The answer in Problem 3 of [1] is negative

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Abstract

It is shown that the answer to Problem 3 of [1] is negative.

Introduction

Problem 3 of [1], which asked about possible finiteness of the gap in a convex relaxation of a non-convex quadratic optimization problem, reads, after fixing the incorrect matrix indexing, as follows.

Problem 3. *Does there exist a finite constant $\gamma > 0$ with the following feature: for any real cyclic n-by-n matrix*

$$H = \begin{bmatrix} h_0 & h_1 & \dots & h_{n-1} \\ h_{n-1} & h_0 & \dots & h_{n-2} \\ & & \ddots & \\ h_1 & h_2 & \dots & h_0 \end{bmatrix}$$

with $\sigma_{\max}(H) \geq \gamma$ there exists a non-zero real vector x such that $|y_i| \geq |x_i|$ for all $i = 1, \dots, n$, where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = Hx.$$

A positive answer to Problem 3 would imply that the circle criterion is sharp up to a constant independent of the system order, when applied to verify asymptotic stability of a feedback interconnection of a finite order stable LTI SISO system and an uncertain memoryless system of multiplication by a time-varying real gain not exceeding 1 in absolute value.

Unfortunately, the statement from the next section shows that the answer to Problem 3 is negative. The negative answer appears not to imply anything of importance.

The author has heard that other researchers reached the same conclusion, but has no references available. Nevertheless, this paper is not intended for publication as a new original result, but rather to provide information that may be available from other sources.

Main Statement

We will use the notion of *matrices* and *linear operators* on $\mathbf{R}^n$ interchangeably.

Let $S_n : \mathbf{R}^n \to \mathbf{R}^n$ be the cyclic shift operator, defined by $(S_n x)_r = x_{r-1}$ for $r = 1, \dots, n$, where, by the cyclicity rule, $x_0 = x_{n-1}$. Note that a given n-by-n matrix L is cyclic if and only if $S_n H = H S_n$.

Theorem 0.1 *Let $n = 2k+1$ be an odd number. Then $I + S_n$ is invertible, and $H_n = (I+S_n)^{-1}(I-S_n)$ is a cyclic matrix. However*

$$\sigma_{\max}(H_n) = \tan\left(\frac{\pi k}{2k+1}\right)$$

is unbounded as $n \to \infty$, while for any $\theta > 1$ there exist no non-zero real vector x such that $|y_i| \geq |x_i|$ for all $i = 1, \dots, n$, where $y = (1/\theta) H_n x$.

Proof By inspection, S_n has n complex eigenvalues $\lambda_r^S = \exp(2\pi r j/n)$, $r = 0, 1, \dots, n-1$, and n orthogonal complex eigenvectors $v_l = (v_{lr})_{r=1}^n$, where $v_{lr} = \exp(2\pi r l j/n)$. Hence $I + S_n$ is invertible, and H_n has eigenvalues

$$\lambda_r^H = \frac{1 - \exp(2\pi r j/n)}{1 + \exp(2\pi r j/n)} = -j \tan\left(\frac{\pi k}{2k+1}\right) ,$$

and the same set of orthogonal eigenvectors. Therefore the largest singular value of H_n equals the largest absolute value of its eigenvalue, which yields the expression for $\sigma_{\max}(H_n)$ from the statement of the theorem.

Now assume that $y = \theta H_n x$, $\theta < 1$, and $|y_r| \geq |x_r|$ for all r. Then

$$x_r - x_{r-1} = \theta(y_r + y_{r-1}) \;\; \forall\, r.$$

Since $x_r = \delta_r y_r$ where $\delta_r \in [-1, 1]$, we have

$$y_r(\theta - \delta_r) = -y_{r-1}(\theta + \delta_{r-1}) \;\; \forall\, r.$$

Since $\theta \pm \delta_r > 0$, either all y_r are zero, or $(-1)^r y_r$ must have same sign for $r = 0, 1, \ldots, n$. Since n is odd, the second outcome is impossible. Hence $y_r = 0$ for all r, and $x_r = 0$ as well. ■

References

[1] A. Megretski, "How conservative is the circle criterion?", In "Open Problems in Mathematical Systems and Control Theory", ed. by V. D. Blondel, E. D. Sontag, M. Vidyasagar, and J. C. Willems, Springer, 1999, pp. 149-151.