A control problem in game theory

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Note: The background material for this problem is the book by Hofbauer and Sigmund, Evolutionary games and replicator dynamics, Cambridge University Press, 1998.

Review of the replicator equation

Consider a game with n pure strategies and payoff matrix $A \in \mathbb{R}^{n \times n}$. Each pure strategy can be identified with a vector e_i from the standard basis of $\mathbb{R}^n$. A possibly mixed strategy p is identified with a point on the standard simplex S_n, also called the strategy space. The payoff for a player with strategy p whose opponent is using strategy q is given by $p^T Aq$.

For now we would like to study a population consisting of players who can only adopt one of the pure strategies. We are interested in the dynamics of the relative number or *frequencies* of pure strategists within the population. If we define a state vector $x = (x_1, ..., x_n)$ containing these frequencies, then it is clear that the state space can be identified with the standard simplex S_n. Notice that we have made the same identification for the strategy space, but obviously the strategy space and the state space have completely different meanings! To avoid confusion, we reserve letters p and q for points of the strategy space, while we use x and y for points of the state space.

Let's first write down the equation that describes the dynamics of the frequencies -this equation is known as the replicator equation- and explain later where it came from:

$$\dot{x} = \text{diag}(x)(Ax - x^T Ax\mathbf{1}) \tag{1}$$

where $\text{diag}(x)$ is a diagonal matrix containing the components of x on its diagonal and $\mathbf{1}$ is a vector with all components equal to 1. Notice that the simplex S_n is an invariant set, as it should be.

So where did (1) come from? The underlying assumption is that the growth (or death) rates of e_i strategists are given by the following difference:

expected payoff for an e_i strategist $-$ average fitness of the whole population.

If the current state of the population is x, then the expected payoff for an e_i strategist is given by $(Ax)_i$ and the average fitness of the population by $x^T Ax$. This leads to the replicator equation given above.

Of particular interest in game theory are concepts such as Nash equilibria and Evolutionary Stable States (ESS). Theorem 7.2.1 in the book summarizes why. It states that Nash equilibria *are* equilibria of the system, that stable equilibria must be Nash; and maybe most importantly, that an ESS is asymptotically stable.

Taking control of the game

Suppose that we want to modify the system behavior by introduction of a new kind of player in the population. The new player adopts a mixed strategy $p \in S_n$ defined as follows:

$$p = \sum_{i=1}^{n} p_i e_i, \ p_i \geq 0 \text{ for all } i \text{ and } \sum_{i=1}^{n} p_i = 1.$$

The p_i should be interpreted as design or control parameters.

Possibly some of the following problems could be of interest. You are not at all required to solve all of them; simply pick the most attractive one.

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1. Find the replicator equation for the modified population. Denoting the frequency of players adopting the new strategy by x_{n+1} and introducing a new state vector $y^T = (x^T, x_{n+1})$, show that:

$$\dot{y} = \mathrm{diag}(y)(A^* y - y^T A^* y \mathbf{1}) \tag{2}$$

with

$$A^* = \begin{pmatrix} A & Ap \\ p^T A & p^T Ap \end{pmatrix}$$

Notice that this is an $n+1$ dimensional system, defined on S_{n+1}. Although parametrized, it is still a replicator equation, so in particular all known results for replicator equations apply.

2. Consider what happens when you apply this to the 'standard games' such as the Hawk-Dove game or the Prisoner's Dilemma. You could perform simulations and see what the implications are of choosing different design parameters. Obviously you should compare the new system behavior to the old one. If you can prove certain things this would even be better, see for instance the next item. A more challenging game to consider is the Rock-Scissors-Paper game.

3. What happens to equilibria, Nash equilibria and ESS of the old system?

4. Entering the realm of control theory even further, you could interpret the p_i as feedback variables and see what happens with simple feedback laws. (but keep in mind that there are constraints on the control variables!) Again, the standard games could serve as toy models.