 Remark. In Section 4.2 we will show that assumption (iii) is equivalent to the vanishing of the equilibrium density $\frac{d\nu_2(x)}{dx}$ at the right endpoint b, where ν_2 is the unique measure which minimizes $I_{V_2}(\nu)$, see (4.6), among all signed measures ν, supported on $[a, b]$ and having zero mass, $\nu([a, b]) = 0$.

4.5 Remark. The case where the left (instead of the right) endpoint of the support is singular can be transformed to our case by considering the external field $V_{s,t}(-x)$.

4.6 Remark. If we let $t = 0$ and s vary around 0, one typically observes the transition from the regular one-interval case to the singular case and back to the regular one-interval case. For $s = 0$ and t around 0, one can observe the transition from the regular one-interval case to the regular two-interval case. Letting $s, t \neq 0$, we can observe one of the above described transitions, or the critical transition where a type II singular point moves to the endpoint b, where it becomes a type III singular point before moving on as a type I singular point.

Further, to describe our results, we have to introduce constants c, c_1, and c_2,

$$c = \left(\frac{15}{2} h_0''(b)\sqrt{b-a}\right)^{2/7} > 0, \tag{4.30}$$

$$c_1 = \frac{h_1(b)}{c^{1/2}(b-a)^{1/2}}, \qquad c_2 = -\frac{h_2'(b)}{c^{3/2}(b-a)^{1/2}}, \tag{4.31}$$

where h_0 is the real analytic function appearing in (4.24), and where the functions h_j, $j = 1, 2$, are defined as,

$$h_j(x) = -\frac{1}{\pi} \fint_a^b \sqrt{(b-u)(u-a)} V_j'(u) \frac{du}{u-x}, \qquad \text{for } x \in [a, b]. \tag{4.32}$$

Universality of the double scaling limit

Our main result is the following.

4.7 Theorem. *Let $V_{s,t} = V_0 + sV_1 + tV_2$ be such that the assumptions* 4.1 *above are satisfied. We take a double scaling limit where we let $n \to \infty$ and at the same time $s, t \to 0$, in such a way that*

$$c_1 \cdot \lim n^{6/7} s = s_0 \in \mathbb{R}, \qquad c_2 \cdot \lim n^{4/7} t = t_0 \in \mathbb{R}, \tag{4.33}$$

where the constants c_1 and c_2 are defined by (4.31)*. Then, the 2-point kernel $K_n^{(s,t)}$ satisfies the following universality result,*

$$\lim \frac{1}{cn^{2/7}} K_n^{(s,t)} \left(b + \frac{u}{cn^{2/7}}, b + \frac{v}{cn^{2/7}}\right) = \mathbb{K}^{\text{crit,III}}(u, v; s_0, t_0), \tag{4.34}$$

uniformly for u, v in compact subsets of $\mathbb{R}$. Here, $\mathbb{K}^{\text{crit,III}}$ is built out of the functions Φ_1 and Φ_2 defined in Section 4.1.3*,*

$$\mathbb{K}^{\text{crit,III}}(u, v; s, t) = \frac{\Phi_1(u; s, t)\Phi_2(v; s, t) - \Phi_1(v; s, t)\Phi_2(u; s, t)}{-2\pi i(u - v)}. \tag{4.35}$$

4.8 Remark. Since $y(s, t)$ has no poles for $s, t \in \mathbb{R}$ (see Chapter 3), the kernel $\mathbb{K}^{\text{crit,III}}(u, v; s, t)$ exists for all real u, v, s, and t. Furthermore, using a similar argument as in the proof of Lemma 3.9 (ii), one can show that $e^{\pi i/4}\Phi_1$ and $e^{\pi i/4}\Phi_2$ are real. It then follows that $\mathbb{K}^{\text{crit,III}}(u, v; s, t)$ is real for real u, v, s, and t.

4.9 Remark. It is possible to give an integral formula for $\mathbb{K}^{\text{crit,III}}$. Using the fact that $\binom{\Phi_1}{\Phi_2}$ satisfies the second differential equation of the Lax pair (4.15), we have that

$$\frac{\partial \Phi_1}{\partial s}(\zeta; s, t) = \Phi_2(\zeta; s, t), \qquad \text{and} \qquad \frac{\partial \Phi_2}{\partial s}(\zeta; s, t) = (\zeta - 2y(s, t))\Phi_1(\zeta; s, t).$$

Using (4.35) this yields,

$$\frac{\partial \mathbb{K}^{\mathrm{crit,III}}}{\partial s}(u,v;s,t) = \frac{1}{2\pi i}\Phi_1(u;s,t)\Phi_1(v;s,t).$$

Now, since $\lim_{s\to-\infty}\mathbb{K}^{\mathrm{crit,III}}(u,v;s,t)=0$, which can be shown using a Deift/Zhou steepest descent method argument [36], it then follows that $\mathbb{K}^{\mathrm{crit,III}}$ has the following integral formula,

$$\mathbb{K}^{\mathrm{crit,III}}(u,v;s,t) = \frac{1}{2\pi i}\int_{-\infty}^{s}\Phi_1(u;\sigma,t)\Phi_1(v;\sigma,t)d\sigma. \tag{4.36}$$

4.10 Remark. Theorem 4.7 can be generalized to the case where the support of ν_0 (the equilibrium measure in external field V_0) consists of more than one interval. Then, the proof becomes much more technical, although the main ideas remain the same. We comment in Remark 4.23 on the modifications that have to be made in the multi-interval case.

Recurrence coefficients for orthogonal polynomials

It is well-known [95] that the orthonormal polynomials $p_k = p_k^{(n,s,t)}$ satisfy a three-term recurrence relation of the form,

$$xp_k(x) = a_{k+1}p_{k+1}(x) + b_k p_k(x) + a_k p_{k-1}(x), \tag{4.37}$$

where $a_k = a_k^{(n,s,t)} > 0$ and $b_k = b_k^{(n,s,t)} \in \mathbb{R}$ (we suppress the s and t dependence for brevity). In the generic case where V_0 has no singular points, the recurrence coefficients for $s=t=0$ have the following asymptotics, see e.g. [14, 27],

$$a_n^{(n,0,0)} = \frac{b-a}{4} + \mathcal{O}(n^{-1}), \qquad b_n^{(n,0,0)} = \frac{b+a}{2} + \mathcal{O}(n^{-1}), \qquad \text{as } n\to\infty. \tag{4.38}$$

For singular potentials V_0, the constant terms in the expansions (4.38) remain the same, but the error terms behave differently [14, 23]. In our case of interest, where we have a type III singular (edge) point with $m=1$, the error term is of order $\mathcal{O}(n^{-2/7})$, and the coefficient of the $n^{-2/7}$ term is expressed in terms of the special solution y of the P_I^2 equation discussed in Section 4.1.3.

4.11 Theorem. *Let $V_{s,t}$ be such that the assumptions* 4.1 *above are satisfied. Consider the three-term recurrence relation* (4.37) *satisfied by the orthonormal polynomials $p_k = p_k^{(n,s,t)}$ with respect to the weight function $e^{-nV_{s,t}}$. Then, in the double scaling limit where $n\to\infty$ and $s,t\to 0$ such that*

$$\lim c_1 n^{6/7}s = s_0 \in \mathbb{R}, \qquad \lim c_2 n^{4/7}t = t_0 \in \mathbb{R}, \tag{4.39}$$

with c_1 and c_2 given by (4.31)*, we have*

$$a_n^{(n,s,t)} = \frac{b-a}{4} + \frac{1}{2c}\, y(c_1 n^{6/7} s, c_2 n^{4/7} t) n^{-2/7} + \mathcal{O}(n^{-3/7}), \tag{4.40}$$

$$= \frac{b-a}{4} + \frac{1}{2c}\, y(s_0, t_0) n^{-2/7}(1 + o(1)),$$

and

$$b_n^{(n,s,t)} = \frac{b+a}{2} + \frac{1}{c}\, y(c_1 n^{6/7} s, c_2 n^{4/7} t) n^{-2/7} + \mathcal{O}(n^{-3/7}) \tag{4.41}$$

$$= \frac{b+a}{2} + \frac{1}{c}\, y(s_0, t_0) n^{-2/7}(1 + o(1)),$$

where the constant c is given by (4.30)*, and where y is the special solution of the P_I^2 equation discussed in Section* 4.1.3*.*

4.12 Remark. For polynomials which are orthogonal on certain complex contours, it can occur that the equilibrium density vanishes like a power 3/2. Asymptotics for the recurrence coefficients in this case were obtained in [41]. Here the role of y is played by special solutions of the Painlevé I equation.

4.1.5 Outline of the rest of the chapter

We prove our results by characterizing the orthogonal polynomials via the well-known 2×2 matrix-valued Fokas-Its-Kitaev Riemann-Hilbert (RH) problem [47] and applying the Deift/Zhou steepest descent method [36] to analyze this RH problem asymptotically. This approach has been used many times before, see e.g. [22, 23, 27, 32, 33, 41, 75, 98, 99].

An important step in the Deift/Zhou steepest descent method is the construction of so-called g-functions associated with equilibrium measures. Those equilibrium measures will be constructed in Section 4.2. In order to deal with the deformations $V_{s,t}$ of $V_{0,0}$, we use modified equilibrium problems where we allow the measures to be negative, which was also done in Chapters 1 and 2. Another modification of the equilibrium problem is that we choose the support of the equilibrium measure fixed, instead of allowing it to choose its own support.

In Section 4.3, we perform the Deift/Zhou steepest descent analysis to the RH problem Y for orthogonal polynomials. Via a series of transformations $Y \mapsto T \mapsto S \mapsto R$ we want to arrive at a RH problem for R which is normalized at infinity (i.e. $R(z) \to I$ as $z \to \infty$) and with jumps uniformly close to the identity matrix. Then, R itself is close to the identity matrix. By unfolding the series of transformations we then get the asymptotics for Y. The key step in this method will be the local analysis near the endpoints a and b. Near the regular endpoint a, we construct (in Section 4.3.5) a parametrix built out of Airy functions. Due to the

modified equilibrium measures, which have a fixed support, we also need to make a technical modification in the construction of the Airy parametrix, compared with the parametrix as used e.g. in [27]. To construct the local parametrix near the singular endpoint b (in Section 4.3.6) we use a model RH problem associated with the special solution y of the P_I^2 equation as discussed in Section 4.1.3.

The results of Section 4.3 will be used in Section 4.4 to prove the universality result for the correlation kernel (see Theorem 4.7) and in Section 4.5 to determine the asymptotics for the recurrence coefficients (see Theorem 4.11).

4.2 Equilibrium measures

We consider external fields $V_{s,t} = V_0 + sV_1 + tV_2$ which satisfy the Assumptions 4.1 in the beginning of Section 4.1.4. In order to perform the Deift/Zhou steepest descent analysis to the RH problem for orthogonal polynomials one would expect to use the equilibrium measure $\mu_{s,t}$ in external field $V_{s,t}$ minimizing $I_{V_{s,t}}(\mu)$, see (4.6), among all probability measures μ on $\mathbb{R}$. However, as in Chapters 1 and 2, it will be more convenient to use modified equilibrium measures $\nu_{s,t}$ which we allow to be negative.

The aim of this section is to find measures $\nu_{s,t}$ (depending on the parameters $s, t \in \mathbb{R}$) supported on the interval $[a, b] \subset \mathbb{R}$ (where $[a, b]$ is the support of the equilibrium measure ν_0 in external field V_0), such that $\nu_{s,t}([a, b]) = 1$, and such that they satisfy the following condition: there exist $\ell_{s,t} \in \mathbb{R}$ such that for every $\delta > 0$ there are $\varepsilon, \kappa > 0$ sufficiently small such that for $s, t \in [-\varepsilon, \varepsilon]$,

$$2\int \log|x-u|d\nu_{s,t}(u) - V_{s,t}(x) = \ell_{s,t}, \qquad \text{for } x \in [a, b]. \tag{4.42}$$

$$2\int \log|x-u|d\nu_{s,t}(u) - V_{s,t}(x) < \ell_{s,t} - \kappa, \text{ for } x \in \mathbb{R} \setminus [a-\delta, b+\delta]. \tag{4.43}$$

We seek $\nu_{s,t}$ in the following form,

$$\nu_{s,t} = \nu_0 + s\nu_1 + t\nu_2, \tag{4.44}$$

where ν_0 is the equilibrium measure in external field V_0 minimizing $I_{V_0}(\nu)$, see (4.6), among all probability measures ν on $\mathbb{R}$. From Assumption 4.1 (ii) we know that ν_0 can be written as follows

$$d\nu_0(x) = \psi_{0,+}(x)\chi_{[a,b]}(x)dx, \tag{4.45}$$

where $\chi_{[a,b]}$ is the indicator function of the set $[a, b]$, and where $\psi_{0,+}$ is the +boundary value of the function

$$\psi_0(z) = \frac{1}{2\pi i}R(z)h_0(z), \qquad \text{for } z \in \mathcal{V} \setminus [a, b], \tag{4.46}$$

with h_0 analytic in the neighborhood $\mathcal{V}$ of the real line, and with

$$R(z) = \big((z-a)(z-b)\big)^{1/2}, \qquad \text{for } z \in \mathbb{C} \setminus [a,b]. \tag{4.47}$$

Here, we take the principal branch of the square root so that R is analytic in $\mathbb{C} \setminus [a,b]$. Further, since a is a regular (edge) point and since b is a type III singular (edge) point with $m = 1$, we have

$$h_0(a) > 0, \qquad h_0(b) = h_0'(b) = 0, \qquad \text{and} \qquad h_0''(b) > 0. \tag{4.48}$$

Since V_0 is assumed to have no other singular points besides b, we know (cf. (4.8) and (4.9)) that ν_0 satisfies the following condition: there exists $\ell_0 \in \mathbb{R}$ such that

$$2\int \log|x-u|d\nu_0(u) - V_0(x) = \ell_0, \qquad \text{for } x \in [a,b], \tag{4.49}$$

$$2\int \log|x-u|d\nu_0(u) - V_0(x) < \ell_0, \qquad \text{for } x \in \mathbb{R} \setminus [a,b]. \tag{4.50}$$

We will now construct the two measures ν_1 and ν_2. In order to do this we introduce the following auxiliary (analytic) functions,

$$h_j(z) = \frac{1}{2\pi i}\oint_\gamma R(\xi)V_j'(\xi)\frac{d\xi}{\xi - z}, \qquad \text{for } z \in \mathcal{V} \text{ and } j = 1,2, \tag{4.51}$$

where γ is a positively oriented contour in $\mathcal{V}$ with $[a,b]$ and z in its interior, and where R is given by (4.47). Observe that, using the fractional residue theorem, one has,

$$h_j(x) = -\frac{1}{\pi i}\fint_a^b R_+(u)V_j'(u)\frac{du}{u-x}, \qquad \text{for } x \in [a,b], \tag{4.52}$$

where the integral is a Cauchy principal value integral. So, h_j is real on $[a,b]$. Observe that by Assumption 4.1 (iii) and (4.52),

$$h_2(b) = 0. \tag{4.53}$$

4.13 Lemma. *Define two signed measures ν_1 and ν_2 supported on $[a,b]$ as*

$$d\nu_j(x) = \psi_{j,+}(x)\chi_{[a,b]}dx, \qquad j = 1,2, \tag{4.54}$$

where $\chi_{[a,b]}$ is the indicator function of the set $[a,b]$, and where $\psi_{j,+}$ is the $+$boundary value of the function

$$\psi_j(z) = \frac{1}{2\pi i}\frac{h_j(z)}{R(z)}, \qquad \textit{for } z \in \mathcal{V} \setminus [a,b]. \tag{4.55}$$

Here, h_j is given by (4.51)*, see also* (4.52) *for its expression on* $[a,b]$*, and R is given by* (4.47)*. Then, ν_j has zero mass, i.e.*

$$\nu_j([a,b]) = \int_a^b \psi_{j,+}(u)du = 0, \tag{4.56}$$

and there exist constants $\ell_j \in \mathbb{R}$ such that

$$2\int \log|x-u|d\nu_j(u) - V_j(x) = \ell_j, \qquad \textit{for } x \in [a,b]. \tag{4.57}$$

Proof. Define, for $j = 1, 2$, the auxiliary functions

$$F_j(z) = \frac{1}{2\pi i R(z)} \int_a^b R_+(u) V_j'(u) \frac{du}{u-z}, \qquad \text{for } z \in \mathbb{C} \setminus [a,b], \tag{4.58}$$

which, by standard techniques and by (4.51) and (4.55), are equal to

$$\begin{aligned} F_j(z) &= \frac{1}{2}V_j'(z) - \frac{1}{4\pi i R(z)} \oint_\gamma R(\xi) V_j'(\xi) \frac{d\xi}{\xi - z} \\ &= \frac{1}{2}V_j'(z) - \pi i \psi_j(z), \qquad\qquad \text{for } z \in \mathcal{V} \setminus [a,b], \end{aligned}$$

where γ is a positively oriented contour in $\mathcal{V}$ with $[a,b]$ and z in its interior. This, together with the fact that $\psi_{j,+} = -\psi_{j,-}$ on (a,b), yields

$$F_{j,+}(x) - F_{j,-}(x) = -2\pi i \psi_{j,+}(x), \qquad \text{for } x \in [a,b], \tag{4.59}$$

$$F_{j,+}(x) + F_{j,-}(x) = V_j'(x), \qquad \text{for } x \in [a,b]. \tag{4.60}$$

Since F_j is analytic in $\mathbb{C} \setminus [a,b]$ and since, by (4.58), $F_j(z) = \mathcal{O}(z^{-2})$ as $z \to \infty$, a standard complex analysis argument, shows that

$$\frac{1}{2\pi i} \int_a^b \frac{F_{j,+}(u) - F_{j,-}(u)}{u-z} ds = F_j(z), \qquad \text{for } z \in \mathbb{C} \setminus [a,b].$$

By (4.59), this yields,

$$F_j(z) = -\int_a^b \frac{\psi_{j,+}(u)}{u-z} du = -z^{-1} \int_a^b \psi_{j,+}(u)du + \mathcal{O}(z^{-2}), \qquad \text{as } z \to \infty.$$

Comparing this with the fact that $F_j(z) = \mathcal{O}(z^{-2})$ as $z \to \infty$, we obtain that $\int_a^b \psi_{j,+}(u)du = 0$, which proves (4.56).

It remains to prove (4.57). It is straightforward to check that,

$$F_j(z) = -\int_a^b \frac{\psi_{j,+}(u)}{u-z} du = -\pi i \psi_j(z) + \frac{1}{2} \oint_\gamma \frac{\psi_j(\xi)}{\xi - z} d\xi, \qquad \text{for } z \in \mathcal{V} \setminus [a,b],$$

so that, using the fractional residue theorem,

$$F_{j,\pm}(x) = -\pi i \psi_{j,\pm}(x) - \fint_a^b \frac{\psi_{j,+}(u)}{u-x} du, \qquad \text{for } x \in [a,b].$$

From (4.60) and the fact that $\psi_{j,+} + \psi_{j,-} = 0$ on $[a,b]$ this yields,

$$\frac{d}{dx}\left(2\int \log|x-u| d\nu_j(u) + V_j(x)\right) = 2\fint_a^b \frac{\psi_{j,+}(u)}{u-x} du + \big(F_{j,+}(x) + F_{j,-}(x)\big) = 0. \quad (4.61)$$

This proves (4.57). □

4.14 Corollary. *Let $\nu_{s,t} = \nu_0 + s\nu_1 + t\nu_t$. Then, $d\nu_{s,t}(x) = \psi_{s,t,+}(x)\chi_{[a,b]}dx$, where*

$$\psi_{s,t} = \psi_0 + s\psi_1 + t\psi_2, \qquad \text{on } \mathcal{V} \setminus [a,b], \quad (4.62)$$

with ψ_0 given by (4.46) and ψ_1 and ψ_2 given by (4.55). So, $\nu_{s,t}$ is supported on $[a,b]$ and has mass one, i.e. $\nu_{s,t}([a,b]) = 1$. Further, there exist constants $\ell_{s,t} \in \mathbb{R}$ such that for any $\delta > 0$ there are $\varepsilon, \kappa > 0$ sufficiently small such that for $s,t \in [-\varepsilon,\varepsilon]$ the conditions (4.42) and (4.43) are satisfied.

Proof. Since $\nu_{s,t} = \nu_0 + s\nu_1 + t\nu_t$, from (4.56), and from the fact that $\nu_0([a,b]) = 1$ it is clear that $\nu_{s,t}([a,b]) = 1$. Next, with $\ell_{s,t} = \ell_0 + s\ell_1 + t\ell_2$, we have

$$2\int \log|x-u| d\nu_{s,t}(u) - V_{s,t}(x) - \ell_{s,t} = I_0(x) + sI_1(x) + tI_2(x) \quad (4.63)$$

where

$$I_j(x) = 2\int \log|x-u| d\nu_j(u) - V_j(x) - \ell_j, \qquad j = 1,2,3.$$

Then, condition (4.42) follows from (4.49) and (4.57). Now, by using (4.50) and the fact that $I_0(x) \to -\infty$ as $|x| \to \infty$, there exists $\kappa > 0$ such that

$$I_0 < -\frac{3}{2}\kappa, \qquad \text{on } \mathbb{R} \setminus [a-\delta, b+\delta]. \quad (4.64)$$

Further, one can check that I_1 and I_2 are bounded on $\mathbb{R} \setminus [a-\delta, b+\delta]$, and thus there exists $\varepsilon > 0$ such that for $s,t \in [-\varepsilon,\varepsilon]$,

$$sI_1 + tI_2 < \frac{1}{2}\kappa, \qquad \text{on } \mathbb{R} \setminus [a-\delta, b+\delta]. \quad (4.65)$$

Inserting (4.64) and (4.65) into (4.63) we obtain condition (4.43). □

4.15 Remark. The measure ν_1 (ν_2) is the equilibrium measure that minimizes $I_{V_1}(\nu)$ ($I_{V_2}(\nu)$) among all signed measures ν, supported on $[a,b]$ with $\nu([a,b]) = 0$. The measures $\nu_{s,t}$ on the other hand minimize $I_{V_{s,t}}(\nu)$ among all signed measures supported on $[a,b]$ with $\nu([a,b]) = 1$.

Observe that since ν_0 has a strictly positive density on (a,b) (since ν_0 has no type II singular points) we have for any $\delta > 0$ that $\nu_{s,t}$ is positive on $(a+\delta, b-\delta)$ for s,t sufficiently small.

4.3 Steepest descent analysis of the Riemann-Hilbert problem

4.3.1 RH problem for orthogonal polynomials

For each fixed n, s, and t, we consider the Fokas-Its-Kitaev Riemann-Hilbert problem [47] characterizing the orthogonal polynomials $p_k^{(n,s,t)}$ with respect to the weight functions $e^{-nV_{s,t}}$. We seek a 2×2 matrix-valued function $Y(z) = Y(z; n, s, t)$ (we suppress the n, s, and t dependence for brevity) that satisfies the following conditions.

RH problem for Y:

(a) $Y : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{2\times 2}$ is analytic.

(b) Y possesses continuous boundary values for $x \in \mathbb{R}$ denoted by $Y_+(x)$ and $Y_-(x)$, where $Y_+(x)$ and $Y_-(x)$ denote the limiting values of $Y(z')$ as z' approaches x from above and below, respectively, and

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & e^{-nV_{s,t}(x)} \\ 0 & 1 \end{pmatrix}, \qquad \text{for } x \in \mathbb{R}. \tag{4.66}$$

(c) Y has the following asymptotic behavior at infinity

$$Y(z) = \left(I + \mathcal{O}(z^{-1})\right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}, \qquad \text{as } z \to \infty. \tag{4.67}$$

The unique solution of the RH problem is given by, for $z \in \mathbb{C} \setminus \mathbb{R}$,

$$Y(z) = \begin{pmatrix} \kappa_n^{-1} p_n(z) & \dfrac{\kappa_n^{-1}}{2\pi i} \displaystyle\int_{\mathbb{R}} \frac{p_n(u) e^{-nV_{s,t}(u)}}{u - z}\, du \\ -2\pi i \kappa_{n-1} p_{n-1}(z) & -\kappa_{n-1} \displaystyle\int_{\mathbb{R}} \frac{p_{n-1}(u) e^{-nV_{s,t}(u)}}{u - z}\, du \end{pmatrix}, \tag{4.68}$$

where $p_k = p_k^{(n,s,t)}$ is the k-th degree orthonormal polynomial with respect to the varying weight $e^{-nV_{s,t}}$, and where $\kappa_k = \kappa_k^{(n,s,t)} > 0$ is the leading coefficient of p_k. The solution (4.68) is due to Fokas, Its, and Kitaev [47], see also [27, 32, 33].

It is now possible to write the 2-point kernel $K_n^{(s,t)}$, see (4.3), in terms of Y. Indeed using the Christoffel-Darboux formula for orthogonal polynomials and the fact that $\det Y \equiv 1$ (which follows easily from (4.66), (4.67), and Liouville's theorem), we get

$$K_n^{(s,t)}(x,y) = e^{-\frac{n}{2}V_{s,t}(x)} e^{-\frac{n}{2}V_{s,t}(y)} \frac{1}{2\pi i(x-y)} \begin{pmatrix} 0 & 1 \end{pmatrix} Y_\pm^{-1}(y) Y_\pm(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (4.69)$$

So, in order to prove Theorem 4.7, we need to analyze the RH problem for Y asymptotically. We do this by applying the Deift/Zhou steepest descent method [36] to this RH problem.

4.3.2 Normalization of the RH problem at infinity: $Y \mapsto T$

In order to normalize the RH problem for Y at infinity, the equilibrium measures $\nu_{s,t}$, introduced in Section 4.2 play a key role. Consider the log-transform $g_{s,t}$ of $\nu_{s,t}$,

$$g_{s,t}(z) = \int_a^b \log(z-u) d\nu_{s,t}(u), \qquad \text{for } z \in \mathbb{C} \setminus (-\infty, b]. \quad (4.70)$$

Here, we take the principal branch of the logarithm so that $g_{s,t}$ is analytic in $\mathbb{C} \setminus (-\infty, b]$. We now give properties of $g_{s,t}$ which are crucial in the following. From (4.70) and condition (4.42) it follows that

$$g_{s,t,+}(x) + g_{s,t,-}(x) - V_{s,t}(x) - \ell_{s,t} = 0, \qquad \text{for } x \in [a,b]. \quad (4.71)$$

Another crucial property is that

$$g_{s,t,+}(x) - g_{s,t,-}(x) = 2\pi i \int_x^b d\nu_{s,t}(u), \qquad \text{for } x \in \mathbb{R}, \quad (4.72)$$

so that since $\nu_{s,t}$ is supported on $[a,b]$ and has mass one (see Corollary 4.14),

$$g_{s,t,+}(x) - g_{s,t,-}(x) = \begin{cases} 2\pi i, & \text{for } x < a, \\ 0, & \text{for } x > b. \end{cases} \quad (4.73)$$

Now, we are ready to perform the first transformation $Y \mapsto T$. Define the matrix-valued function T as

$$T(z) = e^{-\frac{1}{2}n\ell_{s,t}\sigma_3} Y(z) e^{-ng_{s,t}(z)\sigma_3} e^{\frac{1}{2}n\ell_{s,t}\sigma_3}, \qquad \text{for } z \in \mathbb{C} \setminus \mathbb{R}, \quad (4.74)$$

where $\ell_{s,t}$ is the constant that appears in the variational conditions (4.42) and (4.43), and where $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ denotes the third Pauli-matrix. Using (4.71), (4.73), the RH conditions for Y, and the fact that $g_{s,t}(z) = \log z + \mathcal{O}(1/z)$ as $z \to \infty$, it is straightforward to check that T is a solution to the following RH problem.

RH problem for T:

(a) $T : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{2\times 2}$ is analytic.

(b) $T_+(x) = T_-(x) v_T(x)$ for $x \in \mathbb{R}$, with

$$v_T = \begin{cases} \begin{pmatrix} e^{-n(g_{s,t,+}-g_{s,t,-})} & 1 \\ 0 & e^{n(g_{s,t,+}-g_{s,t,-})} \end{pmatrix}, & \text{on } (a,b), \\ \begin{pmatrix} 1 & e^{n(g_{s,t,+}+g_{s,t,-}-V_{s,t}-\ell_{s,t})} \\ 0 & 1 \end{pmatrix}, & \text{on } \mathbb{R} \setminus (a,b). \end{cases} \tag{4.75}$$

(c) $T(z) = I + \mathcal{O}(1/z), \qquad \text{as } z \to \infty.$

4.16 Remark. From (4.72) we see that the diagonal entries of v_T on (a,b) are rapidly oscillating for large n. Further, using condition (4.43) and (4.70), we see that $v_T - I$ decays exponentially on $\mathbb{R} \setminus [a-\delta, b+\delta]$.

4.3.3 Opening of the lens: $T \mapsto S$

Here, we will transform the oscillatory diagonal entries of the jump matrix v_T on (a,b) into exponentially decaying off-diagonal entries. This step is referred to as the opening of the lens.

Introduce a scalar function $\phi_{s,t}$ as,

$$\phi_{s,t}(z) = -\pi i \int_z^b \psi_{s,t}(\xi) d\xi, \qquad \text{for } z \in \mathcal{V} \setminus [a,b], \tag{4.76}$$

where the path of integration does not cross the real line, and where $\psi_{s,t}$ is defined by (4.62).

The important feature of the function $\phi_{s,t}$ is that by (4.72), $\phi_{s,t,+}$ and $\phi_{s,t,-}$ are purely imaginary on (a,b) and satisfy,

$$\begin{aligned} -2\phi_{s,t,+}(x) &= 2\phi_{s,t,-}(x) \\ &= 2\pi i \int_x^b d\nu_{s,t}(u) = g_{s,t,+}(x) - g_{s,t,-}(x), \quad \text{for } x \in (a,b), \end{aligned} \tag{4.77}$$

which means that $-2\phi_{s,t}$ and $2\phi_{s,t}$ provide analytic extensions of $g_{s,t,+} - g_{s,t,-}$ into the upper half-plane and lower half-plane, respectively. Further, $2g_{s,t}+2\phi_{s,t}-V_{s,t} - \ell_{s,t}$ is analytic in $\mathcal{V} \setminus (-\infty, b]$ and satisfies by (4.77) and (4.71),

$$2g_{s,t,\pm} + 2\phi_{s,t,\pm} - V_{s,t} - \ell_{s,t} = g_{s,t,+} + g_{s,t,-} - V_{s,t} - \ell_{s,t} = 0, \qquad \text{on } (a,b),$$

so that by the identity theorem,

$$2g_{s,t} - V_{s,t} - \ell_{s,t} = -2\phi_{s,t}, \qquad \text{on } \mathcal{V} \setminus (-\infty, a]. \tag{4.78}$$

Using (4.73) this yields,

$$\begin{aligned} g_{s,t,+} + g_{s,t,-} - V_{s,t} - \ell_{s,t} &= 2g_{s,t,-} - V_{s,t} - \ell_{s,t} + (g_{s,t,+} - g_{s,t,-}) \\ &= -2\phi_{s,t,-} + 2\pi i, \qquad \text{on } (-\infty, a). \end{aligned} \tag{4.79}$$

Inserting (4.77), (4.78), and (4.79) into (4.75), the jump matrix for T can be written in terms of $\phi_{s,t}$ as

$$v_T(x) = \begin{cases} \begin{pmatrix} e^{2n\phi_{s,t,+}(x)} & 1 \\ 0 & e^{2n\phi_{s,t,-}(x)} \end{pmatrix}, & \text{for } x \in (a,b), \\ \begin{pmatrix} 1 & e^{-2n\phi_{s,t,-}(x)} \\ 0 & 1 \end{pmatrix}, & \text{for } x \in \mathbb{R} \setminus (a,b). \end{cases} \tag{4.80}$$

It is straightforward to check, using the fact that $\phi_{s,t,+} + \phi_{s,t,-} = 0$ on (a,b), that v_T has on the interval (a,b) the following factorization,

$$v_T = \begin{pmatrix} 1 & 0 \\ e^{2n\phi_{s,t,-}} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{2n\phi_{s,t,+}} & 1 \end{pmatrix}, \qquad \text{on } (a,b), \tag{4.81}$$

and the opening of the lens is based on this factorization. Observe that, since $\operatorname{Re}\phi_{s,t,\pm}(x) = 0$ and $\operatorname{Im}\phi_{s,t,\pm}(x) = \mp \int_x^b d\nu_{s,t}(u)$ for $x \in (a,b)$ (see (4.77)), and since $\nu_{s,t}$ is positive on $(a+\delta, b-\delta)$ for $\delta > 0$ and s,t sufficiently small (see Remark 4.15), it follows (as in [27]) from the Cauchy-Riemann conditions that

$$\operatorname{Re}\phi_{s,t}(z) < 0, \qquad \text{for } |\operatorname{Im} z| \neq 0 \text{ small and } a+\delta < \operatorname{Re} z < b-\delta. \tag{4.82}$$

We deform the RH problem for T into a RH problem for S by opening a lens as shown in Figure 4.1, so that we obtain a contour Σ. For now, we choose the lens to be contained in $\mathcal{V}$, but we will specify later how we choose the lens exactly. Let

$$S(z) = \begin{cases} T(z), & \text{for } z \text{ outside the lens.} \\ T(z) \begin{pmatrix} 1 & 0 \\ -e^{2n\phi_{s,t}(z)} & 1 \end{pmatrix}, & \text{for } z \text{ in the upper part of the lens,} \\ T(z) \begin{pmatrix} 1 & 0 \\ e^{2n\phi_{s,t}(z)} & 1 \end{pmatrix}, & \text{for } z \text{ in the lower part of the lens.} \end{cases} \tag{4.83}$$

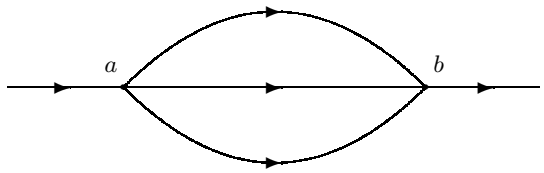

Figure 4.1 *The lens* Σ

Then, using (4.81) and the RH conditions for T, one can check that S is the unique solution of the following RH problem.

RH problem for S:

(a) $S : \mathbb{C} \setminus \Sigma \to \mathbb{C}^{2\times 2}$ is analytic.

(b) $S_+(z) = S_-(z) v_S(z)$ for $z \in \Sigma$, with

$$v_S(z) = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \text{for } z \in (a,b), \\ \begin{pmatrix} 1 & 0 \\ e^{2n\phi_{s,t}(z)} & 1 \end{pmatrix}, & \text{for } z \in \Sigma \cap \mathbb{C}_\pm, \\ \begin{pmatrix} 1 & e^{-2n\phi_{s,t,-}(z)} \\ 0 & 1 \end{pmatrix}, & \text{for } z \in \mathbb{R} \setminus (a,b). \end{cases} \tag{4.84}$$

(c) $S(z) = I + \mathcal{O}(1/z), \qquad \text{as } z \to \infty.$

4.17 Remark. On the lips of the lens (away from a and b) and on $\mathbb{R}\setminus[a-\delta, b+\delta]$, it follows from (4.82) and (4.43) that the jump matrix for S converges exponentially fast to the identity matrix as $n \to \infty$. This convergence is uniform as long as we stay away from small disks surrounding the endpoints a and b. Near these endpoints we have to construct local parametrices.

4.3.4 Parametrix $P^{(\infty)}$ for the outside region

From Remark 4.17, we expect that the leading order asymptotics of Y will be determined by a solution $P^{(\infty)}$ of the following RH problem.

RH problem for $P^{(\infty)}$:

(a) $P^{(\infty)} : \mathbb{C} \setminus [a,b] \to \mathbb{C}^{2\times 2}$ is analytic.

(b) $P_+^{(\infty)}(x) = P_-^{(\infty)}(x)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \text{for } x \in (a,b).$

(c) $P^{(\infty)}(z) = I + \mathcal{O}(1/z), \qquad \text{as } z \to \infty.$

It is well known, see for example [27, 33], that $P^{(\infty)}$ given by

$$P^{(\infty)}(z) = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\left(\frac{z-b}{z-a}\right)^{\sigma_3/4}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}^{-1}, \qquad \text{for } z \in \mathbb{C}\setminus[a,b], \tag{4.85}$$

is a solution to the above RH problem. Note that $P^{(\infty)}$ is independent of the parameters s, t and n.

4.3.5 Parametrix $P^{(a)}$ near the regular endpoint a

Here, we do the local analysis near the regular endpoint a. Let $U_{\delta,a} = \{z \in \mathbb{C} : |z-a| < \delta\}$ be a small disk with center a and radius $\delta > 0$ sufficiently small such that the disk lies in the region $\mathcal{V}$ where $V_{s,t}$ is analytic. We seek a 2×2 matrix-valued function $P^{(a)}$ (depending on the parameters n, s, and t) in the disk $U_{\delta,a}$ with the same jumps as S and which matches with $P^{(\infty)}$ on the boundary $\partial U_{\delta,a}$ of the disk. We thus seek a 2×2 matrix-valued function that satisfies the following RH problem.

RH problem for $P^{(a)}$:

(a) $P^{(a)} : U_{\delta,a}\setminus\Sigma \to \mathbb{C}^{2\times 2}$ is analytic.

(b) $P_+^{(a)}(z) = P_-^{(a)}(z)v_S(z)$ for $z \in \Sigma \cap U_{\delta,a}$, where v_S is given by (4.84).

(c) $P^{(a)}$ satisfies the matching condition

$$P^{(a)}(z)(P^{(\infty)})^{(-1)}(z) = I + \mathcal{O}(n^{-1/7}), \tag{4.86}$$

as $n \to \infty$ and $s, t \to 0$ such that (4.33) holds, uniformly for $z \in \partial U_{\delta,a}\setminus\Sigma$.

Airy model RH problem

We will construct $P^{(a)}$ by introducing an auxiliary 2×2 matrix-valued function $A(\zeta; r)$ with jumps (in the variable ζ) on an oriented contour $\Gamma = \bigcup_j \Gamma_j$, shown in Figure 4.2, consisting of four straight rays

$$\Gamma_1 : \arg\zeta = 0, \qquad \Gamma_2 : \arg\zeta = \frac{6\pi}{7}, \qquad \Gamma_3 : \arg\zeta = \pi, \qquad \Gamma_4 : \arg\zeta = -\frac{6\pi}{7}.$$

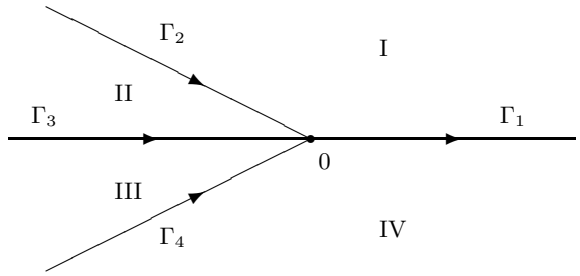

Figure 4.2 *The oriented contour* Γ. *The four straight rays* $\Gamma_1, \ldots, \Gamma_4$ *divide the complex plane into four regions* $\mathrm{I}, \mathrm{II}, \mathrm{III}$ *and* IV.

These four rays divide the complex plane into four regions $\mathrm{I}, \mathrm{II}, \mathrm{III}$, and IV, also shown in Figure 4.2. Let,

$$A(\zeta; r) = \sqrt{2\pi} e^{-\frac{\pi i}{12}} \times \begin{cases} \begin{pmatrix} \mathrm{Ai}(\zeta^{(r)}) & \mathrm{Ai}(\omega^2 \zeta^{(r)}) \\ \mathrm{Ai}'(\zeta^{(r)}) & \omega^2 \mathrm{Ai}'(\omega^2 \zeta^{(r)}) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3}, & \zeta \in \mathrm{I}, \\ \begin{pmatrix} \mathrm{Ai}(\zeta^{(r)}) & \mathrm{Ai}(\omega^2 \zeta^{(r)}) \\ \mathrm{Ai}'(\zeta^{(r)}) & \omega^2 \mathrm{Ai}'(\omega^2 \zeta^{(r)}) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, & \zeta \in \mathrm{II}, \\ \begin{pmatrix} \mathrm{Ai}(\zeta^{(r)}) & -\omega^2 \mathrm{Ai}(\omega \zeta^{(r)}) \\ \mathrm{Ai}'(\zeta^{(r)}) & -\mathrm{Ai}'(\omega \zeta^{(r)}) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & \zeta \in \mathrm{III}, \\ \begin{pmatrix} \mathrm{Ai}(\zeta^{(r)}) & -\omega^2 \mathrm{Ai}(\omega \zeta^{(r)}) \\ \mathrm{Ai}'(\zeta^{(r)}) & -\mathrm{Ai}'(\omega \zeta^{(r)}) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3}, & \zeta \in \mathrm{IV}, \end{cases} \tag{4.87}$$

with $\omega = e^{\frac{2\pi i}{3}}$, with $\zeta^{(r)} = \zeta + r$ (for brevity), and with Ai the Airy function. It is well-known, see e.g. [27, 33], that A satisfies the following RH problem.

RH problem for A:

(a) A is analytic for $\zeta \in \mathbb{C} \setminus \Gamma$ and for r in $\mathbb{C}$.

(b) A satisfies the following jump relations on Γ,

$$A_+(\zeta) = A_-(\zeta)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_3, \tag{4.88}$$

$$A_+(\zeta) = A_-(\zeta)\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_1, \tag{4.89}$$

$$A_+(\zeta) = A_-(\zeta)\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_2 \cup \Gamma_4. \tag{4.90}$$

(c) A has the following asymptotic behavior at infinity,

$$\begin{aligned} A(\zeta; r) &= (\zeta + r)^{-\frac{\sigma_3}{4}} N \left[I + \mathcal{O}\left((\zeta + r)^{-3/2} \right) \right] e^{-\frac{2}{3}(\zeta+r)^{3/2}\sigma_3} \\ &= \zeta^{-\frac{\sigma_3}{4}} N \left[I - \frac{1}{4} r^2 \zeta^{-1/2} \sigma_3 + \frac{1}{32} r^4 \zeta^{-1} I \right. \\ &\qquad \left. + \mathcal{O}(r^6 \zeta^{-3/2}) + \mathcal{O}(r\zeta^{-1}) \right] e^{-(\frac{2}{3}\zeta^{3/2} + r\zeta^{1/2})\sigma_3}, \end{aligned} \tag{4.91}$$

as $\zeta \to \infty$, uniformly for r such that

$$\operatorname{sgn}\big(\operatorname{Im}(\zeta + r)\big) = \operatorname{sgn}\big(\operatorname{Im}\zeta\big), \qquad \text{and} \qquad |r| < |\zeta|^{1/4}. \tag{4.92}$$

In (4.91), N is given by

$$N = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} e^{-\frac{1}{4}\pi i \sigma_3}. \tag{4.93}$$

Construction of $P^{(a)}$

We seek $P^{(a)}$ in the following form

$$P^{(a)}(z) = E^{(a)}(z)\sigma_3 A\left(n^{2/3} f_a(z); n^{2/3} r_{s,t}(z) \right) \sigma_3 e^{n\phi_{s,t}(z)\sigma_3}, \tag{4.94}$$

where $E^{(a)}$ is an invertible 2×2 matrix-valued function analytic on $U_{\delta,a}$ and where f_a and $r_{s,t}$ are (scalar) analytic functions on $U_{\delta,a}$ which are real on $(a-\delta, a+\delta)$. In addition we take f_a to be a conformal map from $U_{\delta,a}$ onto a convex neighborhood $f_a(U_{\delta,a})$ of 0 such that $f_a(a) = 0$ and $f_a'(a) < 0$.

If those conditions are all satisfied, and if we open the lens Σ (recall that the lens was not yet fully specified) such that

$$f_a(\Sigma \cap (U_{\delta,a} \cap \mathbb{C}_+)) = \Gamma_4 \cap f_a(U_{\delta,a}), \qquad \text{and} \qquad f_a(\Sigma \cap (U_{\delta,a} \cap \mathbb{C}_-)) = \Gamma_2 \cap f_a(U_{\delta,a}),$$

then it is straightforward to verify, using (4.84) and (4.88)–(4.90), that $P^{(a)}$ defined by (4.94) satisfies conditions (a) and (b) of the RH problem for $P^{(a)}$.

Let

$$f_a(z) = \left[\frac{3}{2}\left(-\pi i \int_z^a \psi_0(\xi)d\xi\right)(a-z)^{-3/2}\right]^{2/3}(a-z)$$
$$= -\left(\frac{1}{2}h_0(a)\sqrt{b-a}\right)^{2/3}(z-a) + \mathcal{O}((z-a)^2), \qquad \text{as } z \to a, \quad (4.95)$$

where we have used (4.46), and let

$$r_{s,t}(z) = \left(-\pi i \int_z^a (s\psi_1(\xi) + t\psi_2(\xi))d\xi\right) f_a(z)^{-1/2}. \quad (4.96)$$

Then, f_a is analytic with $f_a(a) = 0$ and $f_a'(a) < 0$, it is real on $(a-\delta, a+\delta)$, and it is a conformal mapping on $U_{\delta,a}$ provided $\delta > 0$ is sufficiently small. Further, it is straightforward to check that $r_{s,t}$ is analytic on $U_{\delta,a}$ and real on $(a-\delta, a+\delta)$, as well. Thus, f_a and $r_{s,t}$ satisfy the above conditions, so that $P^{(a)}$ defined by (4.94), with $E^{(a)}$ any invertible analytic matrix-valued function, satisfies conditions (a) and (b) of the RH problem for $P^{(a)}$.

4.18 Remark. We can use any functions f_a and $r_{s,t}$, satisfying the conditions stated under equation (4.94), to construct the parametrix $P^{(a)}$. However, we have to choose them so as to compensate for the factor $e^{n\phi_{s,t}\sigma_3}$ in (4.94). Using (4.62), (4.76), and the fact that $\int_a^b \psi_{s,t+}(u)du = 1$ we have, for $z \in U_{\delta,a} \setminus [a, a+\delta)$,

$$e^{-n\left(\frac{2}{3}f_a(z)^{3/2} + r_{s,t}(z)f_a(z)^{1/2}\right)\sigma_3} = (-1)^n e^{-n\phi_{s,t}(z)\sigma_3}. \quad (4.97)$$

From this and (4.91) it is clear that our choice of f_a and $r_{s,t}$ will do the job.

It now remains to determine $E^{(a)}$ such that the matching condition (c) holds as well. In order to do this we make use of the following result.

4.19 Proposition. *Let $n \to \infty$ and $s, t \to 0$ such that* (4.33) *holds. Then,*

$$P^{(a)}(z) = (-1)^n E^{(a)}(z)\left(n^{2/3}f_a(z)\right)^{-\sigma_3/4}\sigma_3 N \sigma_3$$
$$\times\left[I - \frac{(n^{4/7}r_{s,t}(z))^2}{4f_a(z)^{1/2}}\sigma_3\, n^{-1/7} + \frac{(n^{4/7}r_{s,t}(z))^4}{32 f_a(z)} I\, n^{-2/7} + \mathcal{O}(n^{-3/7})\right]. \quad (4.98)$$

Proof. We will use the asymptotics (4.91) of A. In order to do this we have to check that condition (4.92) is satisfied for our choice of $\zeta = n^{2/3}f_a(z)$ and $r = n^{2/3}r_{s,t}(z)$.

Obviously $r_{s,t}(z) = \mathcal{O}(n^{-4/7})$ as $n \to \infty$ and $s, t \to 0$ such that (4.33) holds, uniformly for $z \in \partial U_{\delta,a}$. Then, it is straightforward to check that there exists $n_0 \in \mathbb{N}$ sufficiently large, and $\kappa_1, \kappa_2 > 0$ sufficiently small, such that

$$|n^{2/3}r_{s,t}(z)| < |n^{2/3}f_a(z)|^{1/4}, \quad (4.99)$$

for $z \in \partial U_{\delta,a}$ (for a possible smaller δ), for $n \geq n_0$, and for s and t such that $|c_1 n^{6/7} s - s_0| \leq \kappa_1$ and $|c_1 n^{4/7} t - t_0| \leq \kappa_2$.

Further, since f_a and $r_{s,t}$ are analytic near a and real valued on $(a-\delta, a+\delta)$ one can check that

$$\begin{aligned} \operatorname{Im} f_a(z) &= f_a'(\operatorname{Re} z)\operatorname{Im} z + \mathcal{O}((\operatorname{Im} z)^2), && \text{as } z \to a, \\ \operatorname{Im} r_{s,t}(z) &= r_{s,t}'(\operatorname{Re} z) + \mathcal{O}((\operatorname{Im} z)^2), && \text{as } z \to a. \end{aligned}$$

Now, since $f_a'(a) \neq 0$ and $r_{s,t}'(\operatorname{Re} z) = \mathcal{O}(n^{-4/7})$ uniformly for $z \in U_{\delta,a}$ one then can find a constant $C > 0$ such that,

$$|\operatorname{Im} r_{s,t}(z)| < C|\operatorname{Im} z| < |\operatorname{Im} f_a(z)|,$$

for $z \in \partial U_{\delta,a} \setminus (a-\delta, a+\delta)$, for $n \geq n_0$, and for s and t such that $|c_1 n^{6/7} s - s_0| \leq \kappa_1$ and $|c_1 n^{4/7} t - t_0| \leq \kappa_2$ (for a possible smaller δ, κ_1 and κ_2, and for a possible larger n_0). This yields

$$\operatorname{sgn}(\operatorname{Im}(f_a(z) + r_{s,t}(z))) = \operatorname{sgn}(\operatorname{Im} f_a(z)). \tag{4.100}$$

We now have shown that condition (4.92) is satisfied so that we can use the asymptotic behavior (4.91) of A. Using (4.94), (4.91), (4.97), and the fact that $r_{s,t}(z) = \mathcal{O}(n^{-4/7})$ we obtain (4.98). □

From (4.98) and the fact that $r_{s,t} = \mathcal{O}(n^{-4/7})$ it is clear that (in order that the matching condition (c) is satisfied) we have to define $E^{(a)}$ by,

$$E^{(a)} = (-1)^n P^{(\infty)} \sigma_3 N^{-1} \sigma_3 (n^{2/3} f_a)^{\sigma_3/4}. \tag{4.101}$$

Obviously, $E^{(a)}$ is well-defined and analytic in $U_{\delta,a} \setminus (a, a+\delta)$. Further, using condition (b) of the RH problem for $P^{(\infty)}$, equation (4.93), and the fact that $f_{a,-}^{1/4} = i f_{a,+}^{1/4}$ on $(a, a+\delta)$, it is straightforward to check that $E^{(a)}$ has no jump on $(a, a+\delta)$. We then have that $E^{(a)}$ is analytic in $U_{\delta,a}$ except for a possible isolated singularity at a. However, $E^{(a)}$ has at most a square root singularity at a and hence it has to be a removable singularity. Further, since $\det P^{(\infty)} \equiv 1$ and $\det N = 1$ it is clear that $\det E^{(a)} \equiv 1$ and thus $E^{(a)}$ is invertible. This ends the construction of the parametrix near the regular endpoint.

4.3.6 Parametrix $P^{(b)}$ near the critical endpoint b

Here, we do the local analysis near the critical endpoint b. Let $U_{\delta,b} = \{z \in \mathbb{C} : |z-b| < \delta\}$ be a small disk with center b and radius $\delta > 0$ sufficiently small such that the disks $U_{\delta,a}$ and $U_{\delta,b}$ do not intersect. We seek a 2×2 matrix-valued function $P^{(b)}$ (depending on n, s and t) in the disk $U_{\delta,b}$ with the same jumps as S and with matches with $P^{(\infty)}$ on the boundary $\partial U_{\delta,b}$ of the disk. We thus seek a 2×2 matrix valued function that satisfies the following RH problem.

RH problem for $P^{(b)}$:

(a) $P^{(b)} : U_{\delta,b} \setminus \Sigma \to \mathbb{C}^{2\times 2}$ is analytic.

(b) $P_+^{(b)}(z) = P_-^{(b)}(z) v_S(z)$ for $z \in U_{\delta,b} \cap \Sigma$, where v_S is given by (4.84).

(c) $P^{(b)}$ satisfies the matching condition

$$P^{(b)}(z)(P^{(\infty)})^{-1}(z) = I + \mathcal{O}(n^{-1/7}), \tag{4.102}$$

as $n \to \infty$ and $s, t \to 0$ such that (4.33) holds, uniformly for $z \in \partial U_{\delta,b} \setminus \Sigma$.

Due to the singular behavior of the equilibrium measure $d\nu_0(x)$ near b, see Assumptions 4.1 (ii), the Airy parametrix does not fit near b. Instead we use a different model RH problem associated with the P_I^2 equation (4.14).

Model RH problem for the P_I^2 equation

We construct $P^{(b)}$ by introducing the following model RH problem for the special solution y of the P_I^2 equation (4.14) as discussed in Section 4.1.3. This RH problem depends on two complex parameters s, t and has jumps on the oriented contour Γ as defined in Section 4.3.5, see Figure 4.2. We seek a 2×2 matrix-valued function $\Psi(\zeta) = \Psi(\zeta; s, t)$ satisfying the following conditions.

RH problem for Ψ:

(a) Ψ is analytic for $\zeta \in \mathbb{C} \setminus \Gamma$.

(b) Ψ satisfies the following jump relations on Γ,

$$\Psi_+(\zeta) = \Psi_-(\zeta) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_3, \tag{4.103}$$

$$\Psi_+(\zeta) = \Psi_-(\zeta) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_1, \tag{4.104}$$

$$\Psi_+(\zeta) = \Psi_-(\zeta) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_2 \cup \Gamma_4. \tag{4.105}$$

(c) Ψ has the following behavior at infinity,

$$\Psi(\zeta) = \zeta^{-\frac{1}{4}\sigma_3} N \left(I - h\sigma_3 \zeta^{-1/2} + \frac{1}{2} \begin{pmatrix} h^2 & iy \\ -iy & h^2 \end{pmatrix} \zeta^{-1} + \mathcal{O}(\zeta^{-3/2}) \right) \times \quad e^{-\theta(\zeta;s,t)\sigma_3}, \tag{4.106}$$

where $y = y(s,t)$ is the special solution of the P_I^2 equation (4.14) as discussed in Section 4.1.3, where $\frac{\partial h}{\partial s} = -y$, where N is given by (4.93), and where θ is given by (4.21).

4.20 Remark. Note that the only difference between the model RH problem for Airy functions and the one for P_{I}^2 lies in the asymptotic condition (c). In particular, in θ we have an extra factor $\zeta^{7/2}$.

If we fix $s_0, t_0 \in \mathbb{R}$, it was proven in Lemma 3.9 and Proposition 3.11 that there exists a neighborhood $\mathcal{U}$ of s_0 and a neighborhood $\mathcal{W}$ of t_0 such that the RH problem for Ψ is (uniquely) solvable for all $(s,t) \in \mathcal{U} \times \mathcal{W}$. Furthermore, for $(s,t) \in \mathcal{U} \times \mathcal{W}$, Ψ is analytic both in s and t, and condition (c) holds uniformly for (s,t) in compact subsets of $\mathcal{U} \times \mathcal{W}$.

In Section 3.2.3, we have shown that the solution Ψ of the RH problem for Ψ satisfies the Lax pair (4.15)–(4.18). From (4.20), (4.106), and (4.105) we then obtain

$$\begin{pmatrix} \Phi_1(\zeta;s,t) \\ \Phi_2(\zeta;s,t) \end{pmatrix} = \begin{cases} \begin{pmatrix} \Psi_{11}(\zeta;s,t) \\ \Psi_{21}(\zeta;s,t) \end{pmatrix}, & \text{for } 0 < \arg\zeta < 6\pi/7, \\ \begin{pmatrix} \Psi_{11}(\zeta;s,t) \\ \Psi_{21}(\zeta;s,t) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & \text{for } 6\pi/7 < \arg\zeta < \pi. \end{cases} \tag{4.107}$$

Construction of $P^{(b)}$

We seek $P^{(b)}$ in the following form

$$P^{(b)}(z) = E^{(b)}(z)\Psi\left(n^{2/7}f_b(z); n^{6/7}sf_1(z), n^{4/7}tf_2(z)\right)e^{n\phi_{s,t}(z)\sigma_3}, \tag{4.108}$$

where $E^{(b)}$ is an invertible 2×2 matrix-valued function analytic on $U_{\delta,b}$ and where f_b, f_1, and f_2 are (scalar) analytic functions on $U_{\delta,b}$ which are real on $(b-\delta, b+\delta)$. We take f_1 and f_2 to be such that $f_1(b) = c_1$ and $f_2(b) = c_2$ (where c_1 and c_2 are given by (4.31)). Then it is clear from (4.33) that for n sufficiently large and s and t sufficiently small,

$$n^{6/7}sf_1(z) \in \mathcal{U}, \qquad \text{and} \qquad n^{4/7}tf_2(z) \in \mathcal{W}, \qquad \text{for } z \in U_{\delta,b},$$

where $\mathcal{U}$ and $\mathcal{W}$ are the neighborhoods of s_0 and t_0 where Ψ exists. In addition we take f_b to be a conformal map from $U_{\delta,b}$ onto a convex neighborhood $f_b(U_{\delta,b})$ of 0 such that $f_b(b) = 0$ and $f_b'(b) > 0$.

If those conditions are all satisfied, and if we open the lens Σ (recall that the lens was not yet fully specified near b) such that

$$f_b(\Sigma \cap (U_{\delta,b} \cap \mathbb{C}_+)) = \Gamma_2 \cap f_b(U_{\delta,b}), \qquad \text{and} \qquad f_b(\Sigma \cap (U_{\delta,b} \cap \mathbb{C}_-)) = \Gamma_4 \cap f_b(U_{\delta,a}),$$

then it is straightforward to verify, using (4.84) and (4.103)–(4.105), that $P^{(b)}$ defined by (4.108) satisfies conditions (a) and (b) of the RH problem for $P^{(b)}$.

Let

$$f_b(z) = \left[105\left(-\pi i\int_z^b \psi_0(\xi)d\xi\right)(z-b)^{-7/2}\right]^{2/7}(z-b)$$
$$= c(z-b) + \mathcal{O}(z-b)^2, \quad (4.109)$$

as $z \to 0$, where

$$c = \left(\frac{15}{2}h_0''(b)\sqrt{b-a}\right)^{2/7}.$$

To get the expansion of f_b near b we have used (4.46) and the facts that $h_0(b) = h_0'(b) = 0$ (see (4.48)). Further since $h_0''(b) > 0$ we have that $c > 0$. So, we have defined an analytic function f_b with $f_b(b) = 0$ and $f_b'(b) = c > 0$, which is real on $(b-\delta, b+\delta)$, and which is a conformal mapping on $U_{\delta,a}$ provided $\delta > 0$ is sufficiently small.

Next, let f_1 and f_2 be defined by

$$f_1(z) = \left(-\pi i\int_z^b \psi_1(\xi)d\xi\right)f_b(z)^{-1/2}, \quad (4.110)$$

$$f_2(z) = -3\left(-\pi i\int_z^b \psi_2(\xi)d\xi\right)f_b(z)^{-3/2}. \quad (4.111)$$

Since f_b is a conformal mapping in $U_{\delta,b}$ it is clear from (4.55) and (4.47) that f_1 is analytic in $U_{\delta,b}$. To see that f_2 is analytic in $U_{\delta,b}$ as well, we also need to use the extra condition $h_2(b) = 0$ (see (4.53)). Further, f_1 and f_2 are real on $(b-\delta, b+\delta)$ and one can check that,

$$f_1(b) = \frac{h_1(b)}{c^{1/2}(b-a)^{1/2}} = c_1, \qquad f_2(b) = -\frac{h_2'(b)}{c^{3/2}(b-a)^{1/2}} = c_2. \quad (4.112)$$

Thus, f_b, f_1, and f_2 satisfy the above conditions, so that $P^{(b)}$ defined by (4.108), with $E^{(b)}$ any invertible analytic matrix-valued function, satisfies conditions (a) and (b) of the RH problem for $P^{(b)}$.

4.21 Remark. As in Remark 4.18 we note that we could have also used different functions f_b, f_1, and f_2. However, we have to choose them so as to compensate for the factor $e^{n\phi_{s,t}\sigma_3}$ in (4.108). Using (4.21), (4.109), (4.110), (4.111), (4.62), and (4.76), we have for $z \in U_{\delta,b} \setminus (b-\delta, b]$,

$$\theta(n^{2/7}f_b(z); n^{6/7}sf_1(z), n^{4/7}tf_2(z)) = n\phi_{s,t}(z). \quad (4.113)$$

From this and (4.106) it is clear that our choice of f_b, f_1, and f_2 will do the job.

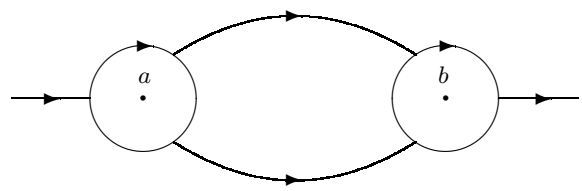

Figure 4.3 *The contour Σ_R after the third and final transformation.*

It now remains to determine $E^{(b)}$ such that the matching condition (c) holds as well. In order to do this we make use of the following proposition (the analogon of Proposition 4.19).

4.22 Proposition. *Let $n \to \infty$ and $s, t \to 0$ such that* (4.33) *holds. Then,*

$$P^{(b)}(z) = E^{(b)}(z) \left(n^{2/7} f_b(z)\right)^{-\sigma_3/4} N \left[I - h f_b(z)^{-1/2} \sigma_3 n^{-1/7} \right.$$
$$\left. + \frac{1}{2} \begin{pmatrix} h^2 & iy \\ -iy & h^2 \end{pmatrix} f_b(z)^{-1} n^{-2/7} + \mathcal{O}(n^{-3/7}) \right]. \quad (4.114)$$

where we have used for brevity the notation

$$h = h(n^{6/7} s f_1(z), n^{4/7} t f_2(z)), \qquad \textit{and} \qquad y = y(n^{6/7} s f_1(z), n^{4/7} t f_2(z)).$$

Proof. This follows easily from (4.108), (4.106), and (4.113) □

From (4.114) it is clear that (in order that the matching condition (c) is satisfied) we have to define $E^{(b)}$ by,

$$E^{(b)} = P^{(\infty)} N^{-1} \left(n^{2/7} f_b\right)^{\sigma_3/4}, \quad (4.115)$$

where N is given by (4.93) and where $P^{(\infty)}$ is the parametrix for the outside region, given by (4.85). Similarly as we have proven that $E^{(a)}$ is an invertible analytic matrix-valued function in U_δ, we can check that $E^{(b)}$ is invertible and analytic in $U_{\delta,b}$. This completes the construction of the parametrix near the singular endpoint.

4.3.7 Final transformation: $S \mapsto R$

Having the parametrix $P^{(\infty)}$ for the outside region and the parametrices $P^{(a)}$ and $P^{(b)}$ near the endpoints a and b, we have all the ingredients to perform the final

transformation of the RH problem. Define

$$R(z) = \begin{cases} S(z)\left(P^{(a)}\right)^{-1}(z), & \text{for } z \in U_{\delta,a}, \\ S(z)\left(P^{(b)}\right)^{-1}(z), & \text{for } z \in U_{\delta,b}, \\ S(z)\left(P^{(\infty)}\right)^{-1}(z), & \text{for } z \in \mathbb{C} \setminus (U_{\delta,a} \cup U_{\delta,b}). \end{cases} \tag{4.116}$$

Then, R has only jumps on the reduced system of contours Σ_R shown in Figure 4.3, and R satisfies the following RH problem. The circles around a and b are oriented clockwise.

RH problem for R:

(a) $R : \mathbb{C} \setminus \Sigma_R \to \mathbb{C}^{2\times 2}$ is analytic.

(b) $R_+(z) = R_-(z)v_R(z)$ for $z \in \Sigma_R$, with

$$v_R = \begin{cases} P^{(a)}\left(P^{(\infty)}\right)^{-1}, & \text{on } \partial U_{\delta,a}, \\ P^{(b)}\left(P^{(\infty)}\right)^{-1}, & \text{on } \partial U_{\delta,b}, \\ P^{(\infty)} v_S \left(P^{(\infty)}\right)^{-1}, & \text{on the rest of } \Sigma_R. \end{cases} \tag{4.117}$$

(c) $R(z) = I + \mathcal{O}(1/z), \qquad \text{as } z \to \infty.$

(d) R remains bounded near the intersection points of Σ_R.

As $n \to \infty$ and $s, t \to 0$ such that (4.33) holds, we have by construction of the parametrices that the jump matrix for R is close to the identity matrix, both in L^2 and L^∞- sense on Σ_R,

$$v_R(z) = \begin{cases} I + \mathcal{O}(n^{-1/7}), & \text{on } \partial U_{\delta,a} \cup \partial U_{\delta,b}, \\ I + \mathcal{O}(e^{-\gamma n}), & \text{on the rest of } \Sigma_R, \end{cases} \tag{4.118}$$

with $\gamma > 0$ some fixed constant. Then, arguments as in [33, 32] guarantee that R itself is close to the identity matrix,

$$R(z) = I + \mathcal{O}(n^{-1/7}), \qquad \text{uniformly for } z \in \mathbb{C} \setminus \Sigma_R, \tag{4.119}$$

as $n \to \infty$ and $s, t \to 0$ such that (4.33) holds. This completes the steepest descent analysis.

4.23 Remark. The Deift/Zhou steepest descent method can be generalized to the case where the support of ν_0 consists of more than one interval. However, there are two (technical) differences. First, in the multi-interval case, the equilibrium

measures ν_1 and ν_2 have densities which are more complicated than in the one-interval case, but it remains possible to give explicit formulae. Consequently, condition (4.27), which expresses the requirement that the density of ν_2 vanishes at the singular edge point, has to be modified. Further, the construction of the outside parametrix $P^{(\infty)}$ is more complicated, since it uses Θ-functions as in [32, Lemma 4.3]. With these modifications the asymptotic analysis can be carried through in the multi-interval case.

4.3.8 Asymptotics for R

For the purpose of proving the universality result for the kernel $K_{n,s,t}$ (Theorem 4.7) it is enough to unfold the series of transformations $Y \mapsto T \mapsto S \mapsto R$ and to use (4.119). This will be done in the next section. However, in order to determine the asymptotics for the recurrence coefficients $a_{n,s,t}$ and $b_{n,s,t}$ (Theorem 4.11) we need to expand the $\mathcal{O}(n^{-1/7})$ term in (4.119).

We show that the jump matrix v_R for R has an expansion of the form,

$$v_R(z) = I + \frac{\Delta_1(z)}{n^{1/7}} + \frac{\Delta_2(z)}{n^{2/7}} + \mathcal{O}(n^{-3/7}), \tag{4.120}$$

as $n \to \infty$ and $s, t \to 0$ such that (4.33) holds, uniformly for $z \in \Sigma_R$, and we will explicitly determine Δ_1 and Δ_2. On $\Sigma_R \setminus (\partial U_{\delta,a} \cup \partial U_{\delta,b})$, the jump matrix is the identity matrix plus an exponentially small term, so that

$$\Delta_1(z) = 0, \qquad \Delta_2(z) = 0, \qquad \text{for } z \in \Sigma_R \setminus (\partial U_{\delta,a} \cup \partial U_{\delta,b}). \tag{4.121}$$

Now, from (4.117), (4.98), and (4.114) we obtain (4.120) with,

$$\Delta_1(z) = -\frac{1}{4}\left(n^{4/7} r_{s,t}(z)\right)^2 f_a(z)^{-1/2} P^{(\infty)}(z)\sigma_3 P^{(\infty)}(z)^{-1}, \qquad \text{for } z \in \partial U_{\delta,a}, \tag{4.122}$$

$$\Delta_1(z) = -h f_0(z)^{-1/2} P^{(\infty)}(z)\sigma_3 P^{(\infty)}(z)^{-1}, \qquad \text{for } z \in \partial U_{\delta,b}, \tag{4.123}$$

and

$$\Delta_2(z) = \frac{1}{32}\left(n^{4/7} r_{s,t}(z)\right)^4 f_a(z)^{-1} I, \qquad \text{for } z \in \partial U_{\delta,a}, \tag{4.124}$$

$$\Delta_2(z) = \frac{1}{2} f_0(z)^{-1} \begin{pmatrix} h^2 & iy \\ -iy & h^2 \end{pmatrix}, \qquad \text{for } z \in \partial U_{\delta,b}, \tag{4.125}$$

where we have used for brevity

$$h = h(n^{6/7} s f_1(z), n^{4/7} t f_2(z)), \qquad y = y(n^{6/7} s f_1(z), n^{4/7} t f_2(z)).$$

Observe that Δ_1 and Δ_2 have an extension to an analytic function in a punctured neighborhood of a and a punctured neighborhood of b with simple poles at a and b.

As in [33, Theorem 7.10] we obtain from (4.120) that R satisfies,

$$R(z) = I + \frac{R^{(1)}(z)}{n^{1/7}} + \frac{R^{(2)}(z)}{n^{2/7}} + \mathcal{O}(n^{-3/7}), \tag{4.126}$$

as $n \to \infty$ and $s, t \to 0$ such that (4.33) holds, which is valid uniformly for $z \in \mathbb{C} \setminus (\partial U_{\delta,a} \cup \partial U_{\delta,b})$. We have that

$$R^{(1)} \text{ and } R^{(2)} \text{ are analytic on } \mathbb{C} \setminus (\partial U_{\delta,a} \cup \partial U_{\delta,b}), \tag{4.127}$$

$$R^{(1)}(z) = \mathcal{O}(1/z), \qquad R^{(2)}(z) = \mathcal{O}(1/z), \qquad \text{as } z \to \infty. \tag{4.128}$$

We will now compute the functions $R^{(1)}$ and $R^{(2)}$ explicitly.

Determination of $R^{(1)}$:

Expanding the jump relation $R_+ = R_- v_R$ using (4.120) and (4.126), and collecting the terms with $n^{-1/7}$ we find

$$R_+^{(1)}(z) = R_-^{(1)}(z) + \Delta_1(z), \qquad \text{for } z \in \partial U_{\delta,a} \cup U_{\delta,b}.$$

This together with (4.127) and (4.128) gives an additive RH problem for $R^{(1)}$. Recall that Δ_1 is analytic in a neighborhood of $z = a$ and $z = b$ except for simple poles at a and b. So,

$$\Delta_1(z) = \frac{A^{(1)}}{z-a} + \mathcal{O}(1), \qquad \text{as } z \to a, \qquad \Delta_1(z) = \frac{B^{(1)}}{z-b} + \mathcal{O}(1), \qquad \text{as } z \to b,$$

for certain matrices $A^{(1)}$ and $B^{(1)}$. We then see by inspection that

$$R^{(1)}(z) = \begin{cases} \dfrac{A^{(1)}}{z-a} + \dfrac{B^{(1)}}{z-b}, & \text{for } z \in \mathbb{C} \setminus (\overline{U}_{\delta,a} \cup \overline{U}_{\delta,b}), \\[2ex] \dfrac{A^{(1)}}{z-a} + \dfrac{B^{(1)}}{z-b} - \Delta_1(z), & \text{for } z \in U_{\delta,a} \cup U_{\delta,b}, \end{cases} \tag{4.129}$$

solves the additive RH problem for $R^{(1)}$. In now remains to determine $A^{(1)}$ and $B^{(1)}$. This can be done by expanding the formulas (4.122) and (4.123) near $z = a$ and $z = b$, respectively. We then find after a straightforward calculation (using also the fact that $f_1(b) = c_1$ and $f_2(b) = c_2$, see (4.112),

$$A^{(1)} = \frac{1}{8}\sqrt{b-a}\,(n^{4/7} r_{s,t}(a))^2 (-f_a'(a))^{-1/2} \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \tag{4.130}$$

$$B^{(1)} = \frac{1}{2} h \sqrt{b-a}\, f_b'(b)^{-1/2} \begin{pmatrix} -1 & i \\ i & 1 \end{pmatrix}, \tag{4.131}$$

where we used h to denote $h(c_1 n^{6/7} s, c_2 n^{4/7} t)$ for brevity.

Determination of $R^{(2)}$:

Next, expanding the jump relation $R_+ = R_- v_R$ using (4.120) and (4.126), and collecting the terms with $n^{-2/7}$ we find

$$R_+^{(2)}(z) = R_-^{(2)}(z) + R_-^{(1)}(z)\Delta_1(z) + \Delta_2(z), \qquad \text{for } z \in \partial U_{\delta,a} \cup U_{\delta,b}.$$

This together with (4.127) and (4.128) gives an additive RH problem for $R^{(2)}$. Since $R_-^{(1)}$ is the boundary value of the restriction of $R^{(1)}$ to the disks $U_{\delta,a}$ and $U_{\delta,b}$ and since Δ_1 and Δ_2 are analytic in a neighborhood of a and b, except for simple poles at a and b, we have

$$R^{(1)}(z)\Delta_1(z) + \Delta_2(z) = \frac{A^{(2)}}{z-a} + \mathcal{O}(1), \qquad \text{as } z \to a,$$

$$R^{(1)}(z)\Delta_1(z) + \Delta_2(z) = \frac{B^{(2)}}{z-b} + \mathcal{O}(1), \qquad \text{as } z \to b,$$

for certain matrices $A^{(2)}$ and $B^{(2)}$. As in the determination of $R^{(1)}$ we then see by inspection that

$$R^{(2)}(z) = \begin{cases} \dfrac{A^{(2)}}{z-a} + \dfrac{B^{(2)}}{z-b}, & \text{for } z \in \mathbb{C} \setminus (\overline{U}_{\delta,a} \cup \overline{U}_{\delta,b}), \\ \dfrac{A^{(2)}}{z-a} + \dfrac{B^{(2)}}{z-b} - R^{(1)}(z)\Delta_1(z) - \Delta_2(z), & \text{for } z \in U_{\delta,a} \cup U_{\delta,b}, \end{cases} \tag{4.132}$$

solves the additive RH problem for $R^{(2)}$. The determination of $A^{(2)}$ and $B^{(2)}$ is more complicated than the determination of $A^{(1)}$ and $B^{(1)}$. It involves $R^{(1)}$ and $R^{(1)}(b)$ for which we need to determine also the constant terms in the expansions of Δ_1 near $z = a$ and $z = b$. After a straightforward (but rather long calculation) we find,

$$A^{(2)} = \frac{(n^{4/7} r_{s,t}(a))^4}{32(-f_a'(a))} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} + \frac{(n^{4/7} r_{s,t}(a))^2 h}{8(-f_a'(a))^{1/2} f_b'(b)^{1/2}} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}, \tag{4.133}$$

$$B^{(2)} = \frac{y + h^2}{2f_b'(b)} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} + \frac{(n^{4/7} r_{s,t}(a))^2 h}{8(-f_a'(a))^{1/2} f_b'(b)^{1/2}} \begin{pmatrix} -1 & i \\ -i & -1 \end{pmatrix}, \tag{4.134}$$

where we used h and y to denote $h(c_1 n^{6/7} s, c_2 n^{4/7} t)$ and $y(c_1 n^{6/7} s, c_2 n^{4/7} t)$ for brevity.

4.4 Universality of the double scaling limit

Here, we will prove the univarsality result for the 2-point correlation kernel $K_n^{(s,t)}$. We do this by unfolding the series of transformations $Y \mapsto T \mapsto S \mapsto R$.

Proof of Theorem 4.7. From equations (4.69), (4.74), and (4.78), the reader can verify that the 2-point kernel $K_n^{(s,t)}$ can be written for $x, y \in \mathbb{R}$ as follows, cf. Chapters 1 and 2,

$$K_n^{(s,t)}(x,y) = e^{-n\phi_{s,t,+}(x)} e^{-n\phi_{s,t,+}(y)} \frac{1}{2\pi i(x-y)} \begin{pmatrix} 0 & 1 \end{pmatrix} T_+^{-1}(y) T_+(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

From (4.83) and the fact that $S_+ = RP_+^{(b)}$ on $(b-\delta, b+\delta)$, see (4.116), we have

$$T_+ = \begin{cases} RP_+^{(b)}, & \text{on } (b, b+\delta), \\ RP_+^{(b)} \begin{pmatrix} 1 & 0 \\ e^{2n\phi_{s,t,+}} & 1 \end{pmatrix}, & \text{on } (b-\delta, b). \end{cases}$$

Inserting this in the previous equation for $K_n^{(s,t)}$ we arrive at,

$$K_n^{(s,t)}(x,y) = e^{-n\phi_{s,t,+}(x)} e^{-n\phi_{s,t,+}(y)} \frac{1}{2\pi i(x-y)} \times \begin{pmatrix} 0 & 1 \end{pmatrix} \widehat{P}^{-1}(y) R^{-1}(y) R(x) \widehat{P}(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (4.135)$$

for $x \in (b-\delta, b+\delta)$, where

$$\widehat{P} = \begin{cases} P_+^{(b)}, & \text{on } (b, b+\delta), \\ P_+^{(b)} \begin{pmatrix} 1 & 0 \\ e^{2n\phi_{s,t,+}} & 1 \end{pmatrix}, & \text{on } (b-\delta, b). \end{cases} \quad (4.136)$$

Further, we define

$$\widehat{\Psi} = \begin{cases} \Psi_+ & \text{on } \mathbb{R}_+, \\ \Psi_+ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & \text{on } \mathbb{R}_-, \end{cases} \quad (4.137)$$

where Ψ is the solution of the RH problem for Ψ, see Section 4.3.6. By (4.107), we have that $\widehat{\Psi}_{11} = \Phi_1$ and $\widehat{\Psi}_{21} = \Phi_2$. Using (4.108), (4.136), and (4.137) a straightforward calculation yields,

$$\widehat{P}(x) = E^{(b)}(x) \widehat{\Psi}\left(n^{2/7} f_b(x); n^{6/7} s f_1(x), n^{4/7} t f_2(x)\right) e^{n\phi_{s,t,+}(x)\sigma_3}, \quad \text{for } x \in (b-\delta, b+\delta).$$

Inserting this into (4.135) we then obtain,

$$K_n^{(s,t)}(x,y) = \frac{1}{2\pi i(x-y)} \begin{pmatrix} 0 & 1 \end{pmatrix} \widehat{\Psi}^{-1}\left(n^{2/7} f_b(y); n^{6/7} s f_1(y), n^{4/7} t f_2(y)\right)$$
$$\times \ (E^{(b)})^{-1}(y) R^{-1}(y) R(x) E^{(b)}(x)$$
$$\times \ \widehat{\Psi}\left(n^{2/7} f_b(x); n^{6/7} s f_1(x), n^{4/7} t f_2(y)\right) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (4.138)$$

for $x \in (b-\delta, b+\delta)$.

Now, we introduce for the sake of brevity some notation. Let

$$u_n = b + \frac{u}{cn^{2/7}}, \qquad v_n = b + \frac{v}{cn^{2/7}}, \quad (4.139)$$

with

$$c = f_b'(b) = \left(\frac{15}{2} h_0''(b)\sqrt{b-a}\right)^{2/7}. \quad (4.140)$$

We then have,

$$\lim_{n\to\infty} n^{2/7} f_b(u_n) = u, \qquad \text{and} \qquad \lim_{n\to\infty} n^{2/7} f_b(v_n) = v. \quad (4.141)$$

Furthermore, since $f_1(b) = c_1$ and $f_2(b) = c_2$ (see (4.112)) we have in the limit as $n \to \infty$ and $s, t \to 0$ such that (4.33) holds,

$$\lim n^{6/7} s f_1(u_n) = s_0, \qquad \lim n^{6/7} s f_1(v_n) = s_0, \quad (4.142)$$
$$\lim n^{4/7} t f_2(u_n) = t_0, \qquad \lim n^{4/7} t f_2(v_n) = t_0. \quad (4.143)$$

Now, a similar argument as in [77] shows that

$$\lim E_b^{-1}(v_n) R(v_n)^{-1} R(u_n) E_b(u_n) = I. \quad (4.144)$$

Inserting (4.141)–(4.144) into (4.138) and using the fact that $\widehat{\Psi}_{11} = \Phi_1$ and $\widehat{\Psi}_{21} = \Phi_2$ it is then straightforward to obtain

$$\lim \frac{1}{cn^{2/7}} K_n^{(s,t)}(u_n, v_n)$$
$$= \frac{1}{2\pi i(u-v)} \begin{pmatrix} 0 & 1 \end{pmatrix} \widehat{\Psi}^{-1}(v; s_0, t_0) \widehat{\Psi}(u; s_0, t_0) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$= \frac{-1}{2\pi i(u-v)} \left(\Phi_1(u; s_0, t_0)\Phi_2(v; s_0, t_0) - \Phi_1(v; s_0, t_0)\Phi_2(u; s_0, t_0)\right),$$

where we take the limit $n \to \infty$ and $s, t \to 0$ such that (4.33) holds. This completes the proof of Theorem 4.7. □

4.5 Asymptotics of the recurrence coefficients

We will now determine the asymptotics of $a_n^{(n,s,t)}$ and $b_n^{(n,s,t)}$ as $n \to \infty$ and $s, t \to 0$ such that (4.33) holds. In order to do this, we make use of the following result, see e.g. [27, 33]. Let Y be the unique solution of the RH problem for Y. There exist 2×2 constant (independent of z but depending on n, s and t) matrices Y_1 and Y_2 such that

$$Y(z) \begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} = I + \frac{Y_1}{z} + \frac{Y_2}{z^2} + \mathcal{O}(1/z^3), \qquad \text{as } z \to \infty, \tag{4.145}$$

and

$$a_n^{(n,s,t)} = \sqrt{(Y_1)_{12}(Y_1)_{21}}, \qquad b_n^{(n,s,t)} = (Y_1)_{11} + \frac{(Y_2)_{12}}{(Y_1)_{12}}. \tag{4.146}$$

We need to determine the constant matrices Y_1 and Y_2. For large $|z|$ it follows from (4.74), (4.83), and (4.116), that

$$Y(z) = e^{\frac{1}{2}n\ell_{s,t}\sigma_3} R(z) P^{(\infty)}(z) e^{ng_{s,t}(z)\sigma_3} e^{-\frac{1}{2}n\ell_{s,t}\sigma_3}. \tag{4.147}$$

So, in order to determine Y_1 and Y_2, we need the asymptotic behavior as $z \to \infty$ of $P^{(\infty)}(z), e^{ng_{s,t}(z)\sigma_3}$, and $R(z)$.

Asymptotic behavior of $P^{(\infty)}(z)$ as $z \to \infty$:

Expanding the factor $((z-b)/(z-a))^{\sigma_3/4}$ in (4.85) at $z = \infty$ it is clear that,

$$P^{(\infty)}(z) = I + \frac{P_1^{(\infty)}}{z} + \frac{P_2^{(\infty)}}{z} + \mathcal{O}(1/z^3), \qquad \text{as } z \to \infty, \tag{4.148}$$

with

$$P_1^{(\infty)} = \frac{i}{4}(b-a) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad P_2^{(\infty)} = \frac{i}{8}(b^2 - a^2) \begin{pmatrix} * & 1 \\ -1 & * \end{pmatrix}. \tag{4.149}$$

Asymptotic behavior of $e^{ng_{s,t}(z)\sigma_3}$ as $z \to \infty$:

By (4.70) we have

$$e^{ng_{s,t}(z)\sigma_3} \begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} = I + \frac{G_1}{z} + \frac{G_2}{z^2} + \mathcal{O}(1/z^3), \qquad \text{as } z \to \infty, \tag{4.150}$$

with

$$G_1 = -n \int_a^b u d\nu_{s,t}(u) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad G_2 = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}. \tag{4.151}$$

Asymptotic behavior of $R(z)$ as $z \to \infty$:

As in [33] the matrix-valued function R has the following asymptotic behavior at infinity,

$$R(z) = I + \frac{R_1}{z} + \frac{R_2}{z^2} + \mathcal{O}(1/z^3), \qquad \text{as } z \to \infty. \tag{4.152}$$

The compatibility with (4.126), (4.129), and (4.132) yields that

$$R_1 = \left(A^{(1)} + B^{(1)}\right) n^{-1/7} + \left(A^{(2)} + B^{(2)}\right) n^{-2/7} + \mathcal{O}(n^{-3/7}), \tag{4.153}$$

$$R_2 = \left(aA^{(1)} + bB^{(1)}\right) n^{-1/7} + \left(aA^{(2)} + bB^{(2)}\right) n^{-2/7} + \mathcal{O}(n^{-3/7}), \tag{4.154}$$

as $n \to \infty$ and $s, t \to 0$ such that (4.33) holds. Here, $A^{(1)}$, $B^{(1)}$, $A^{(2)}$, and $B^{(2)}$ are given by (4.130), (4.131), (4.133), and (4.134), respectively.

Now, we are ready to determine the asymptotics of the recurrence coefficients.

Proof of Theorem 4.11. Note that by (4.147), (4.148), (4.150) and (4.152),

$$Y_1 = e^{\frac{1}{2}n\ell_{s,t}\sigma_3} \left[P_1^{(\infty)} + G_1 + R_1\right] e^{-\frac{1}{2}n\ell_{s,t}\sigma_3} \tag{4.155}$$

and

$$Y_2 = e^{\frac{1}{2}n\ell_{s,t}\sigma_3} \left[P_2^{(\infty)} + G_2 + R_2 + R_1 P_1^{(\infty)} + \left(P_1^{(\infty)} + R_1\right) G_1\right] \times \; e^{-\frac{1}{2}n\ell_{s,t}\sigma_3} \tag{4.156}$$

We start with the recurrence coefficient $a_n^{(n,s,t)}$. Inserting (4.155) into (4.146), and using the facts that $(P_1^{(\infty)})_{12} = -(P_1^{(\infty)})_{21} = i(b-a)/4$ (by (4.149)) and $(G_1)_{12} = (G_1)_{21} = 0$ (by (4.151)), we obtain

$$a_n^{(n,s,t)} = \left[\left(\frac{b-a}{4}\right)^2 + i\frac{b-a}{4}\left((R_1)_{21} - (R_1)_{12}\right) + (R_1)_{12}(R_1)_{21}\right]^{1/2}. \tag{4.157}$$

Now, from the formula (4.153) for R_1 and the formulas (4.130), (4.131), (4.133), and (4.134) for $A^{(1)}, B^{(1)}, A^{(2)}$, and $B^{(2)}$, we have

$$(R_1)_{21} - (R_1)_{12} = -i\left[\frac{y}{f_b'(b)} + \left(\frac{(n^{4/7} r_{s,t}(a))^2}{4(-f_a'(a))^{1/2}} + \frac{h}{f_b'(b)}\right)^2\right] n^{-2/7} + \mathcal{O}(n^{-3/7}),$$

and

$$(R_1)_{12}(R_1)_{21} = -\frac{b-a}{4}\left(\frac{(n^{4/7} r_{s,t}(a))^2}{4(-f_a'(a))^{1/2}} + \frac{h}{f_b'(b)}\right)^2 n^{-2/7} + \mathcal{O}(n^{-3/7}).$$

Note that we have used y to denote $y(c_1 n^{6/7} s, c_2 n^{4/7} t)$ for brevity. Inserting the latter two equations into (4.157) and using the fact that $f_b'(b) = c$ (by (4.109)) we then obtain (4.40).

We will now consider the recurrence coefficient $b_n^{(n,s,t)}$. Inserting (4.155) and (4.156) into (4.146), and using the facts that $(P_1^{(\infty)})_{11} = (P_1^{(\infty)})_{22} = 0$, $(G_1)_{12} = (G_2)_{12} = 0$, $(G_1)_{11} + (G_1)_{22} = 0$, and $R_1 = \mathcal{O}(n^{-1/7})$, we obtain

$$\begin{aligned} b_n^{(n,s,t)} &= (R_1)_{11} + \frac{(P_2^{(\infty)})_{12} + (R_1)_{11}(P_1^{(\infty)})_{12} + (R_2)_{12}}{(P_1^{(\infty)})_{12} + (R_1)_{12}} \\ &= (R_1)_{11} + \left[\frac{(P_2^{(\infty)})_{12}}{(P_1^{(\infty)})_{12}} + (R_1)_{11} + \frac{(R_2)_{12}}{(P_1^{(\infty)})_{12}}\right] \\ &\qquad \times \left[1 - \frac{(R_1)_{12}}{(P_1^{(\infty)})_{12}} + \left(\frac{(R_1)_{12}}{(P_1^{(\infty)})_{12}}\right)^2 + \mathcal{O}(n^{-3/7})\right]. \end{aligned}$$

Since $(P_1^{(\infty)})_{12} = i(b-a)/4$, $(P_2^{(\infty)})_{12} = i(b^2 - a^2)/8$, $R_1 = \mathcal{O}(n^{-1/7})$, and $R_2 = \mathcal{O}(n^{-1/7})$ we then obtain after a straightforward calculation and combining terms,

$$\begin{aligned} b_n^{(n,s,t)} &= \frac{b+a}{2} + \left(2(R_1)_{11} + 2i\frac{b+a}{b-a}(R_1)_{12} - \frac{4i}{b-a}(R_2)_{12}\right) \\ &\quad \times \left(1 + \frac{4i}{b-a}(R_1)_{12}\right) - \frac{4i}{b-a}(R_1)_{11}(R_1)_{12} + \mathcal{O}(n^{-3/7}). \end{aligned} \tag{4.158}$$

Now, from (4.153), (4.154), (4.130), (4.131), (4.133), and (4.134) we have

$$\begin{aligned} &2(R_1)_{11} + 2i\frac{b+a}{b-a}(R_1)_{12} - \frac{4i}{b-a}(R_2)_{12} \\ &\qquad = 2i\left[A_{12}^{(2)} - B_{12}^{(2)}\right] n^{-2/7} + \mathcal{O}(n^{-3/7}) \\ &\qquad = \left(\frac{y}{f_b'(b)} + \frac{h^2}{f_b'(b)} - \frac{(n^{4/7} r_{s,t}(a))^4}{16(-f_a'(a))}\right) n^{-2/7} + \mathcal{O}(n^{-3/7}), \end{aligned}$$

and

$$\begin{aligned} (R_1)_{11}(R_1)_{12} &= -i\left[\left(A_{12}^{(1)}\right)^2 - \left(B_{12}^{(1)}\right)^2\right] n^{-2/7} + \mathcal{O}(n^{-3/7}) \\ &= -i\frac{b-a}{4}\left(\frac{h^2}{f_b'(b)} - \frac{(n^{4/7} r_{s,t}(a))^4}{16(-f_a'(a))}\right) n^{-2/7} + \mathcal{O}(n^{-3/7}). \end{aligned}$$

Inserting the latter two equations into (4.158) and using the facts that $(R_1)_{12} = \mathcal{O}(n^{-1/7})$ and $f_b'(b) = c$ we obtain (4.41). So, the theorem is proven. $\square$

Chapter 5

Universality of a double scaling limit near a soft edge approaching the hard edge

5.1 Introduction and statement of result

5.1.1 Universality in unitary random matrix ensembles

We consider unitary random matrix models on the space of positive-definite $n \times n$ Hermitian matrices, with a probability measure of the form

$$\frac{1}{Z_{n,N}}(\det M)^{\alpha} \exp(-N \operatorname{Tr} V(M)) dM, \qquad \alpha > -1. \tag{5.1}$$

Here $Z_{n,N}$ is a normalization constant and the confining potential V is real analytic on $[0, +\infty)$ with enough growth at infinity,

$$\frac{V(x)}{\log(x^2+1)} \to +\infty \quad \text{ as } x \to +\infty. \tag{5.2}$$

Because the eigenvalues of a random matrix in this model are restricted to be positive, 0 is called the *hard edge* of the random matrix model. The limiting mean eigenvalue distribution of the random matrix model (5.1), as $n, N \to \infty$ in such a way that $n/N \to 1$, is the equilibrium measure μ_V which minimizes the weighted energy

$$I_V(\mu) = \iint \log \frac{1}{|x-y|} d\mu(x) d\mu(y) + \int V(x) d\mu(x) \tag{5.3}$$

among all probability measures on $[0,+\infty)$. The measure μ_V is uniquely characterized by the following variational conditions for some constant ℓ_V,

$$2\int \log|x-y|d\mu_V(y) - V(x) + \ell_V = 0 \quad \text{as } x \in \operatorname{supp}\mu_V, \tag{5.4}$$

$$2\int \log|x-y|d\mu_V(s) - V(x) + \ell_V \leq 0 \quad \text{as } x \in [0,+\infty)\setminus \operatorname{supp}\mu_V. \tag{5.5}$$

The support S_V of μ_V is bounded and consists in general of a finite union of intervals. The measure μ_V has a density ψ_V which can be written [31] in the form

$$\psi_V(x) = \sqrt{Q_V^-(x)}, \tag{5.6}$$

where Q_V is a real analytic function on $(0,+\infty)$ and Q_V^- denotes the negative part of Q_V. In 0, Q_V has an isolated singularity which is at most a simple pole. Points where Q_V has a zero of odd multiplicity (which are endpoints of the support S_V) are called *soft edges*.

Statistics of the eigenvalues are determined by the two-point kernel

$$K_{n,N}(x,y) = x^{\frac{\alpha}{2}}y^{\frac{\alpha}{2}}e^{-\frac{N}{2}V(x)}e^{-\frac{N}{2}V(y)}\sum_{j=0}^{n-1}p_j^{(N)}(x)p_j^{(N)}(y), \tag{5.7}$$

where $p_j^{(N)}$ is the j-th degree orthonormal polynomial with respect to the weight $x^\alpha e^{-NV(x)}$ on $[0,+\infty)$. Scaling limits of the kernel show universal behavior. Near a soft edge $x^* \neq 0$ where the density ψ_V vanishes like a square root (this means that x^* is a simple zero of Q_V), it is known (see e.g. [30]) that for some constant c,

$$\lim_{n\to\infty}\frac{1}{(cn)^{2/3}}K_{n,n}\left(x^* + \frac{u}{(cn)^{2/3}}, x^* + \frac{v}{(cn)^{2/3}}\right) = \frac{\operatorname{Ai}(u)\operatorname{Ai}'(v) - \operatorname{Ai}'(u)\operatorname{Ai}(v)}{u-v}$$

where Ai is the Airy function. If $0 \in \operatorname{supp}\psi_V$ and if the density ψ_V tends to infinity at the origin (which means that Q_V has a simple pole in 0), the following holds for some constant c [76],

$$\lim_{n\to\infty}\frac{1}{(cn)^2}K_{n,n}\left(\frac{u}{(cn)^2}, \frac{v}{(cn)^2}\right) = \mathbb{J}_\alpha(u,v),$$

where the Bessel kernel $\mathbb{J}_\alpha$ is built out of Bessel functions of the first kind of order α,

$$\mathbb{J}_\alpha(u,v) = \frac{J_\alpha(\sqrt{u})\sqrt{v}J_\alpha'(\sqrt{v}) - J_\alpha(\sqrt{v})\sqrt{u}J_\alpha'(\sqrt{u})}{2(u-v)}.$$

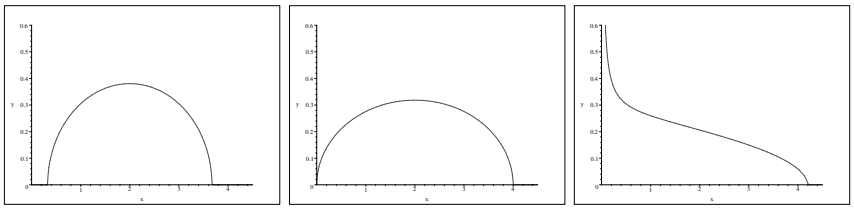

Figure 5.1 *The density of* $\mu_{V/c}$ *for* $V(x) = \frac{x^2}{2} - 2x$ *with* c *equal to* 0.7, 1 *and* 1.2, *respectively.*

We will focus on the case where a soft edge coincides with the hard edge at the origin, so $Q_V(0) = 0$ and $Q_V'(0) \neq 0$. We assume furthermore that there are no singular points, which means that strict inequality in (5.5) holds, that Q_V has simple zeros at the endpoints of supp ψ_V, and that Q_V does not vanish in the interior of supp ψ_V. These conditions are generic [74].

We will consider a double scaling limit of the eigenvalue correlation kernel, where we let $n, N \to \infty$ and at the same time we let $t = n/N \to 1$ at an appropriate speed. For $n/N \to c > 0$, the limiting mean eigenvalue distribution is the equilibrium measure $\mu_{V/c}$ in external field V/c. For c close to 1 and $c < 1$, 0 lies outside the support of $\mu_{V/c}$ and the left endpoint of the support is then a soft edge, see [74]. For c close to 1 and $c > 1$, the density $\psi_{V/c}$ tends to infinity at the origin, see Figure 5.1. If we let $c \nearrow 1$, we observe the soft edge approaching the hard edge. For $c > 1$, the soft edge disappears and the density blows up at the hard edge. The limiting kernel in the transitional case $c \approx 1$ will be associated with the Hastings-McLeod solution of the Painlevé II equation, and can be expressed in terms of the kernel $\mathbb{K}_\alpha^{\mathrm{crit,II}}$ that already appeared in Chapter 1 and Chapter 2. This is the main result of this chapter, see Theorem 5.1.

There is a connection between random matrix models of the form (5.1) and random matrix models on the space of Hermitian $n \times n$ matrices, with probability measure of the form

$$\frac{1}{\hat{Z}_{n,N}} |\det M|^{2\alpha \pm 1} e^{-\frac{N}{2} \operatorname{Tr} V(M^2)}, \qquad 2\alpha \pm 1 > -1. \tag{5.8}$$

Therefore we will use the results we obtained in Chapter 2 and we will translate them to results for the random matrix ensembles of the form (5.1). Another strategy would be (as in Chapters 1, 2, and 4) to apply the Deift/Zhou steepest descent method on the RH problem for the orthogonal polynomials on $[0, +\infty)$ related to the matrix model. Then we would have to construct a local parametrix near the origin, which would be built out of the ψ-functions associated with the Hastings-McLeod solution of Painlevé II. However the approach we present here is much shorter and more direct.

5.1.2 The general Painlevé II equation

We recall briefly some notions which were already defined in Section 2.1.4. The general Painlevé II equation, depending on a parameter α, is the following differential equation,

$$q'' = sq + 2q^3 - \alpha. \tag{5.9}$$

For $\alpha \neq 0$, the Hastings-McLeod solution q_α is the unique solution of the Painlevé II equation which has the following asymptotics,

$$q_\alpha(s) \sim \frac{\alpha}{s}, \qquad \text{as } s \to +\infty, \tag{5.10}$$

$$q_\alpha(s) \sim \sqrt{\frac{-s}{2}}, \qquad \text{as } s \to -\infty. \tag{5.11}$$

If $\alpha = 0$, the Hastings-McLeod solution q_α is characterized by the asymptotic condition

$$q_\alpha(s) \sim \mathrm{Ai}(s) \qquad \text{as } s \to +\infty. \tag{5.12}$$

From Theorem 2.1, we know that q_α has no real poles for $\alpha > -1/2$.

For $\alpha > -1/2$ and $s \in \mathbb{R}$, we abbreviate $q = q_\alpha(s)$ and $r = q'_\alpha(s)$, where q_α is the Hastings-McLeod solution, and we define $\begin{pmatrix} \Phi_{\alpha,1}(\zeta;s) \\ \Phi_{\alpha,2}(\zeta;s) \end{pmatrix}$ as the unique solution of the equation

$$\frac{d}{d\zeta}\begin{pmatrix} \Phi_{\alpha,1} \\ \Phi_{\alpha,2} \end{pmatrix} = \begin{pmatrix} -4i\zeta^2 - i(s+2q^2) & 4\zeta q + 2ir + \alpha/\zeta \\ 4\zeta q - 2ir + \alpha/\zeta & 4i\zeta^2 + i(s+2q^2) \end{pmatrix}\begin{pmatrix} \Phi_{\alpha,1} \\ \Phi_{\alpha,2} \end{pmatrix}, \tag{5.13}$$

with asymptotics

$$e^{i(\frac{4}{3}\zeta^3 + s\zeta)}\begin{pmatrix} \Phi_{\alpha,1}(\zeta;s) \\ \Phi_{\alpha,2}(\zeta;s) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathcal{O}(\zeta^{-1}), \tag{5.14}$$

uniformly as $\zeta \to \infty$ in the sector $\varepsilon < \arg\zeta < \pi - \varepsilon$ for any $\varepsilon > 0$. Since q_α has no real poles for $\alpha > -1/2$, $\Phi_{\alpha,1}$ and $\Phi_{\alpha,2}$ are well-defined for real s. The functions $\Phi_{\alpha,1}$ and $\Phi_{\alpha,2}$ extend to analytic functions on $\mathbb{C} \setminus (-i\infty, 0]$, with branch points in 0. We denote these extensions also by $\Phi_{\alpha,1}$ and $\Phi_{\alpha,2}$. In particular we are interested in their values on the real line.

The Painlevé II kernel appearing in Chapter 2 is the following function, for real u, v, and s.

$$\begin{aligned} &\mathbb{K}_\alpha^{\mathrm{crit,II}}(u,v;s) = \\ &\quad - e^{\frac{1}{2}\pi i\alpha[\mathrm{sgn}(u)+\mathrm{sgn}(v)]}\frac{\Phi_{\alpha,1}(u;s)\Phi_{\alpha,2}(v;s) - \Phi_{\alpha,1}(v;s)\Phi_{\alpha,2}(u;s)}{2\pi i(u-v)}. \end{aligned} \tag{5.15}$$

5.1.3 Statement of result

Before we formulate our results, we need to define an equilibrium measure which plays a role in the double scaling limit. Define ω as the unique minimizer of

$$I(\omega) = \iint \log \frac{1}{|x-y|} d\omega(x) d\omega(y) \tag{5.16}$$

among all probability measures on S_V. We write w for the density of ω. In the one-interval case where $S_V = [0, b]$, the equilibrium density w is given by $w(x) = \frac{1}{\pi\sqrt{x(b-x)}}$ for $x \in (0, b)$.

We are now ready to formulate the main result of this chapter.

5.1 Theorem. *Fix $\alpha > -1$. Let V be real analytic on $[0, +\infty)$ such that $\lim_{x\to+\infty} \frac{V(x)}{\log(x^2+1)} = +\infty$. Let ψ_V be the limiting mean eigenvalue density of the random matrix model* (5.1) *as $n, N \to \infty$ in such a way that $n/N \to 1$. Assume that $0 \in S_V = \operatorname{supp} \psi_V$ with $\psi_V(0) = 0$ and $\psi_V'(0) \neq 0$, and assume also that there are no singular points in the external field V. Now let $n, N \to \infty$ such that the limit*

$$\lim_{n,N\to\infty} (2n)^{2/3} \left(\frac{n}{N} - 1\right) = L$$

exists with $L \in \mathbb{R}$. Let $K_{n,N}$ be the correlation kernel (5.7) *for the eigenvalues of the random matrix model* (5.1)*. Then there exist constants $c > 0$ and $s \in \mathbb{R}$ such that*

$$\begin{aligned} &\lim_{n,N\to\infty} \frac{1}{(2cn)^{2/3}} K_{n,N}\left(\frac{u}{(2cn)^{2/3}}, \frac{v}{(2cn)^{2/3}}\right) \\ &\qquad = \frac{1}{2}(uv)^{-1/4}\left(\mathbb{K}^{\mathrm{crit,II}}_{\alpha+\frac{1}{2}}(\sqrt{u}, \sqrt{v}; s) + \mathbb{K}^{\mathrm{crit,II}}_{\alpha+\frac{1}{2}}(\sqrt{u}, -\sqrt{v}; s)\right), \end{aligned} \tag{5.17}$$

uniformly for u, v in compact subsets of $(0, +\infty)$. If $\alpha > 0$, the limit (5.17) *can also be written as*

$$\begin{aligned} &\lim_{n,N\to\infty} \frac{1}{(2cn)^{2/3}} K_{n,N}\left(\frac{u}{(2cn)^{2/3}}, \frac{v}{(2cn)^{2/3}}\right) \\ &\qquad = \frac{1}{2}(uv)^{-1/4}\left(\mathbb{K}^{\mathrm{crit,II}}_{\alpha-\frac{1}{2}}(\sqrt{u}, \sqrt{v}; s) - \mathbb{K}^{\mathrm{crit,II}}_{\alpha-\frac{1}{2}}(\sqrt{u}, -\sqrt{v}; s)\right). \end{aligned} \tag{5.18}$$

Explicit formulas for the constants c and s are

$$c = \frac{\pi\sqrt{-Q_V'(0)}}{4} \tag{5.19}$$

and

$$s = L\frac{\pi}{c^{1/3}} \lim_{x\searrow 0} \sqrt{x} w(x), \tag{5.20}$$

where w is the density of the equilibrium measure of S_V.

5.2 Remark. For $\alpha > 0$, we find a remarkable relation between $\mathbb{K}^{\text{crit,II}}_{\alpha+\frac{1}{2}}$ and $\mathbb{K}^{\text{crit,II}}_{\alpha-\frac{1}{2}}$ by (5.17) and (5.18). However this is not completely surprising since $q_{\alpha+\frac{1}{2}}$ and $q_{\alpha-\frac{1}{2}}$ are also related by a Bäcklund transformation [53, 81].

The orthonormal polynomials $p_k^{(N)}$ with respect to the weight $x^\alpha e^{-NV(x)}$ on $[0,+\infty)$ satisfy a three-term recurrence relation of the form

$$xp_k^{(N)}(x) = a_{k+1}^{(N)}p_{k+1}^{(N)}(x) + b_k^{(N)}p_k^{(N)}(x) + a_k^{(N)}p_{k-1}^{(N)}(x). \tag{5.21}$$

We have the following result about the asymptotics of $a_n^{(N)}$ and $b_n^{(N)}$ when $n, N \to \infty$ in such a way that $n/N - 1 = \mathcal{O}(n^{-2/3})$.

5.3 Theorem. *Let V satisfy the conditions of Theorem* 5.1 *and suppose in addition that* supp $\psi_V = [0,b]$ *consists of a single interval. Then, if we let $n, N \to \infty$ in such a way that $n/N - 1 = \mathcal{O}(n^{-2/3})$, we have*

$$a_n^{(N)} = \frac{b}{4} + \mathcal{O}(n^{-2/3}), \tag{5.22}$$

$$b_n^{(N)} = \frac{b}{2} + \mathcal{O}(n^{-2/3}). \tag{5.23}$$

5.1.4 Outline of the rest of the chapter

In Section 5.2 we give formulae which express the limiting mean eigenvalue density μ_V in terms of the limiting mean eigenvalue density of (5.8), and we express the orthogonal polynomials $p_k^{(N)}$ on $\mathbb{R}^+$ in terms of orthogonal polynomials on the real line. This allows us to find a formula for the kernel $K_{n,N}$ in terms of the correlation kernel for the random matrix model (5.8). In Section 5.3, we prove Theorem 5.1. This result will basically follow from Theorem 2.2, using the formulae obtained in Section 5.2. In Section 5.4, we compute asymptotics for the recurrence coefficients of the orthogonal polynomials with respect to the weight $x^\alpha e^{-NV(x)}$ on $\mathbb{R}^+$, using the asymptotics for the recurrence coefficients of orthogonal polynomials with respect to the weight $|x|^{2\alpha\pm1}e^{-\frac{N}{2}V(x^2)}$ on the real line, which we computed in Chapter 2 (see Theorem 2.7).

5.2 Connection between random matrix models

5.2.1 Equilibrium measures μ_V and ν_V

We define μ_V to be the limiting mean eigenvalue distribution of (5.1), and ν_V to be the limiting mean eigenvalue distribution of the matrix ensemble (5.8) as $n, N \to \infty$ in such a way that $n/N \to 1$. Then ν_V is the equilibrium measure which minimizes the weighted energy

$$I_{V(x^2)/2}(\nu) = \iint \log\frac{1}{|x-y|}d\nu(x)d\nu(y) + \int \frac{V(x^2)}{2}d\nu(x) \tag{5.24}$$

among all probability measures on $\mathbb{R}$. The measure ν_V is uniquely characterized by the following variational conditions for some constant $\hat{\ell}_V$:

$$2\int \log|x-y|d\nu_V(y) - V(x^2)/2 + \hat{\ell}_V = 0 \qquad \text{as } x \in \operatorname{supp}\nu_V, \tag{5.25}$$

$$2\int \log|x-y|d\nu_V(y) - V(x^2)/2 + \hat{\ell}_V \leq 0 \qquad \text{as } x \in \mathbb{R}\setminus\operatorname{supp}\nu_V. \tag{5.26}$$

We write ϕ_V for the density of ν_V.

5.4 Lemma. (i) *Let ψ_V be the density of μ_V and ϕ_V the density of ν_V. Then*

$$\phi_V(x) = |x|\psi_V(x^2) \qquad \textit{for } x \in \mathbb{R}. \tag{5.27}$$

(ii) *Let w be the equilibrium density of the set S_V and let $\hat{w}$ be the equilibrium density of $\{x : x^2 \in S_V\}$. Then*

$$\hat{w}(x) = |x|w(x^2) \qquad \textit{for } x \in \mathbb{R}. \tag{5.28}$$

Proof. We give the proof of part (i), the proof of part (ii) is completely similar. We first show that $|x|\psi_V(x^2)$ is indeed a probability density. This follows from the positivity of $|x|\psi_V(x^2)$ and the fact that

$$\begin{aligned}\int_{-\infty}^{+\infty} |x|\psi_V(x^2)dx &= 2\int_0^{+\infty} x\psi_V(x^2)dx \\ &= \int_0^{+\infty} \psi_V(u)du = 1.\end{aligned}$$

To prove part (i) of the lemma, it is now sufficient to show that the measure $|x|\psi_V(x^2)dx$ satisfies variational conditions (5.25) and (5.26). We have

$$\begin{aligned}&\int_{-\infty}^{+\infty} \log|x-y||y|\psi_V(y^2)dy \\ &\qquad = \int_0^{+\infty} \log|x-y|y\psi_V(y^2)dy + \int_0^{+\infty} \log|x+y|y\psi_V(y^2)dy \\ &\qquad = \int_0^{+\infty} \log|x^2-y^2|y\psi_V(y^2)dy \\ &\qquad = \frac{1}{2}\int_0^{+\infty} \log|x^2-u|\psi_V(u)du.\end{aligned}$$

This equation implies (5.25) and (5.26) by (5.4) and (5.5). These variational conditions characterize uniquely the probability measure ν_V. □

5.2.2 Orthogonal polynomials $p_n^{(N)}$, $P_n^{(N)}$, and $Q_n^{(N)}$

Let $p_n^{(N)}$ be the orthonormal polynomial of degree n with respect to the weight $w(x) = x^\alpha e^{-NV(x)}$ on $[0, +\infty)$, and let $P_n^{(N)}$ and $Q_n^{(N)}$ be the orthonormal polynomials with respect to the weights $W_1(x) = |x|^{2\alpha+1} e^{-\frac{N}{2}V(x^2)}$ and $W_2(x) = |x|^{2\alpha-1} e^{-\frac{N}{2}V(x^2)}$ on the real line. Note that $p_n^{(N)}$ and $P_n^{(N)}$ are well-defined for $\alpha > -1$, while $Q_n^{(N)}$ is only defined for $\alpha > 0$.

5.5 Lemma. *We have*

(i) $Q_n^{(N)}(x) = xP_{n-1}^{(N)}(x)$ *if* $\alpha > 0$.

(ii) $P_{2n}^{(2N)}(x) = p_n^{(N)}(x^2)$ *if* $\alpha > -1$,
$Q_{2n+1}^{(2N)}(x) = xp_n^{(N)}(x^2)$ *if* $\alpha > 0$.

Proof. Part (i) of the lemma follows directly from the orthogonality conditions for $P_n^{(N)}$ and $Q_n^{(N)}$. For part (ii) we argue as follows. Since W_1 and W_2 are even, the even degree orthogonal polynomials are even and the odd degree orthogonal polynomials are odd. Together with part (i) of the lemma, this implies that there exists a polynomial $\widetilde{p}_n^{(N)}$ of degree n such that

$$P_{2n}^{(2N)}(x) = \widetilde{p}_n^{(N)}(x^2), \qquad Q_{2n+1}^{(2N)}(x) = x\widetilde{p}_n^{(N)}(x^2) \tag{5.29}$$

For $k < n$, we have

$$\begin{aligned} 0 &= \int_{-\infty}^{+\infty} P_{2n}^{(2N)}(x) x^{2k} |x|^{2\alpha+1} e^{-NV(x^2)} dx \\ &= 2\int_0^{+\infty} \tilde{p}_n^{(N)}(x^2) x^{2k} x^{2\alpha+1} e^{-NV(x^2)} dx \\ &= \int_0^{+\infty} \tilde{p}_n^{(N)}(u) u^k u^\alpha e^{-NV(u)} du. \end{aligned} \tag{5.30}$$

In the above equation we read exactly the orthogonality conditions for $p_n^{(N)}$. For $k = n$, the normalization of the polynomials follows exactly in the same way. We can conclude that $p_n^{(N)} = \widetilde{p}_n^{(N)}$. Now part (ii) of the lemma follows from (5.29). □

5.2.3 Correlation kernels $K_{n,N}$, $K^+_{n,N}$, and $K^-_{n,N}$

We write $K^\pm_{n,N}$ for the two-point kernels of the random matrix ensembles (5.8). This means that

$$K^+_{2n,2N}(x,y) = |x|^{\alpha+\frac{1}{2}}|y|^{\alpha+\frac{1}{2}}e^{-\frac{N}{2}V(x^2)}e^{-\frac{N}{2}V(y^2)}\sum_{j=0}^{2n-1}P_j^{(2N)}(x)P_j^{(2N)}(y), \quad \text{if } \alpha > -1, \tag{5.31}$$

$$K^-_{2n,2N}(x,y) = |x|^{\alpha-\frac{1}{2}}|y|^{\alpha-\frac{1}{2}}e^{-\frac{N}{2}V(x^2)}e^{-\frac{N}{2}V(y^2)}\sum_{j=0}^{2n-1}Q_j^{(2N)}(x)Q_j^{(2N)}(y), \quad \text{if } \alpha > 0. \tag{5.32}$$

In the following lemma, we show that the kernel $K_{n,N}$, in which we are mainly interested, can be expressed in terms of the kernels $K^\pm_{2n,2N}$.

5.6 Lemma. *For $\alpha > -1$,*

$$K_{n,N}(x,y) = \frac{1}{2}(xy)^{-\frac{1}{4}}\left(K^+_{2n,2N}(\sqrt{x},\sqrt{y}) + K^+_{2n,2N}(\sqrt{x},-\sqrt{y})\right), \tag{5.33}$$

and for $\alpha > 0$,

$$K_{n,N}(x,y) = \frac{1}{2}(xy)^{-\frac{1}{4}}\left(K^-_{2n,2N}(\sqrt{x},\sqrt{y}) - K^-_{2n,2N}(\sqrt{x},-\sqrt{y})\right). \tag{5.34}$$

Proof. From (5.31), we know that for $x, y \geq 0$ and $\alpha > -1$,

$$\begin{aligned} K^+_{2n,2N}(\sqrt{x},\sqrt{y}) + K^+_{2n,2N}(\sqrt{x},-\sqrt{y}) &= (xy)^{\frac{\alpha}{2}+\frac{1}{4}}e^{-\frac{N}{2}(V(x)+V(y))} \\ &\times \left(\sum_{j=0}^{2n-1}P_j^{(2N)}(\sqrt{x})P_j^{(2N)}(\sqrt{y}) + \sum_{j=0}^{2n-1}P_j^{(2N)}(\sqrt{x})P_j^{(2N)}(-\sqrt{y})\right). \end{aligned} \tag{5.35}$$

Since the weight $|x|^{2\alpha+1}e^{-NV(x^2)}$ is even, we have that

$$P_{2j}^{(2N)}(x) = P_{2j}^{(2N)}(-x), \qquad P_{2j+1}^{(2N)}(x) = -P_{2j+1}^{(2N)}(-x),$$

so that the odd j-terms in the sums of (5.35) cancel out, while the even j-terms are equal in both sums. We obtain by Lemma 5.5 that

$$\begin{aligned} &K^+_{2n,2N}(\sqrt{x},\sqrt{y}) + K^+_{2n,2N}(\sqrt{x},-\sqrt{y}) \\ &\quad = 2(xy)^{\frac{\alpha}{2}+\frac{1}{4}}e^{-\frac{N}{2}(V(x)+V(y))}\sum_{j=0}^{n-1}P_{2j}^{(2N)}(\sqrt{x})P_{2j}^{(2N)}(\sqrt{y}) \\ &\quad = 2(xy)^{\frac{1}{4}}K_{n,N}(x,y). \end{aligned}$$

This proves (5.33). In a similar way we find for $\alpha > 0$,

$$\begin{aligned} &K^-_{2n,2N}(\sqrt{x},\sqrt{y}) - K^-_{2n,2N}(\sqrt{x},-\sqrt{y}) \\ &\quad = 2(xy)^{\frac{\alpha}{2}-\frac{1}{4}} e^{-\frac{N}{2}(V(x)+V(y))} \sum_{j=0}^{n-1} Q^{(2N)}_{2j+1}(\sqrt{x}) Q^{(2N)}_{2j+1}(\sqrt{y}) \\ &\quad = 2(xy)^{\frac{1}{4}} K_{n,N}(x,y), \end{aligned}$$

and this gives us (5.34). □

5.3 Universality near an endpoint approaching 0 in a double scaling limit

Proof of Theorem 5.1. Fix $\alpha > -1$. Consider an external field V such that 0 is a soft edge point of supp ψ_V, and assume that there are no singular points. Then by Lemma 5.4,

$$\phi_V(x) = |x|\psi_V(x^2) = |x|\sqrt{Q^-_V(x^2)},$$

which means that

$$\phi_V(0) = \phi'_V(0) = 0, \qquad \phi''_V(0) = 2\sqrt{-Q'_V(0)} > 0. \tag{5.36}$$

It follows from Lemma 5.4 that there are no singular points in the external field $V(x^2)/2$ (in fact, the absence of type I singular points follows from the proof of the lemma). The conditions of Theorem 2.2 are satisfied. We take a double scaling limit where we let $n, N \to \infty$ such that the limit

$$\lim (2n)^{2/3}\,(n/N - 1) = L$$

exists with $L \in \mathbb{R}$. Define constants

$$c = \frac{\pi\phi''_V(0)}{8} = \frac{\pi\sqrt{-Q'_V(0)}}{4}, \tag{5.37}$$

and

$$s = L\frac{\pi}{c^{1/3}}\hat{w}(0) = L\frac{\pi}{c^{1/3}} \lim_{x\searrow 0} \sqrt{x}w(x), \tag{5.38}$$

where $\hat{w}$ is the equilibrium density of the support of ϕ_V and w is the equilibrium density of the support S_V of ψ_V. Then by Theorem 2.2 we get, for $u, v > 0$,

$$\lim \frac{1}{(2cn)^{1/3}} K^+_{2n,2N}\left(\frac{\sqrt{u}}{(2cn)^{1/3}}, \frac{\pm\sqrt{v}}{(2cn)^{1/3}}\right) = \mathbb{K}^{\mathrm{crit,II}}_{\alpha+\frac{1}{2}}(\sqrt{u}, \pm\sqrt{v}; s),$$

and for $\alpha > 0$ we also have

$$\lim \frac{1}{(2cn)^{1/3}} K^{-}_{2n,2N}\left(\frac{\sqrt{u}}{(2cn)^{1/3}}, \frac{\pm\sqrt{v}}{(2cn)^{1/3}}\right) = \mathbb{K}^{\mathrm{crit,II}}_{\alpha-\frac{1}{2}}(\sqrt{u}, \pm\sqrt{v}; s).$$

Moreover these results hold uniformly for u, v in compact subsets of $(0, +\infty)$. Using Lemma 5.6, the proof of Theorem 5.1 follows immediately. □

5.4 Asymptotics for the recurrence coefficients

Orthonormal polynomials satisfy a three-term recurrence relation

$$xp_k^{(N)}(x) = a_{k+1}^{(N)} p_{k+1}^{(N)}(x) + b_k^{(N)} p_k^{(N)}(x) + a_k^{(N)} p_{k-1}^{(N)}(x), \tag{5.39}$$

$$xP_k^{(N)}(x) = A_{k+1}^{(N)} P_{k+1}^{(N)}(x) + B_k^{(N)} P_k^{(N)}(x) + A_k^{(N)} P_{k-1}^{(N)}(x), \tag{5.40}$$

$$xQ_k^{(N)}(x) = C_{k+1}^{(N)} Q_{k+1}^{(N)}(x) + D_k^{(N)} Q_k^{(N)}(x) + C_k^{(N)} Q_{k-1}^{(N)}(x). \tag{5.41}$$

Since the weights W_1 and W_2 are even, $B_n^{(N)} = D_n^{(N)} = 0$, and since $Q_n^{(N)} = xP_{n-1}^{(N)}$, it follows that $C_{n+1}^{(N)} = A_n^{(N)}$. In what follows, we abbreviate for convenience

$$p_k = p_k^{(N)}, \qquad a_k = a_k^{(N)}, \qquad b_k = b_k^{(N)},$$

$$P_k = P_k^{(2N)}, \qquad A_k = A_k^{(2N)}, \qquad B_k = B_k^{(2N)},$$

$$Q_k = Q_k^{(2N)}, \qquad C_k = C_k^{(2N)}, \qquad D_k = D_k^{(2N)}.$$

5.7 Lemma. *We have*

$$a_n = A_{2n}A_{2n-1} = C_{2n+1}C_{2n}, \tag{5.42}$$

$$b_n = A_{2n+1}^2 + A_{2n}^2 = C_{2n+2}^2 + C_{2n+1}^2, \tag{5.43}$$

Proof. By (5.39), we have

$$x^2 p_n(x^2) = a_{n+1}p_{n+1}(x^2) + b_n p_n(x^2) + a_n p_{n-1}(x^2),$$

which yields by Lemma 5.5

$$x^2 P_{2n}(x) = a_{n+1}P_{2n+2}(x) + b_n P_{2n}(x) + a_n P_{2n-2}(x).$$

Using (5.40) and the fact that $B_k = 0$, we find

$$xA_{2k+1}P_{2k+1}(x) + xA_{2k}P_{2k-1}(x) = a_{k+1}P_{2k+2}(x) + b_k P_{2k}(x) + a_k P_{2k-2}(x).$$

Again using (5.40) twice, we obtain

$$\begin{aligned} A_{2k+1}A_{2k+2}P_{2k+2}(x) + A_{2k+1}^2 P_{2k}(x) + A_{2k}^2 P_{2k}(x) + A_{2k}A_{2k-1}P_{2k-2}(x) \\ = a_{k+1}P_{2k+2}(x) + b_k P_{2k}(x) + a_k P_{2k-2}(x). \end{aligned}$$

Since the orthogonal polynomials are linearly independent, we find (5.42) and (5.43). □

We now collected all the ingredients for the proof of Theorem 5.3.

Proof of Theorem 5.3. As in the proof of Theorem 5.1, it follows that the external field $V(x^2)/2$ satisfies the necessary conditions for Theorem 2.7. It then follows from this theorem that, if we let $n, N \to \infty$ in such a way that $n/N - 1 = \mathcal{O}(n^{-2/3})$,

$$A_n^{(N)} = \frac{\sqrt{b}}{2} + (-1)^{n+1} \frac{1}{2c^{1/3}} q_{\alpha+\frac{1}{2}}(s_{n,N}) n^{-1/3} + \mathcal{O}(n^{-2/3}), \quad \text{as } \alpha > -1, \tag{5.44}$$

$$C_n^{(N)} = \frac{\sqrt{b}}{2} + (-1)^{n+1} \frac{1}{2c^{1/3}} q_{\alpha-\frac{1}{2}}(s_{n,N}) n^{-1/3} + \mathcal{O}(n^{-2/3}), \quad \text{as } \alpha > 0, \tag{5.45}$$

with

$$s_{n,N} = n^{2/3} \frac{\pi}{c} \phi_{\frac{n}{N}V}(0). \tag{5.46}$$

Here $\phi_{\frac{n}{N}V}$ is the modified equilibrium density (which we introduced in Chapter 1) which is allowed to be negative close to 0.

From (5.42), we find that

$$\begin{aligned} a_n &= A_{2n} A_{2n-1} \\ &= \frac{b}{4} + \frac{1}{2^{4/3} c^{1/3}} (q_{\alpha+\frac{1}{2}}(s_{2n,2N}) - q_{\alpha+\frac{1}{2}}(s_{2n-1,2N})) n^{-1/3} + \mathcal{O}(n^{-2/3}) \\ &= \frac{b}{4} + \mathcal{O}(n^{-2/3}) \qquad \text{as } n \to \infty, \end{aligned} \tag{5.47}$$

and by (5.43),

$$\begin{aligned} b_n &= A_{2n+1}^2 + A_{2n}^2 \\ &= \frac{b}{2} + \mathcal{O}(n^{-2/3}) \qquad \text{as } n \to \infty. \end{aligned} \tag{5.48}$$

□

It would be interesting to calculate the $\mathcal{O}(n^{-2/3})$-terms in (5.47) and (5.48) explicitly. If we would know the coefficients of the $\mathcal{O}(n^{-2/3})$-terms in the large N expansions (5.44) for $A_{n,N}$ or (5.45) for $C_{n,N}$, this would allow us to compute the $\mathcal{O}(n^{-2/3})$ in (5.47) and (5.48) too. However these coefficients have only been computed in the literature for the case where $V(x^2)/2$ is a quartic polynomial [14] and for $2\alpha \pm \frac{1}{2} = 0$.

Chapter 6

Relations between limiting kernels

6.1 Introduction and statement of results

In this chapter, we show that the different limiting kernels that appeared in the previous chapters of this thesis are related to each other. The regular kernels $\mathbb{K}^{\text{bulk}}$ and $\mathbb{K}^{\text{edge}}$ can be obtained as limits of the critical kernels $\mathbb{K}^{\text{crit,II}}$ and $\mathbb{K}^{\text{crit,III}}$. Also the Bessel kernel $\mathbb{J}_\alpha^0$ can be seen as a limit of the kernel $\mathbb{K}_\alpha^{\text{crit,II}}$. This is not surprising if we keep the random matrix ensembles in mind where the critical kernels $\mathbb{K}_\alpha^{\text{crit,II}}$ and $\mathbb{K}^{\text{crit,III}}$ appear. These critical ensembles have a singular point whose nature can change from one regime to another when parameters in the external field change. When we let the deformation parameter(s) vary, we can observe the transition from the critical scaling regime to a regular scaling regime. We show in this chapter that these transitions are not only visible in the random matrix ensembles, but that they can also be observed on the level of the limiting kernels.

The kernel $\mathbb{K}_\alpha^{\text{crit,II}}$ describes the opening of a gap when we let $s \to -\infty$, and for $s \to +\infty$, it describes the case where a singular point is absorbed in the bulk (cf. Figure 1.1). For $s \to +\infty$, this leads us to the sine kernel $\mathbb{K}^{\text{bulk}}$ when $\alpha = 0$ and to the Bessel kernel $\mathbb{J}_\alpha^0$ for general $\alpha > -1/2$. When $s \to -\infty$, two regular endpoints appear near the singular point. Following those endpoints brings us to the Airy kernel $\mathbb{K}^{\text{edge}}$. For the critical kernel $\mathbb{K}^{\text{crit,III}}$ similar observations can be made. If we deform the external field, the singular point can for example turn into a point in the bulk (which leads to the sine kernel) or a regular edge point (which leads to the Airy kernel).

In the random matrix ensembles the mean distance between eigenvalues depends on the regime, this is for example $\mathcal{O}(n^{-1})$ in the bulk, $\mathcal{O}(n^{-2/3})$ at the re-

gular edge of the spectrum, $\mathcal{O}(n^{-1/3})$ near a type II singular point with quadratic vanishing, and $\mathcal{O}(n^{-2/7})$ near a type I singular point where vanishing with a power $5/2$ occurs. This induces a re-scaling of the variables u, v in the double scaling limits near singular points. On the level of the limiting kernels, we need to re-scale u and v in a similar fashion in order to see the transition from one kernel to another.

We have the following results.

6.1 Theorem. (i) *For $u, v \in \mathbb{R}$,*

$$\lim_{s\to+\infty} \frac{1}{s}\mathbb{K}_0^{\mathrm{crit,II}}\left(\frac{u}{s}, \frac{v}{s}; s\right) = \mathbb{K}^{\mathrm{bulk}}(u, v).$$

(ii) *For $u, v \in \mathbb{R} \setminus \{0\}$, we have*

$$\lim_{s\to+\infty} \frac{1}{s}\mathbb{K}_\alpha^{\mathrm{crit,II}}\left(\frac{u}{s}, \frac{v}{s}; s\right) = \mathbb{J}_\alpha^0(u, v).$$

(iii) *For $u, v \in \mathbb{R}$,*

$$\lim_{s\to-\infty} \frac{1}{2^{7/6}|s|^{1/2}}\mathbb{K}_0^{\mathrm{crit,II}}\left(\mp\frac{1}{\sqrt{2}}|s|^{1/2} \pm \frac{u}{2^{7/6}|s|^{1/2}}, \right.$$
$$\left. \mp\frac{1}{\sqrt{2}}|s|^{1/2} \pm \frac{v}{2^{7/6}|s|^{1/2}}; s\right) = \mathbb{K}^{\mathrm{edge}}(u, v).$$

(iv) *For $u, v, t \in \mathbb{R}$,*

$$\lim_{s\to\pm\infty} \left(\frac{4}{3}\right)^{4/9} \frac{1}{|s|^{4/9}}\mathbb{K}^{\mathrm{crit,III}}\left(|s|^{1/3}z_0 + \left(\frac{4}{3}\right)^{4/9}\frac{u}{|s|^{4/9}}, \right.$$
$$\left. |s|^{1/3}z_0 + \left(\frac{4}{3}\right)^{4/9}\frac{v}{|s|^{4/9}}; s, t\right) = \mathbb{K}^{\mathrm{edge}}(u, v),$$

where z_0 is defined as the real zero of

$$z_0^3 = -\mathrm{sgn}(s)48 + 24z_0|s|^{-2/3}t, \qquad \text{for } s \neq 0.$$

(v) *Set $c = \frac{|g_+'(0)|}{\pi}$, with g defined by* (3.92) *and* (3.95)*. Then, for $u, v, t \in \mathbb{R}$,*

$$\lim_{s\to-\infty} \frac{1}{c|s|^{5/6}}\mathbb{K}^{\mathrm{crit,III}}\left(\frac{u}{c|s|^{5/6}}, \frac{v}{c|s|^{5/6}}; s, t\right) = \mathbb{K}^{\mathrm{bulk}}(u, v).$$

Outline of the rest of the chapter

We will prove Theorem 6.1 by applying the Deift/Zhou steepest descent method on the model RH problems associated with the special solutions of the Painlevé II equation and the P_I^2 equation. We use similar techniques as in [37]. In Section 6.2 we recall the different model RH problems that are relevant to prove Theorem 6.1. In Section 6.3 we prove part (i) and part (ii) of the theorem. Here we need to construct a local parametrix near the origin by using a model RH problem for Bessel functions. In Section 6.4 we prove part (iii) of the theorem. The main difficulties here are the construction of an appropriate g-function, a contour deformation, and the construction of a local parametrix using Airy functions. In Section 6.5 finally, we prove part (iv) and part (v) of the theorem. Here we do not need to perform the Deift/Zhou steepest descent method since we already did the required asymptotic analysis in Chapter 3. We only need to use the results from Chapter 3 in an appropriate way in order to prove Theorem 6.1 (iv) and (v).

6.2 Model RH problems and related kernels

6.2.1 Model RH problem for Painlevé II

From Chapter 2 (see Section 2.2.1), we recall the RH problem associated with the Hastings-McLeod solution of the Painlevé II equation with parameter α. We use a slight modification of the RH problem stated in Chapter 2, by using a more general jump contour. However, it is easily checked that the RH problem we consider here is equivalent to the one considered in Chapter 2 by analytic continuation. Let $\Sigma = \bigcup_j \Sigma_j$ be a contour consisting of four curves Σ_j connecting 0 with infinity, with Σ_1 contained in the sector $\epsilon < \arg \zeta < \pi/3 - \epsilon$ for some $\epsilon > 0$, with Σ_2 contained in the sector $2\pi/3 + \epsilon < \arg \zeta < \pi - \epsilon$, Σ_3 contained in $\pi + \epsilon < \arg \zeta < 4\pi/3 - \epsilon$, and Σ_4 contained in the sector $7\pi/3 + \epsilon < \arg \zeta < 2\pi - \epsilon$. The contour Σ divides the complex plane in four regions $S_1, \ldots, S_4$ as shown in Figure 6.1. We will need this freedom in the choice of the contour later on. For $\alpha > -1/2$ and $s \in \mathbb{R}$, we write $\Psi_\alpha^{\mathrm{II}}$ for the unique 2×2 matrix-valued function satisfying the following conditions.

RH problem for $\Psi_\alpha^{\mathrm{II}}$:

(a) $\Psi_\alpha^{\mathrm{II}}$ is analytic in $\mathbb{C} \setminus \Sigma$.

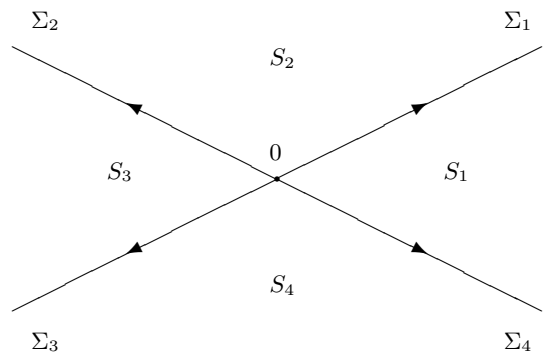

Figure 6.1 *The contour* Σ.

(b) $\Psi_\alpha^{\mathrm{II}}$ satisfies the following jump relations on $\Sigma \setminus \{0\}$,

$$\Psi_{\alpha,+}^{\mathrm{II}}(\zeta) = \Psi_{\alpha,-}^{\mathrm{II}}(\zeta) \begin{pmatrix} 1 & 0 \\ e^{-\pi i\alpha} & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \Sigma_1, \tag{6.1}$$

$$\Psi_{\alpha,+}^{\mathrm{II}}(\zeta) = \Psi_{\alpha,-}^{\mathrm{II}}(\zeta) \begin{pmatrix} 1 & 0 \\ -e^{\pi i\alpha} & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \Sigma_2, \tag{6.2}$$

$$\Psi_{\alpha,+}^{\mathrm{II}}(\zeta) = \Psi_{\alpha,-}^{\mathrm{II}}(\zeta) \begin{pmatrix} 1 & e^{-\pi i\alpha} \\ 0 & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \Sigma_3, \tag{6.3}$$

$$\Psi_{\alpha,+}^{\mathrm{II}}(\zeta) = \Psi_{\alpha,-}^{\mathrm{II}}(\zeta) \begin{pmatrix} 1 & -e^{\pi i\alpha} \\ 0 & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \Sigma_4. \tag{6.4}$$

(c) $\Psi_\alpha^{\mathrm{II}}$ has the following behavior at infinity,

$$\Psi_\alpha^{\mathrm{II}}(\zeta) = (I + \mathcal{O}(1/\zeta))e^{-i\theta^{\mathrm{II}}(\zeta;s)\sigma_3}, \qquad \text{as } \zeta \to \infty, \tag{6.5}$$

with

$$\theta^{\mathrm{II}}(\zeta; s) = \frac{4}{3}\zeta^3 + s\zeta. \tag{6.6}$$

Here $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ denotes the third Pauli matrix.

(d) $\Psi_\alpha^{\mathrm{II}}$ has the following behavior near the origin. If $\alpha < 0$,

$$\Psi_\alpha^{\mathrm{II}}(\zeta) = \mathcal{O}\begin{pmatrix} |\zeta|^\alpha & |\zeta|^\alpha \\ |\zeta|^\alpha & |\zeta|^\alpha \end{pmatrix}, \qquad \text{as } \zeta \to 0, \tag{6.7}$$

and if $\alpha \geq 0$,

$$\Psi_\alpha^{\mathrm{II}}(\zeta) = \begin{cases} \mathcal{O}\begin{pmatrix} |\zeta|^{-\alpha} & |\zeta|^{-\alpha} \\ |\zeta|^{-\alpha} & |\zeta|^{-\alpha} \end{pmatrix}, & \text{as } \zeta \to 0,\ \zeta \in S_1 \cup S_3, \\ \mathcal{O}\begin{pmatrix} |\zeta|^{\alpha} & |\zeta|^{-\alpha} \\ |\zeta|^{\alpha} & |\zeta|^{-\alpha} \end{pmatrix}, & \text{as } \zeta \to 0,\ \zeta \in S_2, \\ \mathcal{O}\begin{pmatrix} |\zeta|^{-\alpha} & |\zeta|^{\alpha} \\ |\zeta|^{-\alpha} & |\zeta|^{\alpha} \end{pmatrix}, & \text{as } \zeta \to 0,\ \zeta \in S_4. \end{cases} \tag{6.8}$$

Recall from Chapter 2 and in particular from (2.62) that, if we define $\Phi_\alpha^{\mathrm{II}}$ by

$$\Phi_\alpha^{\mathrm{II}}(\zeta; s) = \begin{cases} \Psi_\alpha^{\mathrm{II}}(\zeta; s) \begin{pmatrix} 1 & 0 \\ e^{-\pi i \alpha} & 1 \end{pmatrix}, & \text{for } \zeta > 0, \\ \Psi_\alpha^{\mathrm{II}}(\zeta; s) \begin{pmatrix} 1 & 0 \\ e^{\pi i \alpha} & 1 \end{pmatrix}, & \text{for } \zeta < 0, \end{cases} \tag{6.9}$$

then the kernel $\mathbb{K}_\alpha^{\mathrm{crit,II}}$ is given by

$$\begin{aligned} &\mathbb{K}_\alpha^{\mathrm{crit,II}}(u, v; s) \\ &= -e^{\frac{1}{2}\pi i \alpha[\mathrm{sgn}(u)+\mathrm{sgn}(v)]} \frac{\Phi_{\alpha,11}^{\mathrm{II}}(u; s)\Phi_{\alpha,21}^{\mathrm{II}}(v; s) - \Phi_{\alpha,11}^{\mathrm{II}}(v; s)\Phi_{\alpha,21}^{\mathrm{II}}(u; s)}{2\pi i(u - v)} \\ &= \frac{e^{\frac{1}{2}\pi i \alpha[\mathrm{sgn}(u)+\mathrm{sgn}(v)]}}{2\pi i(u - v)} \begin{pmatrix} 0 & 1 \end{pmatrix} \Phi_\alpha^{\mathrm{II}}(v; s)^{-1} \Phi_\alpha^{\mathrm{II}}(u; s) \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{aligned} \tag{6.10}$$

6.2.2 Model RH problem for Bessel functions

The model RH problem for Bessel functions can be stated in a very similar way as the one for the Painlevé II equation. We write B_α for the function satisfying exactly the same RH conditions as $\Psi_\alpha^{\mathrm{II}}$, but with $\theta^{\mathrm{II}}(\zeta; s)$ replaced by ζ, so that

$$B_\alpha(\zeta) = (I + \mathcal{O}(1/\zeta))e^{-i\zeta\sigma_3}, \qquad \text{as } \zeta \to \infty, \tag{6.11}$$

For convenience, we choose the jump contour of B_α to be $\Sigma' = \cup_{j=1}^4 \Sigma'_j$, with

$$\Sigma'_1 : \arg\zeta = \frac{\pi}{6}, \qquad \Sigma'_2 : \arg\zeta = \frac{5\pi}{6}, \qquad \Sigma'_3 : \arg\zeta = -\frac{5\pi}{6}, \qquad \Sigma'_4 : \arg\zeta = -\frac{\pi}{6}.$$

The function B_α can be constructed explicitly in terms of Bessel and Hankel functions (see e.g. [1] for more details about these functions) in a similar way as in [98, Section 4]. If we define $\widehat{B}_\alpha$ by, cf. (6.9),

$$\widehat{B}_\alpha(\zeta) = \begin{cases} \widehat{B}_\alpha(\zeta) \begin{pmatrix} 1 & 0 \\ e^{-\pi i \alpha} & 1 \end{pmatrix}, & \text{for } \zeta > 0, \\ \widehat{B}_\alpha(\zeta) \begin{pmatrix} 1 & 0 \\ e^{\pi i \alpha} & 1 \end{pmatrix}, & \text{for } \zeta < 0, \end{cases} \tag{6.12}$$

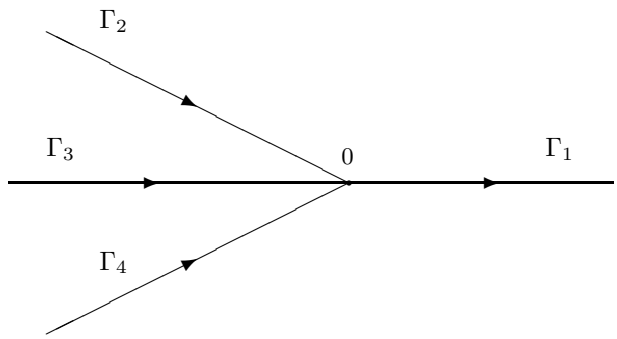

Figure 6.2 *The oriented contour* Γ.

the kernel $\mathbb{J}_\alpha^0$ is given by

$$\begin{aligned}\mathbb{J}_\alpha^0(u,v) &= -e^{\frac{1}{2}\pi i\alpha[\mathrm{sgn}(u)+\mathrm{sgn}(v)]}\frac{\widehat{B}_{\alpha,11}(u)\widehat{B}_{\alpha,21}(v)-\widehat{B}_{\alpha,11}(v)\widehat{B}_{\alpha,21}(u)}{2\pi i(u-v)}\\ &= \frac{e^{\frac{1}{2}\pi i\alpha[\mathrm{sgn}(u)+\mathrm{sgn}(v)]}}{2\pi i(u-v)}\begin{pmatrix}0 & 1\end{pmatrix}\widehat{B}_\alpha^{-1}(v)\widehat{B}_\alpha(u)\begin{pmatrix}1\\0\end{pmatrix}. \qquad (6.13)\end{aligned}$$

6.2.3 Model RH problem for P_I^2

We recall the RH problem associated with the pole-free solution of the P_I^2 equation from Chapter 3 and Chapter 4, see (4.103)-(4.106). Let $\Gamma = \bigcup_{j=1}^4 \Gamma_j$ be the contour consisting of four curves connecting 0 with infinity, with $\Gamma_1 = \mathbb{R}^+$, $\Gamma_3 = \mathbb{R}^-$, with Γ_2 contained in the sector $5\pi/7+\epsilon < \arg\zeta < \pi-\epsilon$ for some $\epsilon > 0$, and with Γ_4 contained in the sector $\pi+\epsilon < \arg\zeta < 9\pi/7-\epsilon$. Note that, as in Section 6.2.1, this contour is more general than the one considered in Chapter 3 and Chapter 4, but we can find the solution satisfying the RH problem on the deformed contour by continuing the solution in an analytic way. The contour Γ is oriented as shown in Figure 6.2. We write Ψ^{III} for the unique solution of the following RH problem.

RH problem for $\Psi^{\mathrm{P_I^2}}$:

(a) Ψ^{III} is analytic in $\mathbb{C}\setminus\Gamma$.

(b) Ψ^{III} satisfies the following constant jump relations on Γ,

$$\Psi_+^{\mathrm{III}}(\zeta) = \Psi_-^{\mathrm{III}}(\zeta) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_1, \tag{6.14}$$

$$\Psi_+^{\mathrm{III}}(\zeta) = \Psi_-^{\mathrm{III}}(\zeta) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_2 \cup \Gamma_4, \tag{6.15}$$

$$\Psi_+^{\mathrm{III}}(\zeta) = \Psi_-^{\mathrm{III}}(\zeta) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \text{for } \zeta \in \Gamma_3. \tag{6.16}$$

(c) Ψ^{III} has the following behavior at infinity,

$$\Psi^{\mathrm{III}}(\zeta) = \zeta^{-\frac{1}{4}\sigma_3} N (I + \mathcal{O}(1/\zeta^{1/2})) e^{-\theta^{\mathrm{III}}(\zeta;x,T)\sigma_3}, \qquad \text{as } \zeta \to \infty, \tag{6.17}$$

where N and θ^{III} are given by

$$N = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} e^{-\frac{1}{4}\pi i \sigma_3}, \quad \theta^{\mathrm{III}}(\zeta; x, T) = \frac{1}{105}\zeta^{7/2} - \frac{T}{3}\zeta^{3/2} + x\zeta^{1/2}.$$

From (4.137) we know that, if we define Φ^{III} by

$$\Phi^{\mathrm{III}}(\zeta; s, t) = \begin{cases} \Psi_+^{\mathrm{III}}(\zeta; s, t) & \text{for } \zeta > 0, \\ \Psi_+^{\mathrm{III}}(\zeta; s, t) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & \text{for } \zeta < 0, \end{cases} \tag{6.18}$$

the kernel $\mathbb{K}^{\mathrm{crit,III}}$ is given by

$$\begin{aligned} \mathbb{K}^{\mathrm{crit,III}}(u, v; s, t) &= \frac{\Phi_{11}^{\mathrm{III}}(u; s, t)\Phi_{21}^{\mathrm{III}}(v; s, t) - \Phi_{11}^{\mathrm{III}}(v; s, t)\Phi_{21}^{\mathrm{III}}(u; s, t)}{-2\pi i(u - v)} \\ &= \frac{1}{2\pi i(u - v)} \\ &\qquad \times \begin{pmatrix} 0 & 1 \end{pmatrix} \Phi^{\mathrm{III}}(v; s, t)^{-1} \Phi^{\mathrm{III}}(u; s, t) \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{aligned} \tag{6.19}$$

6.2.4 Model RH problem for Airy function

The model RH problem for the Airy function is very similar to the one for the special solution of the P_I^2 equation. We let A be the function satisfying exactly the same RH conditions as Ψ^{III} except for the asymptotic condition (c) which is now the following,

$$\Psi^{\mathrm{III}}(\zeta) = \zeta^{-\frac{1}{4}\sigma_3} N (I + \mathcal{O}(1/\zeta^{3/2})) e^{-\theta^{\mathrm{Ai}}(\zeta;x,T)\sigma_3}, \tag{6.20}$$

with $\theta^{\mathrm{Ai}}(\zeta) = \frac{2}{3}\zeta^{3/2}$. Here we make the particular choice $\Gamma' = \cup_{j=1}^{4}\Gamma'_j$ for the jump contour, with

$$\Gamma'_1 : \arg\zeta = 0, \quad \Gamma'_2 : \arg\zeta = \frac{6\pi}{7}, \quad \Gamma'_3 : \arg\zeta = \pi, \quad \Gamma'_4 : \arg\zeta = -\frac{6\pi}{7},$$

If we define $\widehat{A}$, cf. (6.18), by

$$\widehat{A}(\zeta) = \begin{cases} A_+(\zeta) & \text{for } \zeta > 0, \\ A_+(\zeta)\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & \text{for } \zeta < 0, \end{cases} \tag{6.21}$$

the kernel $\mathbb{K}^{\mathrm{edge}}$ is given by, see e.g. [27],

$$\begin{aligned} \mathbb{K}^{\mathrm{edge}}(u,v) &= \frac{\widehat{A}_{11}(u)\widehat{A}_{21}(v) - \widehat{A}_{11}(v)\widehat{A}_{21}(u)}{-2\pi i(u-v)} \\ &= \frac{1}{2\pi i(u-v)}\begin{pmatrix} 0 & 1 \end{pmatrix}\widehat{A}^{-1}(v)\widehat{A}(u)\begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{aligned} \tag{6.22}$$

6.2 Remark. The RH solution A is closely related with the function M defined in Section 3.2.2. To be precise,

$$A(\zeta) = M(\zeta)e^{-\frac{2}{3}\zeta^{3/2}\sigma_3} \tag{6.23}$$

6.3 Proof of Theorem 6.1 (i) and (ii)

Assume that $s > M$ with $M > 0$ sufficiently large, and consider a contour Σ as described in Section 6.2.1. We will specify the exact form of the contour later. We now apply the Deift/Zhou steepest descent method on the RH problem for $\Psi_\alpha^{\mathrm{II}}$ when $s \to +\infty$.

RH problem for S

We write S for the function

$$S(\zeta) = \Psi_\alpha^{\mathrm{II}}(s^{1/2}\zeta)e^{i\theta^{\mathrm{II}}(s^{1/2}\zeta;s)\sigma_3} \qquad \text{for } \zeta \in \mathbb{C}\setminus\widehat{\Sigma},\ \widehat{\Sigma} = s^{-1/2}\Sigma. \tag{6.24}$$

Then it follows from the RH conditions (6.1)-(6.8) for $\Psi_\alpha^{\mathrm{II}}$ that S satisfies the RH problem

(a) S is analytic in $\mathbb{C}\setminus\widehat{\Sigma}$,

(b) $S_+(\zeta) = S_-(\zeta)v_S(\zeta)$ as $\zeta \in \widehat{\Sigma}$,

(c) $S(\zeta) = I + \mathcal{O}(\zeta^{-1})$ as $\zeta \to \infty$,

(d) S has the same behavior as $\Psi_\alpha^{\mathrm{II}}$ near the origin,

where v_S is given by

$$v_S(\zeta) = \begin{pmatrix} 1 & 0 \\ e^{-\pi i\alpha} e^{2i\theta^{\mathrm{II}}(s^{1/2}\zeta;s)} & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \widehat{\Sigma}_1 := s^{-1/2}\Sigma_1, \tag{6.25}$$

$$v_S(\zeta) = \begin{pmatrix} 1 & 0 \\ -e^{\pi i\alpha} e^{2i\theta^{\mathrm{II}}(s^{1/2}\zeta;s)} & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \widehat{\Sigma}_2 := s^{-1/2}\Sigma_2, \tag{6.26}$$

$$v_S(\zeta) = \begin{pmatrix} 1 & e^{-\pi i\alpha} e^{-2i\theta^{\mathrm{II}}(s^{1/2}\zeta;s)} \\ 0 & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \widehat{\Sigma}_3 := s^{-1/2}\Sigma_3, \tag{6.27}$$

$$v_S(\zeta) = \begin{pmatrix} 1 & -e^{\pi i\alpha} e^{-2i\theta^{\mathrm{II}}(s^{1/2}\zeta;s)} \\ 0 & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \widehat{\Sigma}_4 := s^{-1/2}\Sigma_4. \tag{6.28}$$

Note that, since $\theta^{\mathrm{II}}(s^{1/2}\zeta;s) = s^{3/2}(\frac{4}{3}\zeta^3 + \zeta)$, the jump matrix v_S tends to I exponentially fast when $s \to +\infty$, except for ζ near 0, where we need to construct a local parametrix. We do this by using the Bessel model RH problem.

Local parametrix near 0

We construct the local parametrix in a sufficiently small disk $U_\delta = \{\zeta : |\zeta| < \delta\}$ using the function B_α which we defined in Section 6.3. Define the parametrix P as follows,

$$P(\zeta) = B_\alpha(s^{3/2} f(\zeta)) e^{i\theta^{\mathrm{II}}(s^{1/2}\zeta;s)\sigma_3} \qquad \text{for } \zeta \in U_\delta. \tag{6.29}$$

Here f is the conformal mapping given by

$$f(\zeta) = \frac{4}{3}\zeta^3 + \zeta. \tag{6.30}$$

Now we can choose the contour $\widehat{\Sigma}$ (and thus also $\Sigma = s^{1/2}\widehat{\Sigma}$) by requiring that P and S have their jumps along the same curves in U_δ. Therefore we require $\widehat{\Sigma}$ to satisfy

$$f(\widehat{\Sigma} \cap U_\delta) = \Sigma' \cap f(U_\delta),$$

where Σ' is the jump contour for B_α. Outside U_δ we choose $\widehat{\Sigma}$ to be independent of s. Note that, for δ small, we can still choose the contour $\Sigma = s^{1/2}\widehat{\Sigma}$ according to the conditions given in Section 6.2.1. We constructed P in such a way that it has exactly the same jumps as S has in U_δ and the same behavior at the origin. Furthermore (6.30) ensures that $P(\zeta) = I + \mathcal{O}(s^{-3/2})$ for $\zeta \in \partial U_\delta$ as $s \to +\infty$. We have that P satisfies the following conditions,

(a) P is analytic in $U_\delta \setminus \widehat{\Sigma}$,

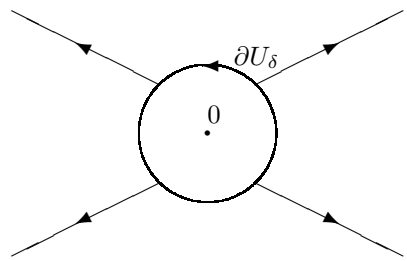

Figure 6.3 *The contour* Σ_R.

(b) $P_+(\zeta) = P_-(\zeta)v_S(\zeta)$ as $\zeta \in \widehat{\Sigma} \cap U_\delta$,

(c) $P(\zeta) = I + \mathcal{O}(s^{-3/2})$ for $\zeta \in \partial U_\delta$, as $s \to +\infty$,

(d) P has the same behavior than S has for $\zeta \to 0$.

RH problem for R

Now we define R by

$$R(\zeta; s) = \begin{cases} S(\zeta) & \text{for } \zeta \in \mathbb{C} \setminus U_\delta, \\ S(\zeta)P(\zeta)^{-1} & \text{for } \zeta \in U_\delta. \end{cases} \tag{6.31}$$

Then R is analytic outside a contour Σ_R as shown in Figure 6.3, with a removable singularity at the origin. We can conclude from condition (c) of the RH problem for P and from the exponential decay of v_S outside U_δ that R satisfies a RH problem of the following form:

(a) $R : \mathbb{C} \setminus \Sigma_R \to \mathbb{C}^{2\times 2}$ is analytic on $\mathbb{C} \setminus \Sigma_R$,

(b) R satisfies the following jump conditions for some $\gamma > 0$,

$$R_+(\zeta) = R_-(\zeta)(I + \mathcal{O}(s^{-3/2})) \qquad \text{for } \zeta \in \partial U_\delta, \tag{6.32}$$

$$R_+(\zeta) = R_-(\zeta)(I + \mathcal{O}(e^{-\gamma s^{3/2}})) \qquad \text{for } \zeta \in \Sigma_R \setminus \partial U_\delta, \tag{6.33}$$

(c) $R(\zeta) = I + \mathcal{O}(\zeta^{-1})$ as $\zeta \to \infty$.

From those RH conditions it follows that

$$R(\zeta) = I + \mathcal{O}(s^{-3/2}) \qquad \text{as } s \to +\infty, \tag{6.34}$$

uniformly for $\zeta \in \mathbb{C} \setminus \Sigma_R$.

Proof of Theorem 6.1 (i) and (ii). Inserting (6.31), (6.29), and (6.24) in the asymptotics (6.34) for R, it follows that for $\zeta \in [-\delta, \delta] \setminus \{0\}$,

$$\Psi^{\mathrm{II}}_{\alpha,+}(s^{1/2}\zeta; s) B_{\alpha,+}\left(s^{3/2} f(\zeta)\right)^{-1} = I + \mathcal{O}(s^{-3/2}) \qquad \text{as } s \to +\infty.$$

By (6.9) and (6.12) this yields

$$\Phi^{\mathrm{II}}_{\alpha}(s^{1/2}\zeta; s) \widehat{B}_{\alpha}\left(s^{3/2} f(\zeta)\right)^{-1} = I + \mathcal{O}(s^{-3/2}) \qquad \text{as } s \to +\infty.$$

Setting

$$\zeta = u s^{-3/2}, \qquad u \in \mathbb{R} \setminus \{0\},$$

we obtain using (6.30) that

$$\lim_{s \to +\infty} \Phi^{\mathrm{II}}_{\alpha}(\frac{u}{s}; s) = \widehat{B}_{\alpha}(u).$$

Now using (6.10) and (6.13), we find easily that, for $u, v \in \mathbb{R} \setminus \{0\}$,

$$\lim_{s \to \infty} \frac{1}{s} \mathbb{K}^{\mathrm{crit,II}}_{\alpha}\left(\frac{u}{s}, \frac{v}{s}; s\right) = \mathbb{J}^{0}_{\alpha}(u, v),$$

which proves Theorem 6.1 (ii). In the case $\alpha = 0$, $\mathbb{J}^{0}_{\alpha} = \mathbb{K}^{\mathrm{bulk}}$ which brings us to Theorem 6.1 (i). □

6.4 Proof of Theorem 6.1 (iii)

Assume that $s < -M$ with $M > 0$ sufficiently large. We consider a contour Σ which is slightly different from the ones described in Section 6.2.1. We require that $[0, \frac{1}{\sqrt{2}}|s|^{1/2}] \subset \Sigma_1, \Sigma_4$ and $[-\frac{1}{\sqrt{2}}|s|^{1/2}, 0] \subset \Sigma_2, \Sigma_3$, as shown in Figure 6.4. However the RH problem for Ψ^{II}_0 is equivalent to the RH problem as it is stated in Section 6.2.1. We also partly reverse orientations of the contour so that all pieces of the contour are directed from left to right. Choosing this jump contour, it is straightforward to check that the jumps for Ψ^{II}_0 are given as indicated in Figure 6.4. We now apply the Deift/Zhou steepest descent method on the RH problem for Ψ^{II}_0 when $s \to -\infty$.

RH problem for S

An important step in the asymptotic analysis is the construction of an appropriate g-function, see also [37]. We define g by

$$g(\zeta) = 2^{2/3}(\zeta^2 - \frac{1}{2}). \tag{6.35}$$

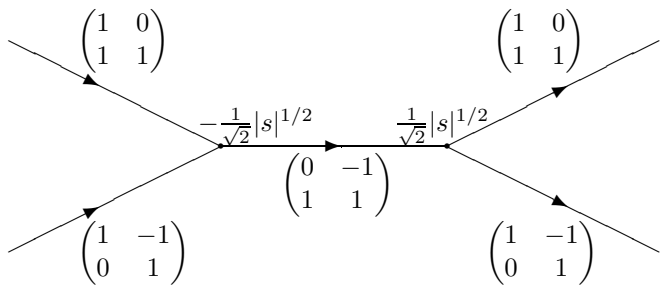

Figure 6.4 *The modified contour Σ and the jump matrices for Ψ^{III}.*

Here we choose the branch cut of $g^{3/2}$ on $[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]$ and we define $g^{3/2}$ in such a way that

$$\frac{2}{3}|s|^{3/2} g(\zeta)^{3/2} = \theta^{\mathrm{II}}(|s|^{1/2}\zeta) + \mathcal{O}(\zeta^{-1}) \qquad \text{as } \zeta \to \infty. \tag{6.36}$$

Now we write $\widehat{\Sigma} = |s|^{-1/2}\Sigma$ (which we will choose independent of s) and we define S as follows,

$$S(\zeta) = \begin{cases} \Psi_0^{\mathrm{II}}(|s|^{1/2}\zeta) e^{\frac{2}{3}i|s|^{3/2} g(\zeta)^{3/2}\sigma_3} & \text{for } \zeta \in \mathbb{C} \setminus \widehat{\Sigma},\ \operatorname{Im}\zeta > 0 \\ \Psi_0^{\mathrm{II}}(|s|^{1/2}\zeta) e^{\frac{2}{3}i|s|^{3/2} g(\zeta)^{3/2}\sigma_3} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \text{for } \zeta \in \mathbb{C} \setminus \widehat{\Sigma},\ \operatorname{Im}\zeta < 0. \end{cases} \tag{6.37}$$

Then S satisfies a RH problem of the form

(a) S is analytic in $\mathbb{C} \setminus \widehat{\Sigma}$,

(b) $S_+(\zeta) = S_-(\zeta) v_S(\zeta)$ as $\zeta \in \widehat{\Sigma}$,

(c) $S(\zeta) = \begin{cases} I + \mathcal{O}(\zeta^{-1}) & \text{as } \operatorname{Im}\zeta > 0 \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \mathcal{O}(\zeta^{-1}) & \text{as } \operatorname{Im}\zeta < 0 \end{cases} \qquad \text{as } \zeta \to \infty.$

Condition (c) follows from (6.36), and using the jump conditions for Ψ_0^{II} we check that the jump matrix v_S is given by

$$v_S(\zeta) = \begin{pmatrix} 1 & 0 \\ e^{\frac{4}{3}i|s|^{3/2}g(\zeta)^{3/2}} & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \widehat{\Sigma}_1 = |s|^{-1/2}\Sigma_1 \setminus [0, \tfrac{1}{\sqrt{2}}], \tag{6.38}$$

$$v_S(\zeta)) = \begin{pmatrix} 1 & 0 \\ e^{\frac{4}{3}i|s|^{3/2}g(\zeta)^{3/2}} & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \widehat{\Sigma}_2 = |s|^{-1/2}\Sigma_2 \setminus [-\tfrac{1}{\sqrt{2}}, 0], \tag{6.39}$$

$$v_S(\zeta) = \begin{pmatrix} 1 & 0 \\ e^{-\frac{4}{3}i|s|^{3/2}g(\zeta)^{3/2}} & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \widehat{\Sigma}_3 = |s|^{-1/2}\Sigma_3 \setminus [-\tfrac{1}{\sqrt{2}}, 0], \tag{6.40}$$

$$v_S(\zeta) = \begin{pmatrix} 1 & 0 \\ e^{-\frac{4}{3}i|s|^{3/2}g(\zeta)^{3/2}} & 1 \end{pmatrix}, \qquad \text{for } \zeta \in \widehat{\Sigma}_4 = |s|^{-1/2}\Sigma_4 \setminus [0, \tfrac{1}{\sqrt{2}}]. \tag{6.41}$$

$$v_S(\zeta) = \begin{pmatrix} 1 & e^{-\frac{4}{3}i|s|^{3/2}g_+(\zeta)^{3/2}} \\ 0 & 1 \end{pmatrix}, \quad \text{for } \zeta \in [-\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}]. \tag{6.42}$$

$$v_S(\zeta) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \text{for } \zeta \in (-\infty, -\tfrac{1}{\sqrt{2}}) \cup (\tfrac{1}{\sqrt{2}}, +\infty). \tag{6.43}$$

Now we choose $\widehat{\Sigma}$ in such a way that the following important inequalities hold,

$$\operatorname{Im} g(\zeta)^{3/2} > 0 \qquad \text{for } \zeta \in \widehat{\Sigma}_1 \cup \widehat{\Sigma}_2, \tag{6.44}$$

$$\operatorname{Im} g(\zeta)^{3/2} < 0 \qquad \text{for } \zeta \in \widehat{\Sigma}_3 \cup \widehat{\Sigma}_4, \tag{6.45}$$

$$\operatorname{Im} g_+(\zeta)^{3/2} < 0 \qquad \text{for } \zeta \in [-\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}]. \tag{6.46}$$

Consequently the jump matrix v_S tends to I exponentially fast when $s \to -\infty$, except for $\zeta \in (-\infty, -\frac{1}{\sqrt{2}}) \cup (\frac{1}{\sqrt{2}}, +\infty)$ and for ζ close to the points $\pm\frac{1}{\sqrt{2}}$. We need to construct an outside parametrix and local parametrices near the special points $\pm\frac{1}{\sqrt{2}}$.

Outside parametrix

We construct the outside parametrix, which we will call $\widehat{P}^{(\infty)}$, using the outside parametrix $P^{(\infty)}$ corresponding to the interval $[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]$, as defined in the previous chapters, see e.g. (4.85). We define

$$\widehat{P}^{(\infty)}(\zeta) = \begin{cases} \sigma_3 P^{(\infty)}(\zeta)\sigma_3, & \text{as } \operatorname{Im}\zeta > 0 \\ \sigma_3 P^{(\infty)}(\zeta) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \sigma_3, & \text{as } \operatorname{Im}\zeta < 0, \end{cases} \tag{6.47}$$

so that it has the same behavior as S at infinity and the same jumps on $(-\infty, -\frac{1}{\sqrt{2}}) \cup (\frac{1}{\sqrt{2}}, +\infty)$:

- $\widehat{P}^{(\infty)}$ is analytic in $\mathbb{C} \setminus \left((-\infty, -\frac{1}{\sqrt{2}}) \cup (\frac{1}{\sqrt{2}}, +\infty)\right)$,
- $\widehat{P}_+^{(\infty)}(\zeta) = \widehat{P}_-^{(\infty)}(\zeta) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ for $\zeta \in (-\infty, -\frac{1}{\sqrt{2}}) \cup (\frac{1}{\sqrt{2}}, +\infty)$,
- $S(\zeta)\widehat{P}^{(\infty)}(\zeta)^{-1} = I + \mathcal{O}(\zeta^{-1})$ as $\zeta \to \infty$.

Local parametrices near $-\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}$

We first construct the parametrix near $-\frac{1}{\sqrt{2}}$ using the solution A of the Airy model RH problem. We will take P of the form

$$P(\zeta) = E(\zeta)A(-|s|g(z))e^{\frac{2}{3}|s|^{3/2}(-g(\zeta))^{3/2}\sigma_3} \qquad \text{for } \zeta \in U_\delta^-, \tag{6.48}$$

where $U_\delta^- := \{\zeta : |\zeta + \frac{1}{\sqrt{2}}| < \delta\}$, $\delta > 0$ is a sufficiently small constant, E is analytic in U_δ^-, and g is defined by (6.35), so that it is a conformal mapping with

$$g\left(-\frac{1}{\sqrt{2}}\right) = 0, \qquad g'\left(-\frac{1}{\sqrt{2}}\right) = -2^{7/6} < 0, \qquad g(\mathbb{R}) \subset \mathbb{R}.$$

We take the branch cut of the power 3/2 along $(-\infty, 0]$, so that $(-g(\zeta))^{3/2}$ is analytic in $\mathbb{C} \setminus ((-\infty, -\frac{1}{\sqrt{2}}) \cup (\frac{1}{\sqrt{2}}, +\infty))$. As in the case where $s \to +\infty$, we can now specify the contour $\widehat{\Sigma}$ by requiring that P and S have their jumps along the same curves near the origin. We define $\widehat{\Sigma} \cap U_\delta^\pm$ by

$$-g(\widehat{\Sigma} \cap U_\delta^-) = \Gamma' \cap -g(U_\delta^-), \qquad \widehat{\Sigma} = -\widehat{\Sigma}$$

where Γ' is the jump contour for the Airy model RH problem where A has its jumps. Now we define E in such a way that P matches with $P^{(\infty)}$ at ∂U_δ^-, by

$$E(\zeta) = \widehat{P}^{(\infty)}(\zeta)N^{-1}(-|s|g(\zeta))^{\frac{\sigma_3}{4}}. \tag{6.49}$$

Then E is analytic and it is straightforward, using the jump conditions for A and the asymptotic behavior (6.20), to check that P satisfies the following conditions,

(a) P is analytic in $\overline{U_\delta^-} \setminus \widehat{\Sigma}$,

(b) $P_+(\zeta) = P_-(\zeta)v_S(\zeta)$ as $\zeta \in \widehat{\Sigma} \cap U_\delta^-$,

(c) $P(\zeta)\widehat{P}^{(\infty)}(\zeta)^{-1} = I + \mathcal{O}(|s|^{-3/2})$ for $\zeta \in \partial U_\delta^-$, as $s \to -\infty$.

Using the symmetries in the RH problem for S, we can construct the parametrix near $\frac{1}{\sqrt{2}}$ easily using the parametrix near $-\frac{1}{\sqrt{2}}$. We define

$$P(\zeta) = \sigma_3 P(-\zeta)\sigma_3, \qquad \text{for } \zeta \in U_\delta^+ := -U_\delta^-, \tag{6.50}$$

so that

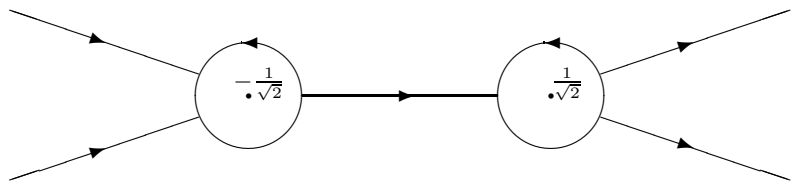

Figure 6.5 *The jump contour for R.*

(a) P is analytic in $\overline{U}_\delta^+ \setminus \widehat{\Sigma}$,

(b) $P_+(\zeta) = P_-(\zeta) v_S(\zeta)$ as $\zeta \in \widehat{\Sigma} \cap U_\delta^+$,

(c) $P(\zeta)\widehat{P}^{(\infty)}(\zeta)^{-1} = I + \mathcal{O}(|s|^{-3/2})$ for $\zeta \in \partial U_\delta^+$, as $s \to -\infty$.

RH problem for R

Now we define R by

$$R(\zeta; s) = \begin{cases} S(\zeta)\widehat{P}^{(\infty)}(\zeta)^{-1} & \text{for } \zeta \in \mathbb{C} \setminus U_\delta^\pm, \\ S(\zeta)P(\zeta)^{-1} & \text{for } \zeta \in U_\delta^\pm. \end{cases} \tag{6.51}$$

Then R satisfies the following RH conditions on a contour Σ_R as shown in Figure 6.5,

(a) $R : \mathbb{C} \setminus \Sigma_R \to \mathbb{C}^{2\times 2}$ is analytic on $\mathbb{C} \setminus \Sigma_R$,

(b) there exists $\gamma > 0$ such that R satisfies the following jump conditions,

$$R_+(\zeta) = R_-(\zeta)(I + \mathcal{O}(|s|^{-3/2})) \qquad \text{for } \zeta \in \partial U_\delta, \tag{6.52}$$

$$R_+(\zeta) = R_-(\zeta)(I + \mathcal{O}(e^{-\gamma|s|^{3/2}})) \qquad \text{for } \zeta \in \Sigma_R \setminus \partial U_\delta, \tag{6.53}$$

(c) $R(\zeta) = I + \mathcal{O}(\zeta^{-1})$ as $\zeta \to \infty$.

From these conditions it again follows that

$$R(\zeta) = I + \mathcal{O}(|s|^{-3/2}), \tag{6.54}$$

uniformly for $\zeta \in \mathbb{C} \setminus \Sigma_R$.

Proof of Theorem 6.1 (iii). It follows from (6.54), (6.37), and (6.48) that for $\zeta \in (-\frac{1}{\sqrt{2}} - \delta, -\frac{1}{\sqrt{2}} + \delta)$,

$$\Psi_{0,+}^{\mathrm{II}}(|s|^{1/2}\zeta; s) = (I + \mathcal{O}(|s|^{-3/2}))E(\zeta)A_+\left(-|s|g(\zeta)\right),$$

and by (6.9) and (6.21)

$$\Phi_0^{\mathrm{II}}(|s|^{1/2}\zeta; s) = (I + \mathcal{O}(|s|^{-3/2}))E(\zeta)\widehat{A}\left(-|s|g(\zeta)\right).$$

Set

$$\zeta = -\frac{1}{\sqrt{2}} - \frac{u}{g'(\frac{1}{\sqrt{2}})}|s|^{-1} = -\frac{1}{\sqrt{2}} + \frac{1}{2^{7/6}}u|s|^{-1}, \qquad u \in \mathbb{R}.$$

If we let $s \to -\infty$, we find

$$\lim_{s\to-\infty} E(-\frac{1}{\sqrt{2}})^{-1}\Phi_0^{\mathrm{II}}(\zeta; s) = \widehat{A}(u).$$

Using (6.10) and (6.22), we find that

$$\lim_{s\to\infty} \frac{1}{2^{7/6}|s|^{1/2}}\mathbb{K}^{\mathrm{crit,II}}\left(-\frac{1}{\sqrt{2}}|s|^{1/2} + \frac{1}{2^{7/6}}u|s|^{-1/2}, -\frac{1}{\sqrt{2}}|s|^{1/2} + \frac{1}{2^{7/6}}v|s|^{-1/2}; s\right) = \mathbb{K}^{\mathrm{edge}}(u, v).$$

Doing exactly the same calculations for ζ near $+\frac{1}{\sqrt{2}}$, we also find

$$\lim_{s\to\infty} \frac{1}{2^{7/6}|s|^{1/2}}\mathbb{K}^{\mathrm{crit,II}}\left(\frac{1}{\sqrt{2}}|s|^{1/2} - \frac{1}{2^{7/6}}u|s|^{-1/2}, \frac{1}{\sqrt{2}}|s|^{1/2} - \frac{1}{2^{7/6}}v|s|^{-1/2}; s\right) = \mathbb{K}^{\mathrm{edge}}(u, v).$$

□

6.5 Proof of Theorem 6.1 (iv) and (v)

We will use the asymptotic analysis done in Section 3.3 in order to prove part (iv) and part (v) of Theorem 6.1.

Proof of Theorem 6.1 (iv) We assume $s, t \in \mathbb{R}$ with $|s| \geq M$ for a sufficiently large constant $M > 0$. If we define Y as in Section 3.3 by (3.86), we can check using (3.53) and (6.19) that the kernel $\mathbb{K}^{\mathrm{crit,III}}$ is given by

$$\mathbb{K}^{\mathrm{crit,III}}(|s|^{1/3}u, |s|^{1/3}v; s, t) = \frac{1}{2\pi i|s|^{1/3}(u-v)}\begin{pmatrix}0 & 1\end{pmatrix}\widehat{Y}(v)^{-1}\widehat{Y}(u)\begin{pmatrix}1\\0\end{pmatrix}, \quad (6.55)$$

where we write $\widehat{Y}$ for the function defined by

$$\widehat{Y}(\zeta) = \begin{cases} Y_+(\zeta) & \text{for } \zeta > z_0, \\ Y_+(\zeta)\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}, & \text{for } \zeta < z_0, \end{cases} \quad (6.56)$$

In Chapter 3 we proved, see (3.124), that

$$R(\zeta) = I + \mathcal{O}(s^{-1}) \qquad \text{as } s \to \pm\infty, \tag{6.57}$$

where, for ζ close to $\hat{z}_0 = -\mathrm{sgn}(s)2.6^{1/3}$, R was defined by (see (3.120), (3.106), and (3.113)),

$$\begin{aligned} R(\zeta) &= S(\zeta)P(\zeta)^{-1} \\ &= \begin{pmatrix} 1 & 0 \\ \star & 1 \end{pmatrix} Y(\zeta) e^{|s|^{7/6} g(\zeta)\sigma_3} M(|s|^{7/9} f(\zeta))^{-1} E(\zeta)^{-1} \qquad (6.58) \\ &= \begin{pmatrix} 1 & 0 \\ \star & 1 \end{pmatrix} Y(\zeta) A(|s|^{7/9} f(\zeta))^{-1} E(\zeta)^{-1}. \qquad (6.59) \end{aligned}$$

This last step follows from (6.23) and Proposition 3.16. Here E is analytic near z_0, $\star$ denotes an entry independent of ζ which is unimportant for now, and f is a conformal mapping with (see (3.112), (3.92), and (3.102))

$$f(z_0) = 0, \qquad f'(z_0) = \left(\frac{3}{4}\right)^{4/9} + \mathcal{O}(|s|^{-2/3}) \quad \text{as } s \to \pm\infty. \tag{6.60}$$

Recall that z_0 is defined as the real zero of (3.94) and that $z_0 = \hat{z}_0 + \mathcal{O}(|s|^{-2/3})$ as $|s| \to \infty$. If we set

$$\zeta = z_0 + \left(\frac{4}{3}\right)^{4/9} \frac{u}{|s|^{7/9}},$$

(6.57) and (6.59) lead by (6.60) to

$$\lim_{s\to\pm\infty} E(z_0)^{-1} \begin{pmatrix} 1 & 0 \\ \star & 1 \end{pmatrix} Y_+ \left(z_0 + \left(\frac{4}{3}\right)^{4/9} \frac{u}{|s|^{7/9}} \right) = A_+(u). \tag{6.61}$$

By (6.56) and (6.21) we find

$$\lim_{s\to\pm\infty} E(z_0)^{-1} \begin{pmatrix} 1 & 0 \\ \star & 1 \end{pmatrix} \widehat{Y} \left(z_0 + \left(\frac{4}{3}\right)^{4/9} \frac{u}{|s|^{7/9}} \right) = \widehat{A}(u). \tag{6.62}$$

Now we can use (6.55) and (6.22) to conclude that

$$\begin{aligned} \lim_{s\to\pm\infty} \left(\frac{4}{3}\right)^{4/9} \frac{1}{|s|^{4/9}} \mathbb{K}^{\mathrm{crit,III}} \Bigg(& |s|^{1/3} z_0 + \left(\frac{4}{3}\right)^{4/9} \frac{u}{|s|^{4/9}}, \\ & |s|^{1/3} z_0 + \left(\frac{4}{3}\right)^{4/9} \frac{v}{|s|^{4/9}}; s, t \Bigg) = \mathbb{K}^{\mathrm{edge}}(u, v). \end{aligned}$$

This proves Theorem 6.1 (iv). □

Proof of Theorem 6.1 (v) We assume $s, t \in \mathbb{R}$ with $s \leq -M$ for a sufficiently large constant $M > 0$. Then $z_0 > 2$, with z_0 defined as the real zero of (3.94), so that $|\zeta| < 1$ is not contained in the small neighborhood U_δ surrounding $\hat{z}_0$.

In Chapter 3 we proved that, for $\zeta \in \mathbb{C} \setminus U_\delta$,

$$\begin{aligned} R(\zeta) &= S(\zeta)P^{(\infty)}(\zeta)^{-1} \\ &= \begin{pmatrix} 1 & 0 \\ \star & 1 \end{pmatrix} Y(\zeta) e^{|s|^{7/6} g(\zeta)\sigma_3} P^{(\infty)}(\zeta)^{-1} \\ &= I + \mathcal{O}(|s|^{-1}) \qquad \text{as } s \to -\infty. \end{aligned} \tag{6.63}$$

Here g is given by (3.92) and (3.95). Now for the particular choice

$$\zeta = \frac{\pi u}{|g'_+(0)|\,|s|^{7/6}},$$

(6.63) implies together with (6.56) that

$$\lim_{s \to -\infty} e^{|s|^{7/6} g_+(0)\sigma_3} P^{(\infty)}(0)^{-1} \begin{pmatrix} 1 & 0 \\ \star & 1 \end{pmatrix} \widehat{Y}(\zeta) = e^{-i\pi u \sigma_3} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

After simple calculation, we find using (6.55) that

$$\begin{aligned} &\lim_{s \to -\infty} \frac{\pi}{|g'_+(0)|\,|s|^{5/6}} \mathbb{K}^{\mathrm{crit,III}} \left(\frac{\pi u}{|g'_+(0)|\,|s|^{5/6}}, \frac{\pi v}{|g'_+(0)|\,|s|^{5/6}} \right) \\ &\qquad = \frac{e^{i\pi(u-v)} - e^{-i\pi(u-v)}}{2i\pi(u-v)} = \mathbb{K}^{\mathrm{bulk}}(u, v). \end{aligned}$$

□

Conclusion and outlook for future research

We conclude with a list containing the main new results we proved in this thesis. These results concern random matrices, orthogonal polynomials, and Painlevé equations.

We obtained the following results in random matrix theory:

- universality of a double scaling limit in unitary random matrix ensembles near type II singular points with quadratic vanishing of the limiting mean eigenvalue density (Theorem 1.1),
- universality of a double scaling limit in multi-critical unitary random matrix ensembles where a spectral singularity coincides at the origin with a type II singular point with quadratic vanishing (Theorem 2.2),
- universality of double scaling limits near type III singular (edge) points where the limiting mean eigenvalue vanishes with a power 5/2 (Theorem 4.7),
- universality of a double scaling limit in unitary random matrix ensembles where the eigenvalues are restricted to be positive, in the critical case where the soft edge approaches the hard edge (Theorem 5.1).

In the theory of orthogonal polynomials, we obtained the following two results:

- asymptotics in a double scaling limit for the recurrence coefficients of orthogonal polynomials with respect to the weight $|x|^{2\alpha}e^{-NV(x)}$, in the presence of a type II singular point with quadratic vanishing (Theorem 2.7),
- asymptotics in double scaling limits for the recurrence coefficients of orthogonal polynomials with respect to the weight $e^{-nV_{s,t}(x)}$, in the presence of a type III singular point where vanishing with a power 5/2 occurs (Theorem 4.11).

Finally we proved the following results about Painlevé equations,

- the Hastings-McLeod solution of the general Painlevé II equation has no poles on the real line for $\alpha > -1/2$ (Theorem 2.1),
- there exists a real solution to the P_I^2 equation with no poles on the real line (Theorem 3.1 (i)); in addition, we compute asymptotics for this pole-free solution (Theorem 3.1 (ii)).

Several problems related to topics covered by this thesis, remain open. We first discuss some open problems concerning random matrices.

- In Chapter 1 and Chapter 2, we obtained a double scaling limit of the two-point kernel near singular points of type II where the limiting mean eigenvalue density vanishes quadratically. If the vanishing is of order $2m$ with $m > 1$, the limiting kernels are different. It is believed [2] that they will be built out of functions associated with a special solution of the m-th member of the Painlevé II hierarchy, but this is not proven rigorously yet.
- A similar problem holds near type III singular points. In Chapter 4, we established universality near points where the limiting mean eigenvalue density vanishes with a power $5/2$, but no similar results have been proven when there is higher order vanishing, with a power $2m + \frac{1}{2}$ for $m > 1$. Here a limiting kernel associated with the $2m$-th member of the Painlevé I hierarchy is expected [18].
- Recently the case of a type I singular point outside the spectrum has also been studied [43], but no rigorous results have been obtained yet. It would be interesting to study double scaling limits of the two-point kernel in this case as well.
- Near regular endpoints, fluctuations of the largest eigenvalue of large random matrices are given by the Tracy-Widom distribution. It is not known yet how these fluctuations can be described near singular endpoints.
- The same problem is open in random matrix ensembles for which the (regular or singular) right edge point of the spectrum coincides with a spectral singularity at the origin.

Also in the theory of Painlevé equations, many interesting questions remain open. We mention two problems which are closely related with this thesis.

- It is not proven that there exist real pole-free solutions of higher members of the Painlevé II hierarchy. However it should be possible to generalize the techniques used in Chapter 2 to prove this. A natural candidate for these pole-free solutions are the ones that are expected to appear in random matrix theory near type II singular points with higher order vanishing.

- It is known that no real solution of the Painlevé I equation is pole-free on the real line. However we expect that all even members of the Painlevé I hierarchy have a pole-free solution since the arguments given in Chapter 3 should be extendable to all even members of the hierarchy. This is also supported by the expectation that special solutions of the even members of the Painlevé I hierarchy arise near singular endpoints in random matrix models.

Nederlandse samenvatting

Inleiding

We gebruiken in deze thesis Riemann-Hilbert problemen met als doel resultaten te vinden die zich situeren in de theorie van random matrices, orthogonale veeltermen en Painlevé vergelijkingen.

Een eerste techniek die we gebruiken is de Deift/Zhou steilste-afdalingsmethode toegepast op het Fokas-Its-Kitaev Riemann-Hilbert probleem dat orthogonale veeltermen karakteriseert. Met deze methode vinden we universaliteit van dubbelgeschaalde limieten in bepaalde kritieke random matrix ensembles. Daarenboven vinden we de asymptotiek voor recursiecoëfficiënten van orthogonale veeltermen die verband houden met de bestudeerde random matrix ensembles.

Een tweede techniek is deze van het 'vanishing lemma'. Hierdoor kunnen we de oplosbaarheid aantonen van sommige Riemann-Hilbert problemen die te maken hebben met Painlevé vergelijkingen. De oplosbaarheid van deze Riemann-Hilbert problemen leidt tot het bestaan van poolvrije oplossingen van Painlevé vergelijkingen.

Random matrices

Random matrices werden door Wishart geïntroduceerd in wiskundige statistiek in 1928 [102]. In de jaren '50 en '60 begonnen ook wiskundige natuurkundigen als Wigner, Dyson en Mehta random matrices te bestuderen [42, 82, 101]. Gedurende de laatste tientallen jaren zijn random matrices opgedoken in verscheidene wiskundige en natuurkundige gebieden.

- Er zijn opmerkelijke gelijkenissen tussen eigenwaarden van random matrices en de niet-triviale nulpunten van de Riemann-zeta functie [84, 58, 90].

- Partitiefuncties voor random matrix ensembles zijn genererende functies voor bepaalde combinatorische grootheden die van belang zijn in kwantumveldentheorie [19, 38, 56].

- Er zijn verbanden tussen random matrices en Brownse bewegingen, random permutaties, random betegelingen, last passage percolation, polynucleaire groeimodellen en andere modellen in statistische fysica [6, 7, 8, 9, 10, 64, 87].
- Bepaalde wachttijden voor bussen en voor het instappen in vliegtuigen kunnen gemodelleerd worden door middel van random matrices [5, 70, 4].

Een random matrix ensemble bestaat uit een ruimte van matrices uitgerust met een kansmaat. De statistische eigenschappen van de eigenwaarden van random matrices zijn van groot belang. Een opmerkelijk feit is dat lokale eigenschappen van de eigenwaarden dezelfde zijn voor grote klassen van random matrix ensembles, terwijl de globale eigenschappen sterk afhangen van de gekozen kansmaat van het ensemble. Dit belangrijke fenomeen noemt men *universaliteit*. We zullen ons beperken tot de studie van Hermitische matrices, hoewel ensembles van unitaire, reële symmetrische en quaternionische zelf-duale matrices ook te vinden zijn in de literatuur. Een algemene referentie voor random matrices is het boek van Mehta [83]. Een overzicht van de geschiedenis van random matrices is ook te vinden in [51]. In [27] worden random matrices bekeken vanuit een perspectief dat heel dicht aanleunt bij het onze, gebruik makende van een Riemann-Hilbert aanpak.

Het grondigst bestudeerde ensemble van Hermitische matrices is allicht het Gaussish Unitaire Ensemble (GUE). De kansmaat op de ruimte van Hermitische $n \times n$ matrices in het GUE is

$$\frac{1}{Z_n} \exp(-\operatorname{Tr} M^2) dM,$$

waarbij Z_n een normalisatieconstante is en dM de Lebesgue maat

$$dM = \prod_{i=1}^{n} d\, M_{ii} \prod_{i<j} d \operatorname{Re} M_{ij} d \operatorname{Im} M_{ij}.$$

In dit ensemble zijn de elementen van de random matrices onafhankelijke (de eis dat de matrices Hermitisch zijn buiten beschouwing gelaten) Gaussische toevalsveranderlijken en de kansmaat is invariant onder conjugatie met unitaire matrices. Deze laatste eigenschap vinden we terug in een grote klasse van random matrix ensembles die we *unitaire ensembles* noemen. Andere unitair invariante verdelingen zijn bvb.

$$\frac{1}{Z_n} \exp(-n \operatorname{Tr} V(M)) dM, \tag{N.1}$$

waarbij V een reëel analytische functie is die voldoende snel groeit naar oneindig toe. De factor n in de exponent blijkt handig te zijn wanneer de grootte n van de matrices toeneemt. Door deze factor bestaat de *limiet van de gemiddelde eigenwaardedichtheid* als $n \to \infty$. In het Gaussische geval waar $V(x) = x^2$ wordt

deze dichtheid gegeven door de Wigner halve-cirkelwet. De limiet van de gemiddelde eigenwaardeverdeling, die sterk afhangt van de gekozen potentiaal V, kan gekarakteriseerd worden als de unieke maat die de logaritmische energie in extern veld V

$$I_V(\mu) = \iint \log \frac{1}{|x-y|} d\mu(x) d\mu(y) + \int V(x) d\mu(x), \tag{N.2}$$

minimaliseert onder alle kansmaten μ op $\mathbb{R}$. De eerste term in (N.2) drukt een afstoting tussen de eigenwaarden uit, terwijl de tweede term de eigenwaarden verhindert om uit te dijen naar oneindig. Door de factor n in de exponent van (N.1) is er een juiste balans tussen deze twee verschijnselen zodat de limietverdeling gedragen wordt op een compacte verzameling.

Bruikbare statistische informatie over de eigenwaarden is bevat in de zogenaamde *tweepuntskern*

$$K_n(x,y) = e^{-\frac{n}{2}V(x)} e^{-\frac{n}{2}V(y)} \sum_{k=0}^{n-1} p_k^{(n)}(x) p_k^{(n)}(y), \tag{N.3}$$

waarbij we $p_k^{(n)}$ noteren voor de orthonormale veelterm van graad k tegenover het gewicht $e^{-nV(x)}$ op $\mathbb{R}$. Dit verband tussen random matrices en orthogonale veeltermen geldt ook voor meer algemene ensembles dan degene met een kansmaat van de vorm (N.1), zie [83]. De limiet van de gemiddelde eigenwaardedichtheid ψ_V als $n \to \infty$ kan teruggevonden worden uit de kern door de formule

$$\psi_V(x) = \lim_{n\to\infty} \frac{1}{n} K_n(x,x)$$

en de m-punts correlatiefuncties $\mathcal{R}_m$ worden gegeven door

$$\mathcal{R}_m(x_1, \ldots, x_m) = \det(K_n(x_i, x_j))_{1 \le i,j \le m}. \tag{N.4}$$

'Gap'-waarschijnlijkheden en de verdeling van de grootste eigenwaarde kunnen ook uitgedrukt worden in functie van K_n, en de gezamenlijke eigenwaardedichtheid van de eigenwaarden is gegeven door

$$\frac{1}{n!} \det(K_n(x_i, x_j))_{1 \le i,j \le n}.$$

Het blijkt dat lokaal geschaalde limieten van de tweepuntskern nabij een punt x^* universeel zijn: de limietkern hangt af van het schalingsregime, maar verder niet van de potentiaal V of van de positie van x^*. De twee reguliere schalingsregimes zijn de volgende,

- bulk schaling, rond punten waar de limietdichtheid positief is,

- randschaling, rond randpunten van het spectrum (dit is de drager van de limiet van de gemiddelde eigenwaardeverdeling).

In het geval waar x^* in het bulk-gebied van het spectrum ligt, is de limietkern de *sinuskern*,

$$\lim_{n\to\infty} \frac{1}{n\psi_V(x^*)} K_n\left(x^* + \frac{u}{n\psi_V(x^*)}, x^* + \frac{v}{n\psi_V(x^*)}\right) = \mathbb{K}^{\text{bulk}}(u,v),$$

waar $\mathbb{K}^{\text{bulk}}$ gegeven wordt door

$$\mathbb{K}^{\text{bulk}}(u,v) = \frac{\sin \pi(u-v)}{\pi(u-v)}. \tag{N.5}$$

Dit is bewezen door Bleher en Its [13] in het geval waar V een vierdegraadsveelterm is en door Deift en anderen [27, 32, 33] voor algemene reëel analytische V, zie ook [86]. Nabij een randpunt x^* waar de dichtheid als een wortel naar 0 gaat, duikt de *Airykern* op in de limiet voor grote n, zie bvb. [30],

$$\lim_{n\to\infty} \frac{1}{(cn)^{2/3}} K_n\left(x^* + \frac{u}{(cn)^{2/3}}, x^* + \frac{v}{(cn)^{2/3}}\right) = \mathbb{K}^{\text{edge}}(u,v),$$

waar c een constante is die afhangt van V en $\mathbb{K}^{\text{edge}}$ wordt gegeven door

$$\mathbb{K}^{\text{edge}}(u,v) = \frac{\operatorname{Ai}(x)\operatorname{Ai}'(y) - \operatorname{Ai}(y)\operatorname{Ai}'(x)}{x-y}. \tag{N.6}$$

De Airy functie Ai is een speciale functie gekarakteriseerd door de differentiaalvergelijking $\operatorname{Ai}_{xx} = x\operatorname{Ai}$ en de asymptotiek

$$\operatorname{Ai}(x) \sim \frac{e^{-\frac{2}{3}x^{3/2}}}{2\sqrt{\pi}x^{1/4}} \qquad \text{als } x \to +\infty.$$

Merk op dat er een verschil in schaling is tussen de bulk, waar de schaling gebeurt met $1/n$, en de rand, waar de schaling gebeurt met $1/n^{2/3}$. Deze schalingen komen overeen met de gemiddelde afstand tussen opeenvolgende eigenwaarden, die van de orde $1/n$ is in de bulk en van de orde $1/n^{2/3}$ aan de rand.

In kritieke random matrix ensembles kunnen *singuliere punten* voorkomen. We onderscheiden hier nog drie andere schalingsregimes afhankelijk van het type van het singuliere punt. Singuliere punten worden als volgt geclassificeerd, zie bvb. [33, 74],

- type I singuliere punten, dit zijn singuliere punten gelegen buiten de drager van de limiet van de gemiddelde eigenwaardeverdeling,
- type II singuliere punten, dit zijn punten in het inwendige van het spectrum waar de limietdichtheid 0 wordt,

- type III singuliere punten, dit zijn randpunten van het spectrum waar de limietdichtheid sneller naar 0 gaat dan in het reguliere gevan.

Nabij singuliere punten gelden de bovenvermelde universaliteitsresultaten niet meer en zijn de limietkernen niet meer gegeven door de sinuskern en de Airykern. Desalniettemin zouden lokaal geschaalde limieten in zekere zin universeel moeten blijven. Kritieke random matrix ensembles met singuliere punten hebben de eigenschap dat het aantal intervallen in de drager kan veranderen wanneer we het extern veld V lichtjes wijzigen. De aanwezigheid van een type II singulier punt wijst bijvoorbeeld op het sluiten of openen van een 'gap', of in andere woorden de overgang waar twee intervallen samenvloeien tot een enkel interval. Ook type I en type III singuliere punten wijzen op een mogelijke verandering in het aantal intervallen waar de eigenwaarden zich bevinden. Om deze overgangen ten volle te beschrijven, is het nuttig om het gewicht $w_0 = e^{-nV}$ te deformeren tot een gewicht w_t dat afhangt van een parameter t. In *dubbelgeschaalde limieten* waar we $n \to \infty$ laten gaan en ter zelfder tijd $t \to 0$ aan een gepaste snelheid, zouden we limietkernen moeten vinden die afhangen van een deformatieparameter.

Unitaire random matrix ensembles van de vorm (1) kunnen aangepast worden zodat er een zogenaamde spectrale singulariteit in de oorsprong toegevoegd wordt. We krijgen dan kansmaten (die nog steeds unitair invariant zijn) van de vorm

$$\frac{1}{Z_n}|\det M|^{2\alpha}\exp(-n\operatorname{Tr}V(M))dM, \qquad \alpha > -1/2. \tag{N.7}$$

De factor $|\det M|^{2\alpha}$ geeft aanleiding tot een extra afstoting tussen de eigenwaarden dicht bij de oorsprong als $\alpha > 0$ en tot een aantrekking als $\alpha < 0$. Deze factor verandert de limiet van de gemiddelde eigenwaardeverdeling niet, maar we krijgen wel een ander lokaal gedrag van de eigenwaarden dicht bij 0. De tweepuntskernen die horen bij zulke random matrix ensembles kunnen uitgedrukt worden als in (3), maar dan in functie van de veeltermen die orthogonaal zijn ten opzichte van het gewicht $|x|^{2\alpha}e^{-nV(x)}$. Door de spectrale singulariteit in de oorsprong zijn lokaal geschaalde limieten nu anders rond de oorsprong. Bijvoorbeeld, als de limiet van de gemiddelde eigenwaardedichtheid positief is rond 0, leidt dit (in plaats van tot de sinuskern) tot de Bessel kern [77] die uitgedrukt kan worden als

$$\mathbb{J}^o_\alpha(u,v) = \pi\sqrt{u}\sqrt{v}\frac{J_{\alpha+\frac{1}{2}}(\pi u)J_{\alpha-\frac{1}{2}}(\pi v) - J_{\alpha+\frac{1}{2}}(\pi v)J_{\alpha-\frac{1}{2}}(\pi u)}{2(u-v)}. \tag{N.8}$$

Hier zijn $J_{\alpha\pm\frac{1}{2}}$ Bessel functies van orde $\alpha \pm \frac{1}{2}$.

In de laatste soort unitaire ensembles die we beschouwen, zijn de eigenwaarden beperkt tot $[0, +\infty)$ door een kansmaat te leggen op de positief-definiete Hermitische matrices,

$$\frac{1}{Z_n}(\det M)^\alpha\exp(-n\operatorname{Tr}V(M))dM, \qquad \alpha > -1. \tag{N.9}$$

De bijhorende orthogonale veeltermen zijn in dit geval orthogonaal op $[0, +\infty)$ ten opzichte van het gewicht $x^\alpha e^{-nV(x)}$. Lokaal geschaalde limieten van de tweepuntskern zijn verschillend rond 0 omwille van twee redenen. Ten eerste is er opnieuw de afstoting of aantrekking van de eigenwaarden door de factor $(\det M)^\alpha$, en ten tweede is 0 nu een *harde rand* waar de eigenwaarden niet voorbij kunnen. Als de limiet van de gemiddelde eigenwaardedichtheid positief is rond de oorsprong, is de limietkern opnieuw opgebouwd uit Bessel functies [99].

Universaliteit komt niet enkel voor in unitaire random matrix ensembles. Recent zijn er ook universaliteitsresultaten gevonden voor andere random matrix ensembles, zoals orthogonale en symplectische ensembles, zie [29, 30].

Painlevé vergelijkingen

In het begin van de 20e eeuw wou Painlevé gewone tweede orde differentiaalvergelijkingen van de vorm

$$u_{xx} = F(x, u, u_x) \tag{N.10}$$

classificeren, waarbij F een rationale functie is. Oplossingen van zulke differentiaalvergelijkingen kunnen singulariteiten hebben, waarvan sommige vast (onafhankelijk van de gekozen oplossing, vastgelegd door de vergelijking zelf) en sommige beweegbaar (afhankelijk van de gekozen oplossing). Painlevé had als doel een classificatie te vinden van alle niet-lineaire gewone differentiaalvergelijkingen van de vorm (10) waarvoor de beweegbare singulariteiten beperkt zijn tot polen. Het bleek dat elk van deze differentiaalvergelijkingen kon herleid worden, ofwel tot een reeds gekende differentiaalvergelijking, ofwel tot een vergelijking uit een lijst van zes differentiaalvergelijkingen, dewelke nu gekend zijn als de Painlevé vergelijkingen. Oplossingen hiervan worden Painlevé transcendenten genoemd.

De Painlevé vergelijkingen zijn integreerbaar, wat betekent dat ze uitgedrukt kunnen worden als de compatibiliteitsconditie van een stelsel van lineaire differentiaalvergelijkingen, het Lax paar. Horend bij dit Lax paar, hebben Painlevé vergelijkingen Riemann-Hilbert problemen waarin de zogenaamde monodromiegegevens van het lineaire stelsel bevat zijn. De Painlevé vergelijking karakteriseert de monodromie-bewarende deformaties van het lineaire stelsel [44, 62]. Veralgemeningen van de Lax paren (die bekomen worden door de graad van de veeltermcoëfficiënten van het lineaire stelsel te verhogen) hebben hogere orde differentiaalvergelijkingen als compatibiliteitscondities. Deze hogere orde differentiaalvergelijkingen worden samen met de Painlevé vergelijkingen zelf verzameld in de Painlevé hiërarchieën.

Buiten het feit dat Painlevé vergelijkingen een rijke theorie vormen vanuit een zuiver wiskundig oogpunt, kennen ze ook een verscheidenheid aan toepassingen, bvb. in

- random matrix theorie:
  - 'gap'-waarschijnlijkheden en de verdeling van de grootste eigenwaarde kunnen uitgedrukt worden door middel van Painlevé transcendenten [97, 103],
  - in kritieke ensembles met type II of type III singulariteiten wordt het lokale gedrag van de eigenwaarden rond het singuliere punt beschreven met functies die te maken hebben met Painlevé vergelijkingen [2, 12, 14, 18],
- combinatoriek: op een vergelijkbare manier als voor de grootste eigenwaarde van unitaire random matrices, worden schommelingen in de lengte van een langst stijgende deelrij van grote random permutaties uitgedrukt met een speciale oplossing van de Painlevé II vergelijking [6],
- Hamiltoniaanse verstoringen van hyperbolische vergelijkingen worden in bepaalde gevallen beschreven met Painlevé vergelijkingen [39, 40],
- bepaalde modellen in statistische mechanica, zoals het Ising model, hebben een verband met Painlevé vergelijkingen [11].

Een uitgebreid overzicht, met ook geschiedenis en toepassingen, over Painlevé vergelijkingen en bijhorende Riemann-Hilbert problemen is terug te vinden in het zeer recente boek [45].

Riemann-Hilbert problemen

"Het Riemann-Hilbert probleem" kent haar oorsprong rond 1900, toen het verwees naar het 21e probleem van Hilbert uit zijn befaamde lijst van 23 problemen. Het was pas later dat de term "Riemann-Hilbert (RH) probleem" gebruikt begon te worden in de context waarin wij deze gebruiken, voor een hele klasse van grenswaardeproblemen. Voor ons is een RH probleem typisch als volgt. We zoeken naar een (scalaire of matrixwaardige) functie die aan voorwaarden van de volgende vorm voldoet:

(a) de functie is analytisch buiten een gegeven contour in het complexe vlak,

(b) de grenswaarden van de functie voldoen aan voorgeschreven sprongvoorwaarden over het contour,

(c) enkele extra voorwaarden moeten voldaan zijn, bvb. op oneindig, om een unieke oplossing van het probleem te hebben.

In de jaren '70 en '80 werden RH technieken gebruikt om integreerbare systemen op te lossen. Integreerbare systemen zijn, zoals hierboven reeds vermeld,

niet-lineaire differentiaalvergelijkingen die kunnen uitgedrukt worden als de compatibiliteitsconditie van een stelsel lineaire differentiaalvergelijkingen. Dit lineaire stelsel vergelijkingen wordt het Lax paar van de onderliggende differentiaalvergelijking genoemd. Er is een RH probleem dat oplossingen van het Lax paar karakteriseert, en dit RH probleem bevat ook informatie over de oorspronkelijke niet-lineaire differentiaalvergelijking. Een belangrijk onderwerp in dit opzicht is de oplosbaarheid van RH problemen. In bepaalde gevallen is het bestaan van een oplossing equivalent met een *vanishing lemma* [48, 49, 104], dat zegt dat een gerelateerd homogeen RH probleem enkel de triviale 0-oplossing heeft. Dit gerelateerde RH probleem heeft doorgaans dezelfde sprongvoorwaarden als het oorspronkelijke, maar een verschillend gedrag op oneindig. Het idee achter deze techniek is dat een RH probleem equivalent is met een integraalvergelijking waarmee een bepaalde operator geassocieerd kan worden. De oplosbaarheid van het RH probleem volgt dan van het feit dat deze operator bijectief is. Als de beschouwde operator Fredholm is met index 0, is dit equivalent met de trivialiteit van de kern van de operator, hetgeen op zijn beurt kan vertaald worden naar een vanishing lemma.

In het begin van de jaren '90 introduceerden Fokas, Its en Kitaev [47] een 2×2 matrixwaardig RH probleem waarvoor de unieke oplossing gegeven wordt in functie van orthogonale veeltermen. Het probleem is hier om, gegeven een gewicht w, een 2×2 matrixwaardige functie $Y = Y_n$ te vinden zodat aan volgende voorwaarden voldaan is:

(a) Y is analytisch in $\mathbb{C} \setminus \mathbb{R}$,

(b) $Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix}$ voor $x \in \mathbb{R}$,

(c) $Y(z) = \big(I + O(z^{-1})\big) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$ als $z \to \infty$,

waarbij $Y_+(x)$ ($Y_-(x)$) de limiet is wanneer we $x \in \mathbb{R}$ benaderen vanuit het bovenhalfvlak (benedenhalfvlak). De oplossing van dit RH probleem wordt gegeven door

$$Y(z) = \begin{pmatrix} \kappa_n^{-1} p_n(z) & \kappa_n^{-1} \dfrac{1}{2\pi i} \displaystyle\int_{\mathbb{R}} \dfrac{p_n(s)w(s)}{s-z} ds \\ -2\pi i \kappa_{n-1} p_{n-1}(z) & -\kappa_{n-1} \displaystyle\int_{\mathbb{R}} \dfrac{p_{n-1}(s)w(s)}{s-z} ds \end{pmatrix}, \tag{N.11}$$

waar $p_n(x) = \kappa_n x^n + \cdots$ de n-de graads orthonormale veelterm is tegenover het gewicht w op $\mathbb{R}$, zie ook [27]. Voor de keuze $w = |x|^{2\alpha} e^{-nV(x)}$ kan de correlatiekern voor random matrix ensembles van de vorm (7) ook uitgedrukt worden in functie

van de RH oplossing Y,

$$\begin{aligned}&K_{n,N}(x,y)\\&=|x|^{\alpha}e^{-\frac{1}{2}NV(x)}|y|^{\alpha}e^{-\frac{1}{2}NV(y)}\frac{1}{2\pi i(x-y)}\begin{pmatrix}0 & 1\end{pmatrix}Y_{\pm}^{-1}(y)Y_{\pm}(x)\begin{pmatrix}1\\0\end{pmatrix},\end{aligned}$$

hetgeen het belang van dit RH probleem in random matrix theorie aangeeft.

Nog in het begin van de jaren '90 ontwikkelden Deift en Zhou [36] een krachtige werkwijze om de asymptotiek te vinden voor oplossingen van matrix RH problemen die afhangen van een parameter die naar oneindig gaat. Deze *Deift/Zhou steilste-afdalingsmethode* bestaat uit een aantal transformaties van het RH probleem, om uiteindelijk sprongmatrices te bekomen die dicht bij de eenheidsmatrix liggen. Ze kan gezien worden als een veralgemening van de klassieke steilste-afdalingsmethode om bepaalde integralen te benaderen. In plaats van een integraal moet hier de oplossing van een RH probleem of, equivalent, de oplossing van een matrix-integraalvergelijking benaderd worden.

Deift, Kriecherbauer, McLaughlin, Venakides en Zhou [27, 32, 33] gebruikten de Deift/Zhou steilste-afdalingsmethode om de asymptotiek te bepalen voor het Fokas-Its-Kitaev RH probleem dat orthogonale veeltermen op de reële rechte karakteriseert. Dit leidt ook tot asymptotiek voor de correlatiekern in unitaire random matrix ensembles en tot universele geschaalde limieten van deze kern. De technieken die in deze papers [27, 32, 33] geintroduceerd werden, hebben intussen geleid tot diverse resultaten betreffende de asymptotiek van orthogonale veeltermen en random matrices, zie [28]. Wij maken in de thesis ook gebruik van deze methode om universaliteit in kritieke random matrix ensembles te bekomen. We geven een kort overzicht van de ideeën achter deze analyse.

Het vertrekpunt is het RH probleem voor orthogonale veeltermen tegenover een exponentieel gewicht dat afhangt van een parameter n. Men wil dan de asymptotiek vinden voor de oplossing van dit RH probleem als $n \to \infty$, waaruit dan ook de asymptotiek voor de orthogonale veeltermen en de correlatiekern voor de eigenwaarden volgen. De strategie is om een 'makkelijker' equivalent RH probleem te vinden waarvoor we de oplossing asymptotisch kunnen benaderen. Om een zulk RH probleem te vinden, moeten we verschillende transformaties doorvoeren. Zonder in detail te gaan, geven we de belangrijkste ideeën achter deze transformaties.

De eerste transformatie, de normalisatie van het RH probleem, zorgt voor een geschikt gedrag op oneindig en leidt tot een RH probleem met een oplossing die naar de eenheidsmatrix convergeert naar oneindig toe. De geschikte manier om dit te doen, is door de constructie van een zogenaamde g-functie die te maken heeft met een evenwichtsprobleem. Door deze transformatie worden oscillerende sprongen gecreëerd.

In de tweede transformatie wordt het sprongcontour vervormd tot een lensvormig contour om de oscillerende sprongen om te vormen tot exponentieel dalende

sprongen. Als we even deze exponentieel kleine sprongen en kleine omgevingen van enkele speciale punten buiten beschouwing laten, kan de RH oplossing nu expliciet geconstrueerd worden. Dit noemt de constructie van de 'outside parametrix'.

De outside parametrix bepaalt het asymptotisch gedrag van de RH oplossing buiten kleine omgevingen van de speciale punten, waar het lokale gedrag bepaald wordt door lokale parametrices die moeten geconstrueerd worden. Deze kunnen, in de meeste gevallen, geconstrueerd worden met behulp van Airy functies.

De constructie van parametrices leidt tot een laatste RH transformatie die resulteert in een RH probleem dat genormaliseerd is op oneindig en dat sprongen heeft die uniform dicht bij de eenheidsmatrix liggen als $n \to \infty$. Voor een dergelijk RH probleem is het gekend dat de oplossing ook dicht bij de eenheidsmatrix ligt. Als we nu deze opeenvolging van transformaties omkeren, vinden we de asymptotiek terug van het RH probleem voor orthogonale veeltermen.

In kritieke gevallen kunnen de lokale parametrices niet geconstrueerd worden met Airy functies. Dan moeten de parametrices gebouwd worden met andere soorten functies, zoals bvb. functies die te maken hebben met Painlevé vergelijkingen. Dit is een van de belangrijkste onderwerpen van deze thesis.

We verwijzen naar [52, 27] voor meer informatie over RH problemen en naar [59] voor een historisch overzicht.

N.1 Universaliteit rond een inwendig singulier punt

In het eerste hoofdstuk van de thesis beschouwen we unitaire random matrix ensembles van $n \times n$ Hermitische matrices met een kansverdeling van de vorm

$$Z_{n,N}^{-1} \exp(-N \operatorname{Tr} V(M)) dM. \tag{N.12}$$

Hierbij veronderstellen we een kritieke potentiaal V zodat de limiet van de gemiddelde eigenwaardedichtheid (als $n, N \to \infty$ op een zodanige manier dat $n/N \to 1$) kwadratisch 0 wordt in een inwendig punt van het spectrum. Dit geval is reeds bestudeerd door Bleher en Its [14] in het geval waar V een even vierdegraadsveelterm is,

$$V(x) = \frac{g}{4}x^4 + \frac{t}{2}x^2, \qquad g > 0, \qquad t = t_c = -2\sqrt{g}.$$

De dichtheid ψ_V heeft dan een dubbel nulpunt in de oorsprong. Bleher en Its beschouwen een dubbelgeschaalde limiet waar t afhangt van n en convergeert naar t_c als $n \to \infty$, op een zodanige manier dat $n^{2/3}(t - t_c)$ constant blijft. Met betrekking tot ons model is dit equivalent met het nemen van een dubbelgeschaalde limiet waar $n, N \to \infty$ zodat

$$\lim_{n,N\to\infty} n^{2/3} \left(\frac{n}{N} - 1\right) \tag{N.13}$$

bestaat. Bleher en Its vonden een één-parameterfamilie $\mathbb{K}^{\mathrm{crit,II}}(u,v;s)$ van limietkernen, afhankelijk van $s \in \mathbb{R}$, waarvoor het volgende geldt,

$$\lim_{n,N\to\infty} \frac{1}{(cn)^{1/3}} K_{n,N}\left(\frac{u}{(cn)^{1/3}}, \frac{v}{(cn)^{1/3}}\right) = \mathbb{K}^{\mathrm{crit,II}}(u,v;s), \tag{N.14}$$

waarbij s evenredig is met de waarde van de limiet (N.13).

Hastings-McLeod oplossing van Painlevé II

De kritieke kernen $\mathbb{K}^{\mathrm{crit,II}}(u,v;s)$ zijn opgebouwd uit zogenaamde ψ-functies horend bij de Hastings-McLeod oplossing van de Painlevé II vergelijking [57]. We beschouwen zoals in [44] de volgende lineaire differentiaalvergelijking voor een 2-vector (of 2×2 matrix) $\Psi = \Psi(\zeta;s)$,

$$\frac{d}{d\zeta}\Psi(\zeta;s) = \begin{pmatrix} -4i\zeta^2 - i(s+2q^2) & 4\zeta q + 2ir \\ 4\zeta q - 2ir & 4i\zeta^2 + i(s+2q) \end{pmatrix} \Psi(\zeta;s) \tag{N.15}$$

en

$$\frac{\partial}{\partial s}\Psi(\zeta;s) = \begin{pmatrix} -i\zeta & q \\ q & i\zeta \end{pmatrix} \Psi(\zeta;s). \tag{N.16}$$

De compatibiliteitsconditie voor het Lax paar (N.15)–(N.16) is dat $q = q(s)$ voldoet aan de Painlevé II vergelijking $q'' = sq + 2q^3$ en dat $r = r(s) = q'(s)$. We bekijken hier het geval waar $q(s)$ de Hastings-McLeod oplossing van Painlevé II is, dewelke gekarakteriseerd wordt door de asymptotische voorwaarde

$$q(s) = \mathrm{Ai}(s)(1+o(1)) \qquad \text{as } s \to +\infty.$$

De kritieke kernen $\mathbb{K}^{\mathrm{crit,II}}$ zijn nu van de volgende vorm

$$\mathbb{K}^{\mathrm{crit,II}}(u,v;s) = \frac{-\Phi_1(u;s)\Phi_2(v;s) + \Phi_2(u;s)\Phi_1(v;s)}{2\pi i(u-v)}, \tag{N.17}$$

waarbij $\left(\begin{smallmatrix}\Phi_1\\ \Phi_2\end{smallmatrix}\right)$ een speciale oplossing is van het Lax paar (N.15)–(N.16), met q de Hastings-McLeod oplossing van Painlevé II en $r = q'$. De functies Φ_1 en Φ_2 vormen de unieke oplossingsvector met asymptotiek

$$e^{i(\frac{4}{3}\zeta^3 + s\zeta)} \begin{pmatrix} \Phi_1(\zeta;s) \\ \Phi_2(\zeta;s) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + O\left(\zeta^{-1}\right) \tag{N.18}$$

als $\zeta \to \infty$. Dit geldt uniform voor $\varepsilon \le \arg\zeta \le \pi - \varepsilon$ voor willekeurige $\varepsilon > 0$.

Formulering van resultaat

Het doel van Hoofdstuk 1 is om aan te tonen dat de kern $\mathbb{K}^{\mathrm{crit,II}}(u,v;s)$ een universele limiet is. Wanneer de limiet van de gemiddelde eigenwaardedichtheid

ψ_V kwadratisch 0 wordt in een inwendig punt x^*, dan heeft de correlatiekern $K_{n,N}$ een dubbelgeschaalde limiet gegeven door (N.14), in de veronderstelling dat er, behalve x^*, geen andere singuliere punten zijn.

We hebben niet kunnen aantonen dat de universaliteit van de dubbelgeschaalde limiet ook geldt in de aanwezigheid van andere singuliere punten, hoewel we er sterk van overtuigd zijn dat de universaliteit ook in dit geval gehandhaafd blijft. Het probleem ligt hier in het bestaan van geschikte parametrices rond andere singuliere punten, hetgeen we niet hebben kunnen aantonen in het dubbelgeschaalde regime.

In Stelling N.1 gebruiken we de evenwichtsmaat ω_S van een compacte verzameling $S \subset \mathbb{R}$. Dit is de unieke maat, met dichtheid w_S, die de logaritmische energie

$$I(\mu) = \iint \log \frac{1}{|s-t|} d\mu(s) d\mu(t) \tag{N.19}$$

minimaliseert onder alle Borel kansmaten μ op S.

We formuleren nu ons belangrijkste resultaat uit Hoofdstuk 1.

N.1 Stelling. *Zij V reëel analytisch in $\mathbb{R}$ zodat $\lim_{x\to\pm\infty} \frac{V(x)}{\log(x^2+1)} = +\infty$. Zij ψ_V de limiet van de gemiddelde eigenwaardedichtheid in* (N.12)*, en laat x^* een inwendig singulier punt zijn zodat*

$$\psi_V(x^*) = \psi_V'(x^*) = 0, \qquad \psi_V''(x^*) > 0.$$

Veronderstel verder dat er behalve x^* geen andere singuliere punten zijn. Laat $n, N \to \infty$ op een zodanige manier dat de limiet

$$\lim_{n,N\to\infty} n^{2/3}\left(\frac{n}{N} - 1\right) = L$$

bestaat met $L \in \mathbb{R}$. Zij $K_{n,N}$ de correlatiekern (N.3) *voor de eigenwaarden van het random matrix model* (N.12)*. Dan bestaan er constanten $c > 0$ en $s \in \mathbb{R}$ zodat*

$$\lim_{n,N\to\infty} \frac{1}{(cn)^{1/3}} K_{n,N}\left(x^* + \frac{u}{(cn)^{1/3}}, x^* + \frac{v}{(cn)^{1/3}}\right) = \mathbb{K}^{\mathrm{crit,II}}(u,v;s) \tag{N.20}$$

uniform voor u, v in compacte deelverzamelingen van $\mathbb{R}$.

Expliciete formules voor de constanten c en s zijn

$$c = \frac{\pi \psi_V''(x^*)}{8} \tag{N.21}$$

en

$$s = L \frac{\pi}{c^{1/3}} w_{S_V}(x^*), \tag{N.22}$$

waar w_{S_V} de dichtheid is van de evenwichtsmaat S_V.

Zoals reeds vermeld, hebben Bleher en Its [14] (N.20) bewezen in het geval van een kritieke even vierdegraadsveelterm V. Recent veralgemeende Shcherbina Stelling N.1 voor bepaalde niet-analytische potentialen [93].

Evenwichtsmaten en Riemann-Hilbert problemen

De belangrijkste ingrediënten voor het bewijs van Stelling N.1 zijn evenwichtsmaten en Riemann-Hilbert problemen.

Zoals reeds gezegd, is de limiet van de gemiddelde eigenwaardedichtheid gegeven als een evenwichtsmaat μ_V die de logarithmische energie in extern veld V minimaliseert. In de dubbelgeschaalde limiet is het belangrijk te weten hoe de evenwichtsmaat $\mu_{V/t}$ zich gedraagt als functie van $t \approx 1$. Door de overgang van een spectrum bestaande uit één interval naar twee intervallen, zal $\mu_{V/t}$ echter niet op een 'gladde' manier afhangen van t. Daarom blijkt het handig te zijn een gewijzigde evenwichtsmaat te bekijken. Dit doen we door de evenwichtsdichtheid negatief te laten worden in de buurt van het kritieke punt x^* als $t < 1$.

Het tweede belangrijke element in het bewijs van Stelling N.1 is de karakterisatie van orthogonale veeltermen door middel van het Fokas-Its-Kitaev RH probleem, en de asymptotische analyse van het RH probleem met behulp van de Deift/Zhou steilste-afdalingsmethode. Van speciaal belang voor ons is het werk [6] door Baik, Deift en Johansson over de lengte van de langst stijgende deelrij van een random permutatie van $\{1, 2, \ldots, n\}$. Deze auteurs tonen aan dat de schommelingen van deze toevalsvariabele verdeeld zijn volgens de Tracy-Widom verdeling [97] in de limiet waar $n \to \infty$. Eén van de belangrijke technieken die in dit artikel gebruikt worden, is de asymptotische analyse van een RH probleem op de eenheidscirkel dat verband houdt met een evenwichtsmaat die kwadratisch 0 wordt in het punt -1. Deze situatie is vergelijkbaar met de onze. De auteurs van [6], zie ook de daaropvolgende werken [7, 8, 9, 10], construeren een lokale parametrix rond -1 met behulp van de ψ-functies horend bij de Hastings-McLeod oplossing van Painlevé II. Deze ψ-functies voldoen aan een model RH probleem en de lokale parametrix wordt geconstrueerd door het model RH problem op een geschikte manier af te beelden op een omgeving van -1 zodat de gevraagde sprongvoorwaarden benaderd worden voor grote n.

Wij volgen dezelfde strategie als in [6], maar we introduceren een wijziging in de constructie van de lokale parametrix zodat de gevraagde sprongvoorwaarden *precies* voldaan zijn, in tegenstelling tot [6] waar deze slechts *benaderend* voldaan zijn. Dit vereenvoudigt de argumenten merkelijk en we hebben het gevoel dat dit ook een belangrijke bijdrage van ons werk is.

N.2 Universaliteit rond een multi-kritieke singulariteit

Voor $n \in \mathbb{N}$, $N > 0$ en $\alpha > -1/2$ beschouwen we het unitaire random matrix ensemble

$$Z_{n,N}^{-1} |\det M|^{2\alpha} \exp(-N \operatorname{Tr} V(M))\, dM, \tag{N.23}$$

op de ruimte van $n \times n$ Hermitische matrices M, waarbij $V : \mathbb{R} \to \mathbb{R}$ een reëel analytische functie is die voldoet aan de voorwaarde

$$\lim_{x \to \pm\infty} \frac{V(x)}{\log(x^2+1)} = +\infty. \tag{N.24}$$

Multi-kritieke geval

Het doel van Hoofdstuk 2 is om (N.23) te bestuderen in het kritieke geval waar de limiet van de gemiddelde eigenwaardedichtheid ψ_V een dubbel nulpunt heeft in 0, m.a.w.

$$\psi_V(0) = \psi_V'(0) = 0, \qquad \text{en} \qquad \psi_V''(0) > 0. \tag{N.25}$$

In dit kritieke geval verwacht men een familie van mogelijke limietkernen in een dubbelgeschaalde limiet waar $n, N \to \infty$ en $n/N \to 1$ aan een geschikte snelheid [12]. Omwille van technische redenen nemen we aan dat er buiten 0 geen andere singuliere punten zijn. Als we $t = n/N$ stellen en $n, N \to \infty$ laten gaan op een zodanige manier dat $t \to 1$, dan beschrijft de parameter t de overgang van het geval waar $\psi_V(0) > 0$ (voor $t > 1$) langs het multi-kritieke geval ($t = 1$) naar het geval waar 0 in een 'gap' tussen twee intervallen van het spectrum ligt ($t < 1$). De geschikte dubbelgeschaalde limiet zal zo zijn dat de limiet $\lim_{n,N\to\infty} n^{2/3}(t-1)$ bestaat.

Voor $\alpha = 0$ werd deze dubbelgeschaalde limiet bestudeerd in [6, 14, 15, 22, 93]. Het algemene geval $\alpha > -1/2$ brengt ons tot de algemene Painlevé II vergelijking

$$q'' = sq + 2q^3 - \alpha. \tag{N.26}$$

De Painlevé II vergelijking met parameter α was gesuggereerd in de fysische werken [2, 92]. De limietkernen in de dubbelgeschaalde limiet horen bij een speciale oplossing van (N.26), die we zo meteen beschrijven. We veronderstellen nu dat $\alpha \neq 0$.

Algemene Painlevé II vergelijking

Als we sq en α laten balanceren in de differentiaalvergelijking (N.26), vinden we dat er oplossingen bestaan waarvoor

$$q(s) \sim \frac{\alpha}{s}, \qquad \text{as } s \to +\infty, \tag{N.27}$$

en als we sq en $2q^3$ laten balanceren, zien we dat er ook oplossingen van (N.26) bestaan waarvoor

$$q(s) \sim \sqrt{\frac{-s}{2}}, \qquad \text{as } s \to -\infty. \tag{N.28}$$

Er is juist één oplossing van (N.26) die zowel aan (N.27) als aan (N.28) voldoet, zie [60, 61, 68], en we schrijven q_α voor deze oplossing. Dit is de speciale oplossing die we nodig hebben. Aangezien q_α de natuurlijke veralgemening blijkt te zijn van de Hastings-McLeod oplossing voor $\alpha = 0$, noemen we q_α de Hastings-McLeod oplossing van de algemene Painlevé II vergelijking.

De Hastings-McLeod oplossing is meromorf in s (zoals alle oplossingen van (N.26)) met oneindig veel polen. We hebben nodig dat ze geen polen op de reële rechte heeft. Vanwege het asymptotische gedrag (2.13) en (2.14) weten we dat er geen reële polen zijn voor $|s|$ groot genoeg, maar dit sluit de mogelijkheid niet uit dat er een eindig aantal reële polen zijn. Hoewel er een uitgebreide literatuur over Painlevé vergelijkingen en Painlevé transcendenten voorradig is, hebben we het volgende resultaat niet kunnen terugvinden.

N.2 Stelling. *Zij q_α de Hastings-McLeod oplossing van de algemene Painlevé II vergelijking* (N.26) *met $\alpha > -1/2$. Dan is q_α meromorf zonder reële polen.*

Hoofdresultaat

Om ons hoofdresultaat te beschrijven, herhalen we de notie van ψ-functies horend bij de Painlevé II vergelijking, zie [44]. De Painlevé II vergelijking (N.26) is de compatibiliteitsvoorwaarde voor het volgende systeem van lineaire differentiaalvergelijkingen voor $\Psi = \Psi_\alpha(\zeta; s)$, dat we het Lax paar voor de Painlevé II vergelijking noemen.

$$\frac{\partial \Psi}{\partial \zeta} = A\Psi, \qquad \frac{\partial \Psi}{\partial s} = B\Psi, \tag{N.29}$$

waarbij

$$A = \begin{pmatrix} -4i\zeta^2 - i(s+2q^2) & 4\zeta q + 2ir + \alpha/\zeta \\ 4\zeta q - 2ir + \alpha/\zeta & 4i\zeta^2 + i(s+2q^2) \end{pmatrix}, \qquad B = \begin{pmatrix} -i\zeta & q \\ q & i\zeta \end{pmatrix}. \tag{N.30}$$

De algemene Painlevé II vergelijking vormt samen met de eis dat $r = q'$ precies de compatibiliteitsconditie van (N.29). Dit wil zeggen dat (N.29) een oplossing heeft waar $q = q(s)$ en $r = r(s)$ afhangen van s maar niet van ζ, als en slechts als q voldoet aan Painlevé II en $r = q'$.

Gegeven s, q en r zijn de oplossingen van

$$\frac{\partial}{\partial \zeta} \begin{pmatrix} \Phi_1(\zeta) \\ \Phi_2(\zeta) \end{pmatrix} = A \begin{pmatrix} \Phi_1(\zeta) \\ \Phi_2(\zeta) \end{pmatrix} \tag{N.31}$$

analytisch met een vertakkingspunt in $\zeta = 0$. Voor $\alpha > -1/2$ en $s \in \mathbb{R}$ nemen we $q = q_\alpha(s)$ en $r = q_\alpha'(s)$, waarbij q_α de Hastings-McLeod oplossing van de Painlevé II vergelijking is, en we definiëren $\begin{pmatrix} \Phi_{\alpha,1}(\zeta; s) \\ \Phi_{\alpha,2}(\zeta; s) \end{pmatrix}$ als de unieke oplossing van (N.31)

met asymptotiek

$$e^{i(\frac{4}{3}\zeta^3+s\zeta)}\begin{pmatrix}\Phi_{\alpha,1}(\zeta;s)\\ \Phi_{\alpha,2}(\zeta;s)\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}+\mathcal{O}(\zeta^{-1}), \tag{N.32}$$

uniform als $\zeta\to\infty$ in de sector $\varepsilon<\arg\zeta<\pi-\varepsilon$ voor willekeurige $\varepsilon>0$. Merk op dat dit goed gedefinieerd is voor elke $s\in\mathbb{R}$ vanwege Theorem 2.1.

De functies $\Phi_{\alpha,1}$ en $\Phi_{\alpha,2}$ zijn uitbreidbaar tot analytische functies in $\mathbb{C}\setminus(-i\infty,0]$, en we noteren deze uitbreidingen ook door $\Phi_{\alpha,1}$ en $\Phi_{\alpha,2}$. Hun waarden op de reële rechte komen voor in de limietkern. Het volgende is het hoofdresultaat van Hoofdstuk 2.

N.3 Stelling. *Zij V reëel analytisch in $\mathbb{R}$ zodat* (N.24) *geldt. Veronderstel dat de limiet ψ_V van de gemiddelde eigenwaardedichtheid kwadratisch* 0 *wordt in de oorsprong, d.w.z. $\psi_V(0)=\psi_V'(0)=0$ en $\psi_V''(0)>0$, en dat er behalve* 0 *geen andere singuliere punten zijn. Laat $n,N\to\infty$ op een zodanige manier dat*

$$\lim_{n,N\to\infty} n^{2/3}(n/N-1)=L\in\mathbb{R}$$

bestaat. Definieer constanten

$$c=\left(\frac{\pi\psi_V''(0)}{8}\right)^{1/3} \tag{N.33}$$

en

$$s=2\pi^{2/3}L\left[\psi_V''(0)\right]^{-1/3}w_{S_V}(0), \tag{N.34}$$

waar w_{S_V} de evenwichtsdichtheid van de drager van ψ_V is. Dan geldt de volgende dubbelgeschaalde limiet voor de correlatiekern horend bij het random matrix model (N.23),

$$\lim_{n,N\to\infty}\frac{1}{cn^{1/3}}K_{n,N}\left(\frac{u}{cn^{1/3}},\frac{v}{cn^{1/3}}\right)=\mathbb{K}_\alpha^{\mathrm{crit,II}}(u,v;s), \tag{N.35}$$

uniform voor u,v in compacte deelverzamelingen van $\mathbb{R}\setminus\{0\}$, waar

$$\begin{aligned}\mathbb{K}_\alpha^{\mathrm{crit,II}}(u,v;s)&=-e^{\frac{1}{2}\pi i\alpha[\mathrm{sgn}(u)+\mathrm{sgn}(v)]}\\ &\quad\times\quad\frac{\Phi_{\alpha,1}(u;s)\Phi_{\alpha,2}(v;s)-\Phi_{\alpha,1}(v;s)\Phi_{\alpha,2}(u;s)}{2\pi i(u-v)}.\end{aligned} \tag{N.36}$$

N.4 Opmerking. Voor $\alpha=0$ hebben we de stelling in Hoofdstuk 1 bewezen. Het bewijs voor het algemene geval verloopt in grote lijnen gelijkaardig, maar we hebben de informatie nodig over het bestaan van $q_\alpha(s)$ voor reële s, hetgeen verzekerd is door Theorem N.2.

Recursiecoëfficiënten voor orthogonale veeltermen

Om Stelling 2.2 te bewijzen, zullen we het RH probleem voor orthogonale veeltermen tegenover het gewicht $|x|^{2\alpha}e^{-NV(x)}$ bestuderen. Deze analyse leidt tot asymptotiek voor de kern $K_{n,N}$, maar voorziet ook de ingrediënten om asymptotiek voor de orthogonale veeltermen af te leiden en voor de coëfficiënten in de recursierelatie waaraan de veeltermen gehoorzamen.

Om deze resultaten te kunnen formuleren, introduceren we eerst maten ν_t als volgt, zie ook Hoofdstuk 1 and Sectie 2.3.2. Neem $\delta_0 > 0$ voldoende klein en laat ν_t de logarithmische energie $I_{V/t}(\nu)$ minimaliseren onder alle maten $\nu = \nu^+ - \nu^-$, waar $\nu^\pm$ niet-negatieve maten zijn op $\mathbb{R}$ zodat $\nu(\mathbb{R}) = 1$ en supp $\nu^- \subset [-\delta_0, \delta_0]$. We schrijven ψ_t voor de dichtheid van ν_t.

We beperken onszelf tot het geval waar ψ_V gedragen is op een enkel interval en waar er geen singuliere punten zijn behalve 0. Dan is supp $\psi_V = [a, b]$ en supp $\psi_t = [a_t, b_t]$ voor t dicht bij 1, waarbij a_t en b_t reëel analytische functies in t zijn.

We schrijven $p_k^{(N)}$ voor de n-de graads orthonormale veelterm tegenover het gewicht $|x|^{2\alpha}e^{-NV(x)}$ op $\mathbb{R}$. Deze veeltermen voldoen aan een drieterms recursievergelijking

$$xp_k^{(N)}(x) = a_{k+1}^{(N)}p_{k+1}^{(N)}(x) + b_k^{(N)}p_k^{(N)}(x) + a_k^{(N)}p_{k-1}^{(N)}(x), \tag{N.37}$$

met recursiecoëfficiënten $a_k^{(N)}$ en $b_k^{(N)}$. In de grote-n-ontwikkeling van $a_n^{(N)}$ en $b_n^{(N)}$ merken we oscillaties in de $\mathcal{O}(n^{-1/3})$-term. De amplitude van de oscillaties is evenredig met $q_\alpha(s)$, terwijl in het algemeen de frequentie traag varieert met $t = n/N$.

N.5 Stelling. *Veronderstel dat de voorwaarden van Stelling* N.3 *voldaan zijn en veronderstel ook dat* supp $\psi_V = [a, b]$ *uit één interval bestaat. Bekijk de drieterms recursievergelijking* (N.37) *voor de orthogonale veeltermen* $p_k^{(N)}$ *tegenover het gewicht* $|x|^{2\alpha}e^{-NV(x)}$. *Als we dan* $n, N \to \infty$ *laten gaan zodat* $n/N - 1 = \mathcal{O}(n^{-2/3})$, *krijgen we dat*

$$a_n^{(N)} = \frac{b-a}{4} - \frac{q_\alpha(s_{t,n})\cos(2\pi n\omega_t + 2\alpha\theta)}{2c}n^{-1/3} + \mathcal{O}(n^{-2/3}), \tag{N.38}$$

$$b_n^{(N)} = \frac{b+a}{2} + \frac{q_\alpha(s_{t,n})\sin(2\pi n\omega_t + (2\alpha+1)\theta)}{c}n^{-1/3} + \mathcal{O}(n^{-2/3}), \tag{N.39}$$

waar $t = n/N$, c *gegeven is door* (N.33),

$$s_{t,n} = n^{2/3}\frac{\pi}{c}\psi_t(0), \tag{N.40}$$

$$\theta = \arcsin\frac{b+a}{b-a} \tag{N.41}$$

en

$$\omega_t = \int_0^{b_t} \psi_t(x)dx. \tag{N.42}$$

N.6 Opmerking. In [14] hebben Bleher en Its (2.25) afgeleid in het geval waar $\alpha = 0$ en waar V is een kritieke even vierdegraadsveelterm. Ze berekenden ook de $\mathcal{O}(n^{-2/3})$-term in de grote-n-ontwikkeling voor $a_n^{(N)}$. Voor even V hebben we dat $a = -b$, $\theta = 0$, $\omega_t = 1/2$ en dus is $\cos(2\pi n\omega_t + 2\alpha\theta) = (-1)^n$, zodat (2.25) herleid wordt tot

$$a_n^{(N)} = \frac{b}{2} - \frac{q_\alpha(s_{t,n})(-1)^n}{2c} n^{-1/3} + \mathcal{O}(n^{-2/3}),$$

wat in overeenstemming is met het resultaat uit [14]. Voor even V is de recursiecoëfficiënt $b_n^{(N)}$ gelijk aan 0 hetgeen overeenstemt met (N.39).

N.3 Een reële poolvrije oplossing van de P_I^2 vergelijking

De eerste Painlevé vergelijking is de tweede orde differentiaalvergelijking

$$y_{ss} = 6y^2 + s. \tag{N.43}$$

Deze vergelijking heeft hogere orde veralgemeningen van even orde $2m$ voor $m \geq 1$, die samen met de eerste Painlevé vergelijking zelf verzameld worden in de Painlevé I hiërarchie, zie bvb. [69, 72]. Het tweede lid van de hiërarchie is de vierde orde differentiaalvergelijking

$$s = -\left(\frac{1}{6}y^3 + \frac{1}{24}(y_s^2 + 2yy_{ss}) + \frac{1}{240}y_{ssss}\right), \tag{N.44}$$

en deze vergelijking heeft meromorfe oplossingen in het complexe vlak. In 1990 gaven Brézin, Marinari en Parisi [20] numeriek aan dat er een oplossing y van (N.44) bestaat zonder reële polen, en met asymptotiek

$$y(s) \sim \mp|6s|^{1/3}, \qquad \text{als } s \to \pm\infty. \tag{N.45}$$

Moore [85] toonde het bestaan aan van een unieke reële oplossing van (N.44) met asymptotiek gegeven door (N.45), en hij gaf een argument waarom deze oplossing poolvrij hoort te zijn op de reële rechte.

Een veralgemening van (N.44) kan bekomen worden door een extra veranderlijke t toe te voegen, zoals in het werk van Dubrovin [40], zodat we de volgende differentiaalvergelijking krijgen voor $y = y(s,t)$, die we noteren als de P_I^2 vergelijking (cfr. [66] voor $t = 0$),

$$s = ty - \left(\frac{1}{6}y^3 + \frac{1}{24}(y_s^2 + 2yy_{ss}) + \frac{1}{240}y_{ssss}\right). \tag{N.46}$$

In het recente werk [40] opperde Dubrovin het vermoeden dat er een unieke reële oplossing van (N.46) bestaat zonder reële polen. We bewijzen het bestaan van deze oplossing.

Formulering van resultaten

Onze resultaten uit Hoofdstuk 3 zijn de volgende

N.7 Stelling. *Er bestaat een oplossing $y(s,t)$ van de P_I^2 vergelijking* (N.46) *met de volgende eigenschappen:*

(i) $y(s,t)$ is reëelwaardig en poolvrij voor $s,t \in \mathbb{R}$.

(ii) Voor vaste $t \in \mathbb{R}$ heeft $y(s,t)$ het volgende asymptotisch gedrag,

$$y(s,t) = \frac{1}{2} z_0 |s|^{1/3} + \mathcal{O}(|s|^{-2}), \qquad \text{als } s \to \pm\infty, \tag{N.47}$$

waarbij $z_0 = z_0(s,t)$ de reële oplossing is van

$$z_0^3 = -48\mathrm{sgn}(s) + 24 z_0 |s|^{-2/3} t. \tag{N.48}$$

N.8 Opmerking. Merk op dat z_0 negatief (positief) is voor $s > 0$ ($s < 0$) met de volgende asymptotiek als $s \to \pm\infty$,

$$z_0 = \hat{z}_0 - \mathrm{sgn}(s)\,\frac{2}{3}\,6^{2/3} t |s|^{-2/3} + \mathcal{O}(|s|^{-4/3}), \qquad \hat{z}_0 = -\mathrm{sgn}(s)\,2 \cdot 6^{1/3}, \tag{N.49}$$

zodat de asymptotiek (N.47) voor y kan herschreven worden als, cfr. (3.3)

$$y(s,t) = \mp (6|s|)^{1/3} \mp \frac{1}{3} 6^{2/3} t |s|^{-1/3} + \mathcal{O}(|s|^{-1}), \qquad \text{als } s \to \pm\infty. \tag{N.50}$$

We bewijzen het bestaan van een poolvrije oplossing door de oplosbaarheid van een geassocieerd RH probleem aan te tonen. Dit kunnen we doen door middel van een vanishing lemma. Om de asymptotiek (N.47) van y te vinden, passen we de Deift/Zhou steilste-afdalingsmethode toe op het RH probleem dat gerelateerd is met y.

N.4 Universaliteit rond een singulier randpunt

Voor $n \in \mathbb{N}$ en $s,t \in \mathbb{R}$ bekijken we in Hoofdstuk 4 een unitair random matrix ensemble van de vorm

$$\frac{1}{Z_{n,s,t}} \exp(-n \operatorname{Tr} V_{s,t}(M)) dM \tag{N.51}$$

op de ruimte van $n \times n$ Hermitische matrices. We beschouwen daarbij externe velden $V_{s,t}$ die zodanig zijn dat in het kritieke geval $s = t = 0$, $V_0 = V_{0,0}$ een

type III singulier (rand)punt heeft waar de limiet van de gemiddelde eigenwaardedichtheid $\rho_{0,0}$ naar 0 gaat als een macht 5/2,

$$\rho_{0,0}(x) \sim c|x - x^*|^{5/2}, \qquad \text{as } x \to x^*. \tag{N.52}$$

Bovendien nemen we $V_{s,t}$ van de speciale vorm

$$V_{s,t} = V_0 + sV_1 + tV_2, \tag{N.53}$$

waar V_1 een willekeurige reëel analytische functie is, terwijl V_2 reëel analytisch is en bovendien aan een bepaalde kritieke voorwaarde moet voldoen die we zo meteen verduidelijken.

ψ-functions associated with a special solution of the P_I^2 equation

We bekijken de P_I^2 differentiaalvergelijking uit Hoofdstuk 3,

$$s = ty - \left(\frac{1}{6}y^3 + \frac{1}{24}(y_s^2 + 2yy_{ss}) + \frac{1}{240}y_{ssss}\right). \tag{N.54}$$

Het Lax paar voor de P_I^2 vergelijking is het lineaire stelsel van differentiaalvergelijkingen

$$\frac{\partial \Psi}{\partial \zeta} = U\Psi, \qquad \frac{\partial \Psi}{\partial s} = W\Psi, \tag{N.55}$$

waar

$$U = \frac{1}{240}\begin{pmatrix} -4y_s\zeta - (12yy_s + y_{sss}) & 8\zeta^2 + 8y\zeta + (12y^2 + 2y_{ss} - 120t) \\ U_{21} & 4y_s\zeta + (12yy_s + y_{sss}) \end{pmatrix}, \tag{N.56}$$

$$\begin{aligned} U_{21} = 8\zeta^3 - 8y\zeta^2 - (4y^2 + 2y_{ss} + 120t)\zeta \\ + (16y^3 - 2y_s^2 + 4yy_{ss} + 240s), \end{aligned} \tag{N.57}$$

en

$$W = \begin{pmatrix} 0 & 1 \\ \zeta - 2y & 0 \end{pmatrix}. \tag{N.58}$$

Dit lineaire stelsel (4.15)–(4.18) kan enkel oplosbaar zijn als $y = y(s,t)$ een oplossing van de P_I^2 vergelijking (N.54) is.

We zijn geïnteresseerd in de bijzondere reële poolvrije oplossing y die we in Hoofdstuk 3 bestudeerd hebben, zie ook [20, 40]. Deze oplossing $y = y(s,t)$ wordt gekarakteriseerd door de Stokes multipliers s_1, s_2, s_5 en s_6 die alle gelijk zijn aan 0, zie [66]. Voor $s, t \in \mathbb{R}$ heeft het Lax paar (N.55)–(N.58) horend bij deze

bijzondere keuze van y een unieke oplossing $\left(\begin{smallmatrix}\Phi_1\\ \Phi_2\end{smallmatrix}\right)$ waarvoor de volgende limiet geldt, zie [66] en Hoofdstuk 3,

$$\zeta^{\frac{1}{4}\sigma_3}\begin{pmatrix}\Phi_1(\zeta;s,t)\\ \Phi_2(\zeta;s,t)\end{pmatrix}e^{\theta(\zeta;s,t)} \longrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}e^{-\frac{1}{4}\pi i},$$
$$\text{als } \zeta\to\infty \text{ met } 0<\arg\zeta<6\pi/7, \tag{N.59}$$

waarbij $\sigma_3 = \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ de derde Pauli-matrix is en waarbij θ gegeven is door

$$\theta(\zeta;s,t) = \frac{1}{105}\zeta^{7/2} - \frac{1}{3}t\zeta^{3/2} + s\zeta^{1/2}. \tag{N.60}$$

De functies Φ_1 en Φ_2 zullen opduiken in de universele limietkern rond type III singuliere (rand)punten waar de dichtheid 0 wordt als een macht 5/2.

We werken onder de volgende aannames.

N.3 Aannames.

(i) We veronderstellen externe velden $V_{s,t}$ van de vorm

$$V_{s,t} = V_0 + sV_1 + tV_2, \tag{N.61}$$

waarbij V_0, V_1 en V_2 reëel analytisch zijn en zodanig dat er een $\delta_0 > 0$ bestaat zodat

$$\lim_{|x|\to\infty}\frac{V_{s,t}(x)}{\log(x^2+1)} = +\infty, \qquad \text{uniform voor } s,t\in[-\delta_0,\delta_0]. \tag{N.62}$$

(ii) V_0 is zodanig dat de evenwichtsmaat ν_0 in extern veld V_0 gedragen wordt op een enkel interval $[a,b]\subset\mathbb{R}$, waarbij b een type III singulier (rand)punt is waar de evenwichtsdichtheid 0 wordt als een macht 5/2.

$$d\nu_0(x) = \frac{1}{2\pi}h_0(x)\sqrt{(b-x)(x-a)}\,\chi_{[a,b]}(x)dx, \tag{N.63}$$

met $\chi_{[a,b]}$ de indicatorfunctie van het interval $[a,b]$, en waarbij h_0 reëel analytisch is en voldoet aan

$$h_0(b) = h_0'(b) = 0 \qquad \text{en} \qquad h_0''(b) > 0. \tag{N.64}$$

Daarenboven veronderstellen we dat er geen andere singuliere punten zijn behalve b. In het bijzonder is a een regulier randpunt

(iii) V_2 is zodanig dat de volgende kritieke voorwaarde voldaan is,

$$\int_a^b \sqrt{\frac{u-a}{b-u}}V_2'(u)du = 0. \tag{N.65}$$

N.4 Voorbeeld. Deze aannames zijn geldig voor het volgende voorbeeld, waar V_0, V_1 en V_2 gegeven worden door

$$V_0(x) = \frac{1}{20}x^4 - \frac{4}{15}x^3 + \frac{1}{5}x^2 + \frac{8}{5}x, \qquad V_1(x) = x, \qquad V_2(x) = x^3 - 6x. \tag{N.66}$$

Dan is de evenwichtsmaat ν_0 gedragen op het interval $[-2, 2]$ en gegeven door

$$d\nu_0(x) = \frac{1}{10\pi}(x+2)^{1/2}(x-2)^{5/2}\chi_{[-2,2]}(x)dx. \tag{N.67}$$

We merken ook op dat een type III singulier (rand)punt niet kan voorkomen als V_0 een veelterm van graad lager dan 4 is.

N.9 Opmerking. Het geval waarbij het linker randpunt van de drager (i.p.v. het rechter randpunt) singulier is, kan getransformeerd worden naar ons geval door het extern veld $V_{s,t}(-x)$ te bekijken.

N.10 Opmerking. Als we $t = 0$ nemen en s laten variëren rond 0, zien we typisch de overgang van het reguliere één-interval-geval naar het singuliere geval en dan terug naar het reguliere één-interval-geval. Voor $s = 0$ en t dicht bij 0 kunnen we de overgang zien waar één interval overgaat naar twee intervallen. Als we $s, t \neq 0$ laten zijn, kunnen we één van de bovenvermelde overgangen zien, of de kritieke overgang waarbij een type II singulier punt naar het eindpunt toegaat, waar het een type III singulier punt wordt, alvorens verder te gaan als een type I singulier punt.

Om onze resultaten te kunnen beschrijven, voeren we constanten c, c_1 en c_2 in,

$$c = \left(\frac{15}{2}h_0''(b)\sqrt{b-a}\right)^{2/7} > 0, \tag{N.68}$$

$$c_1 = \frac{h_1(b)}{c^{1/2}(b-a)^{1/2}}, \qquad c_2 = -\frac{h_2'(b)}{c^{3/2}(b-a)^{1/2}}, \tag{N.69}$$

waarbij h_0 de reëel analytische functie is die voorkomt in (N.63) en waar de functies h_j, $j = 1, 2$, gedefinieerd worden als

$$h_j(x) = -\frac{1}{\pi}\fint_a^b \sqrt{(b-u)(u-a)}V_j'(u)\frac{du}{u-x}, \qquad \text{voor } x \in [a, b]. \tag{N.70}$$

Universaliteit van de dubbelgeschaalde limiet

Ons hoofdresultaat in Hoofdstuk 4 is het volgende.

N.11 Stelling. *Zij* $V_{s,t} = V_0 + sV_1 + tV_2$ *zo dat aan Aannames* N.3 *voldaan is. We nemen een dubbelgeschaalde limiet waarbij we* $n \to \infty$ *laten gaan en terzelfdertijd laten we* $s, t \to 0$*, op een zodanige manier dat*

$$c_1 \cdot \lim n^{6/7} s = s_0 \in \mathbb{R}, \qquad c_2 \cdot \lim n^{4/7} t = t_0 \in \mathbb{R}, \tag{N.71}$$

waarbij de constanten c_1 *en* c_2 *gedefinieerd zijn door* (N.69)*. Dan voldoet de tweepuntskern* $K_n^{(s,t)}$ *voor het random matrix model* (N.51) *aan het volgende universaliteitsresultaat,*

$$\lim \frac{1}{cn^{2/7}} K_n^{(s,t)} \left(b + \frac{u}{cn^{2/7}}, b + \frac{v}{cn^{2/7}} \right) = \mathbb{K}^{\mathrm{crit,III}}(u, v; s_0, t_0), \tag{N.72}$$

univorm voor u, v *in compacte deelverzamelingen van* $\mathbb{R}$*. Hierbij is* $\mathbb{K}^{\mathrm{crit,III}}$ *opgebouwd uit de functies* Φ_1 *en* Φ_2 *die hierboven gedefinieerd zijn,*

$$\mathbb{K}^{\mathrm{crit,III}}(u, v; s, t) = \frac{\Phi_1(u; s, t)\Phi_2(v; s, t) - \Phi_1(v; s, t)\Phi_2(u; s, t)}{-2\pi i(u - v)}. \tag{N.73}$$

N.12 Opmerking. Aangezien $y(s, t)$ geen polen heeft voor $s, t \in \mathbb{R}$ (zie Chapter 3), bestaat de kern $\mathbb{K}^{\mathrm{crit,III}}(u, v; s, t)$ voor alle reële u, v, s en t.

Recursiecoëfficiënten voor orthogonale veeltermen

Het is gekend [95] dat de orthonormale veeltermen $p_k = p_k^{(n,s,t)}$ tegenover het gewicht $e^{-nV_{s,t}}$ op $\mathbb{R}$ voldoen aan een drieterms recursievergelijking van de vorm

$$xp_k(x) = a_{k+1}p_{k+1}(x) + b_k p_k(x) + a_k p_{k-1}(x), \tag{N.74}$$

waar $a_k = a_k^{(n,s,t)} > 0$ en $b_k = b_k^{(n,s,t)} \in \mathbb{R}$. In het reguliere geval waar V_0 geen singuliere punten heeft, hebben de recursiecoëfficiënten de volgende asymptotiek voor $s = t = 0$, zie bvb. [14, 27],

$$a_n^{(n,0,0)} = \frac{b-a}{4} + \mathcal{O}(n^{-1}), \qquad b_n^{(n,0,0)} = \frac{b+a}{2} + \mathcal{O}(n^{-1}), \qquad \text{als } n \to \infty. \tag{N.75}$$

Voor kritieke potentialen V_0 blijven de constante termen in (N.75) nog steeds dezelfde, maar de correctietermen zijn verschillend [14, 23]. In ons geval, waar we een type III singulier punt hebben waar de dichtheid 0 wordt als een macht 5/2, is de correctieterm van de orde $\mathcal{O}(n^{-2/7})$, en de coëfficiënt van de $n^{-2/7}$ term kan dan uitgedrukt worden in functie van de speciale poolvrije oplossing y van de P_I^2 vergelijking.

N.13 Stelling. *Veronderstel dat* $V_{s,t}$ *voldoet aan Aannames* N.3*. Beschouw de drieterms recursierelatie* (N.74) *waaraan de orthogonale veeltermen* $p_k = p_k^{(n,s,t)}$ *tegenover het gewicht* $e^{-nV_{s,t}}$ *voldoen. In de dubbelgeschaalde limiet waar* $n \to \infty$ *en* $s, t \to 0$*, zo dat*

$$\lim c_1 n^{6/7} s = s_0 \in \mathbb{R}, \qquad \lim c_2 n^{4/7} t = t_0 \in \mathbb{R}, \tag{N.76}$$

met c_1 en c_2 gegeven door (N.69)*, krijgen we dat*

$$a_n^{(n,s,t)} = \frac{b-a}{4} + \frac{1}{2c}\, y(c_1 n^{6/7} s, c_2 n^{4/7} t) n^{-2/7} + \mathcal{O}(n^{-3/7}), \tag{N.77}$$

$$= \frac{b-a}{4} + \frac{1}{2c}\, y(s_0, t_0) n^{-2/7}(1 + o(1)),$$

en

$$b_n^{(n,s,t)} = \frac{b+a}{2} + \frac{1}{c}\, y(c_1 n^{6/7} s, c_2 n^{4/7} t) n^{-2/7} + \mathcal{O}(n^{-3/7}) \tag{N.78}$$

$$= \frac{b+a}{2} + \frac{1}{c}\, y(s_0, t_0) n^{-2/7}(1 + o(1)),$$

waar de constante c gegeven wordt door (4.30)*, en waar y de bijzondere poolvrije oplossing van de P_I^2 vergelijking is, beschouwd in Hoofdstuk 3.*

We bewijzen de resultaten in Hoofdstuk 4 op een analoge manier als in Hoofdstuk 1 en Hoofdstuk 2, door een asymptotische analyse van het RH probleem dat orthogonale veeltermen karakteriseert met behulp van de Deift/Zhou steilste-afdalingsmethode. Het belangrijkste verschil situeert zich in de constructie van een lokale parametrix rond het singuliere randpunt b. Verder gebruiken we ook gewijzigde evenwichtsmaten, waarbij we de evenwichtsdichtheden negatief laten worden (zoals in Hoofdstuk 1 en Hoofdstuk 2). In tegenstelling tot de evenwichtsmaten uit de eerdere hoofdstukken kiezen we hier ook de drager van de maten vast, onafhankelijk van s en t, omdat dit handig blijkt te zijn in de asymptotische analyse.

N.5 Universaliteit in de overgang van zachte naar harde rand

In Hoofdstuk 5 beschouwen we unitaire random matrix ensembles op de ruimte van positief-definiete $n \times n$ Hermitische matrices, met een kansmaat van de vorm

$$\frac{1}{Z_{n,N}} (\det M)^{\alpha} \exp(-N \operatorname{Tr} V(M)) dM, \qquad \alpha > -1. \tag{N.79}$$

Hierbij is $Z_{n,N}$ een normalisatieconstante en de potentiaal V is reëel analytisch op $[0, +\infty)$ met voldoende groei op oneindig,

$$\frac{V(x)}{\log(x^2+1)} \to +\infty \quad \text{als } x \to +\infty. \tag{N.80}$$

Omdat de eigenwaarden van een random matrix in dit model beperkt zijn tot de positieve reële as, wordt 0 de *harde rand* van het random matrix model genoemd.

Randpunten waar de limiet van de gemiddelde eigenwaardedichtheid naar 0 toegaat, worden *zachte randen* genoemd. We concentreren ons op het geval waar een zachte rand samenvalt met de harde rand in de oorsprong. We veronderstellen ook dat er geen singuliere punten zijn.

We beschouwen een dubbelgeschaalde limiet van de correlatiekern, waar we $n, N \to \infty$ laten gaan en terzelfdertijd laten we $t = n/N \to 1$ aan een aangepaste snelheid. Voor $n/N \to c > 0$ is de limiet van de gemiddelde eigenwaardeverdeling gelijk aan een evenwichtsmaat $\mu_{V/c}$ in extern veld V/c. Voor c dicht bij 1 en $c < 1$ ligt 0 buiten de drager van $\mu_{V/c}$ en dan is het linkse eindpunt van de drager een zachte rand, zie [74]. Voor c dicht bij 1 en $c > 1$, gaat de dichtheid naar oneindig toe rond 0. Als $c \nearrow 1$ zien we de zachte rand die naar de harde rand toegaat. Als $c > 1$ verdwijnt de zachte rand en de dichtheid gaat naar oneindig aan de harde rand. De limietkern in de overgang $c \approx 1$ zal te maken hebben met de Hastings-McLeod oplossing van de Painlevé II vergelijking en kan uitgedrukt worden in functie van de kernen $\mathbb{K}_\alpha^{\mathrm{crit,II}}$ die reeds voorkwamen in Hoofdstuk 1 en Hoofdstuk 2.

Er is een verband tussen random matrix ensembles van de vorm (N.79) en ensembles op de ruimte van Hermitische $n \times n$ matrices met een kansmaat van de vorm

$$\frac{1}{\hat{Z}_{n,N}} |\det M|^{2\alpha \pm 1} e^{-\frac{N}{2} \operatorname{Tr} V(M^2)}, \qquad 2\alpha \pm 1 > -1. \tag{N.81}$$

We zullen de resultaten die we bekomen hebben in Hoofdstuk 1 en Hoofdstuk 2 vertalen naar resultaten voor ensembles van de vorm (N.79). Een andere strategie zou zijn om (zoals in Hoofdstukken 1, 2 en 4) de Deift/Zhou steilste-afdalingsmethode toe te passen op het RH probleem voor de orthogonale veeltermen op $[0, +\infty)$ die verband houden met het matrix model. De werkwijze die we hier voorstellen, is echter veel korter en directer.

Formulering van resulaten

Vooraleer we onze resultaten formuleren, moeten we nog een evenwichtsmaat definiëren die een rol speelt in de dubbelgeschaalde limiet. Definieer ω als de unieke maat die

$$I(\omega) = \iint \log \frac{1}{|x-y|} d\omega(x) d\omega(y) \tag{N.82}$$

minimaliseert onder alle kansmaten op de drager van de limiet van de gemiddelde eigenwaardedichtheid S_V. We noteren w voor de dichtheid van ω. In het geval waar $S_V = [0, b]$ is de evenwichtsdichtheid gegeven door $w(x) = \frac{1}{\pi\sqrt{x(b-x)}}$ voor $x \in (0, b)$.

We formuleren nu het belangrijkste resultaat uit Hoofdstuk 5.

N.14 Stelling. *Neem $\alpha > -1$. Zij V reëel analytisch op $[0,+\infty)$ zodanig dat $\lim_{x\to+\infty} \frac{V(x)}{\log(x^2+1)} = +\infty$. Noem ψ_V de limiet van de gemiddelde eigenwaardedichtheid van het random matrix model* (N.79) *als $n, N \to \infty$ op een zodanige manier dat $n/N \to 1$. Veronderstel verder dat $0 \in S_V = \operatorname{supp} \psi_V$ met $\psi_V(0) = 0$ en $\psi_V'(0) \neq 0$, en veronderstel ook dat er geen singuliere punten zijn in het extern veld V. Laat nu $n, N \to \infty$ zo dat de limiet*

$$\lim_{n,N\to\infty} (2n)^{2/3} \left(\frac{n}{N} - 1\right) = L$$

bestaat met $L \in \mathbb{R}$. Zij $K_{n,N}$ de correlatiekern voor de eigenwaarden van het random matrix model (N.79)*. Dan bestaan er constanten $c > 0$ en $s \in \mathbb{R}$ zodat*

$$\begin{aligned}\lim_{n,N\to\infty} &\frac{1}{(2cn)^{2/3}} K_{n,N}\left(\frac{u}{(2cn)^{2/3}}, \frac{v}{(2cn)^{2/3}}\right) \\ &= \frac{1}{2}(uv)^{-1/4}\left(\mathbb{K}^{\mathrm{crit,II}}_{\alpha+\frac{1}{2}}(\sqrt{u},\sqrt{v};s) + \mathbb{K}^{\mathrm{crit,II}}_{\alpha+\frac{1}{2}}(\sqrt{u},-\sqrt{v};s)\right), \qquad \text{(N.83)}\end{aligned}$$

uniform voor u, v in compacte deelverzamelingen van $(0,+\infty)$. Als $\alpha > 0$ kan de limiet (N.83) *ook herschreven worden als*

$$\begin{aligned}\lim_{n,N\to\infty} &\frac{1}{(2cn)^{2/3}} K_{n,N}\left(\frac{u}{(2cn)^{2/3}}, \frac{v}{(2cn)^{2/3}}\right) \\ &= \frac{1}{2}(uv)^{-1/4}\left(\mathbb{K}^{\mathrm{crit,II}}_{\alpha-\frac{1}{2}}(\sqrt{u},\sqrt{v};s) - \mathbb{K}^{\mathrm{crit,II}}_{\alpha-\frac{1}{2}}(\sqrt{u},-\sqrt{v};s)\right). \qquad \text{(N.84)}\end{aligned}$$

Expliciete formules voor de constanten c en s zijn

$$c = \frac{\pi\sqrt{-Q_V'(0)}}{4} \tag{N.85}$$

en

$$s = L\frac{\pi}{c^{1/3}} \lim_{x\searrow 0} \sqrt{x}w(x), \tag{N.86}$$

waarbij w de evenwichtsdichtheid van S_V is.

N.15 Opmerking. De eenvoudigste voorbeelden waarbij deze kritieke situatie voorkomt, zijn ensembles met een potentiaal V van de vorm $V(x) = \frac{g}{2}x^2 - 2\sqrt{g}x$ met $g > 0$, waarbij ψ_V gegeven wordt door

$$\psi_V(x) = \frac{g}{2\pi}\sqrt{x}\sqrt{\frac{4}{\sqrt{g}} - x}, \qquad \text{for } x \in [0, 4g^{-1/2}].$$

De constanten c, c_1 zijn dan gegeven door

$$c = \frac{g^{3/4}}{4}, \qquad c_1 = \frac{g^{1/4}}{2c^{1/3}}.$$

De orthonormale veeltermen $p_k^{(N)}$ tegenover het gewicht $x^\alpha e^{-NV(x)}$ op $[0,+\infty)$ voldoen aan een drieterms recursierelatie van de vorm

$$xp_k^{(N)}(x) = a_{k+1}^{(N)} p_{k+1}^{(N)}(x) + b_k^{(N)} p_k^{(N)}(x) + a_k^{(N)} p_{k-1}^{(N)}(x). \tag{N.87}$$

We hebben het volgende resultaat betreffende de asymptotiek van de recursiecoëfficiënten $a_n^{(N)}$ en $b_n^{(N)}$ als $n, N \to \infty$ op een zodanige manier dat $n/N - 1 = \mathcal{O}(n^{-2/3})$.

N.16 Stelling. *Veronderstel dat V voldoet aan de voorwaarden van Stelling* N.14 *en veronderstel ook dat* $\operatorname{supp} \psi_V = [0,b]$ *uit een enkel interval bestaat. Als we dan $n, N \to \infty$ laten gaan zo dat $n/N - 1 = \mathcal{O}(n^{-2/3})$, dan krijgen we dat*

$$a_n^{(N)} = \frac{b}{4} + \mathcal{O}(n^{-2/3}), \tag{N.88}$$

$$b_n^{(N)} = \frac{b}{2} + \mathcal{O}(n^{-2/3}). \tag{N.89}$$

N.6 Relaties tussen limietkernen

In Hoofdstuk 6 tonen we aan dat de verschillende limietkernen die doorheen de thesis opgedoken zijn, gerelateerd zijn met elkaar. Zo kunnen de reguliere kernen $\mathbb{K}^{\text{bulk}}$ en $\mathbb{K}^{\text{edge}}$ bekomen worden als limieten van de kritieke kernen $\mathbb{K}^{\text{crit,II}}$ en $\mathbb{K}^{\text{crit,III}}$. Ook de Bessel kern $\mathbb{J}_\alpha^0$ kan gezien worden als een limiet van de kern $\mathbb{K}_\alpha^{\text{crit,II}}$. Dit is niet verwonderlijk als we de random matrix ensembles in gedachten houden waar de kritieke kernen $\mathbb{K}_\alpha^{\text{crit,II}}$ en $\mathbb{K}^{\text{crit,III}}$ voorkomen. Deze kritieke ensembles hebben een singulier punt waarvan het karakter kan veranderen van het ene regime naar een ander, wanneer we parameters in het extern veld wijzigen. Als we de deformatieparameter(s) laten variëren, kunnen we de overgang zien van het kritieke schalingsregime naar een regulier schalingsregime. We tonen in Hoofdstuk 6 aan dat deze overgangen niet enkel zichtbaar in de random matrix ensembles, maar dat we deze ook kunnen zien op het niveau van de limietkernen.

We hebben de volgende resultaten.

N.17 Stelling. (i) *Voor $u, v \in \mathbb{R}$,*

$$\lim_{s\to+\infty} \frac{1}{s} \mathbb{K}_0^{\text{crit,II}}\left(\frac{u}{s}, \frac{v}{s}; s\right) = \mathbb{K}^{\text{bulk}}(u,v).$$

(ii) *Voor $u, v \in \mathbb{R} \setminus \{0\}$ krijgen we*

$$\lim_{s\to+\infty} \frac{1}{s} \mathbb{K}_\alpha^{\text{crit,II}}\left(\frac{u}{s}, \frac{v}{s}; s\right) = \mathbb{J}_\alpha^0(u,v).$$

(iii) *Voor* $u, v \in \mathbb{R}$,

$$\lim_{s\to-\infty} \frac{1}{2^{7/6}|s|^{1/2}} \mathbb{K}_0^{\mathrm{crit,II}} \left(\mp \frac{1}{\sqrt{2}}|s|^{1/2} \pm \frac{u}{2^{7/6}|s|^{1/2}}, \right.$$
$$\left. \mp \frac{1}{\sqrt{2}}|s|^{1/2} \pm \frac{v}{2^{7/6}|s|^{1/2}}; s \right) = \mathbb{K}^{\mathrm{edge}}(u, v).$$

(iv) *Voor* $u, v, t \in \mathbb{R}$,

$$\lim_{s\to\pm\infty} \left(\frac{4}{3}\right)^{4/9} \frac{1}{|s|^{4/9}} \mathbb{K}^{\mathrm{crit,III}} \left(|s|^{1/3} z_0 + \left(\frac{4}{3}\right)^{4/9} \frac{u}{|s|^{4/9}}, \right.$$
$$\left. |s|^{1/3} z_0 + \left(\frac{4}{3}\right)^{4/9} \frac{v}{|s|^{4/9}}; s, t \right) = \mathbb{K}^{\mathrm{edge}}(u, v),$$

waarbij z_0 *het reële nulpunt is van de vergelijking*

$$z_0^3 = -\mathrm{sgn}(s)48 + 24 z_0 |s|^{-2/3} t, \qquad \textit{for } s \neq 0.$$

(v) *Zij* $c = \frac{|g'_+(0)|}{\pi}$, *met* g *gedefinieerd door* (3.92) *en* (3.95). *Dan geldt voor* $u, v, t \in \mathbb{R}$ *dat*

$$\lim_{s\to-\infty} \frac{1}{c|s|^{5/6}} \mathbb{K}^{\mathrm{crit,III}} \left(\frac{u}{c|s|^{5/6}}, \frac{v}{c|s|^{5/6}}; s, t \right) = \mathbb{K}^{\mathrm{bulk}}(u, v).$$

We tonen deze resultaten aan door de Deift/Zhou steilste-afdalingsmethode toe te passen op de model RH problemen geassocieerd met de kritieke kernen.

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