Numerical Algorithms as Discrete-Time Control Systems

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Numerical algorithms as discrete-time control systems

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Introduction

Iterative Algorithms

\[ \Phi : M \rightarrow M \text{ iteration map} \]

\[ x_t \in M, \quad x_t \rightarrow x_{t+1} \in M \]
Introduction

Iterative Algorithms
\( \Phi : \mathcal{M} \rightarrow \mathcal{M} \) iteration map

\[ x_t \in \mathcal{M}, \quad x_t \rightarrow x_{t+1} \in \mathcal{M} \]

Shifted Iterative Algorithms
\( \Phi : \mathcal{M} \times \mathcal{U} \rightarrow \mathcal{M} \) iteration map, \( \mathcal{U} \) set of shift parameters

\[ x_t \in \mathcal{M}, u_t \in \mathcal{U} \quad (x_t, u_t) \rightarrow x_{t+1} \in \mathcal{M} \]
Introduction

Iterative Algorithms
\( \Phi : \mathcal{M} \to \mathcal{M} \) iteration map
\[
x_t \in \mathcal{M}, \quad x_t \rightarrow x_{t+1} \in \mathcal{M}
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Shifted Iterative Algorithms
\( \Phi : \mathcal{M} \times \mathcal{U} \to \mathcal{M} \) iteration map, \( \mathcal{U} \) set of shift parameters
\[
x_t \in \mathcal{M}, u_t \in \mathcal{U} \quad (x_t, u_t) \rightarrow x_{t+1} \in \mathcal{M}
\]

Example: Shifted Inverse Iteration on \( \mathbb{S}^{n-1} \).
\( A \in \mathbb{R}^{n \times n} \), \( \mathcal{M} = \mathbb{S}^{n-1} \), \( \mathcal{U} = \mathbb{R} \setminus \sigma(A) \).

Iteration Step
\[
\Phi : \mathcal{M} \times \mathcal{U} \to \mathcal{M}, \quad \Phi(x, u) = \frac{(A - uI)^{-1}x}{\| (A - uI)^{-1}x \|}
\]
Introduction

Iterative Algorithms
\( \Phi : M \rightarrow M \) iteration map

\[ x_t \in M, \quad x_t \rightarrow x_{t+1} \in M \]

Shifted Iterative Algorithms
\( \Phi : M \times U \rightarrow M \) iteration map, \( U \) set of shift parameters

\[ x_t \in M, u_t \in U \quad (x_t, u_t) \rightarrow x_{t+1} \in M \]

Observation: \((M, U, \Phi)\) describes a discrete-time control System:

\[
\begin{align*}
x_0 & \in M \\
x_{t+1} &= \Phi(x_t, u_t)
\end{align*}
\]
Introduction

Motivation:
Introduction

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- Control theoretical description of numerical algorithms
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- Control theoretical description of numerical algorithms
- Better understanding of algorithms
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- Control theoretical description of numerical algorithms
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- Optimization of shift strategies using control theoretical tools
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Motivation:

- Control theoretical description of numerical algorithms
- Better understanding of algorithms
- Optimization of shift strategies using control theoretical tools
- Justify existing shift strategies
Introduction

Motivation:
- Control theoretical description of numerical algorithms
- Better understanding of algorithms
- Optimization of shift strategies using control theoretical tools
- Justify existing shift strategies
- Creation of new algorithms
Examples

Example I: Shifted Inverse Iteration on Projective Spaces

- $A \in \mathbb{F}^{n \times n}$
- $\mathcal{M} = \mathbb{F} \mathbb{P}^{n-1} = \{\text{Set of one dimensional subspaces of } \mathbb{F}^n\}$
- $\mathcal{U} = \mathbb{F} \setminus \sigma(A)$
- $\Phi : \mathcal{M} \times \mathcal{U} \to \mathcal{M}$ defined by

$$\Phi(\mathcal{X}, u) = (A - uI)^{-1} \mathcal{X}$$
Examples

Example I: Shifted Inverse Iteration on Projective Spaces

- $A \in \mathbb{F}^{n \times n}$
- $\mathcal{M} = \mathbb{F}P^{n-1} = \{\text{Set of one dimensional subspaces of } \mathbb{F}^n\}$
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$$\Phi(\mathcal{X}, u) = (A - uI)^{-1}\mathcal{X}$$

Questions and remarks:

- Can we find feedback controls to reach eigenvectors by arbitrary initial points?
- Can we find feedback controls to reach specific eigenvectors by arbitrary initial points?
Examples

Example II: Shifted Inverse Iteration on Grassmann Manifolds

- $A \in \mathbb{R}^{n \times n}$
- $\mathcal{M} = \text{Grass}(p, n) = \{\text{Set of } p\text{-dimensional subspaces of } \mathbb{R}^n\}$
- $\mathcal{U}$ set of feedback maps $F : \text{ST}(p, n) \to \mathbb{R}^{p \times p}$

\[ \mathcal{U} = \{\forall X \in \text{ST}(p, n), \forall M \in \text{GL}_p(\mathbb{R}) : F(XM) = M^{-1}F(X)M\} \]

- $\Phi : \mathcal{M} \times \mathcal{U} \to \mathcal{M}$ defined by procedure

1) Choose $X \in \text{ST}(p, n)$ such that $\langle X \rangle = \mathcal{X}$
2) Solve $AX^+ - X^+ F(X) = X$
3) $\Phi(\mathcal{X}, F) := \mathcal{X}^+ := \langle X^+ \rangle$
Examples

Example II: Shifted Inverse Iteration on Grassmann Manifolds

Questions and remarks

- The choice $F = R$ with $R(X) = (X^T X)^{-1} X^T A X$ leads to Grassmann Rayleigh Quotient Iteration (Absil, Mahony, Sepulchre, Van Dooren, 2002).

- Does the Grassmann Rayleigh Quotient Iteration has global convergence properties?

- Can we find feedback laws for global convergence?

- Can we find feedback laws for global convergence to a specific eigenspace?
Examples

Example III: Shifted Inverse Iteration on Flag Manifolds

- \( A \in \mathbb{F}^{n \times n} \)
- \( \mathcal{M} = \text{Flag}(\mathbb{F}^n) = \{ \mathcal{V} = (V_1, \ldots, V_n) | V_i \subseteq V_{i+1}, \dim \mathbb{F} V_i = i \} \)
- \( \mathcal{U} = \mathbb{F} \setminus \sigma(A) \)
- \( \Phi : \mathcal{M} \times \mathcal{U} \to \mathcal{M} \) defined by
  \[
  \Phi(\mathcal{V}, u) = ((A - uI)^{-1}V_1, \ldots, (A - uI)^{-1}V_n)
  \]
Examples

Example III: Shifted Inverse Iteration on Flag Manifolds

- \( A \in \mathbb{F}^{n \times n} \)
- \( \mathcal{M} = \text{Flag}(\mathbb{F}^n) = \{ \mathcal{V} = (V_1, \ldots, V_n) | V_i \subseteq V_{i+1}, \dim_\mathbb{F} V_i = i \} \)
- \( \mathcal{U} = \mathbb{F} \setminus \sigma(A) \)
- \( \Phi : \mathcal{M} \times \mathcal{U} \rightarrow \mathcal{M} \) defined by
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  \Phi(\mathcal{V}, u) = ((A - uI)^{-1}V_1, \ldots, (A - uI)^{-1}V_n)
  \]

Questions and Remarks

- Algorithm is closely related to the QR algorithm (on isospectral manifolds).
- Can we steer to arbitrary eigenflags?
Examples

Example IV: Shifted QR Algorithm on Isospectral Manifolds

- $A \in \mathbb{C}^{n \times n}$
- $\mathcal{M} = \{Q^*AQ \mid Q \in U_n(\mathbb{C})\}$
- $\mathcal{U} = \mathbb{C} \setminus \sigma(A)$
- $\Phi : \mathcal{M} \times \mathcal{U} \to \mathcal{M}$ defined by

$$
\Phi(X, u) = (X - uI)^*_{U_n(\mathbb{C})}(X - uI)(X - uI)_{U_n(\mathbb{C})}
$$

where $(X - uI)_{U_n(\mathbb{C})}$ is the unitary factor of the QR decomposition
of $(X - uI)$. 
Examples

Example V: Shifted Inverse Iteration on Hessenberg Manifolds

- $A \in \mathbb{F}^{n \times n}$ regular
- $\mathcal{M} = \text{Hess}_A(\mathbb{F}^n) = \{ \mathcal{V} = (V_1, \ldots, V_n) \in \text{Flag}(\mathbb{F}^n) \mid AV_i \subseteq V_{i+1} \}$
- $\mathcal{U} = \mathbb{F} \setminus \sigma(A)$
- $\Phi : \mathcal{M} \times \mathcal{U} \to \mathcal{M}$ defined by
  \[
  \Phi(\mathcal{V}, u) = \left( (A - uI)^{-1}V_1, \ldots, (A - uI)^{-1}V_n \right)
  \]
Example V: Shifted Inverse Iteration on Hessenberg Manifolds

- $A \in \mathbb{F}^{n \times n}$ regular
- $\mathcal{M} = \text{Hess}_A(\mathbb{F}^n) = \{ \mathcal{V} = (V_1, \ldots, V_n) \in \text{Flag}(\mathbb{F}^n) \mid AV_i \subseteq V_{i+1} \}$
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  \]

Questions and Remarks

- Algorithm is closely related to the QR algorithm on Hessenberg Matrices.
- Can we steer to arbitrary Hessenberg eigenflags?
Examples

Example VI: Restart-Shifts of Krylov Methods

\[ \mathcal{M} = \text{Grass}(p, n) \]
\[ \mathcal{U} = \mathbb{R}^k[t] \]
\[ x_0 \in \mathbb{R}^n \setminus \{0\}, \ K(x_0) = \langle x_0, Ax_0, \ldots, A^{p-1}x_0 \rangle \in \mathcal{M} \]
\[ \Phi : \mathcal{M} \times \mathcal{U} \to \mathcal{M} \text{ defined by} \]
\[ \Phi(K(x), u) = u(A)K(x) = K(u(A)x) \]
Examples

Example VI: Restart-Shifts of Krylov Methods

- $\mathcal{M} = \text{Grass}(p, n)$
- $\mathcal{U} = \mathbb{R}^k[t]$
- $x_0 \in \mathbb{R}^n \setminus \{0\}, K(x_0) = \langle x_0, Ax_0, \ldots, A^{p-1}x_0 \rangle \in \mathcal{M}$
- $\Phi : \mathcal{M} \times \mathcal{U} \to \mathcal{M}$ defined by
  \[ \Phi(K(x), u) = u(A)K(x) = K(u(A)x) \]

Questions and Remarks

- Find shifts to approximate specific eigenspaces (Beattie, Embree, Sorensen, Rossi).
Reachable Sets

System $\Sigma = (\mathcal{M}, \mathcal{U}, \Phi); x_0$.

**Definition:** Reachable set of $x_0 \in \mathcal{M}$

$$R(x_0) := \{ x \text{ which can be reached from } x_0 \text{ in finite many steps} \}$$
Reachable Sets

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**Definition:** $k \in \mathbb{N}$, $\Phi_k : \mathcal{M} \times \mathcal{U}^k \rightarrow \mathcal{M}$

$$\Phi_1(x, u) = \Phi(x, u)$$
$$\Phi_k(x, u_1, \ldots, u_k) = \Phi(\Phi(x, u_1, \ldots, u_{k-1}), u_k)$$
Reachable Sets

System $\Sigma = (\mathcal{M}, \mathcal{U}, \Phi); \ x_0$.

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**Proposition:**

$$R(x_0) := \{ x \in \mathcal{M} \mid \exists N \in \mathbb{N}, \exists u \in \mathcal{U}^N : x = \Phi_N(x_0, u) \}$$
Reachable Sets

System \( \Sigma = (\mathcal{M}, \mathcal{U}, \Phi) \).

**Definition:** Reachable set of \( x_0 \in \mathcal{M} \)

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Reachable Sets

System $\Sigma = (\mathcal{M}, \mathcal{U}, \Phi)$.

**Definition**: Reachable set of $x_0 \in \mathcal{M}$

$$R(x_0) := \{ x \text{ which can be reached from } x_0 \text{ in finite many steps} \}$$

**Definition**: System Semigroup

$$\Gamma_\Sigma := \{ \Phi : \mathcal{M} \to \mathcal{M} \mid \exists N \in \mathbb{N}, \exists u \in \mathcal{U}^N : \Phi = \Phi_N(\cdot, u) \}$$
Reachable Sets

System $\Sigma = (\mathcal{M}, \mathcal{U}, \Phi)$.

**Definition:** Reachable set of $x_0 \in \mathcal{M}$

$$R(x_0) := \{ x \text{ which can be reached from } x_0 \text{ in finite many steps} \}$$

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$$\Gamma_{\Sigma} := \{ \Phi : \mathcal{M} \to \mathcal{M} | \exists N \in \mathbb{N}, \exists u \in \mathcal{U}^N : \Phi = \Phi_N (\cdot, u) \}$$

**Proposition:** The reachable set of $x_0 \in \mathcal{M}$ is always an orbit of the semigroup action $\alpha : \Gamma_{\Sigma} \times \mathcal{M} \to \mathcal{M}, \alpha(\Phi, x) = \Phi(x)$. I.e

$$R(x_0) = \alpha(\Gamma_{\Sigma}, x_0)$$
Reachable Sets

System $\Sigma = (\mathcal{M}, \mathcal{U}, \Phi)$.

Definition: Reachable set of $x_0 \in \mathcal{M}$

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Proposition: The reachable set of $x_0 \in \mathcal{M}$ is always an orbit of the semigroup action $\alpha : \Gamma_\Sigma \times \mathcal{M} \to \mathcal{M}, \alpha(\Phi, x) = \Phi(x)$. I.e

$$R(x_0) = \alpha(\Gamma_\Sigma, x_0)$$

Corollary: If $\Gamma_\Sigma$ is a group the reachable sets form a partition on $\mathcal{M}$. 
Reachable Sets

Example I, III and IV (Scalar Shifted Inverse Iteration)

- \( A \in \mathbb{F}^{n \times n} \) regular, \( \mathcal{M} = \mathbb{F}^{n-1}, \text{Flag}(\mathbb{F}^n), \text{Hess}_A(\mathbb{F}^n) \), \( \mathcal{U} = \mathbb{F} \setminus \sigma(A) \)
- \( \Phi : \mathcal{M} \times \mathcal{U} \rightarrow \mathcal{M} \) defined by \( \Phi(x, u) = (A - uI)^{-1} \cdot x \)
Reachable Sets

Example I, III and IV (Scalar Shifted Inverse Iteration)

- $A \in \mathbb{F}^{n \times n}$ regular, $\mathcal{M} = \mathbb{F}^{n-1}$, $\text{Flag}(\mathbb{F}^n)$, $\text{Hess}_A(\mathbb{F}^n)$, $\mathcal{U} = \mathbb{F} \setminus \sigma(A)$
- $\Phi : \mathcal{M} \times \mathcal{U} \to \mathcal{M}$ defined by $\Phi(x, u) = (A - uI)^{-1} \cdot x$

Proposition: For all Scalar Shifted Inverse Iterations

$$\Gamma_\Sigma := \left\{ \prod_{t=1}^{N} (A - u_tI)^{-1} \mid N \in \mathbb{N}, u_t \in \mathbb{F} \setminus \sigma(A) \right\}$$
Reachable Sets

Example I, III and IV (Scalar Shifted Inverse Iteration)

- $A \in \mathbb{F}^{n \times n}$ regular, $\mathcal{M} = \mathbb{F}P^{n-1}, \text{Flag}(\mathbb{F}^n), \text{Hess}_A(\mathbb{F}^n)$,
  $\mathcal{U} = \mathbb{F} \setminus \sigma(A)$
- $\Phi : \mathcal{M} \times \mathcal{U} \rightarrow \mathcal{M}$ defined by $\Phi(x, u) = (A - uI)^{-1} \cdot x$

**Proposition:** For all Scalar Shifted Inverse Iterations

$$\Gamma_\Sigma := \left\{ \prod_{t=1}^{N} (A - u_tI)^{-1} \mid N \in \mathbb{N}, u_t \in \mathbb{F} \setminus \sigma(A) \right\}$$

**Theorem:** (Helmke, J 2002) For $\mathbb{F} = \mathbb{C}$, $\Gamma_\Sigma$ is a Group. If $A$ is diagonalizable then $\Gamma_\Sigma$ is homeomorphic to $(\mathbb{C}^*)^k$ whereas $k$ is the number of different eigenvalues of $A$. 
Reachable Sets

Example I, III and IV (Scalar Shifted Inverse Iteration)

- \( A \in \mathbb{F}^{n \times n} \) regular, \( \mathcal{M} = \mathbb{F}P^{n-1}, \) Flag(\( \mathbb{F}^n \)), Hess\( _A (\mathbb{F}^n) \), \( \mathcal{U} = \mathbb{F} \setminus \sigma (A) \)
- \( \Phi : \mathcal{M} \times \mathcal{U} \to \mathcal{M} \) defined by \( \Phi(x, u) = (A - uI)^{-1} \cdot x \)

**Proposition:** For all Scalar Shifted Inverse Iterations

\[
\Gamma_\Sigma := \left\{ \prod_{t=1}^{N} (A - u_t I)^{-1} \mid N \in \mathbb{N}, u_t \in \mathbb{F} \setminus \sigma (A) \right\}
\]

**Theorem:** (Helmke, J 2002) For \( \mathbb{F} = \mathbb{C} \), \( \Gamma_\Sigma \) is a Group. If \( A \) is diagonalizable then \( \Gamma_\Sigma \) is homeomorphic to \( (\mathbb{C}^*)^k \) whereas \( k \) is the number of different eigenvalues of \( A \).

**Theorem:** (J 2003) For \( \mathbb{F} = \mathbb{R} \) there exists an open set of Matrices \( S \subset \mathbb{R}^{n \times n} \) such that \( \Gamma_\Sigma \) is not a Group.
Reachable Sets

Example I, Shifted Inverse Iteration on \( CP^{n-1} \)

- \( A \in \mathbb{C}^{n \times n} \), cyclic (i.e: it exists \( v \in \mathbb{C}^n \) such that \( \langle x, Ax, A^2x, \ldots A^{n-1}x \rangle = \mathbb{C}^n \)).

- \( \mathcal{M} = \mathbb{C}P^{n-1} \)

- \( \mathcal{U} = \mathbb{C} \setminus \sigma(A) \)

- \( \Phi : \mathcal{M} \times \mathcal{U} \rightarrow \mathcal{M} \) defined by \( \Phi(x, u) = (A - uI)^{-1} \cdot x \)
Reachable Sets

Example I, Shifted Inverse Iteration on $\mathbb{CP}^{n-1}$

- $A \in \mathbb{C}^{n \times n}$, cyclic (i.e. it exists $v \in \mathbb{C}^n$ such that $\langle x, Ax, A^2 x, \ldots A^{n-1} x \rangle = \mathbb{C}^n$).
- $\mathcal{M} = \mathbb{CP}^{n-1}$
- $\mathcal{U} = \mathbb{C} \setminus \sigma(A)$
- $\Phi : \mathcal{M} \times \mathcal{U} \to \mathcal{M}$ defined by $\Phi(x, u) = (A - uI)^{-1} \cdot x$

**Theorem:** (Helmke, Fuhrmann 2000) Let $\mathbb{F} = \mathbb{C}$ and $A$ be cyclic. There is a bijective correspondence between the closures of the reachable sets $\overline{R(x)}$ and the $A$-invariant subspaces of $\mathbb{C}^n$. 
Controllability

System $\Sigma = (\mathcal{M}, \mathcal{U}, \Phi)$.

**Definition:** System $\Sigma = (\mathcal{M}, \mathcal{U}, \Phi)$ is said to be controllable if there exist $x_0 \in \mathcal{M}$ such that every point in $\mathcal{M}$ can be reached from $x_0$ at least arbitrarily close. (I.e.

$$\exists x_0 \in \mathcal{M} : \overline{R(x_0)} = \mathcal{M}$$
Controllability

System $\Sigma = (\mathcal{M}, \mathcal{U}, \Phi)$.

**Definition:** System $\Sigma = (\mathcal{M}, \mathcal{U}, \Phi)$ is said to be **controllable** if there exist $x_0 \in \mathcal{M}$ such that every point in $\mathcal{M}$ can be reached from $x_0$ at least arbitrarily close. (i.e.

$$\exists x_0 \in \mathcal{M} : \overline{R(x_0)} = \mathcal{M}$$

**Remark:** If $\Sigma = (\mathcal{M}, \mathcal{U}, \Phi)$ is controllable and $\Gamma_\Sigma$ is a group, then every neighbourhood can be reached from every neighbourhood.
Controllability

Example I, Shifted Inverse Iteration on $\mathbb{FP}^{n-1}$

- $A \in \mathbb{F}^{n \times n}$
- $\mathcal{M} = \mathbb{FP}^{n-1}$
- $\mathcal{U} = \mathbb{F} \setminus \sigma(A)$
- $\Phi: \mathcal{M} \times \mathcal{U} \to \mathcal{M}$ defined by $\Phi(x, u) = (A - uI)^{-1} \cdot x$
Controllability

Example I, Shifted Inverse Iteration on $\mathbb{F}P^{n-1}$

- $A \in \mathbb{F}^{n \times n}$
- $\mathcal{M} = \mathbb{F}P^{n-1}$
- $\mathcal{U} = \mathbb{F} \setminus \sigma(A)$
- $\Phi : \mathcal{M} \times \mathcal{U} \to \mathcal{M}$ defined by $\Phi(x, u) = (A - uI)^{-1} \cdot x$

**Theorem:** (Helmke, Fuhrmann 2000)
Let $\mathbb{F} = \mathbb{C}$. Shifted Inverse Iteration on $\mathbb{C}P^{n-1}$ is controllable if and only if $A$ is cyclic (i.e.: it exists $v \in \mathbb{C}^n$ such that $\langle x, A x, A^2 x, \ldots A^{n-1} x \rangle = \mathbb{C}^n$).
Controllability

Example I, Shifted Inverse Iteration on $\mathbb{F}\mathbb{P}^{n-1}$

- $A \in \mathbb{F}^{n \times n}$
- $\mathcal{M} = \mathbb{F}\mathbb{P}^{n-1}$
- $\mathcal{U} = \mathbb{F} \setminus \sigma(A)$
- $\Phi : \mathcal{M} \times \mathcal{U} \to \mathcal{M}$ defined by $\Phi(x,u) = (A - uI)^{-1} \cdot x$

**Theorem:** (Helmke, Fuhrmann 2000)

Let $\mathbb{F} = \mathbb{C}$. Shifted Inverse Iteration on $\mathbb{C}\mathbb{P}^{n-1}$ is controllable if and only if $A$ is cyclic (i.e.: it exists $v \in \mathbb{C}^n$ such that $\langle x, Ax, A^2x, \ldots A^{n-1}x \rangle = \mathbb{C}^n$).

**Corollary:** For $\mathbb{F} = \mathbb{C}$, $A$ cyclic. One can steer nearly every initial point to every specific target point.
Controllability

Example I, Shifted Inverse Iteration on $\mathbb{F}P^{n-1}$

- $A \in \mathbb{F}^{n \times n}$
- $\mathcal{M} = \mathbb{F}P^{n-1}$
- $\mathcal{U} = \mathbb{F} \setminus \sigma(A)$
- $\Phi : \mathcal{M} \times \mathcal{U} \to \mathcal{M}$ defined by $\Phi(x, u) = (A - uI)^{-1} \cdot x$

Theorem: (Helmke, Fuhrmann 2000)
Let $\mathbb{F} = \mathbb{C}$. Shifted Inverse Iteration on $\mathbb{C}P^{n-1}$ is controllable if and only if $A$ is cyclic (i.e.: it exists $v \in \mathbb{C}^n$ such that $\langle x, Ax, A^2x, \ldots A^{n-1}x \rangle = \mathbb{C}^n$).

Corollary: For $\mathbb{F} = \mathbb{C}$, $A$ cyclic. One can steer nearly every initial point to every specific target point.

Corollary: For $\mathbb{F} = \mathbb{C}$ controllability is a generic property of the Shifted Inverse Iteration. i.e.: It holds true for an open and dense set of matrices $\mathcal{S} \subset \mathbb{C}^{n \times n}$. 
Controllability

Example I, Shifted Inverse Iteration on $\mathbb{F}P^{n-1}$

- $A \in \mathbb{F}^{n \times n}$
- $\mathcal{M} = \mathbb{F}P^{n-1}$
- $\mathcal{U} = \mathbb{F} \setminus \sigma(A)$
- $\Phi : \mathcal{M} \times \mathcal{U} \rightarrow \mathcal{M}$ defined by $\Phi(x, u) = (A - uI)^{-1} \cdot x$

Remark: Let $\mathbb{F} = \mathbb{R}$. If $A$ is not cyclic, the Shifted Inverse Iteration on $\mathbb{R}P^{n-1}$ is not controllable. There exist cyclic matrices such that Shifted Inverse Iteration on $\mathbb{R}P^{n-1}$ is not controllable.

Remark: For $\mathbb{F} = \mathbb{R}$ it is unknown if controllability is a generic property of the Shifted Inverse Iteration.
Controllability

Example I, Shifted Inverse Iteration on $FP^{n-1}$

- $A \in F^{n \times n}$
- $\mathcal{M} = FP^{n-1}$
- $\mathcal{U} = F \setminus \sigma(A)$
- $\Phi : \mathcal{M} \times \mathcal{U} \to \mathcal{M}$ defined by $\Phi(x, u) = (A - uI)^{-1} \cdot x$

Remark:
Let $F = \mathbb{R}$. If $A$ is not cyclic, the Shifted Inverse Iteration on $RP^{n-1}$ is not controllable. There exist cyclic matrices such that Shifted Inverse Iteration on $RP^{n-1}$ is not controllable.
Controllability

Example I, Shifted Inverse Iteration on $\mathbb{F} \mathbb{P}^{n-1}$

- $A \in \mathbb{F}^{n \times n}$
- $\mathcal{M} = \mathbb{F} \mathbb{P}^{n-1}$
- $\mathcal{U} = \mathbb{F} \setminus \sigma(A)$
- $\Phi : \mathcal{M} \times \mathcal{U} \rightarrow \mathcal{M}$ defined by $\Phi(x, u) = (A - uI)^{-1} \cdot x$

Remark:
Let $\mathbb{F} = \mathbb{R}$. If $A$ is not cyclic, the Shifted Inverse Iteration on $\mathbb{R} \mathbb{P}^{n-1}$ is not controllable. There exist cyclic matrices such that Shifted Inverse Iteration on $\mathbb{R} \mathbb{P}^{n-1}$ is not controllable.

Remark:
For $\mathbb{F} = \mathbb{R}$ it is unknown if controllability is a generic property of the Shifted Inverse Iteration.
Controllability

Example I\(^{1/2}\), Polynomial-Shift Inverse Iteration on \(\mathbb{FP}^{n-1}\)

- \(A \in \mathbb{F}^{n \times n}, \mathcal{M} = \mathbb{FP}^{n-1}\)
- \(p_\alpha(u) = A - uI, \ p_\beta(v, w) = A^2 + vA + wI\)
- \(\mathcal{U} := \{u \in \mathbb{F}, (v, w) \in \mathbb{F}^2 \mid p_\alpha(u), p_\beta(v, w) \in \text{GL}_n(\mathbb{F})\}\)
- \(\Phi(x, u) = p_{\pi(t)}^{-1}(u)x, \pi(t) \in \{\alpha, \beta\}\)
Controllability

Example $I^\frac{1}{2}$, Polynomial-Shift Inverse Iteration on $\mathbb{FP}^{n-1}$

- $A \in \mathbb{F}^{n \times n}, \mathcal{M} = \mathbb{FP}^{n-1}$
- $p_\alpha(u) = A - uI$, $p_\beta(v, w) = A^2 + vA + wI$
- $\mathcal{U} := \{u \in \mathbb{F}, (v, w) \in \mathbb{F}^2 \mid p_\alpha(u), p_\beta(v, w) \in \text{GL}_n(\mathbb{F})\}$
- $\Phi(x, u) = p_{\pi(t)}^{-1}(u)x$, $\pi(t) \in \{\alpha, \beta\}$

**Theorem:** (J 2003)
Let $\mathbb{F} = \mathbb{R}, \mathbb{C}$. Polynomial-Shift Inverse Iteration on $\mathbb{FP}^{n-1}$ is controllable if and only if $A$ is cyclic.
Controllability

Example $1^1_2$, Polynomial-Shift Inverse Iteration on $\mathbb{FP}^{n-1}$

- $A \in \mathbb{F}^{n \times n}$, $\mathcal{M} = \mathbb{FP}^{n-1}$
- $p_\alpha(u) = A - uI$, $p_\beta(v, w) = A^2 + vA + wI$
- $\mathcal{U} := \{u \in \mathbb{F}, (v, w) \in \mathbb{F}^2 | p_\alpha(u), p_\beta(v, w) \in \text{GL}_n(\mathbb{F})\}$
- $\Phi(x, u) = p_{\pi(t)}^{-1}(u)x$, $\pi(t) \in \{\alpha, \beta\}$

Theorem: (J 2003)
Let $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$. Polynomial-Shift Inverse Iteration on $\mathbb{FP}^{n-1}$ is controllable if and only if $A$ is cyclic.

Corollary: Controllability is a generic property of the Polynomial-Shift Inverse Iteration.
Controllability

Example III, Shifted Inverse Iteration on $\text{Flag}(\mathbb{C}^n)$

- $A \in \mathbb{C}^{n \times n}$
- $\mathcal{M} = \text{Flag}(\mathbb{C}^n)$
- $\mathcal{U} = \mathbb{F} \setminus \sigma(A)$
- $\Phi : \mathcal{M} \times \mathcal{U} \to \mathcal{M}$ defined by $\Phi(\mathcal{V}, u) = (A - uI)^{-1} \cdot \mathcal{V}$

Remark: Let $n > 2$. The Shifted Inverse Iteration on $\text{Flag}(\mathbb{C}^n)$ is not controllable.

Corollary: Let $n > 2$. The Shifted Inverse Iteration on the isospectral manifold $\mathcal{M} = f Q A f Q^2$ is not controllable.
Controllability

Example III, Shifted Inverse Iteration on $\text{Flag}(\mathbb{C}^n)$

- $A \in \mathbb{C}^{n \times n}$
- $\mathcal{M} = \text{Flag}(\mathbb{C}^n)$
- $\mathcal{U} = \mathbb{F} \setminus \sigma(A)$
- $\Phi : \mathcal{M} \times \mathcal{U} \to \mathcal{M}$ defined by $\Phi(\mathcal{V}, u) = (A - uI)^{-1} \cdot \mathcal{V}$

**Remark:**
Let $n > 2$. The Shifted Inverse Iteration on $\text{Flag}(\mathbb{C}^n)$ is not controllable.
Controllability

Example III, Shifted Inverse Iteration on \( \text{Flag}(\mathbb{C}^n) \)

- \( A \in \mathbb{C}^{n \times n} \)
- \( \mathcal{M} = \text{Flag}(\mathbb{C}^n) \)
- \( \mathcal{U} = \mathbb{F} \setminus \sigma(A) \)
- \( \Phi : \mathcal{M} \times \mathcal{U} \rightarrow \mathcal{M} \) defined by \( \Phi(\mathcal{V}, u) = (A - uI)^{-1} \cdot \mathcal{V} \)

Remark:
Let \( n > 2 \). The Shifted Inverse Iteration on \( \text{Flag}(\mathbb{C}^n) \) is not controllable.

Corollary: Let \( n > 2 \). The Shifted Inverse Iteration on the isospectral manifold \( \mathcal{M}_A = \{Q^*AQ \mid Q \in \text{U}_n(\mathbb{C})\} \) is not controllable.
Controllability

Example III, Shifted Inverse Iteration on $\text{Hess}_A(\mathbb{F}^n)$

- $A \in \mathbb{F}^{n \times n}$ regular
- $\mathcal{M} = \text{Hess}_A(\mathbb{F}^n)$
- $\mathcal{U} = \mathbb{F} \setminus \sigma(A)$
- $\Phi : \mathcal{M} \times \mathcal{U} \to \mathcal{M}$ defined by $\Phi(\mathcal{V}, u) = (A - uI)^{-1} \cdot \mathcal{V}$
Controllability

Example III, Shifted Inverse Iteration on $\text{Hess}_A(\mathbb{F}^n)$

- $A \in \mathbb{F}^{n \times n}$ regular
- $\mathcal{M} = \text{Hess}_A(\mathbb{F}^n)$
- $\mathcal{U} = \mathbb{F} \setminus \sigma(A)$
- $\Phi : \mathcal{M} \times \mathcal{U} \rightarrow \mathcal{M}$ defined by $\Phi(\mathcal{V}, u) = (A - uI)^{-1} \cdot \mathcal{V}$

Theorem: (Helmke, J 2002)
Shifted Inverse Iteration on $\text{Hess}_A(\mathbb{F}^n)$ is controllable (for $A$) if and only if Shifted Inverse Iteration on $\mathbb{F}^n\setminus \mathbb{F}I^{n-1}$ is controllable (for $A$).
Controllability

Example III, Shifted Inverse Iteration on $\text{Hess}_A(\mathbb{F}^n)$

- $A \in \mathbb{F}^{n \times n}$ regular
- $\mathcal{M} = \text{Hess}_A(\mathbb{F}^n)$
- $\mathcal{U} = \mathbb{F} \setminus \sigma(A)$
- $\Phi : \mathcal{M} \times \mathcal{U} \to \mathcal{M}$ defined by $\Phi(\mathcal{V}, u) = (A - uI)^{-1} \cdot \mathcal{V}$

**Theorem:** (Helmke, J 2002)
Shifted Inverse Iteration on $\text{Hess}_A(\mathbb{F}^n)$ is controllable (for $A$) if and only if Shifted Inverse Iteration on $\mathbb{F} \mathbb{P}^{n-1}$ is controllable (for $A$).

**Corollary:** Let $\mathbb{F} = \mathbb{C}$. Shifted Inverse Iteration on $\text{Hess}_A(\mathbb{C}^n)$ is controllable if and only if $A$ is cyclic.
Controllability

Example III, Shifted Inverse Iteration on $\text{Hess}_A(\mathbb{F}^n)$

- $A \in \mathbb{F}^{n \times n}$ regular
- $\mathcal{M} = \text{Hess}_A(\mathbb{F}^n)$
- $\mathcal{U} = \mathbb{F} \setminus \sigma(A)$
- $\Phi : \mathcal{M} \times \mathcal{U} \to \mathcal{M}$ defined by $\Phi(\mathcal{V}, u) = (A - uI)^{-1} \cdot \mathcal{V}$

**Theorem:** (Helmke, J 2002)
Shifted Inverse Iteration on $\text{Hess}_A(\mathbb{F}^n)$ is controllable (for $A$) if and only if Shifted Inverse Iteration on $\mathbb{F} \mathbb{P}^{n-1}$ is controllable (for $A$).

**Corollary:** Let $\mathbb{F} = \mathbb{C}$. Shifted Inverse Iteration on $\text{Hess}_A(\mathbb{C}^n)$ is controllable if and only if $A$ is cyclic.

**Corollary:** Let $\mathbb{F} = \mathbb{C}$. Controllability is a generic property of the Shifted Inverse Iteration on $\text{Hess}_A(\mathbb{C}^n)$. 
Current and Future Work

Is shifted Inverse Iteration on $\mathbb{RP}^{n-1}$ resp. $\text{Hess}_A(\mathbb{R}^n)$ generic?
Current and Future Work

- Is shifted Inverse Iteration on $\mathbb{RP}^{n-1}$ resp. $\text{Hess}_A(\mathbb{R}^n)$ generic?

- Characterizations of system semigroups
Current and Future Work

- Is shifted Inverse Iteration on $\mathbb{R}P^{n-1}$ resp. $\text{Hess}_A(\mathbb{R}^n)$ generic?
- Characterizations of system semigroups
- Criteria for controllability
Current and Future Work

- Is shifted Inverse Iteration on $\mathbb{R}P^{n-1}$ resp. $\text{Hess}_A(\mathbb{R}^n)$ generic?
- Characterizations of system semigroups
- Criteria for controllability
- Adherence structure of reachable sets
Current and Future Work

- Is shifted Inverse Iteration on $\mathbb{R}P^{n-1}$ resp. $\text{Hess}_A(\mathbb{R}^n)$ generic?

- Characterizations of system semigroups

- Criteria for controllability

- Adherence structure of reachable sets

- Constructive controllability
Thank you for your attention

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