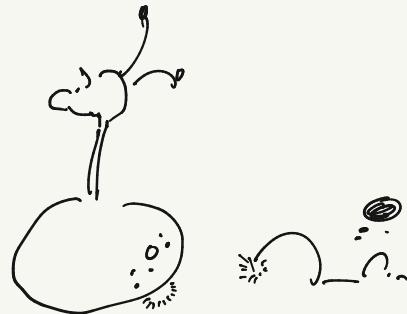


COLOR FLUID DYNAMICS :-)

WHAT
IS
CFD ?



COMPUTATIONAL
FLUID
DYNAMICS :-L

A quoi servent les méthodes numériques ?

End of the Millenium Question

Do the Bubbles in a Glass of Guinness Beer Go Up or Down?

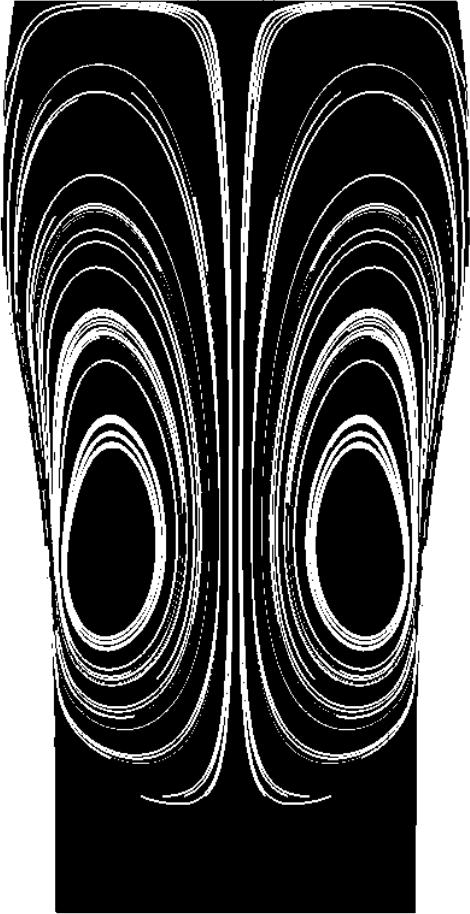


Since time immemorial, mankind has been troubled by natural phenomena that cannot be easily explained. Why do the bubbles in a glass of that venerable dark brew called Guinness appear to travel downward in the glass?

This goes against what we know about the physics of bubbles, or does it?

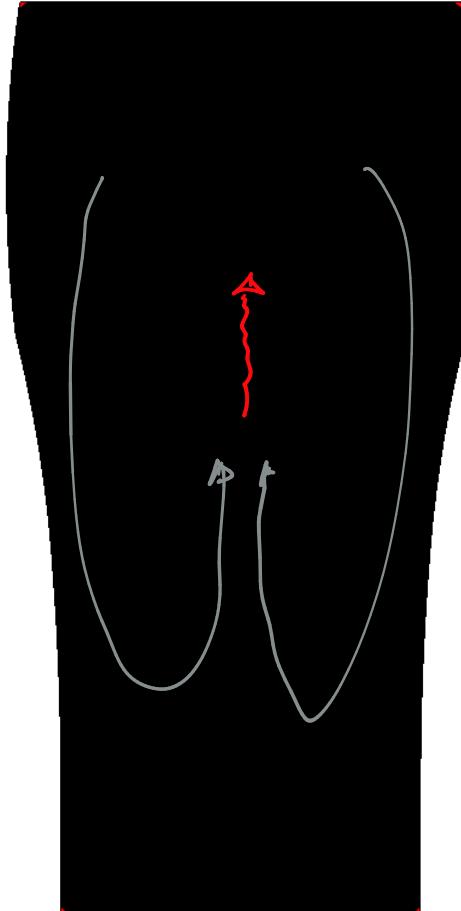
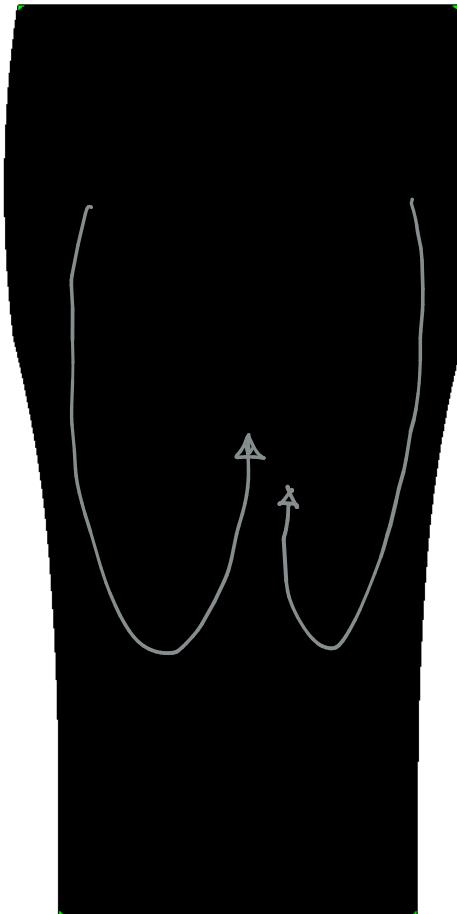
The answer to this most fundamental question has been locked away for centuries, resisting the direct assault of the most determined philosophers and theoreticians without compassion.

But, with the help of Computational Fluid Dynamics, we can provide a categorical and definitive answer: the bubbles go both up and down.



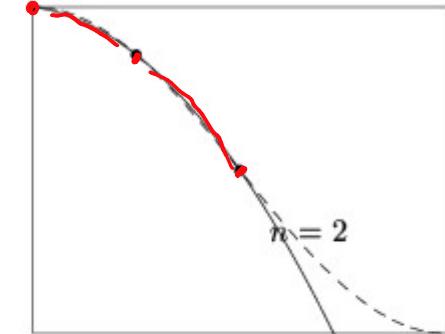
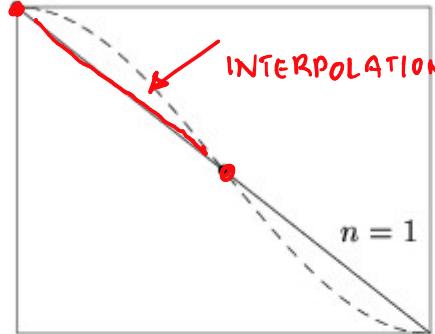
Fletcher et al. (1999)
CANCES-UNSW Australia
Fluent United States

La réponse des scientifiques...



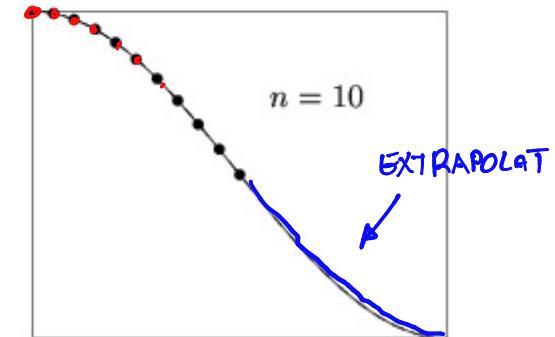
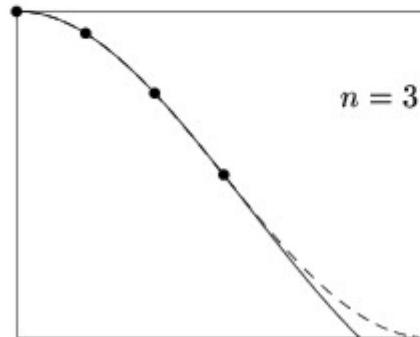
Bubble tracks show that the 1 mm bubbles move steadily upwards while the 60 micron bubbles are dragged downwards near the side of the glass.

Convergence



Convergence de l'interpolation polynomiale de cos(x)

$$e^h(x) = u(x) - u^h(x)$$

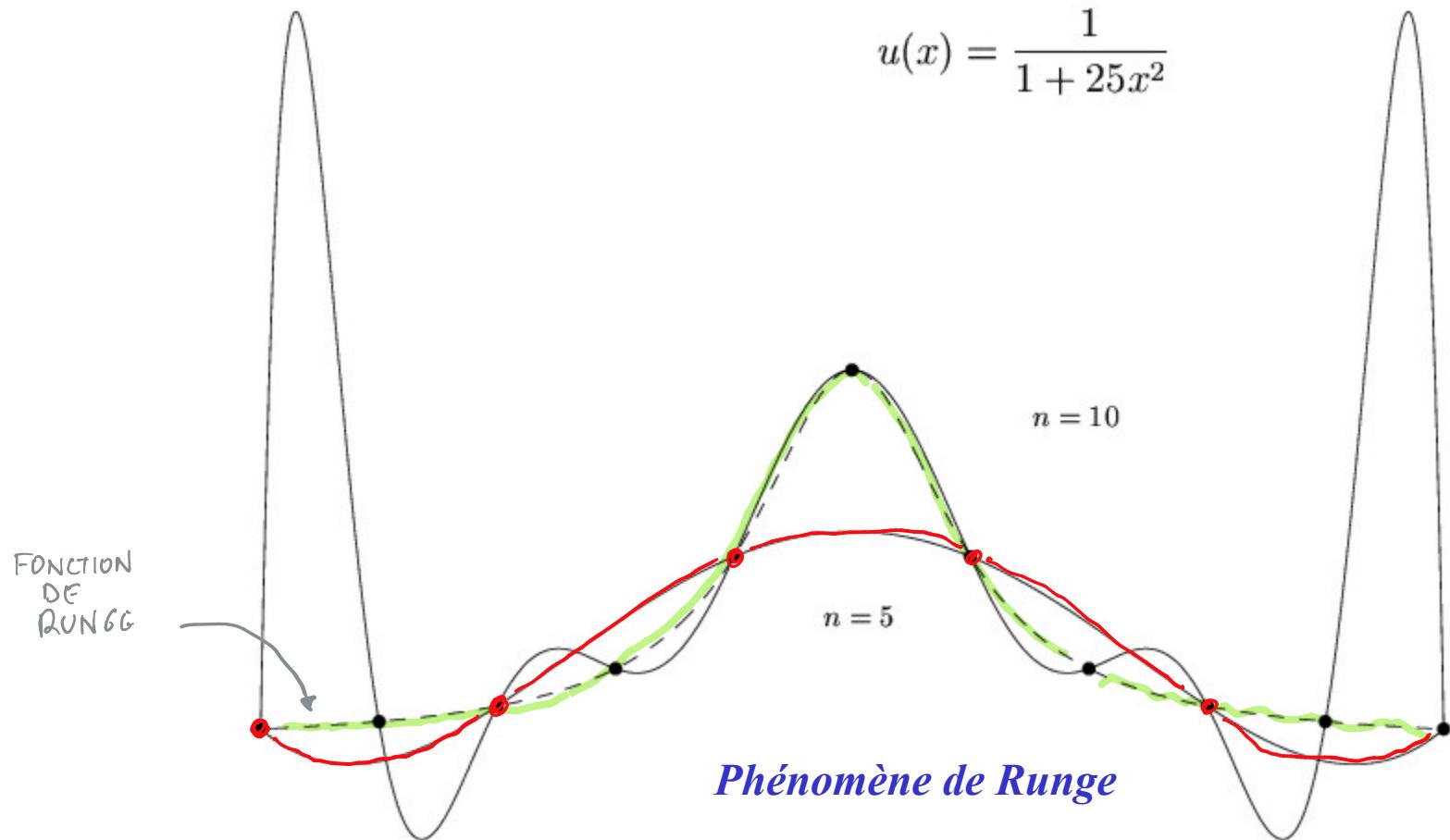


Définition 1.3.

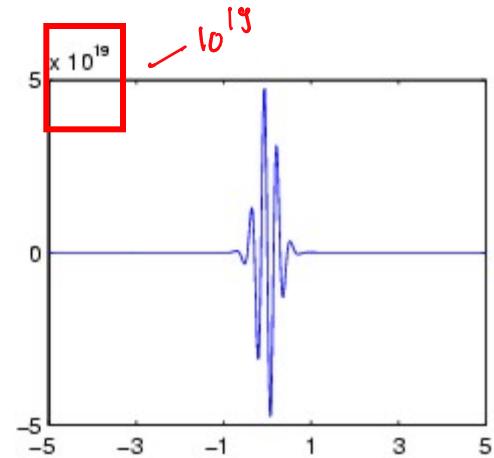
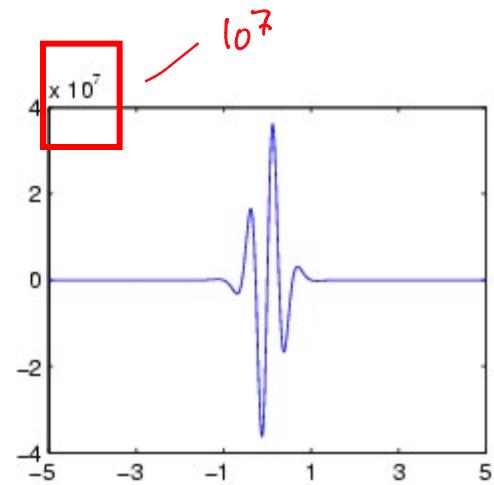
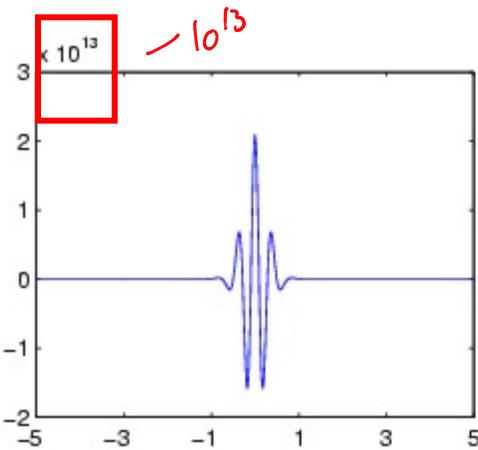
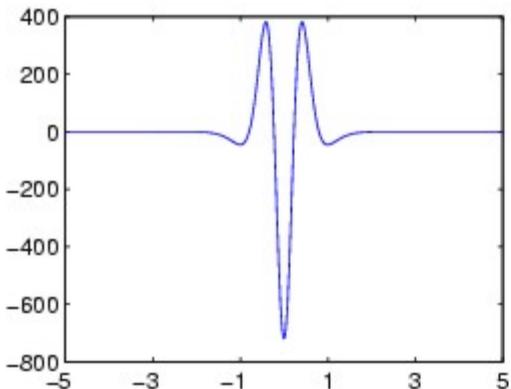
Une interpolation est dite convergente si l'erreur d'interpolation tend vers zéro lorsque le nombre de degrés de liberté, c'est-à-dire n tend vers l'infini :

$$\lim_{n \rightarrow \infty} e^h(x) = 0 \quad \text{pour } x \in [X_0, X_n].$$

L'interpolation polynomiale, parfois cela ne converge pas...



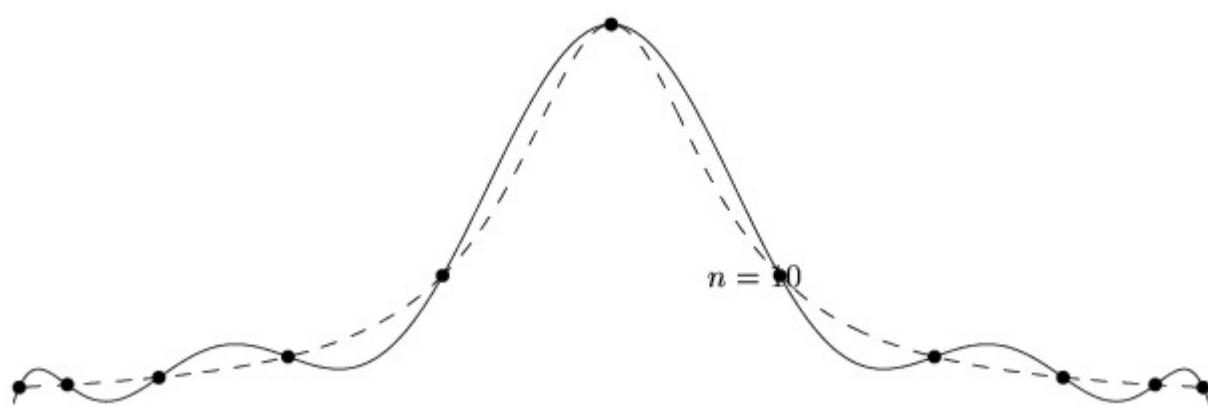
Why
does
it not
work ?



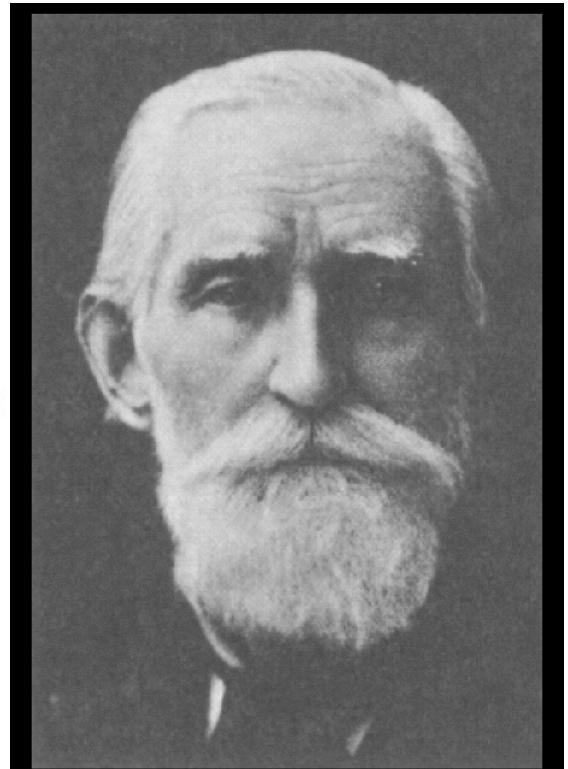
Dérivées d'ordre 6, 11,
16 et 21 de la fonction
de Runge

$$e^h(x) = \frac{u^{(n+1)}(\xi(x))}{(n+1)!} (x - X_0)(x - X_1)(x - X_2) \cdots (x - X_n).$$

Parfois, on peut sauver la mise...



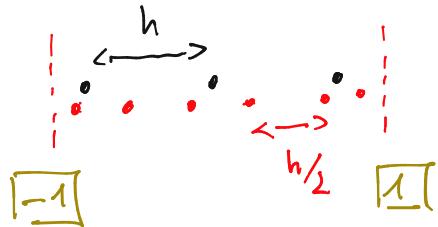
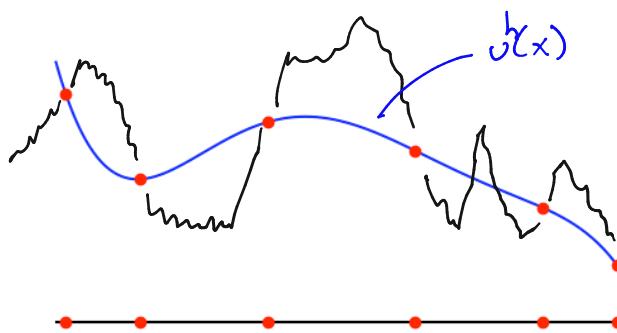
Abscisses de Chebyshev



ПАФНУТИЙ ЛЬВОВИЧ ЧЕБЫШЕВ

Pafnuty Lvovitch Chebyshev (1821-1894)

Les abscisses de Chebychev...



$$\left| \underbrace{v(x) - v^h(x)}_{e^{h(x)}} \right| \leq \left| \frac{v^{(m+1)}(\xi(x))}{(m+1)!} \right| \underbrace{(x-x_0)(x-x_1)\dots(x-x_n)}_{\leq n! h^{n+1}} \leq \frac{n! h^{n+1}}{4}$$

$$\boxed{v^{(m+1)}(\xi(x))} \quad \begin{matrix} m \\ m+1 \end{matrix}$$

$$\boxed{\frac{h^{m+1}}{4(m+1)}} \quad \begin{matrix} m \\ h \end{matrix}$$

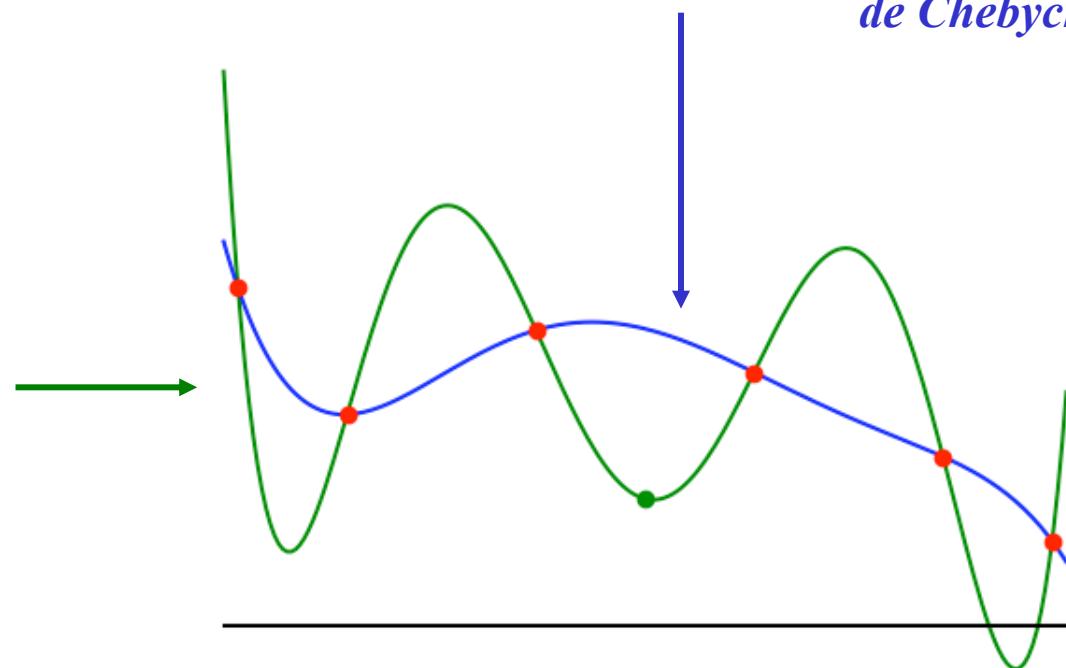
$$X_i = \cos \left[\frac{(2i+1)\pi}{2(m+1)} \right]$$

S'ARRANGER
POUR QUE CE TERME
DIMINUE LE PLUS
EFFICACEMENT POSSIBLE

Interpolons un polynôme par un polynôme...

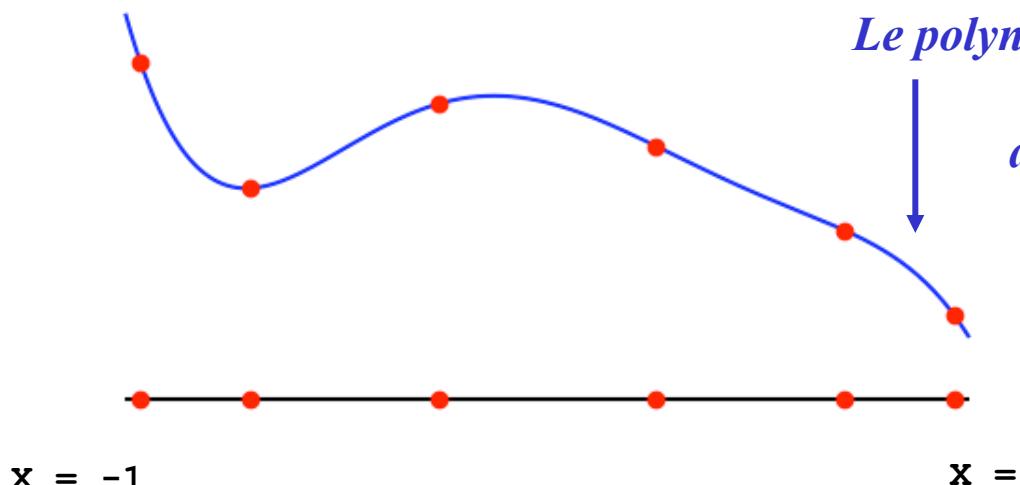
u^h : polynôme de degré cinq
interpolant u aux abscisses
de Chebychev

u : un polynôme
quelconque
de degré six



...c'est bête, je sais :-)

Six points aux abscisses de Chebyshev....



Le polynôme de degré cinq passant par six points aux abscisses de Chebyshev

$$x = -1$$

$$x = 1$$

```
n = 5  
x = cos(pi * (2*arange(0,n+1) + 1)/((n+1) * 2))  
U = [0.2,0.4,0.6,0.7,0.5,0.8]  
  
puh = polyfit(x,U,5)  
x = linspace(-1,1,200)  
uh = polyval(puh,x)  
plt.plot(x,uh,'-b')  
plt.plot(x,U,'or')
```

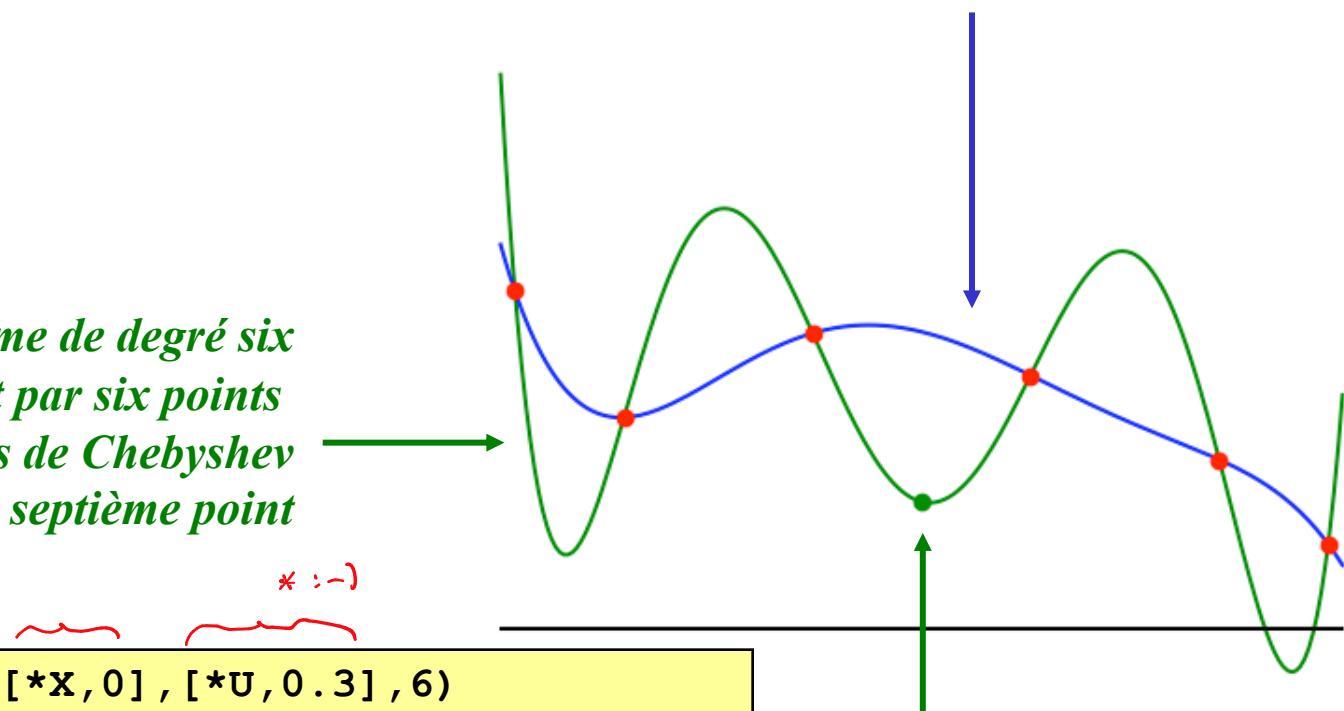
DEGRE
DU POLYNOME

Ajoutons un septième point !

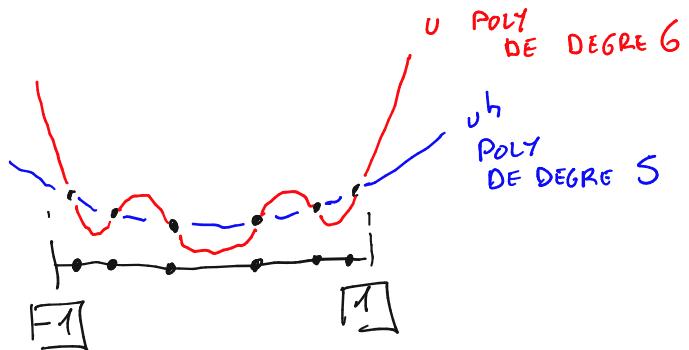
Le polynôme de degré cinq passant par six points aux abscisses de Chebyshev

Le polynôme de degré six passant par six points aux abscisses de Chebyshev et par le septième point

```
pu = polyfit([*x,0],[*U,0.3],6)
u = polyval(pu,x)
plt.plot(x,u,'-g')
plt.plot([0],[0.3],'og')
```



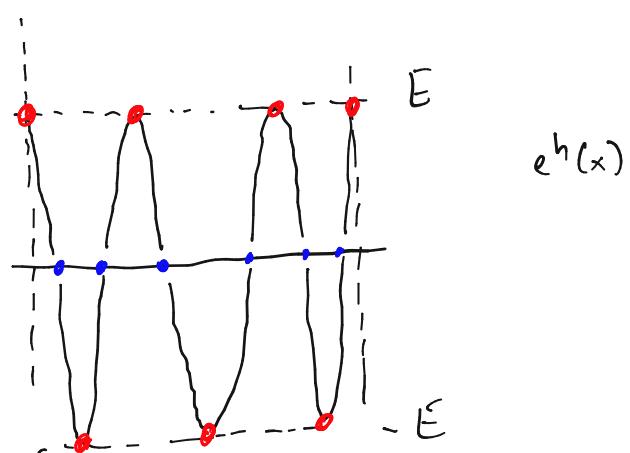
Le septième point

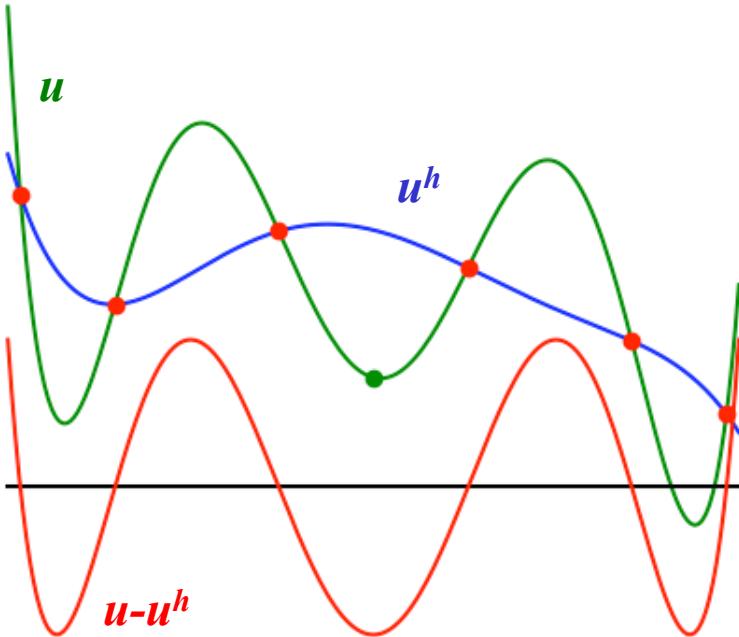


$$a_6 x^6 + a_5 x^5 \dots + a_0 \\ b_5 x^5 \dots + b_0$$

$$p_u = [a_6 \ a_5 \ \dots \ a_0] \\ p_{uh} = [b_5 \ \dots \ b_0]$$

$$p_{eh} = p_u - \underbrace{[0 * p_{uh}]}_{[0 \ b_5 \ b_4 \ \dots \ b_0]}$$

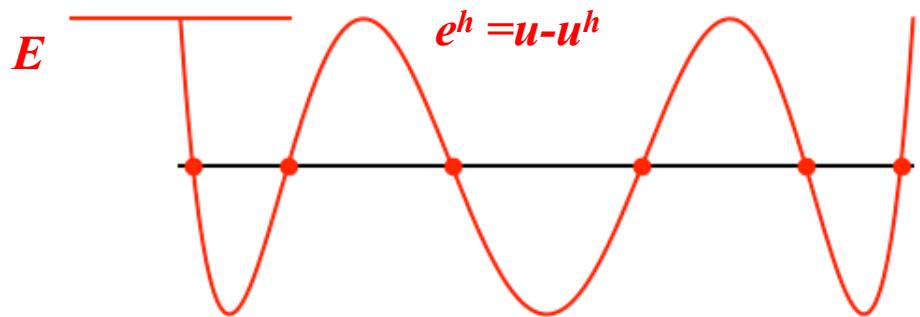




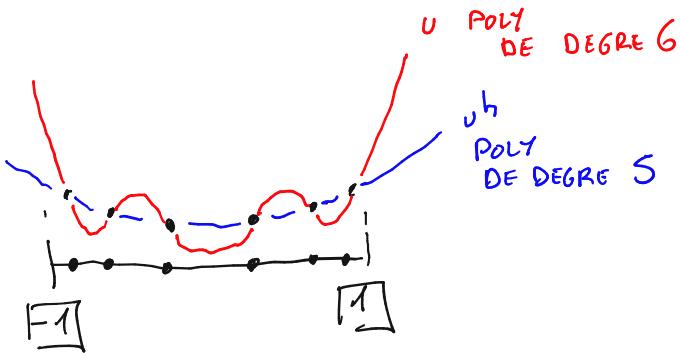
```

error = u - uh
E = error[0]
plt.plot(x,error,'-r');

```



Erreur d'interpolation



$$a_6 x^6 + a_5 x^5 \dots + a_0 \\ b_5 x^5 \dots + b_0$$

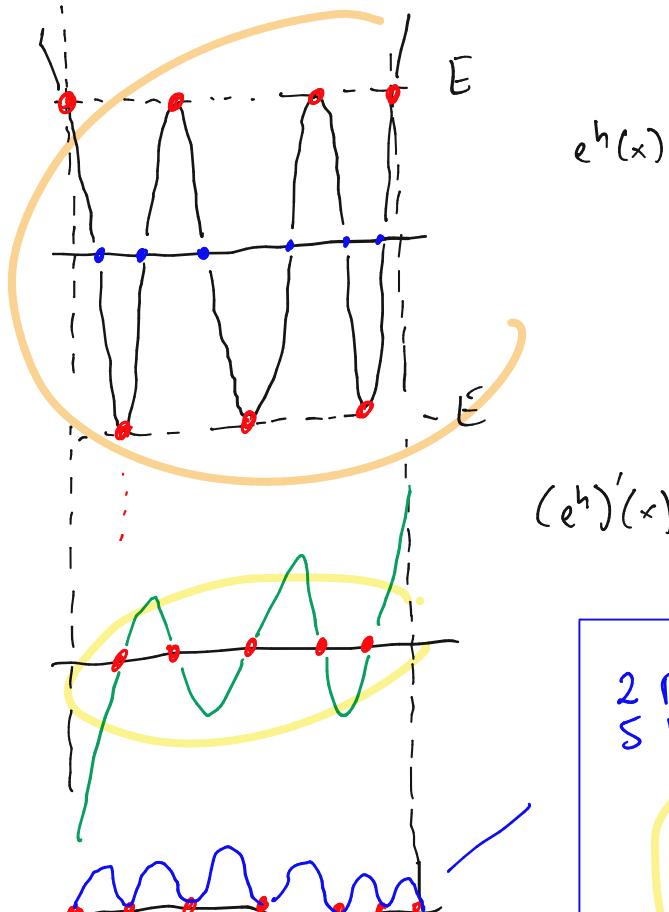
$$pu = [a_6 \ a_5 \ \dots \ a_0] \\ pu_h = [b_5 \ \dots \ b_0]$$

$$peh = pu - \underbrace{[0 * pu_h]}_{[0 \ b_5 \ b_4 \ \dots \ b_0]}$$

$$pdeh = peh * [6 \ 5 \ 4 \ 3 \ 2 \ 1 \ 0]$$

$$pdeh = \underbrace{pdeh}_{[6 \ a_6 \ 5(a_5 - b_5) \ \dots \ (a_1 - b_1)]} [0 : 6]$$

$$[6 \ a_6 \ 5(a_5 - b_5) \ \dots \ (a_1 - b_1)] \times$$



2 RACINES SIMPLES
5 RACINES DOUBLES

$$((e^h)'(x))^2 (1-x^2) = (n+1)^2 \\ \sim x^2 6^2 a_6^2 \times 10$$

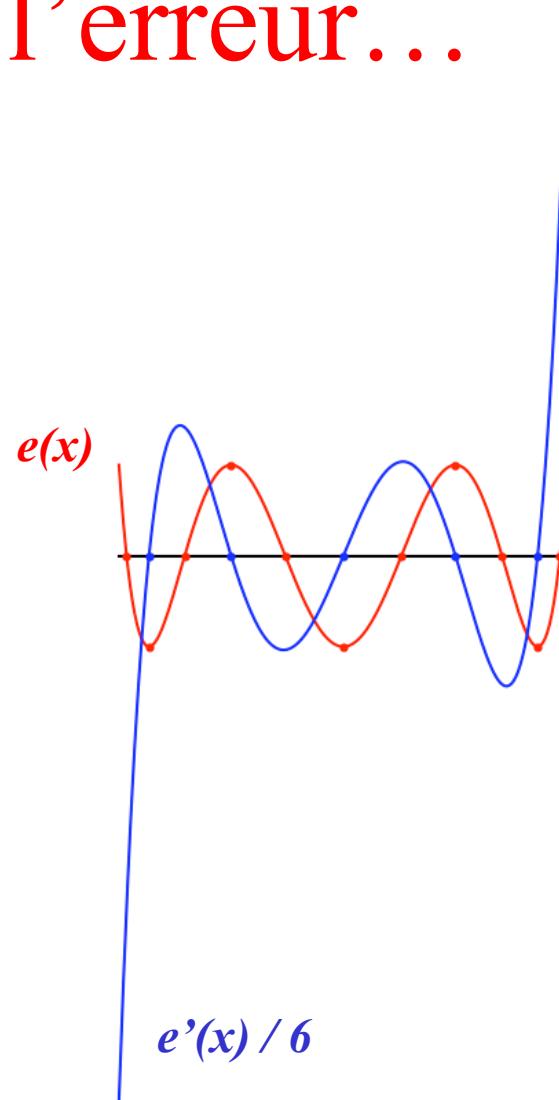
$$(E^2 - (e^h(x))^2)$$

$$- a_6^2 x^{12} (n+1)^2$$

Dérivons et divisons l'erreur...

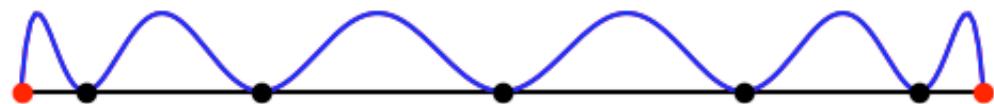
```
peh = pu - [0,*puh]
eh = polyval(peh, x)
plt.plot(x,eh,'-r')

pdeh = peh * [6,5,4,3,2,1,0]
pdeh = pdeh[0:-1]
deh = polyval(pdeh,x)/6
plt.plot(x,deh,'-b')
```



Et encore quelques petites manipulations...

$$(e'(x))^2 (1 - x^2) = (n + 1)^2 (E^2 - e^2(x))$$



```
Xd = roots(pdeh)
Ed = polyval(peh,Xd)
E = Ed[0]

plt.plot(x,E**2-eh**2,'-r')
plt.plot(x,(1-x**2)*(deh**2),'-b')
plt.plot(Xd,zeros(size(Xd)),'ok')
plt.plot([-1,1],[0,0],'or')
```

LA COURBE EN ROUGE

LA COURBE EN BLEU

2 RACINES SIMPLES
5 RACINES DOUBLES

$$\left(\left(e^{\frac{x}{E}} \right)'(x) \right)^2 (1-x^2) = (n+1)^2 (E^2 - (e^{\frac{x}{E}})^2)$$

ZEROES

$$0 = \cos \left[(n+1) \arccos(x) \right]$$

$$\frac{\pi}{2} + i\pi = (n+1) \arccos(x)$$

↓

$$X_i = \cos \left[\frac{\pi(2i+1)}{2(n+1)} \right]$$

$$\begin{aligned} \frac{e^{\frac{x}{E}}}{E} \sqrt{\frac{1}{1-(\frac{e^{\frac{x}{E}}}{E})^2}} &= \pm (n+1) \sqrt{\frac{1}{1-x^2}} \\ \left(\arccos \left(\frac{e^{\frac{x}{E}}}{E} \right) \right)' &= \pm (n+1) \left(\arccos(x) \right)' \\ \arccos \left(\frac{e^{\frac{x}{E}}}{E} \right) &= \pm (n+1) \arccos(x) + C \end{aligned}$$

$e(x) = E \cos \left[(n+1) \arccos(x) \right]$

$\cos e(\pm 1) = E$

$T_{n+1}(x)$ POLYNOME DE CHEBYSHEV :-)

Une petite solution analytique comme le faisaient les anciens...

$$\begin{aligned}(e'(x))^2 (1 - x^2) &= (n+1)^2 (E^2 - e^2(x)) \\ \downarrow \\ \frac{e'(x)}{\sqrt{1 - \left(\frac{e}{E}\right)^2}} &= \pm(n+1) \frac{1}{\sqrt{1-x^2}} \\ \downarrow \\ \arccos\left(\frac{e(x)}{E}\right) &= \pm((n+1) \arccos(x) + C) \\ \downarrow \\ &\text{En vertu de la parité du cosinus !} \\ e(x) &= E \cos((n+1) \arccos(x) + C) \\ \downarrow \\ &\text{En imposant que } e(1) = E \\ e(x) &= E \underbrace{\cos((n+1) \arccos(x))}_{T_{n+1}(x)}\end{aligned}$$

*Polynôme de Chebyshev de degré n+1
Drôle d'expression pour un polynôme, non ?*

Une solution analytique d'une équation différentielle ?

$$(e'(x))^2 (1 - x^2) = (n + 1)^2 (E^2 - e^2(x))$$



$$\frac{e'(x)}{\frac{E}{\sqrt{1 - \left(\frac{e}{E}\right)^2}}} = \pm(n + 1) \frac{1}{\sqrt{1 - x^2}}$$



```
>>> from sympy import *
>>> x = symbols('x')
>>> f = 1/sqrt(1-x**2)
>>> integrate(f)
asin(x)
```

Et si on dérive la primitive...

```
>>> from sympy import *
>>> x = symbols('x')
>>> f = 1/sqrt(1-x**2)
>>> integrate(f)
asin(x)
>>> diff(asin(x))
1/sqrt(-x**2 + 1)
>>> diffacos(x))
-1/sqrt(-x**2 + 1)
```

Les polynômes de Chebyshev $T_{n+1}(x) = \cos((n+1)\arccos(x))$ définis sur l'intervalle $[-1, 1]$ satisfont la relation de récurrence

Théorème 1.2.

$$T_{i+1}(x) = 2x T_i(x) - T_{i-1}(x), \quad i = 1, 2, 3, \dots,$$

avec $T_0(x) = 1$ et $T_1(x) = x$.

Calcul des polynômes de Chebyshev : formule de récurrence

Démonstration : Définissons $\theta = \arccos(x)$ et écrivons :

$$\begin{aligned} T_{i+1}(x) &= \cos((i+1)\theta) \\ &= \cos(\theta) \cos(i\theta) - \sin(\theta) \sin(i\theta) \end{aligned}$$

$$\begin{aligned} T_{i-1}(x) &= \cos((i-1)\theta) \\ &= \cos(\theta) \cos(i\theta) + \sin(\theta) \sin(i\theta) \end{aligned}$$

$$\begin{aligned} T_{i+1}(x) + T_{i-1}(x) &= 2 \cos(\theta) \cos(i\theta) \\ &= 2x T_i(x) \end{aligned}$$

□

Abscisses de Chebyshev

$$0 = \overbrace{\cos((n+1) \arccos(X_i))}^{T_{n+1}(X_i)} \quad i = 0, \dots, n$$

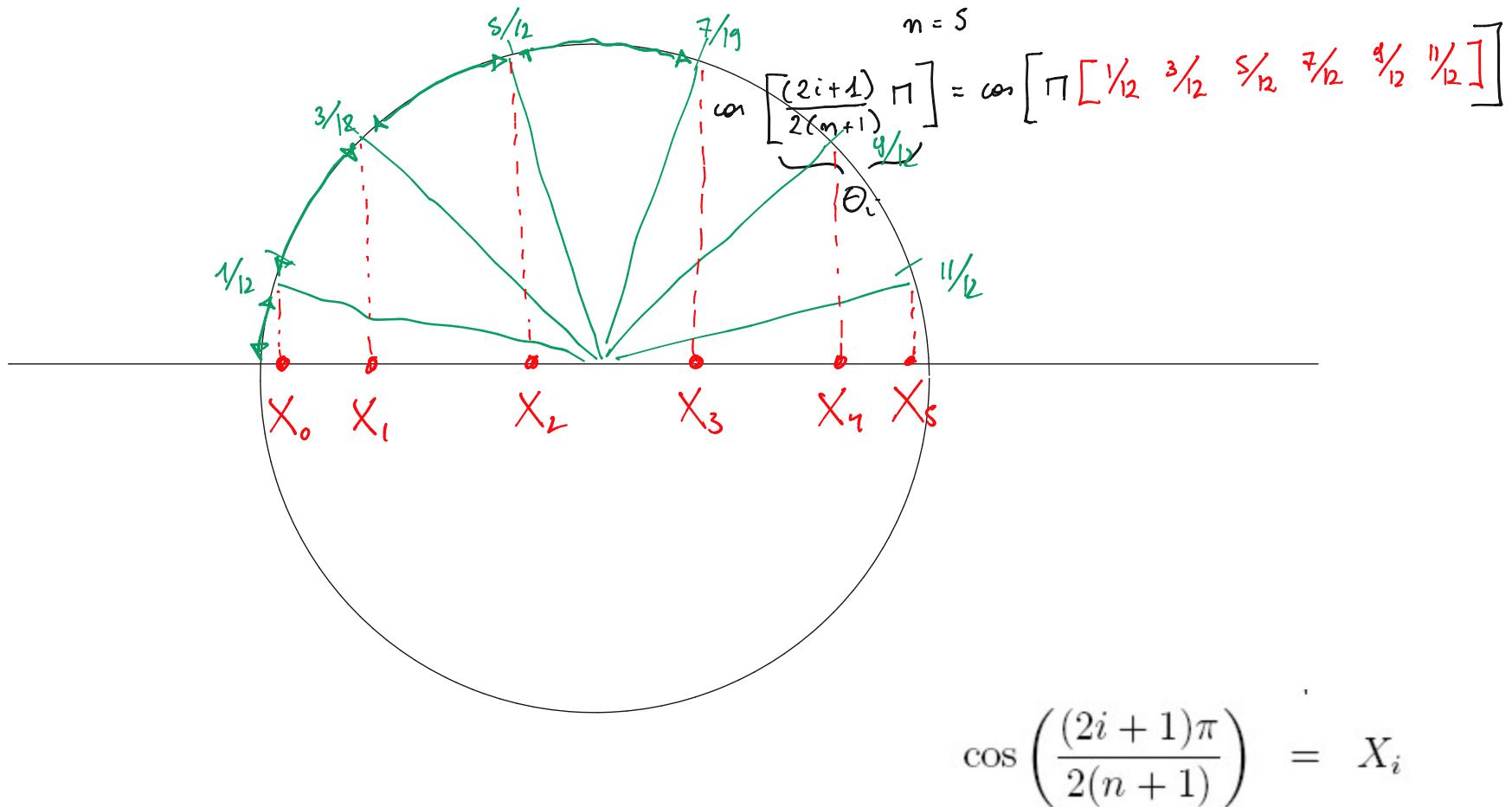


$$\frac{\pi/2 + i\pi}{(n+1)} = \arccos(X_i) \quad i = 0, \dots, n$$

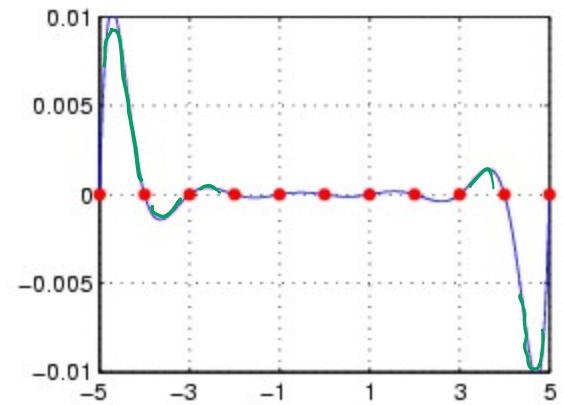
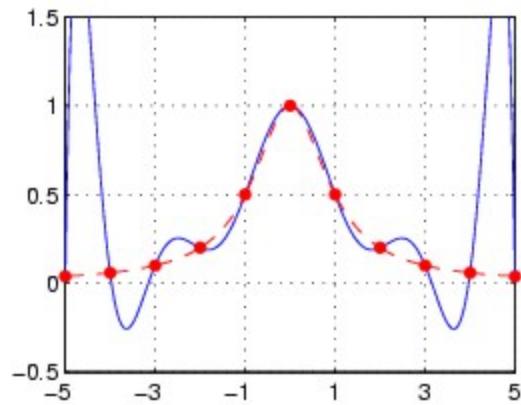
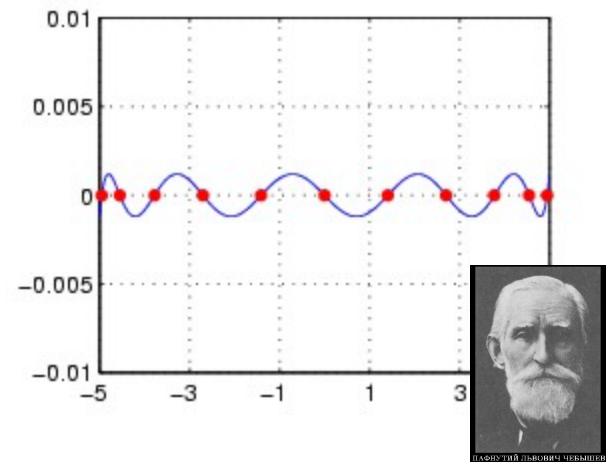
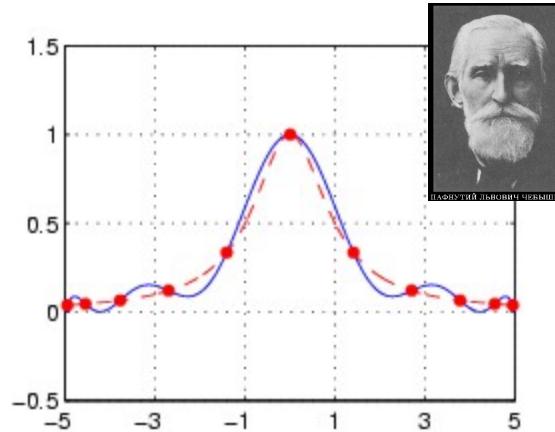


$$\cos\left(\frac{(2i+1)\pi}{2(n+1)}\right) = X_i \quad i = 0, \dots, n$$

Abscisses de Chebyshev



Why does it work ?



$$e^h(x) = \frac{u^{(n+1)}(\xi(x))}{(n+1)!} (x - X_0)(x - X_1)(x - X_2) \cdots (x - X_n).$$

Interpolation polynomiale : bilan

- Pour une fonction $u(x)$ très régulière : **fonction cosinus**

Convergence de l'interpolation polynomiale

- Pour une fonction $u(x)$ suffisamment régulière : **fonction de Runge**

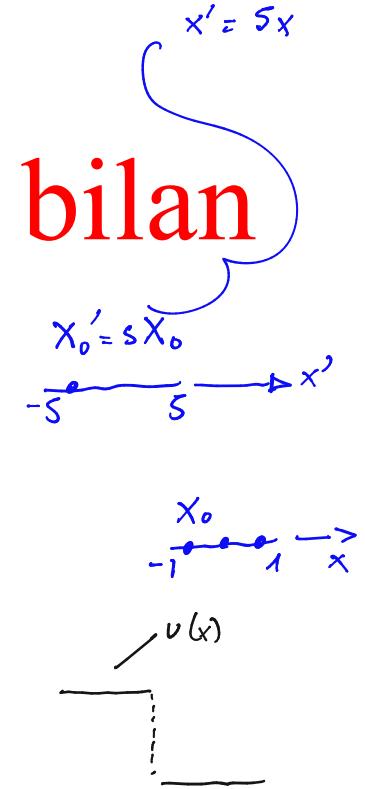
Divergence pour des abscisses équidistantes

Convergence pour les abscisses de Chebyshev

- Pour une fonction $u(x)$ peu régulière : **fonction échelon**

Divergence !

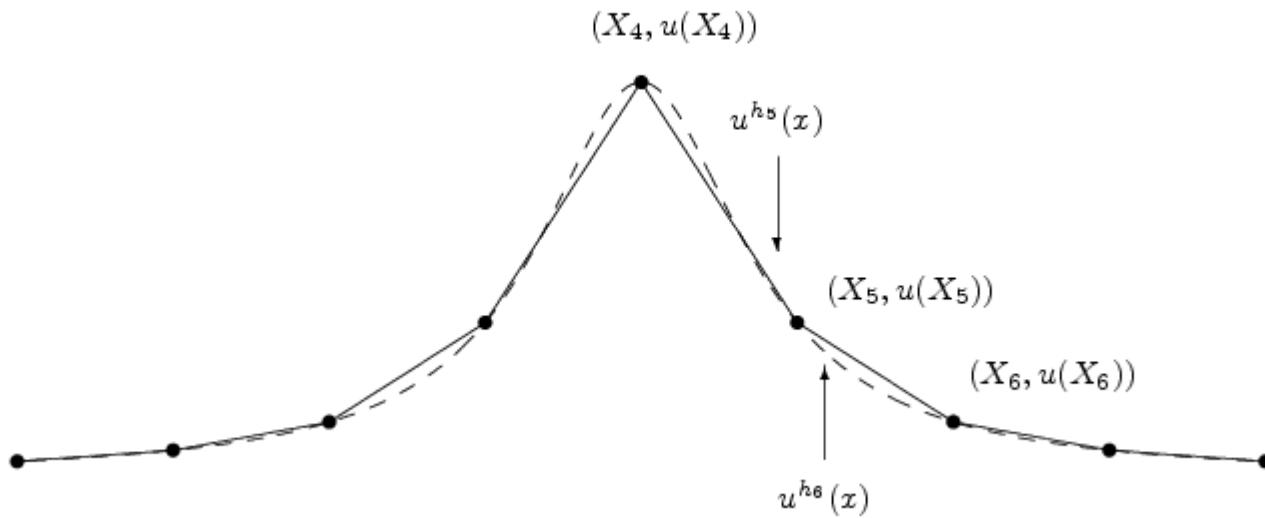
Eviter l'interpolation polynomiale de degré élevé



Idée :

*Utiliser une interpolation par morceaux
composée des polynômes de degré bas !*

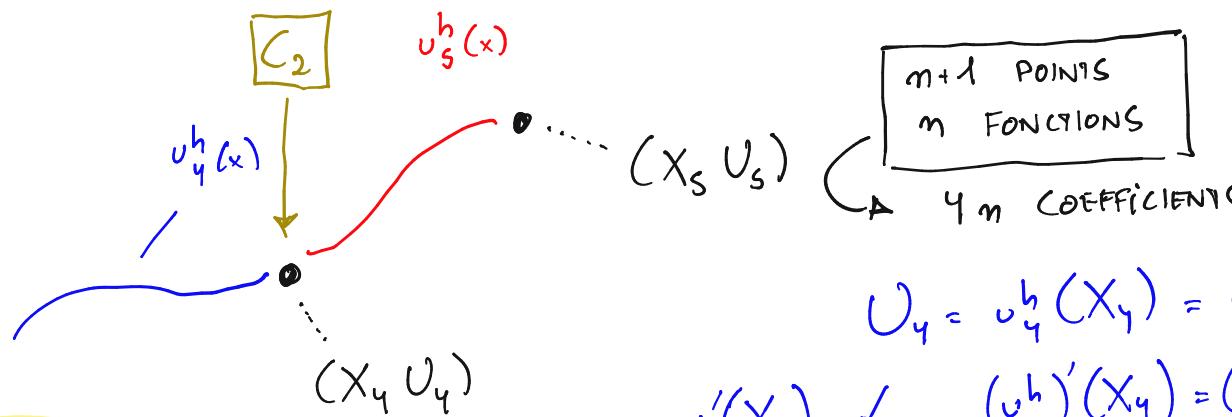
Interpolation linéaire par morceaux



Interpolation par splines cubiques

$$u^{h_i}(x) = a_i + b_i x + c_i x^2 + d_i x^3 \quad i = 1, 2, \dots, n$$

4n coefficients inconnus



$$\begin{aligned} u_h''(x_0) &= 0 \\ u_h''(x_n) &= 0 \end{aligned}$$

4n-2 CONDITIONS

IL MANQUE
2 CONDITIONS

$\begin{array}{l} m+1 \text{ POINTS} \\ m \text{ FONCTIONS} \\ \hline 4n \text{ COEFFICIENTS} \end{array}$

$2(n-1)+2$

$$U_4 = u^h_4(X_4) = u^h_s(X_4)$$

$$u'(X_4) \neq (u^h_4)'(X_4) = (u^h_s)'(X_4)$$

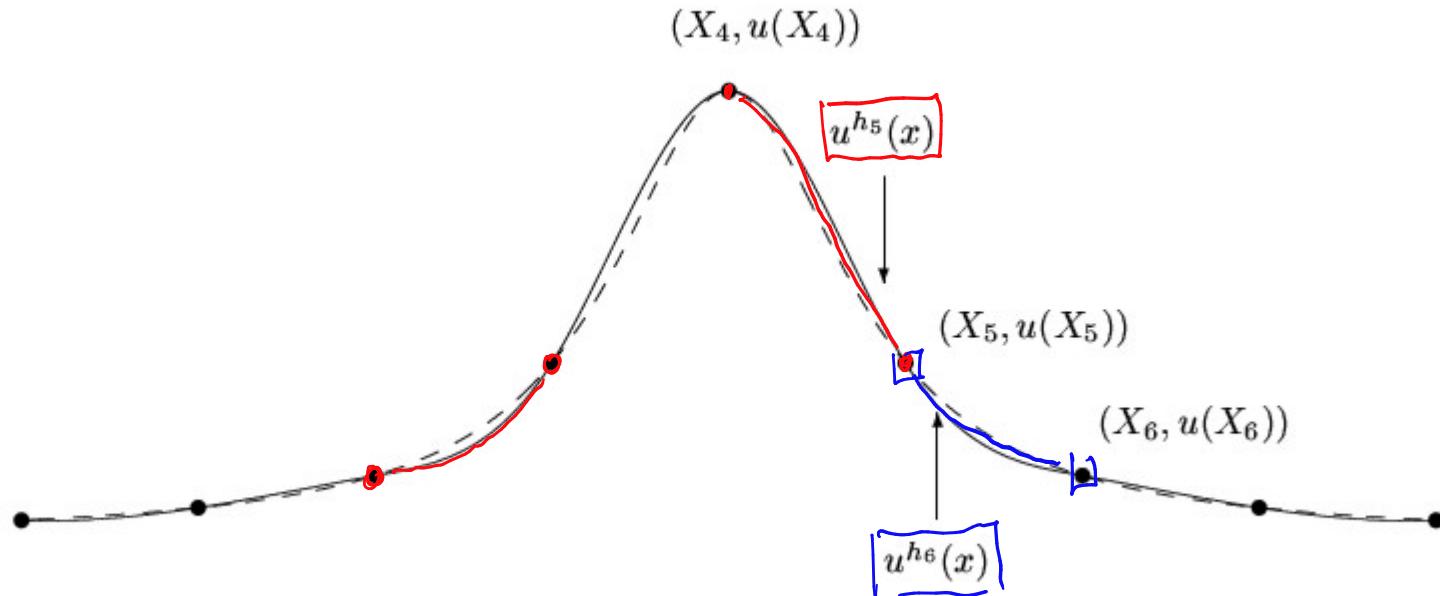
$$u''(X_4) \neq (u^h_4)''(X_4) = (u^h_s)''(X_4)$$

$2(n-1)$

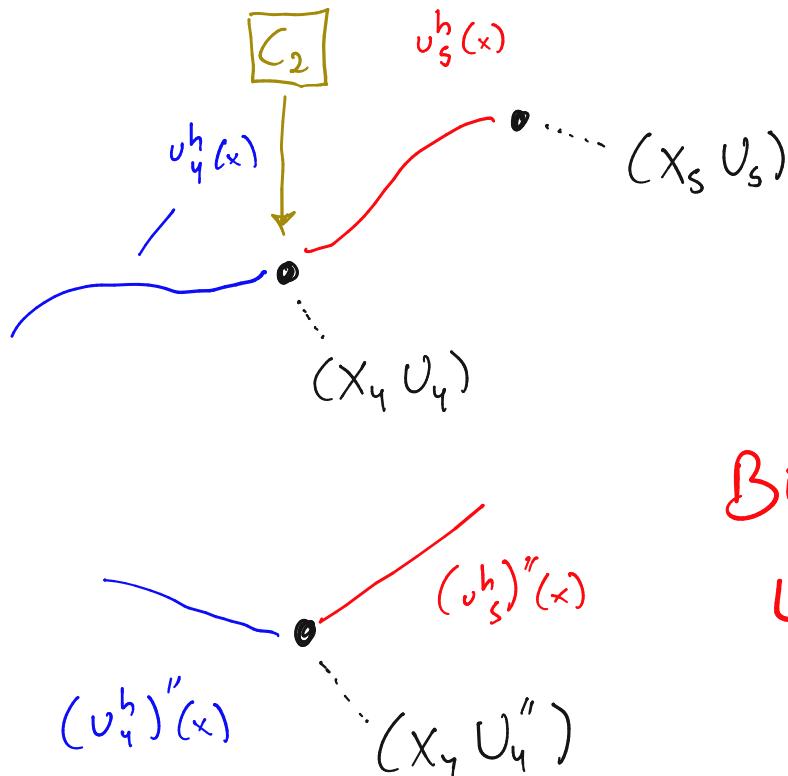
Interpolation par splines cubiques

$$u^{h_i}(x) = a_i + b_i x + c_i x^2 + d_i x^3 \quad i = 1, 2, \dots, n$$

4n coefficients inconnus



Comment trouver les coefficients ?



BE CAREFUL !
 $v_q'' \neq v''(x_q)$

Comment trouver les coefficients ?

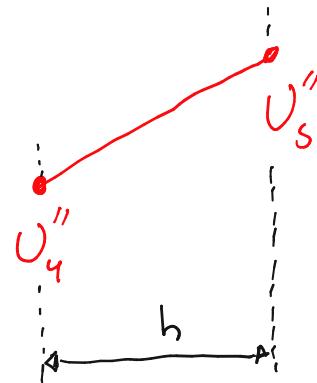
$$\begin{aligned} u^{h_1}(X_0) &= U_0 \\ u^{h_n}(X_n) &= U_n \end{aligned}$$

$$\begin{aligned} u^{h_i}(X_i) &= U_i & i &= 1, \dots, n-1 \\ u^{h_{i+1}}(X_i) &= U_i & i &= 1, \dots, n-1 \end{aligned}$$

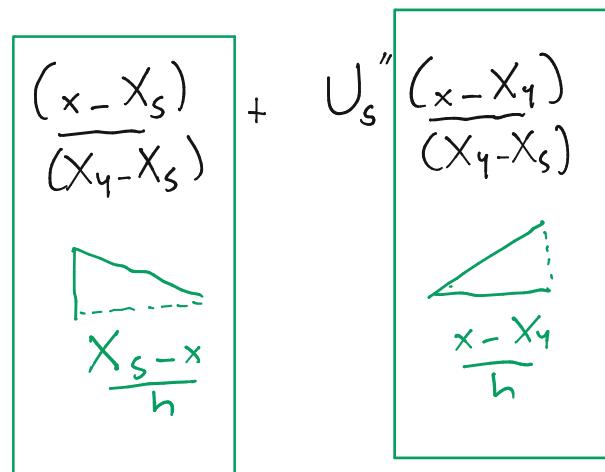
$$\begin{aligned} (u^{h_i})'(X_i) &= (u^{h_{i+1}})'(X_i) & i &= 1, \dots, n-1 \\ (u^{h_i})''(X_i) &= (u^{h_{i+1}})''(X_i) & i &= 1, \dots, n-1 \end{aligned}$$

4n-2 conditions

$$U_q'' \neq v''(X_q)$$



$$(v_s^h)'' = U_q'' \left[\frac{(x - X_s)}{(X_q - X_s)} \right] + U_s'' \left[\frac{(x - X_q)}{(X_q - X_s)} \right]$$

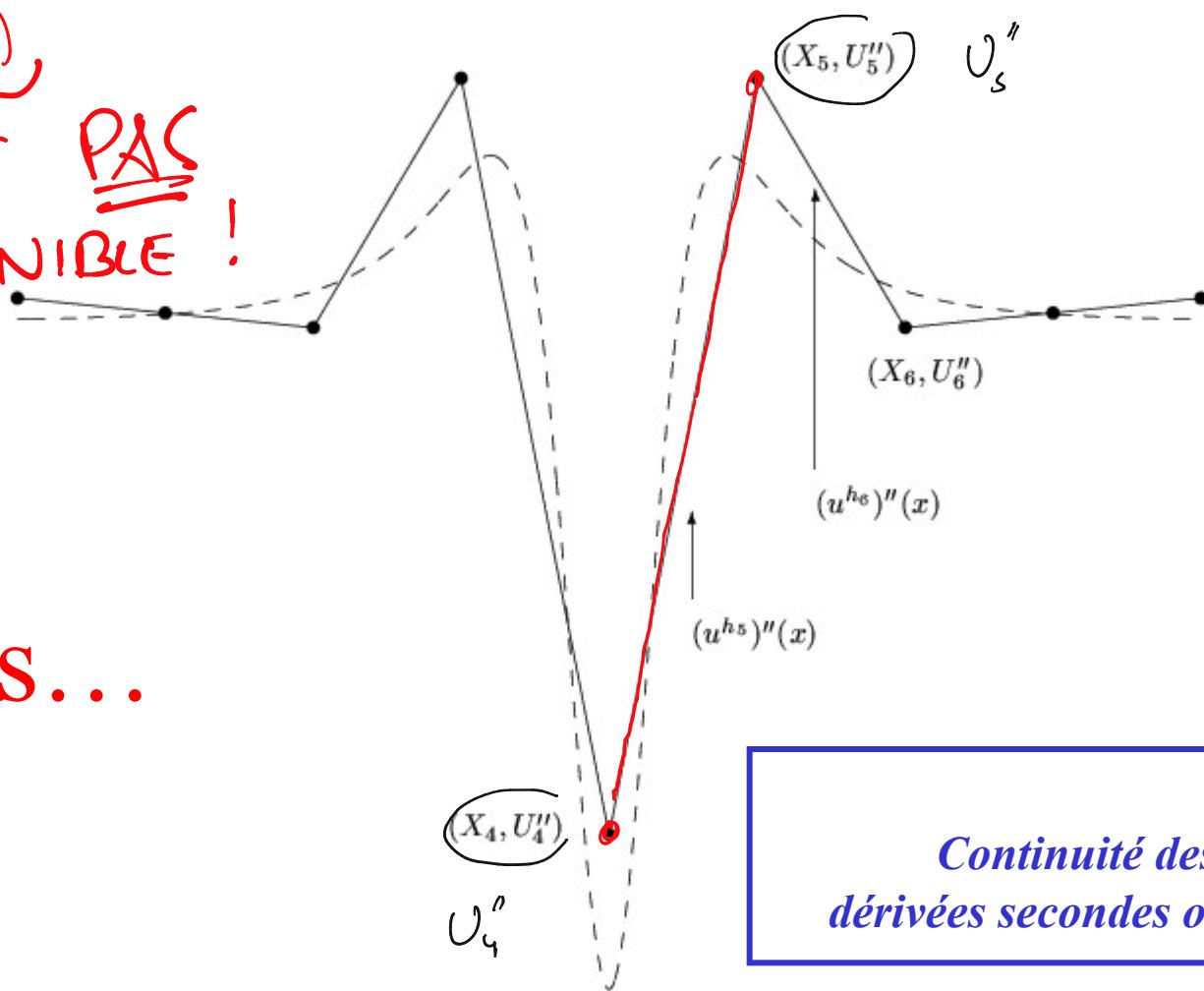


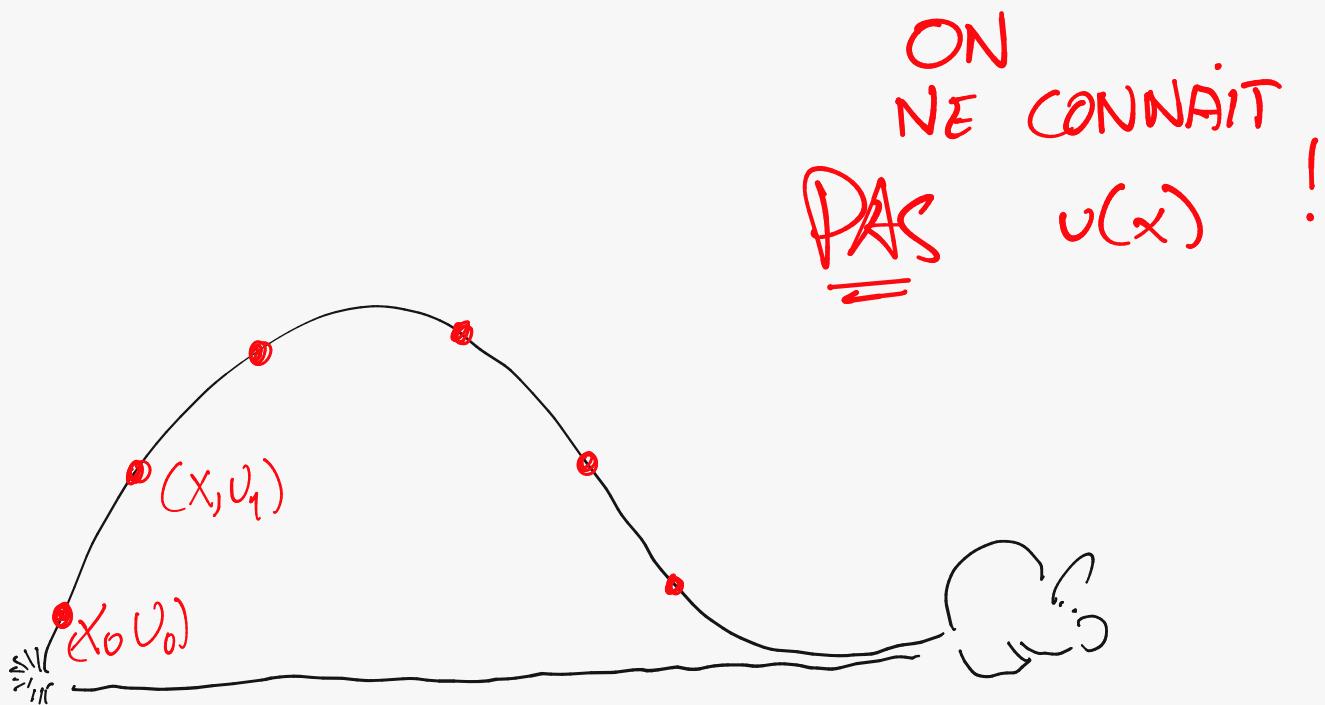
Etape 1

$$(u^{h_i})'' = U_{i-1}'' \frac{(x - X_i)}{(X_{i-1} - X_i)} + U_i'' \frac{(x - X_{i-1})}{(X_i - X_{i-1})}$$

$U_4'' \neq u''(X_4)$
 C'EST PAS
 DISPONIBLE !

Ecrivons...





$$(v_s^h)'' = U_q'' \frac{(x - X_s)}{(X_q - X_s)} + U_s'' \frac{(x - X_q)}{(X_s - X_q)}$$

$$v_s^h(x) = U_q'' \frac{(X_s - x)^3}{6h} + U_s'' \frac{(x - X_q)^3}{6h} + A \frac{(X_s - x)}{h} + B \frac{(x - X_q)}{h}$$

$+ Cx + D$

Etape 2

$$A = U_q - \frac{U_q''}{6} h^2$$

$$B = U_s - \frac{U_s''}{6} h^2$$

$\rightsquigarrow x = X_q$

$\rightsquigarrow x = X_s$

Intégrons...

$$(u^{h_i})'' = U''_{i-1} \frac{(X_i - x)}{h_i} + U''_i \frac{(x - X_{i-1})}{h_i}$$

En intégrant deux fois,

$$(u^{h_i}) = U''_{i-1} \frac{(X_i - x)^3}{6h_i} + U''_i \frac{(x - X_{i-1})^3}{6h_i} + A_i \frac{(X_i - x)}{h_i} + B_i \frac{(x - X_{i-1})}{h_i}$$

$$U_{i-1} - \frac{U''_{i-1} h_i^2}{6}$$

[kg/m²]
[m²]
[kg]

$$\begin{matrix} U & [\text{kg}] \\ X & [\text{m}] \end{matrix}$$

$$U'' [\text{kg/m}^2]$$

$$U_{i-1} = U''_{i-1} \frac{h_i^3}{6h_i} + A_i \frac{h_i}{h_i} \quad \text{et} \quad U_i = U''_i \frac{h_i^3}{6h_i} + B_i \frac{h_i}{h_i}$$

$$A_i = U_{i-1} - \underbrace{\frac{U''_{i-1} h_i^2}{6}}$$

$$B_i = U_i - \frac{U''_i h_i^2}{6}$$

Continuité de la fonction ok

$$v_s^h(x) = U_q'' \frac{(x_s - x)^3}{6h} + U_s'' \frac{(x - x_q)^3}{6h} + A \frac{(x_s - x)}{h} + B \frac{(x - x_q)}{h}$$

$$(v_q^h)'(x_q) = (v_s^h)'(x_q)$$

$$\downarrow = U_q'' \left(-\frac{3h^2}{6h} \right) - \frac{A}{h} + \frac{B}{h}$$

$$\frac{U_q''h}{2} + \left(U_s - U_q \right) - \left(\frac{U_q'' - U_s''}{h} \right) \frac{h^2}{6} = -\frac{U_q''h}{2} + \underbrace{\left(U_s - U_q \right)}_{\text{}} - \underbrace{\left(\frac{U_s'' - U_q''}{h} \right) \frac{h^2}{6}}$$

$$U_s - U_q - \frac{U_s''h}{6} + \frac{2h}{6} U_q''$$

Etape 3

$$A = U_q - \frac{U_q''h^2}{6}$$

$$B = U_s - \frac{U_s''h^2}{6}$$

$$\left[U_s - \frac{2U_q + U_3}{h} \right] = \frac{h}{6} \left[U_s'' + 4U_q'' + U_3'' \right]$$

$$v_s^h(x) = U_q'' \frac{(x_s - x)^3}{6h} + U_s'' \frac{(x - x_q)^3}{6h} + A \frac{(x_s - x)}{h} + B \frac{(x - x_q)}{h}$$

$$(v_q^h)' \quad (v_s^h)' \\ (x_q \quad v_q')$$

$$(v_q^h)'(x_q) = (v_s^h)'(x_q)$$

$$= -3 \frac{U_q''(x_s - x)^2}{6h} \Big|_{x=x_q} - \frac{A}{h} + \frac{B}{h}$$

$$-3 \frac{U_q'' h^2}{6h} = -\frac{h}{2} U_q''$$

$$= -\frac{h}{2} U_q'' + \frac{(U_s - U_q)}{h} - \frac{(U_s'' - U_q'')h}{6}$$

$$\Downarrow \quad U_s - U_q - \frac{U_s''h}{6} - \frac{2hU_q''}{6}$$

$$A = U_q - \frac{U_q''h^2}{6}$$

$$B = U_s - \frac{U_s''h^2}{6}$$

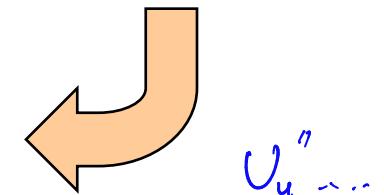
Calculons...

Continuité des dérivées premières ok

$$\begin{aligned}
 (u^{h_i})'(X_i) &= (u^{h_{i+1}})'(X_i) \\
 \frac{U''_i h_i}{2} + \frac{(U_i - U_{i-1})}{h_i} - \frac{(U''_i - U''_{i-1})h_i}{6} &= \frac{U''_{i+1} h_{i+1}}{2} + \frac{(U_{i+1} - U_i)}{h_{i+1}} - \frac{(U''_{i+1} - U''_i)h_{i+1}}{6} \\
 \frac{(2U''_i + U''_{i-1})h_i}{6} + \frac{(U_i - U_{i-1})}{h_i} &= \frac{(U_{i+1} - U_i)}{h_{i+1}} - \frac{(U''_{i+1} + 2U''_i)h_{i+1}}{6}
 \end{aligned}$$

$$\frac{h_i}{6} U''_{i-1} + \frac{2(h_i + h_{i+1})}{6} U''_i + \frac{h_{i+1}}{6} U''_{i+1} = \frac{(U_{i+1} - U_i)}{h_{i+1}} - \frac{(U_i - U_{i-1})}{h_i}$$

$i = 1, \dots, n-1$



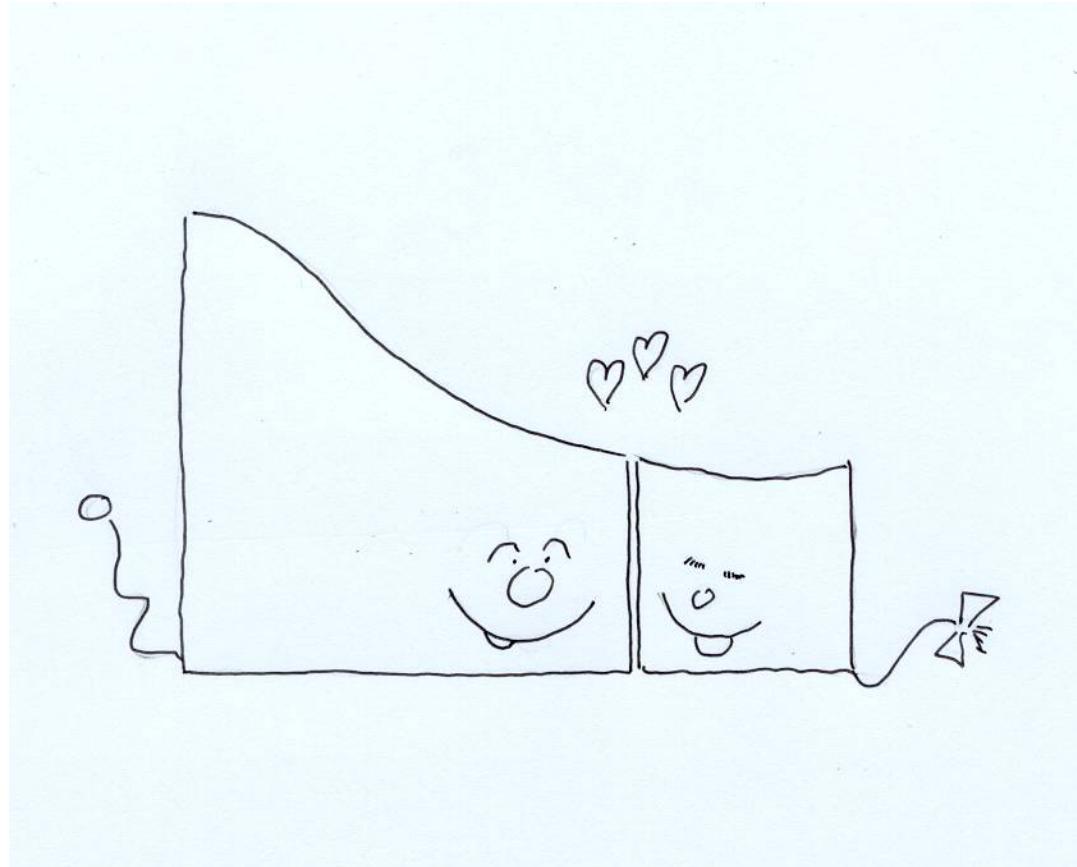
Abscisses équidistantes

$$U''_0 = 0 \quad \text{et} \quad U''_n = 0$$

*2 conditions supplémentaires
Courbe spline naturelle*

$$\frac{h^2}{6} \begin{bmatrix} 1 & 0 & & & & \\ 1 & 4 & 1 & & & \\ & 1 & 4 & 1 & & \\ & & 1 & 4 & 1 & \\ & & & 1 & 4 & 1 \\ & & & & \ddots & \ddots \\ & & & & 1 & 4 & 1 \\ & & & & & 1 & 4 & 1 \\ & & & & & & 0 & 1 \end{bmatrix} \begin{bmatrix} U''_0 \\ U''_1 \\ U''_2 \\ U''_3 \\ U''_4 \\ \vdots \\ U''_{n-2} \\ U''_{n-1} \\ U''_n \end{bmatrix} = \begin{bmatrix} 0 \\ U_0 - 2U_1 + U_2 \\ U_1 - 2U_2 + U_3 \\ U_2 - 2U_3 + U_4 \\ U_3 - 2U_4 + U_5 \\ \vdots \\ \vdots \\ U_{n-2} - 2U_{n-1} + U_n \\ 0 \end{bmatrix}$$

Ou de manière plus
poétique...



```

from numpy import *
from scipy.interpolate import CubicSpline as spline
from matplotlib import pyplot as plt

x = arange(-55,70,10)
U = [3.25, 3.37, 3.35, 3.20, 3.12, 3.02, 3.02,
      3.07, 3.17, 3.32, 3.30, 3.20, 3.10]
x = linspace(X[0],X[-1],100)
uhLag = polyval(polyfit(x,U,len(x)-1),x)
uhSpl = spline(x,U)

plt.plot(x,uhLag,'--r',x,uhSpl(x),'-b')
plt.plot(x,U,'or')

```

Exemple

