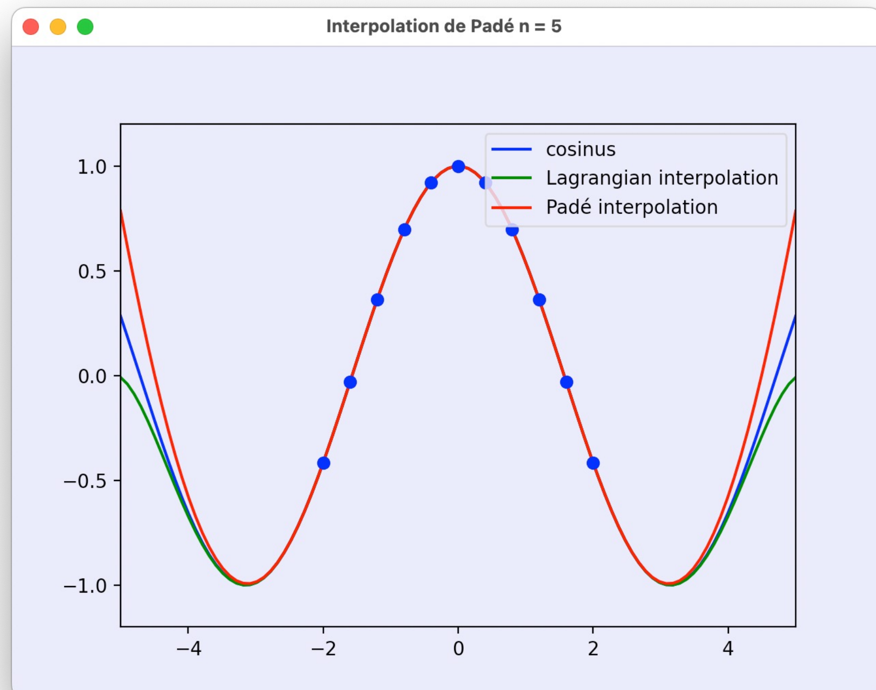
Interpolation de Padé

$$u(x) \approx u^h(x) = \frac{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5}{1 + a_6 x + a_7 x^2 + a_8 x^3 + a_9 x^4 + a_{10} x^5}$$



Homework 1

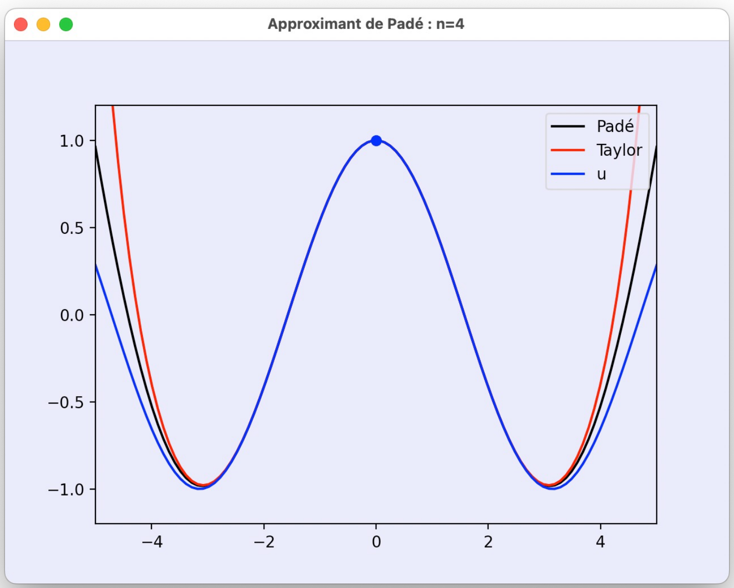
$$u^h(x) = u^t(x) + \mathcal{O}(x^5)$$

$$\frac{a_0 + a_1 x + a_2 x^2}{1 + a_3 x + a_4 x^2} = U_0 + U_0' x + U_0'' \frac{x^2}{2} + U_0''' \frac{x^3}{6} + U_0'''' \frac{x^4}{24} + \mathcal{O}(x^5)$$

En espérant que le dénominateur ne vaille pas zéro :-)

$$a_0 + a_1 x + a_2 x^2 = \Big(1 + a_3 x + a_4 x^2\Big)\Big(U_0 + U_0' x + U_0'' \frac{x^2}{2} + U_0''' \frac{x^3}{6} + U_0'''' \frac{x^4}{24} + \mathcal{O}(x^5)\Big)$$

$$\begin{aligned} a_0 + a_1 x + a_2 x^2 = \; & U_0 + \Big(U_0' + a_3 U_0\Big)\, x \\ & + \Big(\tfrac{1}{2} U_0'' + a_3 U_0' + a_4 U_0\Big)\, x^2 \\ & + \Big(\tfrac{1}{6} U_0''' + a_3 \tfrac{1}{2} U_0'' + a_4 U_0'\Big)\, x^3 \\ & + \Big(\tfrac{1}{24} U_0'''' + a_3 \tfrac{1}{6} U_0''' + a_4 \tfrac{1}{2} U_0''\Big)\, x^4 + \mathcal{O}(x^5) \end{aligned}$$



Homework 2

$$
\begin{aligned}
a_0 + a_1 x + a_2 x^2 \quad = \quad & U_0 + \left(U_0' + a_3 U_0 \right) x \\
& + \left(\tfrac{1}{2} U_0'' + a_3 U_0' + a_4 U_0 \right) x^2 \\
& + \left(\tfrac{1}{6} U_0''' + a_3 \tfrac{1}{2} U_0'' + a_4 U_0' \right) x^3 \\
& + \left(\tfrac{1}{24} U_0'''' + a_3 \tfrac{1}{6} U_0''' + a_4 \tfrac{1}{2} U_0'' \right) x^4 + \mathcal{O}(x^5)
\end{aligned}
$$

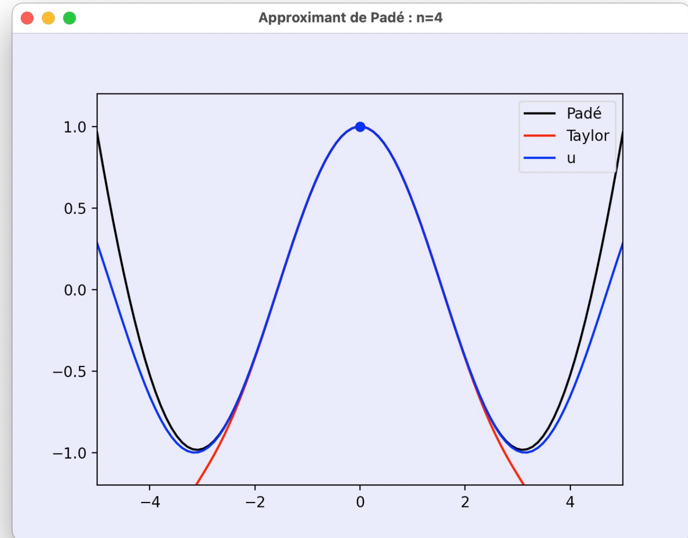
Approximant de Padé

Toujours aussi simple !

On identifie tous les coefficients !

$$
\begin{bmatrix}
1 & & & & \\
 & 1 & & -U_0 & \\
 & & 1 & -U_0' & -U_0 \\
 & & & -U_0''/2 & -U_0' \\
 & & & -U_0'''/6 & -U_0''/2
\end{bmatrix}
\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}
=
\begin{bmatrix} U_0 \\ U_0' \\ U_0''/2 \\ U_0'''/6 \\ U_0''''/24 \end{bmatrix}.
$$

Est-ce vraiment un peu utile ?



11 OPERATIONS

7 OPERATIONS

$$u^h(x) \;=\; \frac{15120 - 6900\,x^2 + 313\,x^4}{15120 - 660\,x^2 + 13\,x^4} \;=\; c_0 + \cfrac{c_1}{c_2 + x^2 + \cfrac{c_3}{c_4 + x^2}}$$

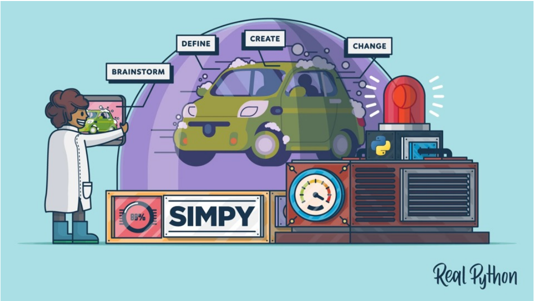
```
======== Value of cos(x)  for x = 1.50 : 0.0707372016677029
======== Taylor expansion of order 8  for x = 1.50 : 0.0707528250558036
 Error : -1.5623388e-05
======== Pade approximation for x = 1.50 : 0.0707561494078349
 Error : -1.8947740e-05
======== Taylor expansion of order 6  for x = 1.50 : 0.0701171875000000
 Error :  6.2001417e-04
```

Comment obtenir
le développement de Taylor ?

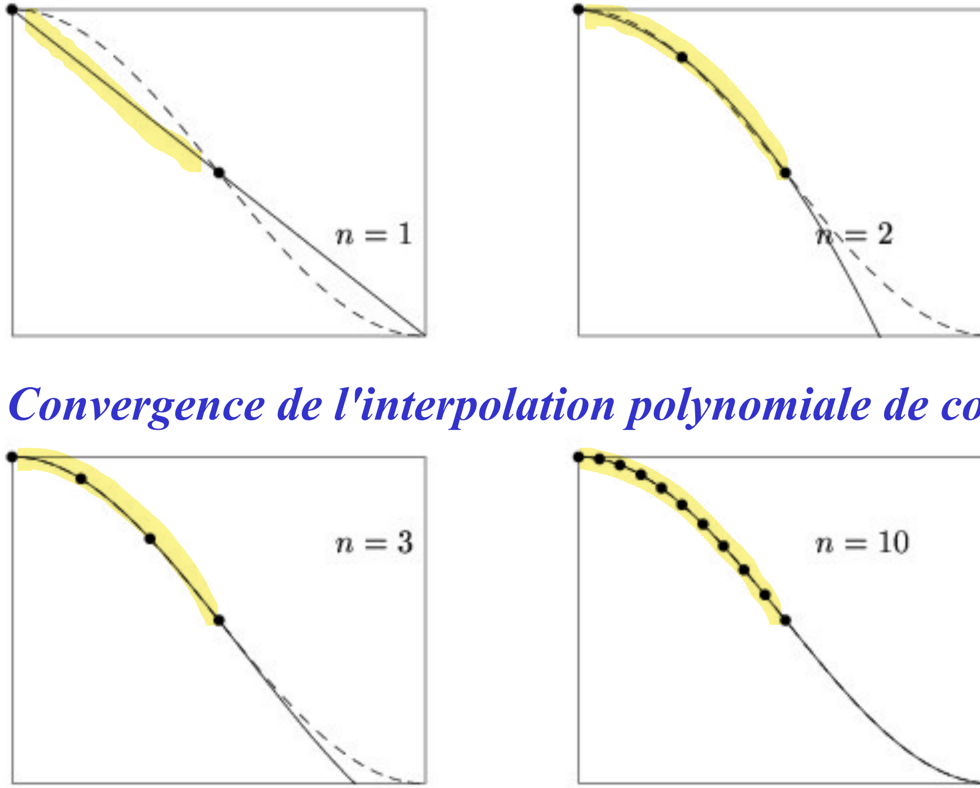
```
def macLaurinCompute(u,x,n,X) :
  ut = 0
  Ut = 0
  dU = np.zeros(n+1)
  for i in range(0,n+1):
    dudx=diff(u,x,i)
    dUdx=dudx.subs(x,0)
    dU[i] = dUdx
    if dUdx != 0:
       term = dUdx*(x**i)/factorial(i)
       Ut += term.subs(x,X)
       ut += term
  return [ut,Ut,dU]
```

```
import numpy as np
from sympy import *

def main() :
  x = symbols('x’); u = cos(x)
  n = 8
  X  = 1.5; U = u.subs(x,X)
  [ut,Ut,dU]  = macLaurinCompute(u,x,n,X)
```



Convergence



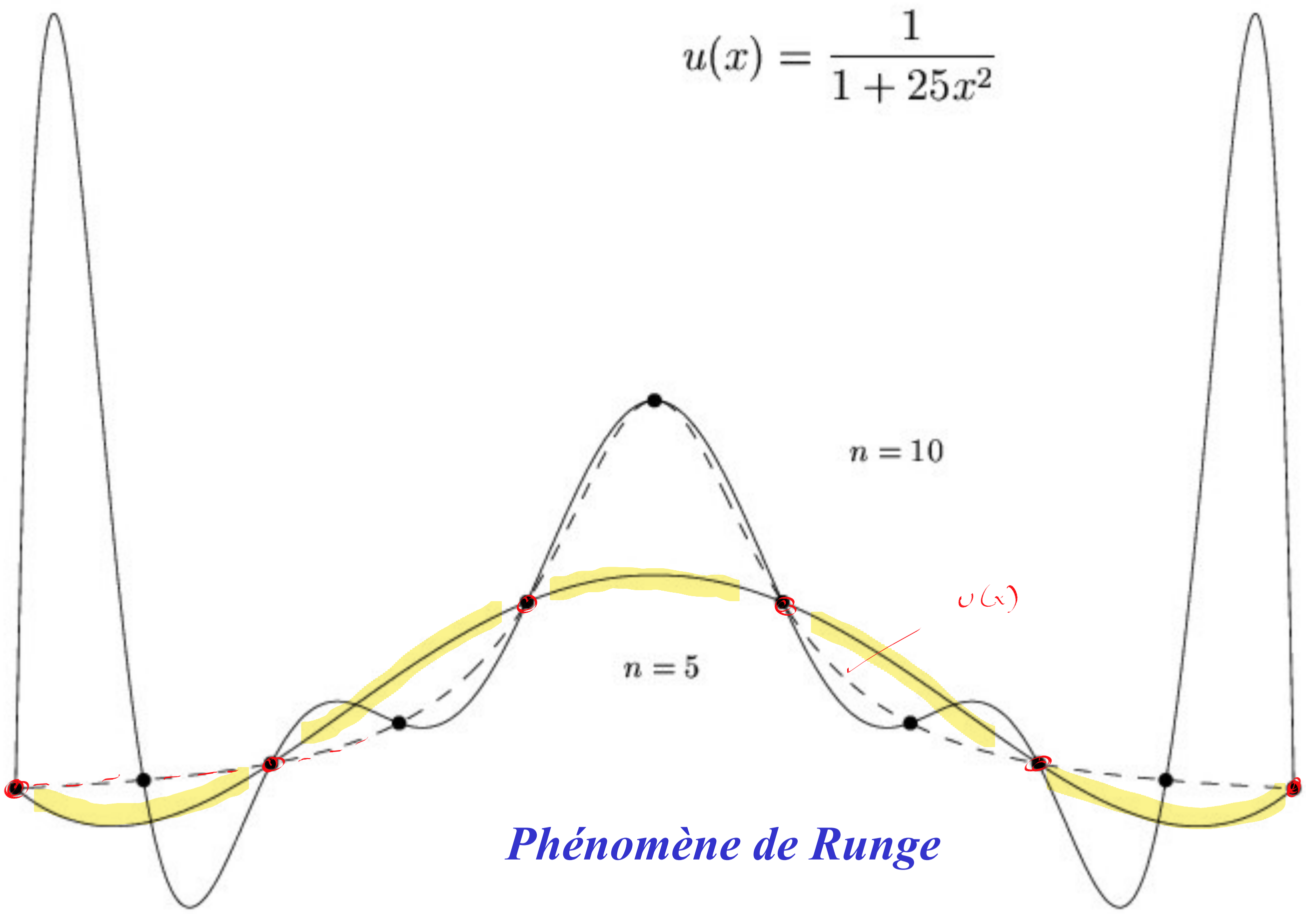
Convergence de l'interpolation polynomiale de cos(x)

$$e^h(x) = u(x) - u^h(x)$$

Définition 1.3. *Une interpolation est dite* convergente *si l'erreur d'interpolation tend vers zéro lorsque le nombre de degrés de liberté, c'est-à-dire n tend vers l'infini :*

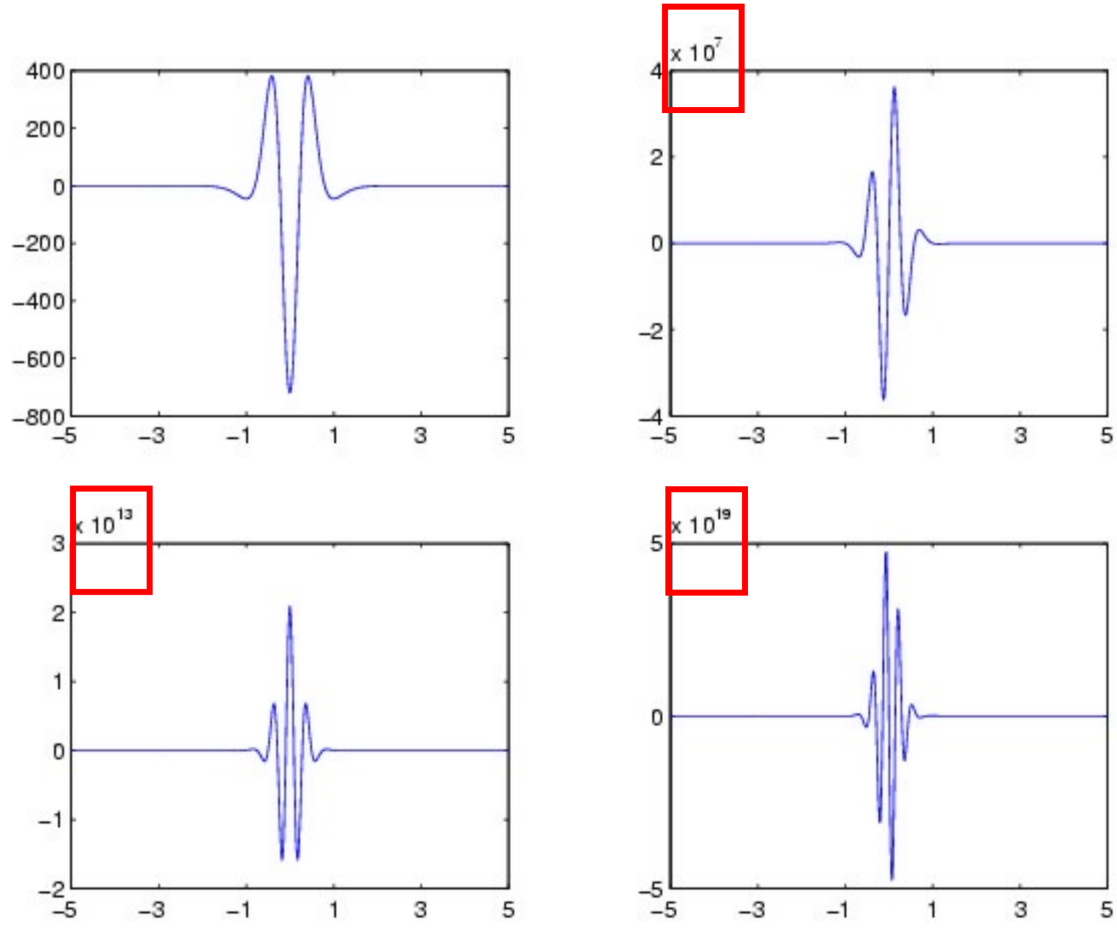
$$\lim_{n\to\infty} e^h(x) = 0 \quad \textit{pour } x \in [X_0, X_n].$$

L'interpolation polynomiale, parfois cela ne converge pas…



Phénomène de Runge

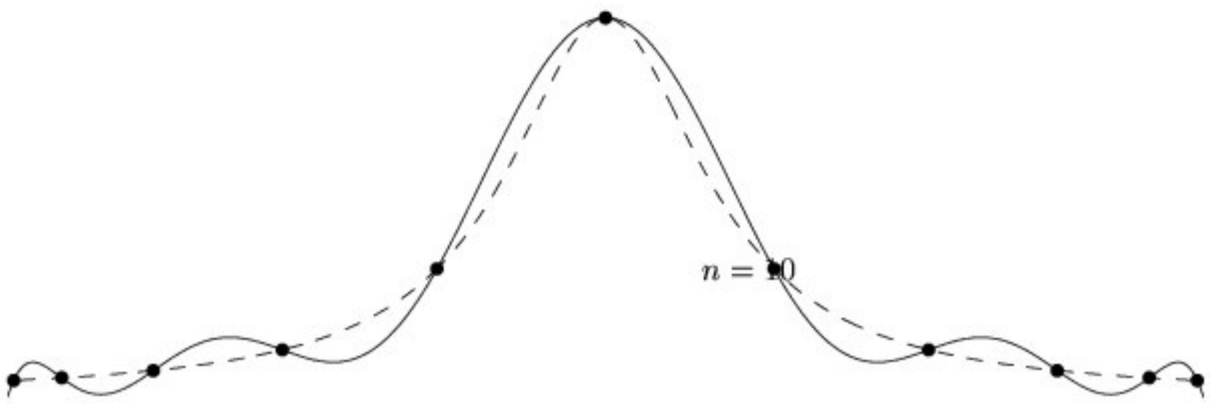
Why does it not work ?



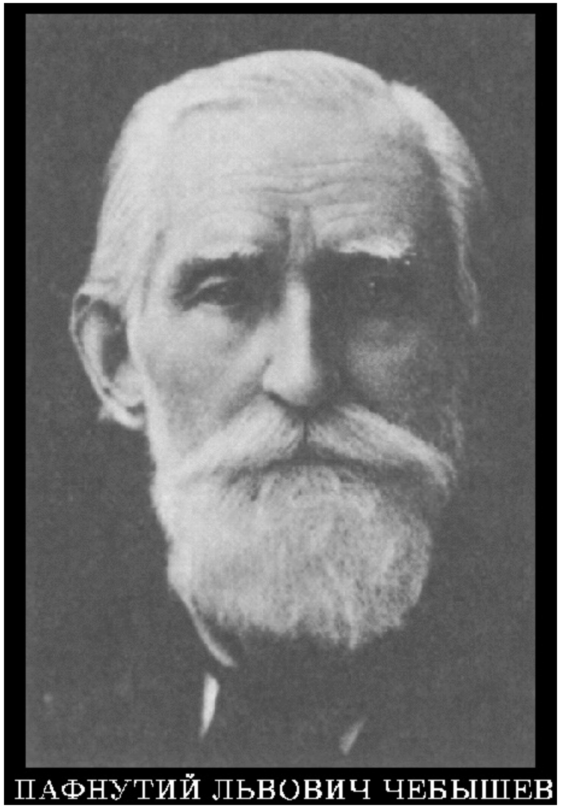
Dérivées d'ordre 6, 11, 16 et 21 de la fonction de Runge

$$e^h(x) = \frac{u^{(n+1)}(\xi(x))}{(n+1)!}(x - X_0)(x - X_1)(x - X_2)\cdots(x - X_n).$$

Parfois, on peut sauver la mise…



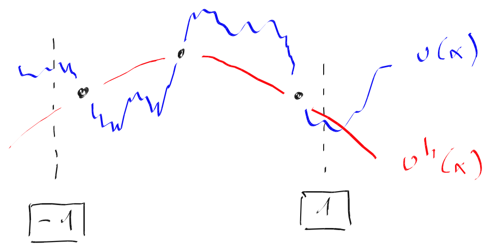
Abscisses de Chebyshev



Pafnuty Lvovitch Chebyshev (1821-1894)

Abscisses de Chebychev

$$\underbrace{|u(x) - u^h(x)|}_{e^h(x)} \leq \left| \frac{u^{(n+1)}(\xi(x))}{(n+1)!} \right| \, \underbrace{|(x-X_0)(x-X_1)\dots(x-X_n)|}_{\leq \frac{n!\, h^{n+1}}{4}}$$



$u^{(n+1)}(\xi(x))$

$n \nearrow$ ↗

$\frac{h^{n+1}}{4(n+1)}$

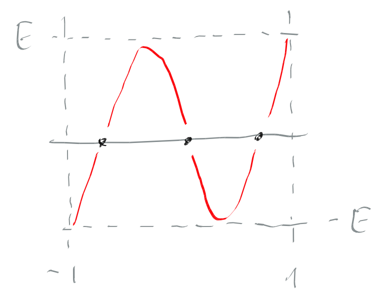
$n \nearrow$

$h \searrow$ ↘

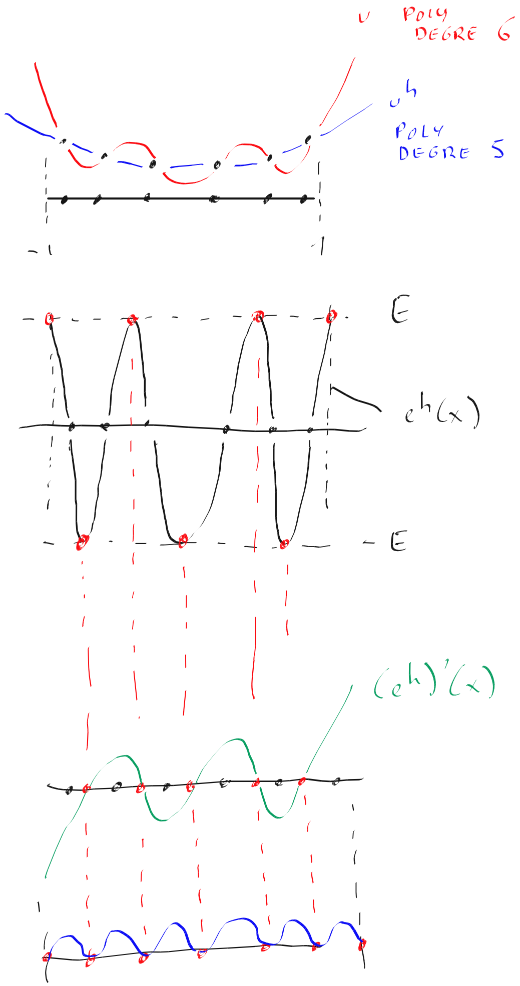
ESTIMATION ASYMPTOTIQUE DE L'ERREUR

$h \to 0$

$$X_i = \cos\left[\frac{(2i+1)\pi}{2(n+1)}\right]$$



DIMINUER CECI LE PLUS EFFICACEMENT POSSIBLE



$a_6 x^6 + a_5 x^5 \dots \quad a_0$

$b_5 x^5 \dots \quad b_0$

$[a_6 \; a_5 \dots \; a_0] = pv$

$[b_5 \dots \; b_0] = pvh$

$peh = pv - [0 \; +pvh]$

$[a_6 \dots \; a_0]$

$b_5 \; b_4 \dots \; b_0$

$[a_6 \; a_5 - b_5 \; \dots \; a_0 - b_0]$

$pdeh = peh * [6 \; 5 \; 4 \; 3 \; 2 \; 1 \; 0]$

$pdeh = pdeh[0:6]$

$[6a_6 \; 5(a_5 - b_5) \; \dots \; (a_1 - b_1)]$

$$\left((e^h)'(x)\right)^2 (1 - x^2) = (n+1)^2 \left(E^2 - (e^h(x))^2\right)$$

$a_6^2 x^{12} + \dots$

x^2

6^2

POLYNOME DE DEGRE 12

POLYNOME DE DEGRE 12

$(6 a_6 x^5 \quad)^2 \dots$

$6^2 a_6^2 x^{12} \dots$

2 racines simples

5 racines doubles

$$\left(\left(e^h\right)'(x)\right)^2 (1-x^2) = (n+1)^2 \left(E^2 - \left(e^h(x)\right)^2\right)$$

$$e' \sqrt{1-x^2} = \pm (n+1)\sqrt{E^2-e^2}$$

$$\frac{e'}{E} \frac{1}{\sqrt{1-\left(\frac{e}{E}\right)^2}} = \pm (n+1) \frac{1}{\sqrt{1-x^2}}$$

$$\left(\arccos\left(\frac{e}{E}\right)\right)' = \pm (n+1) \left(\arccos(x)\right)'$$

$$\arccos\left(\frac{e}{E}\right) = \pm (n+1) \arccos(x) + C$$

CAR $e(\pm 1) = \pm E$

$$e^h(x) = E \cos\left[(n+1) \underbrace{\arccos(x)}_{\theta}\right]$$

$\cos\left[(n+1)\arccos(x)\right]$: $T_{n+1}(x)$

X_i ZEROS DE $T_{n+1}(x)$

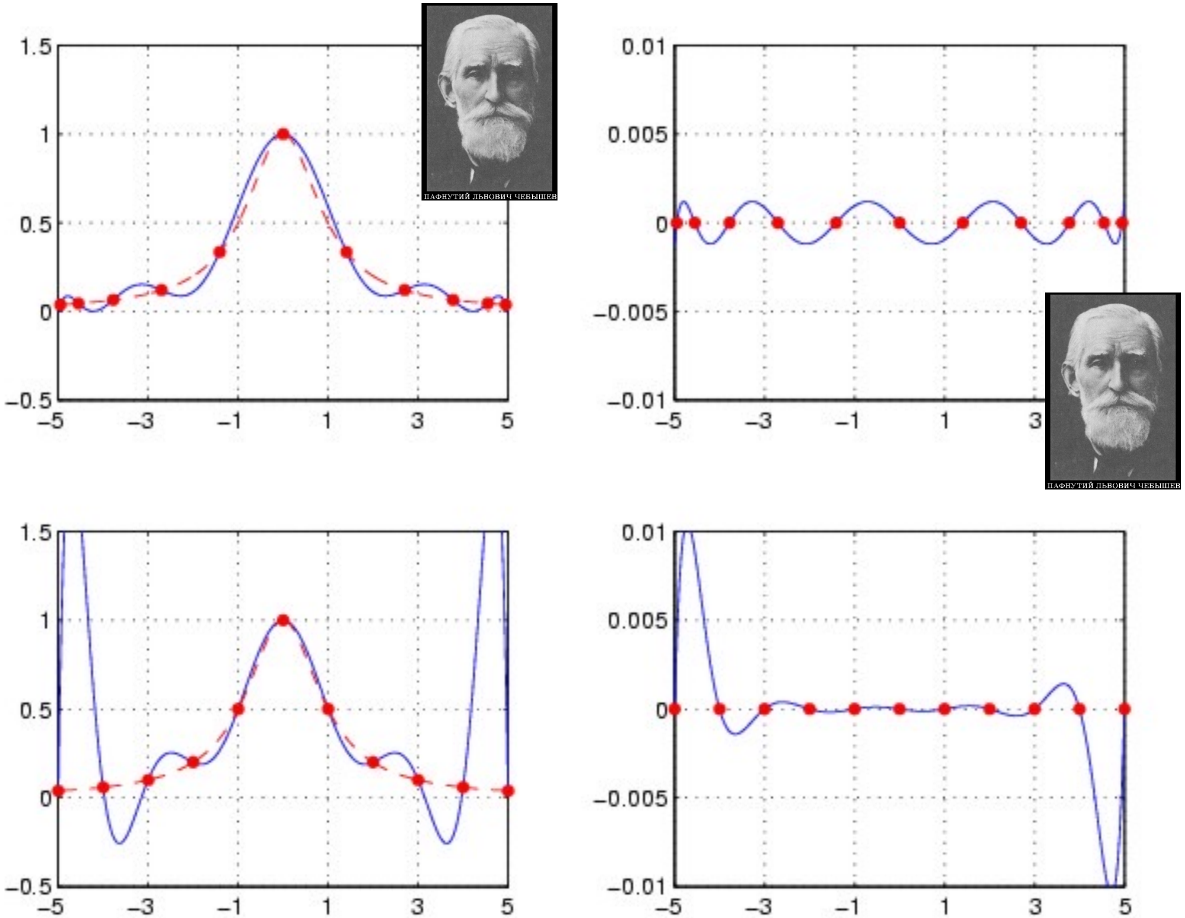
$$0 = \cos\left[(n+1)\arccos(X_i)\right]$$

$$\pi/2 + i\pi = (n+1)\arccos(X_i)$$

$$X_i = \cos\left[\frac{(2i+1)\pi}{2(n+1)}\right]$$

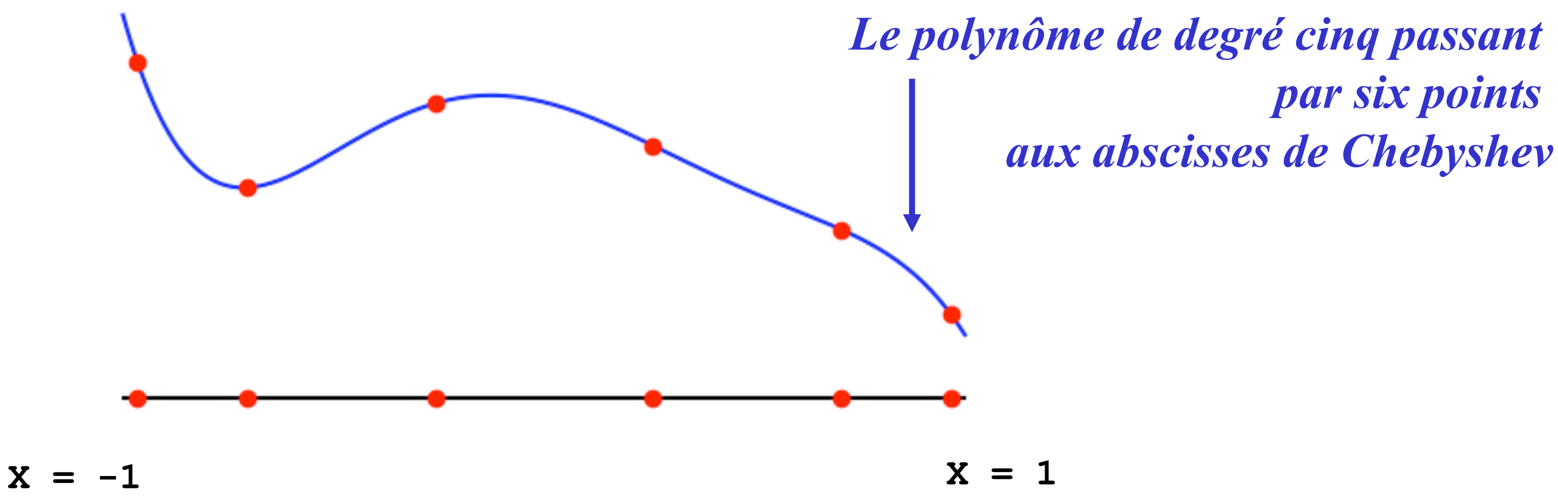
Polynômes de Chebychev

Why does it work ?



$$e^h(x) = \frac{u^{(n+1)}(\xi(x))}{(n+1)!}(x - X_0)(x - X_1)(x - X_2)\cdots(x - X_n).$$

Six points aux abscisses de Chebyshev….



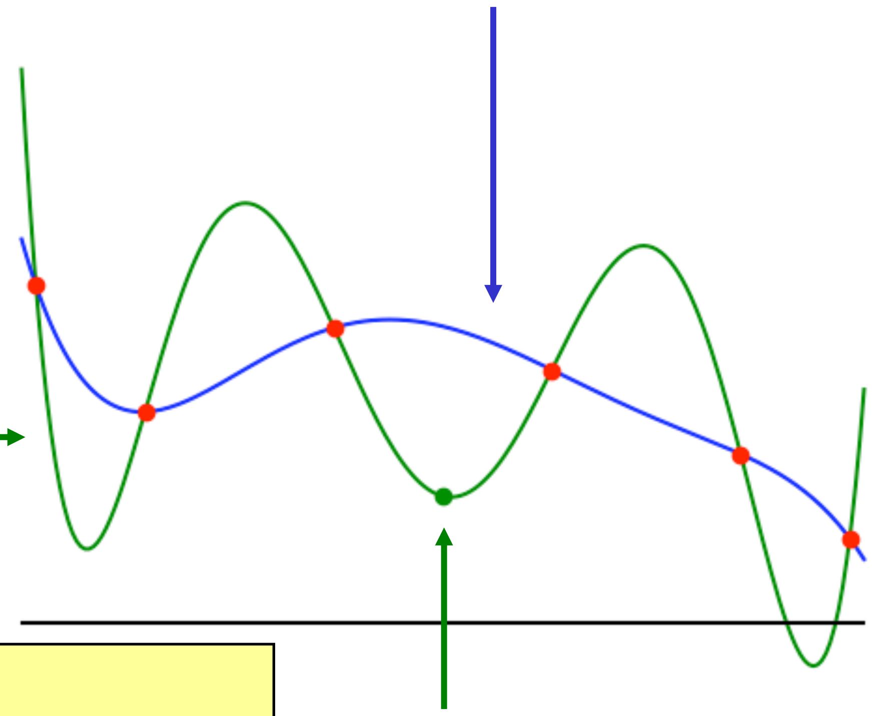

```
n = 5
X = cos(pi * (2*arange(0,n+1) + 1)/((n+1) * 2))
U = [0.2,0.4,0.6,0.7,0.5,0.8]

puh = polyfit(X,U,5)
x = linspace(-1,1,200)
uh = polyval(puh,x)
plt.plot(x,uh,'-b')
plt.plot(X,U,'or')
```

Ajoutons un septième point !

Le polynôme de degré cinq passant par six points aux abscisses de Chebyshev

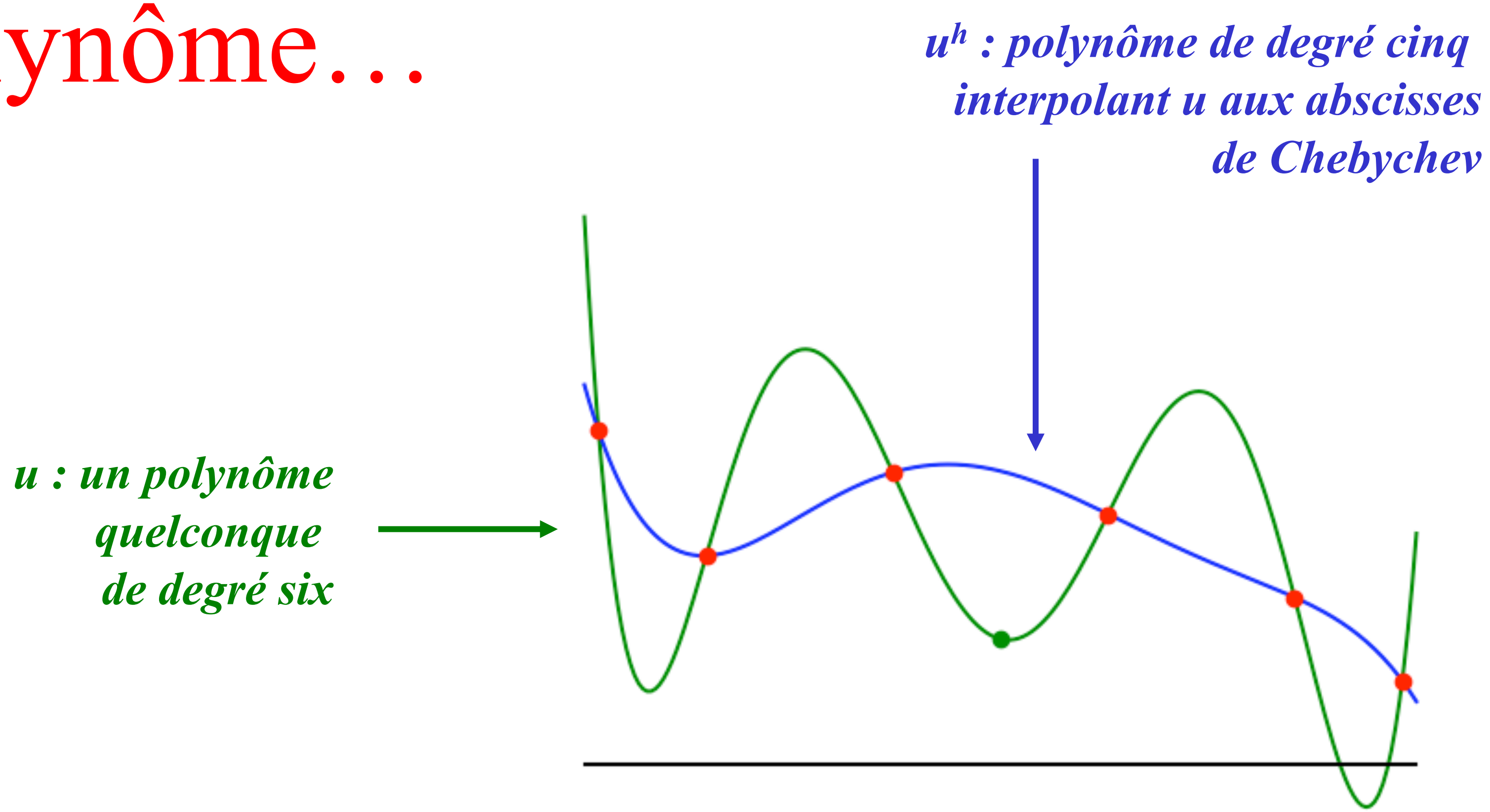
Le polynôme de degré six passant par six points aux abscisses de Chebyshev et par le septième point



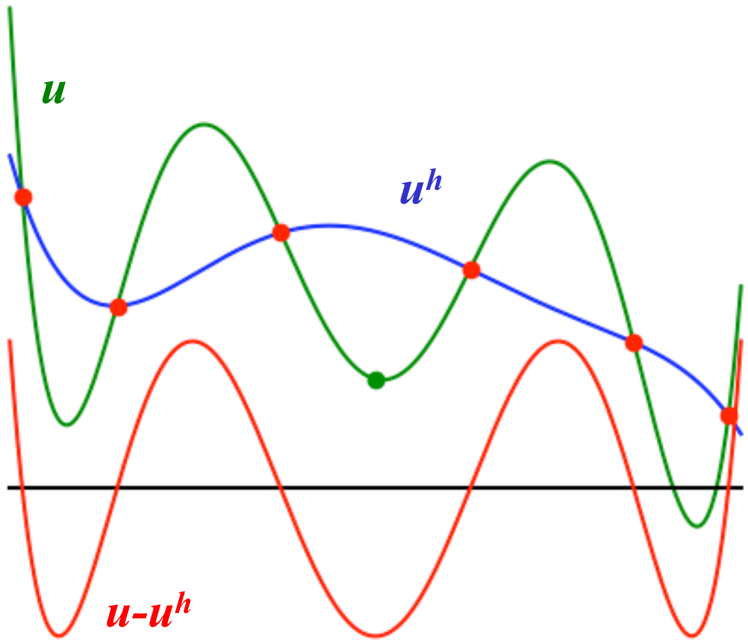
Le septième point

```
pu = polyfit([*X,0],[*U,0.3],6)
u = polyval(pu,x)
plt.plot(x,u,'-g')
plt.plot([0],[0.3],'og')
```

Interpolons un polynôme par un polynôme…

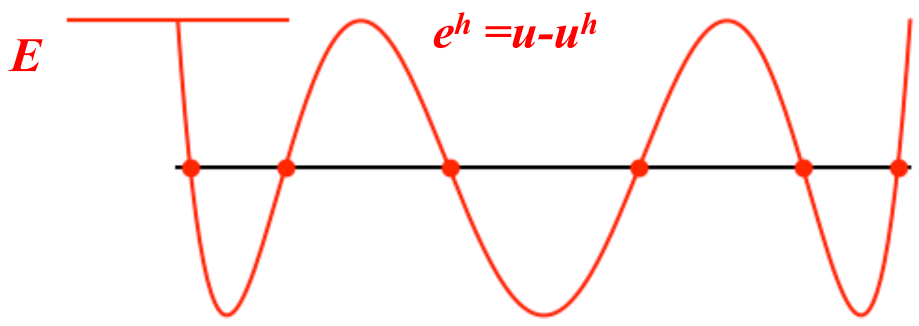


…c'est bête, je sais :-)

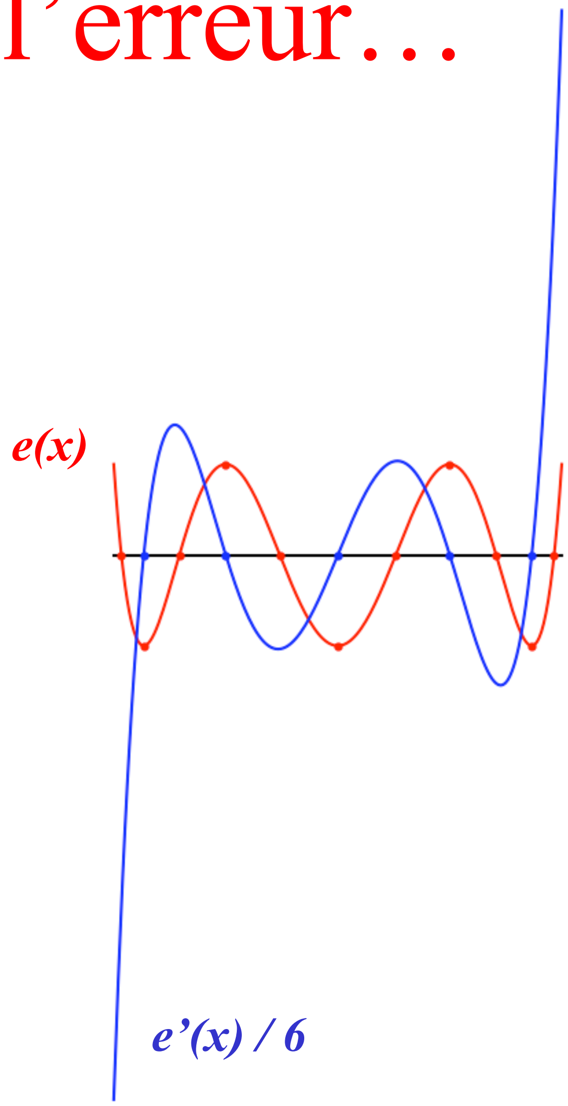



```
error = u - uh
E = error[0]
plt.plot(x,error,'-r');
```

Erreur d'interpolation



Dérivons et divisons l'erreur…

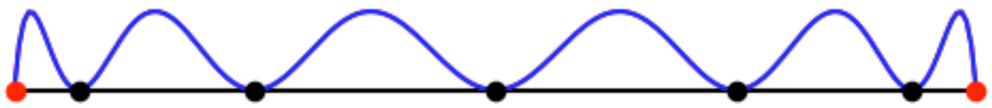



```
peh = pu - [0,*puh]
eh = polyval(peh, x)
plt.plot(x,eh,'-r')

pdeh = peh * [6,5,4,3,2,1,0]
pdeh = pdeh[0:-1]
deh = polyval(pdeh,x)/6
plt.plot(x,deh,'-b')
```

Et encore quelques petites manipulations…

$$\left(e'(x)\right)^2 (1 - x^2) \quad = \quad (n+1)^2 \left(E^2 - e^2(x)\right)$$



```
Xd = roots(pdeh)
Ed = polyval(peh,Xd)
E = Ed[0]

plt.plot(x,E**2-eh**2,'-r')
plt.plot(x,(1-x**2)*(deh**2),'-b')
plt.plot(Xd,zeros(size(Xd)),'ok')
plt.plot([-1,1],[0,0],'or')
```

Une solution analytique d'une équation différentielle ?

$$\left(e'(x)\right)^2 (1 - x^2) \quad = \quad (n+1)^2 \left(E^2 - e^2(x)\right)$$

$$\downarrow$$

$$\frac{\frac{e'(x)}{E}}{\sqrt{1 - \left(\frac{e}{E}\right)^2}} \quad = \quad \pm(n+1)\frac{1}{\sqrt{1 - x^2}}$$

```
>>> from sympy import *
>>> x = symbols('x')
>>> f = 1/sqrt(1-x**2)
>>> integrate(f)
asin(x)
```

Et si on dérive la primitive…

```
>>> from sympy import *
>>> x = symbols('x')
>>> f = 1/sqrt(1-x**2)
>>> integrate(f)
asin(x)
>>> diff(asin(x))
1/sqrt(-x**2 + 1)
>>> diff(acos(x))
-1/sqrt(-x**2 + 1)
```

Une petite solution analytique comme le faisaient les anciens…

$$(e'(x))^2(1-x^2) = (n+1)^2(E^2 - e^2(x))$$

$$\downarrow$$

$$\frac{\frac{e'(x)}{E}}{\sqrt{1-\left(\frac{e}{E}\right)^2}} = \pm(n+1)\frac{1}{\sqrt{1-x^2}}$$

$$\downarrow$$

$$\arccos\left(\frac{e(x)}{E}\right) = \pm\left((n+1)\arccos(x) + C\right)$$

$\downarrow$ En vertu de la parité du cosinus !

$$e(x) = E\cos\left((n+1)\arccos(x) + C\right)$$

$\downarrow$ En imposant que e(1) = E

$$e(x) = E\underbrace{\cos\left((n+1)\arccos(x)\right)}_{T_{n+1}(x)}$$

Polynôme de Chebyshev de degré n+1

Drôle d'expression pour un polynôme, non ?

Théorème 1.2. *Les polynômes de Chebyshev* $T_{n+1}(x) = \cos((n+1)\arccos(x))$ *définis sur l'intervalle* $[-1, 1]$ *satisfont la relation de récurrence*

$$T_{i+1}(x) = 2x\ T_i(x) - T_{i-1}(x), \quad i = 1, 2, 3, \ldots,$$

avec $T_0(x) = 1$ *et* $T_1(x) = x$.

Calcul des polynômes de Chebyshev : formule de récurrence

Démonstration : Définissons $\theta = \arccos(x)$ et écrivons :

$$\begin{aligned} T_{i+1}(x) &= \cos((i+1)\theta) \\ &= \cos(\theta)\cos(i\theta) - \sin(\theta)\sin(i\theta) \end{aligned}$$

$$\begin{aligned} T_{i-1}(x) &= \cos((i-1)\theta) \\ &= \cos(\theta)\cos(i\theta) + \sin(\theta)\sin(i\theta) \end{aligned}$$

$$\begin{aligned} T_{i+1}(x) + T_{i-1}(x) &= 2\cos(\theta)\cos(i\theta) \\ &= 2xT_i(x) \end{aligned}$$

□

Abscisses de Chebyshev

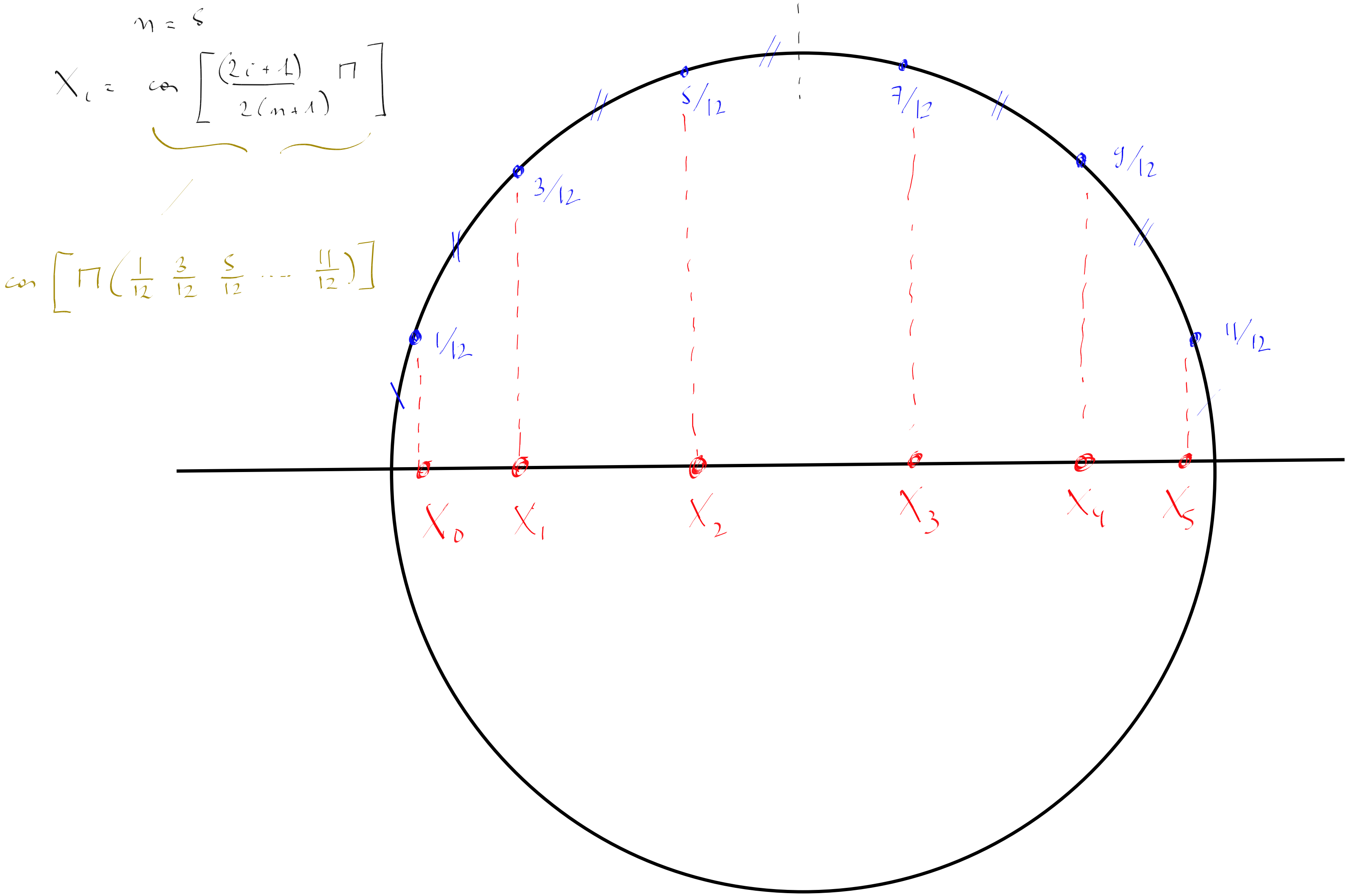
$$0 = \overbrace{\cos\left((n+1)\arccos(X_i)\right)}^{T_{n+1}(X_i)} \qquad i = 0, \ldots, n$$

$$\downarrow$$

$$\frac{\pi/2 + i\pi}{(n+1)} = \arccos(X_i) \qquad i = 0, \ldots, n$$

$$\downarrow$$

$$\cos\left(\frac{(2i+1)\pi}{2(n+1)}\right) = X_i \qquad i = 0, \ldots, n$$



Abscisses
de Chebychev

Interpolation polynomiale : bilan

❑**Pour une fonction u(x) très régulière : fonction cosinus**

Convergence de l'interpolation polynomiale

❑**Pour une fonction u(x) suffisamment régulière : fonction de Runge**

Divergence pour des abscisses équidistantes
Convergence pour les abscisses de Chebyshev

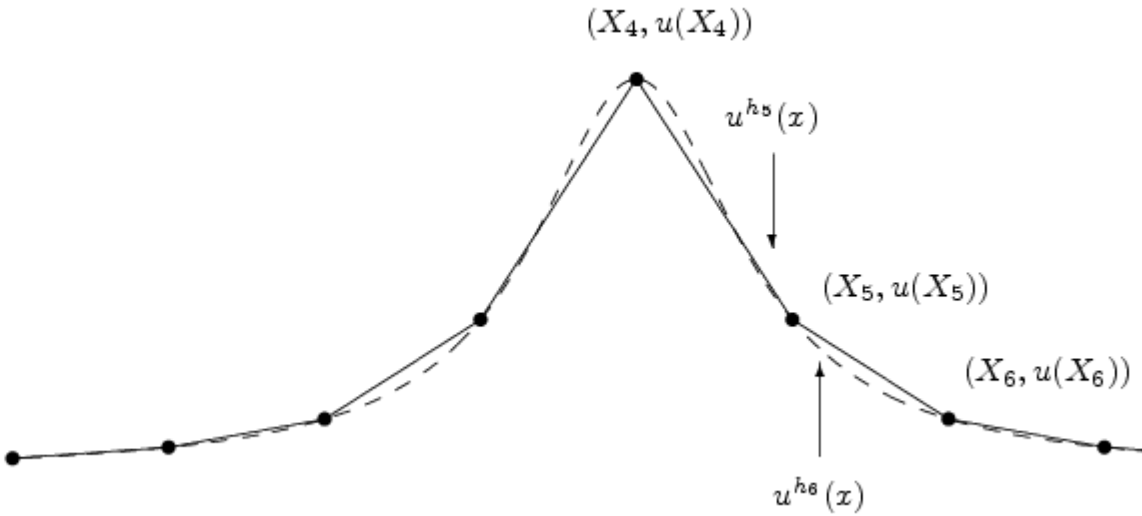
❑**Pour une fonction u(x) peu régulière : fonction échelon**

Divergence !
Eviter l'interpolation polynomiale de degré élevé

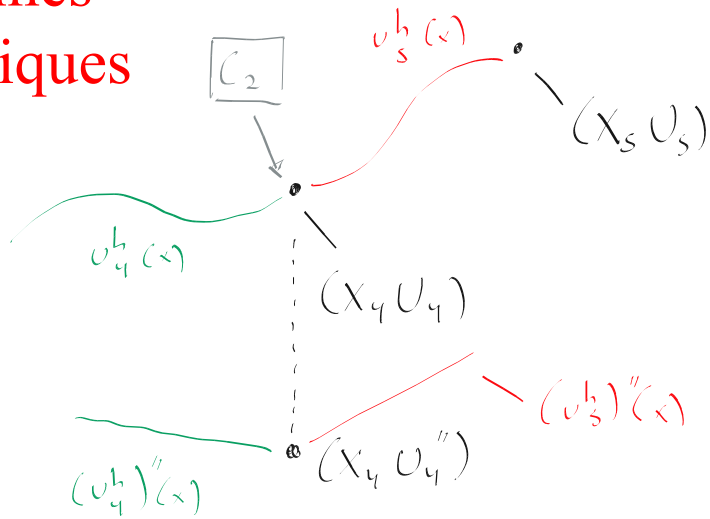
Idée :

Utiliser une interpolation par morceaux composée des polynômes de degré bas !

Interpolation linéaire par morceaux



Splines cubiques



BE CAREFUL

$$U_4'' \neq u''(X_4)$$

$$u_5^h(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0$$

$n+1$ POINTS

n FONCTIONS

→ $4n$ COEFFICIENTS

$$u_4^h(X_4) = U_4 = u_5^h(X_4)$$

$$(u_4^h)'(X_4) = (u_5^h)'(X_4) \neq (u)'(X_4)$$

$$(u_4^h)''(X_4) = (u_5^h)''(X_4) \neq (u)''(X_4)$$

$$2(n-1) + 2(n-1) + 2$$

IL Y A $4n - 2$ CONDITIONS

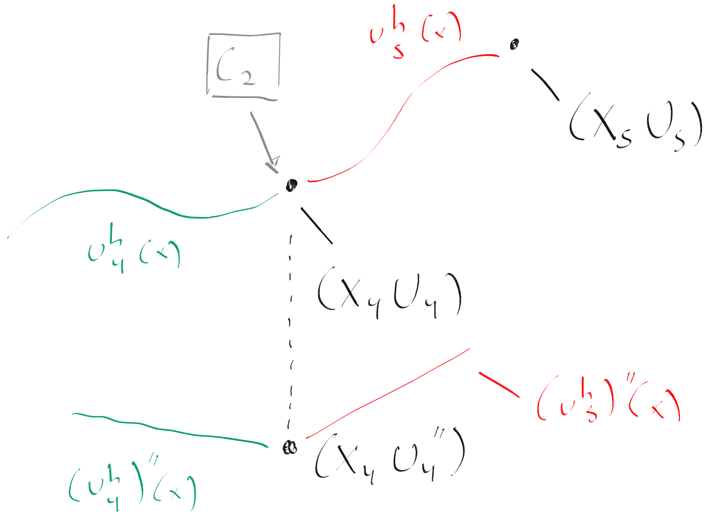
IL MANQUE 2 CONDITIONS

$$u^{h''}(X_0) = u^{h''}(X_n) = 0$$

Etape 1
Dérivée seconde ok

$$(u_5^h)''(x) = U_4'' \frac{(x - X_5)}{(X_4 - X_5)} + U_5'' \frac{(x - X_4)}{(X_5 - X_4)}$$

$$\frac{X_5 - x}{h} \qquad \frac{x - X_4}{h}$$



$$+ Cx + D$$

$$u_5^h = U_4'' \frac{(X_5 - x)^3}{6h} + U_5'' \frac{(x - X_4)^3}{6h} + A \frac{(X_5 - x)}{h} + B \frac{(x - X_4)}{h}$$

$$U_4 = U_4'' \frac{h^2}{6} + A$$

$$A = U_4 - U_4'' \frac{h^2}{6}$$

$$B = U_5 - U_5'' \frac{h^2}{6}$$

Etape 2
Interpolation ok

Etape 3
Dérivée première ok

$$u_5^h = U_4'' \frac{(X_5 - x)^3}{6h} + U_5'' \frac{(x - X_4)^3}{6h} + A\frac{(X_5 - x)}{h} + B\frac{(x - X_4)}{h}$$

$$(u_4^h)'(X_4) = (u_5^h)'(X_4)$$

$$= U_4'' \frac{-3h^2}{6h} - \frac{A}{h} + \frac{B}{h}$$

$$\left[\frac{(X_5 - x)^3}{6h}\right]'_{x=X_4} = \left.\frac{-3(X_5 - x)^2}{6h}\right|_{x=X_4}$$

$$\frac{h}{2}U_4'' + \left[\frac{U_4 - U_3}{h}\right] - \left[\frac{U_4'' - U_3''}{h}\right]\frac{h^2}{6} = -\frac{h}{2}U_4'' + \left[\frac{U_5 - U_4}{h}\right] - \left[\frac{U_5'' - U_4''}{h}\right]\frac{h^2}{6}$$

$$\left[\frac{U_5 - 2U_4 + U_3}{h}\right] = \frac{h}{6}\left[U_5'' + 4U_4'' + U_3''\right]$$

$$A = U_4 - U_4''\frac{h^2}{6}$$

$$B = U_5 - U_5''\frac{h^2}{6}$$

Et le système
à résoudre est finalement…

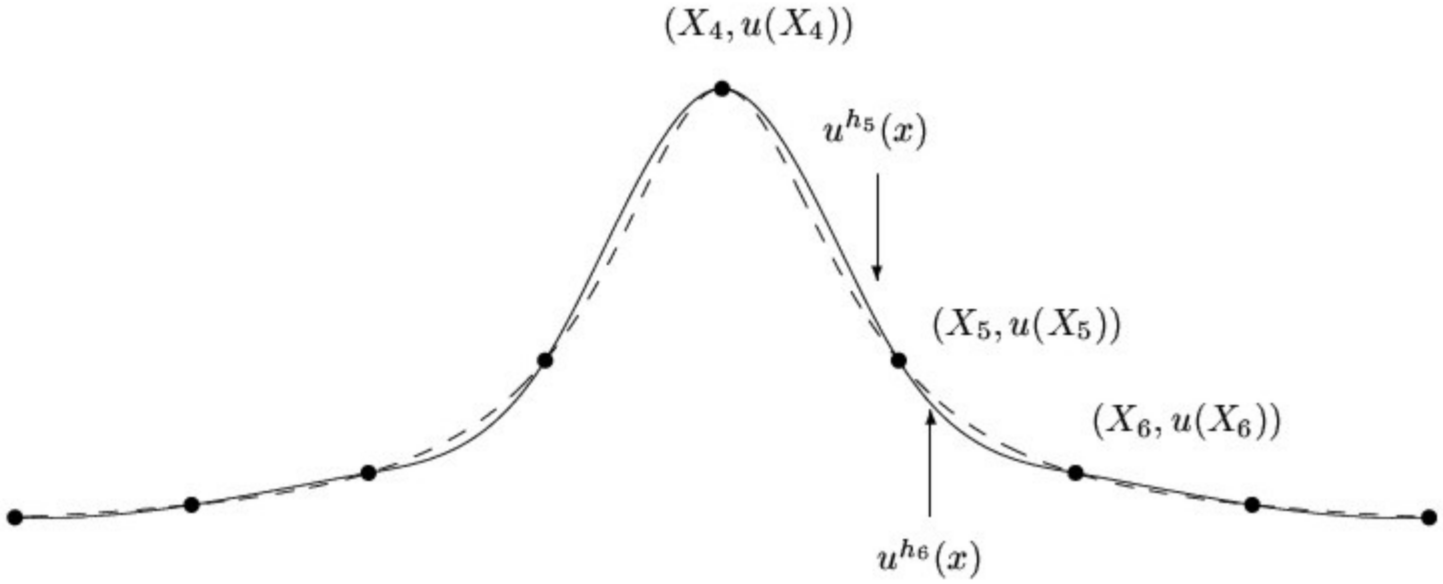
$$\left[\frac{U_5 - 2U_4 + U_3}{h}\right] = \frac{h}{6}\left[U_5'' + 4U_4'' + U_3''\right]$$

$$\frac{h^2}{6}\begin{bmatrix} 1 & & & & & \\ 1 & 4 & 1 & & & \\ & 1 & 4 & 1 & & \\ & & & \ddots & & \\ & & & 1 & 4 & 1 \\ & & & & & 1 \end{bmatrix}\begin{bmatrix} U_0'' \\ \vdots \\ U_4'' \\ \vdots \\ U_n'' \end{bmatrix} = \begin{bmatrix} 0 \\ \\ U_5 - 2U_4 + U_3 \\ \\ 0 \end{bmatrix}$$

Interpolation par splines cubiques

$$u^{h_i}(x) = a_i + b_i x + c_i x^2 + d_i x^3 \qquad i = 1, 2, \ldots, n$$

4n coefficients inconnus



Comment trouver les coefficients ?

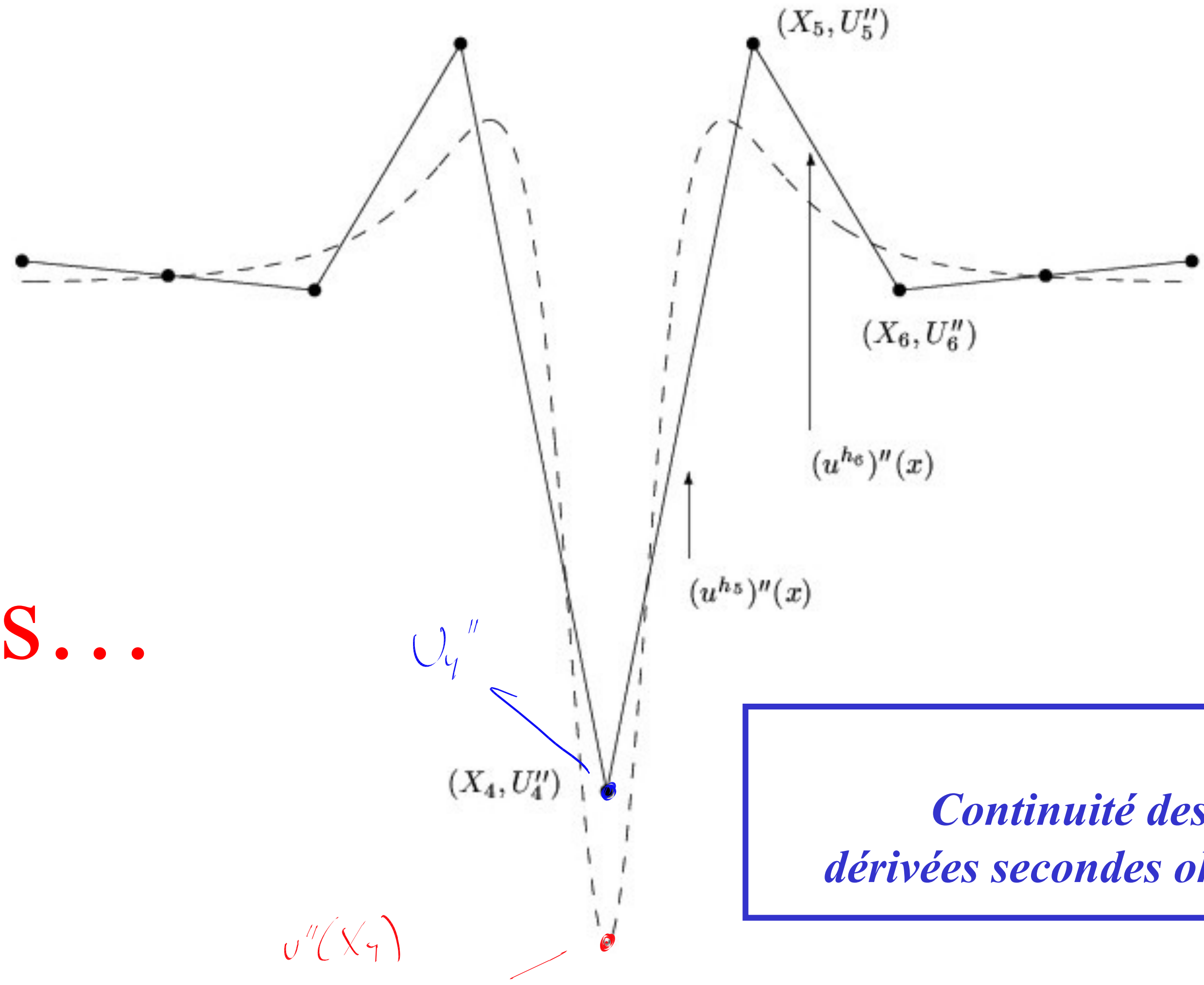
$$\begin{aligned} u^{h_1}(X_0) &= U_0 \\ u^{h_n}(X_n) &= U_n \end{aligned}$$

$$\begin{aligned} u^{h_i}(X_i) &= U_i & i = 1, \ldots, n-1 \\ u^{h_{i+1}}(X_i) &= U_i & i = 1, \ldots, n-1 \end{aligned}$$

$$\begin{aligned} (u^{h_i})'(X_i) &= (u^{h_{i+1}})'(X_i) & i = 1, \ldots, n-1 \\ (u^{h_i})''(X_i) &= (u^{h_{i+1}})''(X_i) & i = 1, \ldots, n-1 \end{aligned}$$

4n-2 conditions

$$(u^{h_i})'' = U''_{i-1}\frac{(x-X_i)}{(X_{i-1}-X_i)} + U''_i\frac{(x-X_{i-1})}{(X_i-X_{i-1})}$$



Ecrivons…

Continuité des dérivées secondes ok

Intégrons…

$$(u^{h_i})'' = U''_{i-1}\frac{(X_i - x)}{h_i} + U''_i\frac{(x - X_{i-1})}{h_i}$$

En intégrant deux fois,

$$(u^{h_i}) = U''_{i-1}\frac{(X_i - x)^3}{6h_i} + U''_i\frac{(x - X_{i-1})^3}{6h_i} + A_i\frac{(X_i - x)}{h_i} + B_i\frac{(x - X_{i-1})}{h_i}$$

$$U_{i-1} = U''_{i-1}\frac{h_i^3}{6h_i} + A_i\frac{h_i}{h_i} \quad \text{et} \quad U_i = U''_i\frac{h_i^3}{6h_i} + B_i\frac{h_i}{h_i}$$

$$A_i = U_{i-1} - \frac{U''_{i-1}\,h_i^2}{6} \qquad B_i = U_i - \frac{U''_i\,h_i^2}{6}$$

Continuité de la fonction ok

Calculons…

Continuité des dérivées premières ok

$$(u^{h_i})'(X_i) = (u^{h_{i+1}})'(X_i)$$

$$\frac{U_i''h_i}{2} + \frac{(U_i - U_{i-1})}{h_i} - \frac{(U_i'' - U_{i-1}'')h_i}{6} = -\frac{U_i''h_{i+1}}{2} + \frac{(U_{i+1} - U_i)}{h_{i+1}} - \frac{(U_{i+1}'' - U_i'')h_{i+1}}{6}$$

$$\frac{(2U_i'' + U_{i-1}'')h_i}{6} + \frac{(U_i - U_{i-1})}{h_i} = \frac{(U_{i+1} - U_i)}{h_{i+1}} - \frac{(U_{i+1}'' + 2U_i'')h_{i+1}}{6}$$

$$\frac{h_i}{6}\,U_{i-1}'' + \frac{2(h_i + h_{i+1})}{6}\,U_i'' + \frac{h_{i+1}}{6}\,U_{i+1}'' = \frac{(U_{i+1} - U_i)}{h_{i+1}} - \frac{(U_i - U_{i-1})}{h_i}$$

$$i = 1, \ldots, n-1$$

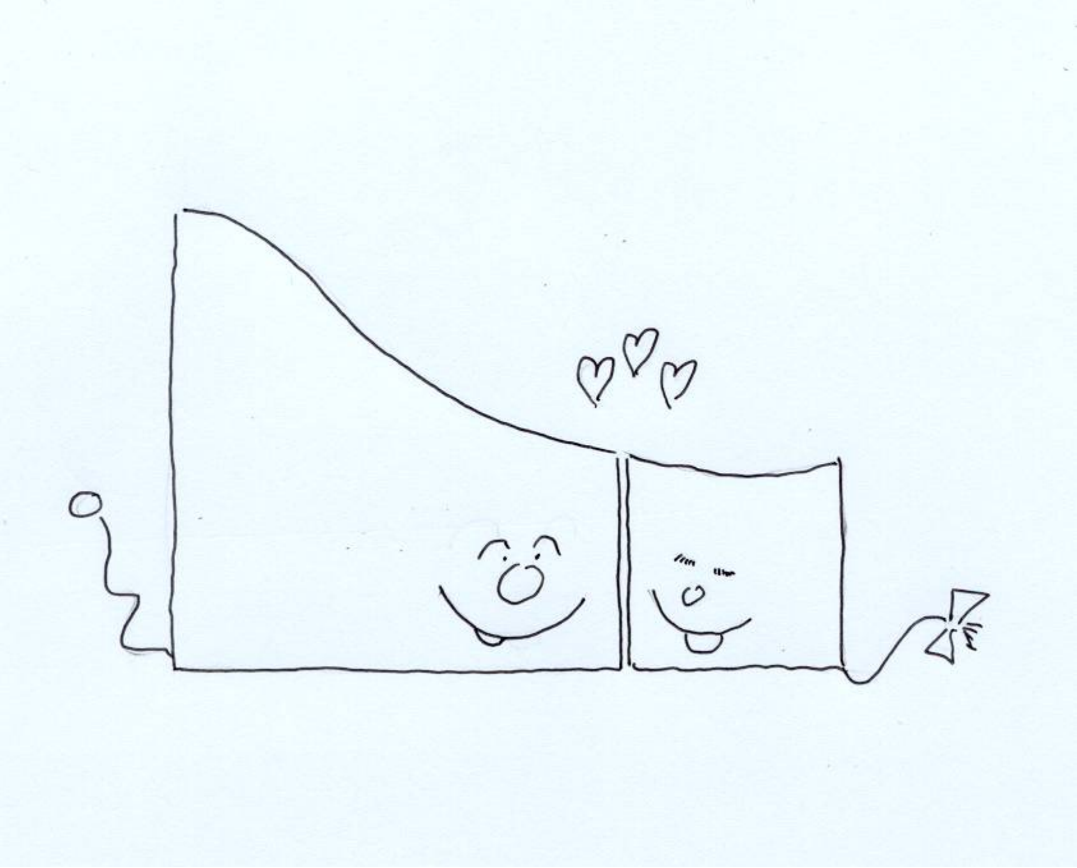
Abscisses équidistantes

$$U_0'' = 0 \quad \text{et} \quad U_n'' = 0$$

2 conditions supplémentaires
Courbe spline naturelle

$$\frac{h^2}{6}\begin{bmatrix} 1 & 0 & & & & & & & \\ 1 & 4 & 1 & & & & & & \\ & 1 & 4 & 1 & & & & & \\ & & 1 & 4 & 1 & & & & \\ & & & 1 & 4 & 1 & & & \\ & & & & \cdot & \cdot & \cdot & & \\ & & & & & 1 & 4 & 1 & \\ & & & & & & 1 & 4 & 1 \\ & & & & & & & 0 & 1 \end{bmatrix}\begin{bmatrix} U_0'' \\ U_1'' \\ U_2'' \\ U_3'' \\ U_4'' \\ \vdots \\ U_{n-2}'' \\ U_{n-1}'' \\ U_n'' \end{bmatrix} = \begin{bmatrix} 0 \\ U_0 - 2U_1 + U_2 \\ U_1 - 2U_2 + U_3 \\ U_2 - 2U_3 + U_4 \\ U_3 - 2U_4 + U_5 \\ \vdots \\ \vdots \\ U_{n-2} - 2U_{n-1} + U_n \\ 0 \end{bmatrix}$$

Ou de manière plus poétique…

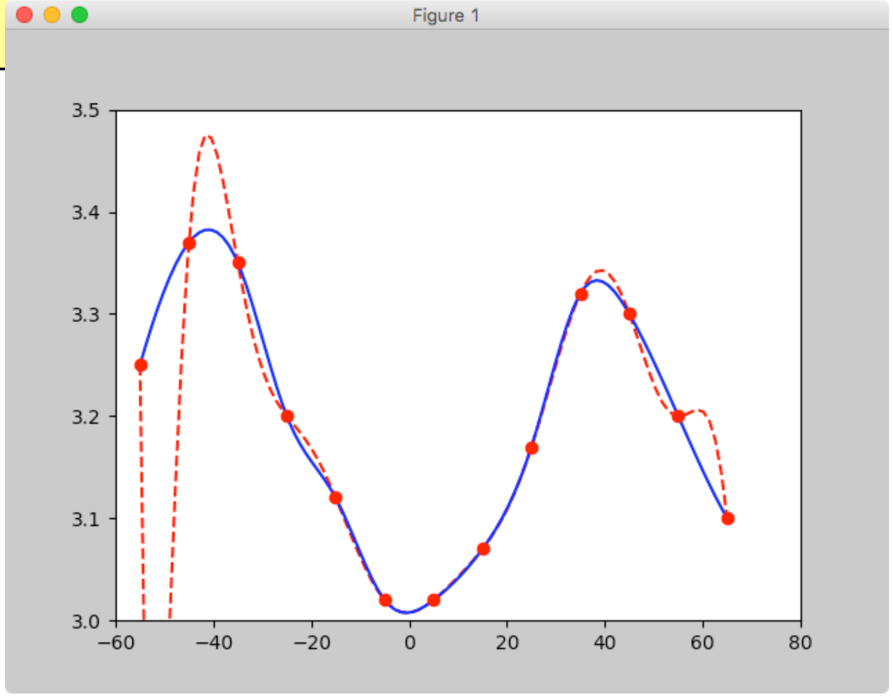


```
from numpy import *
from scipy.interpolate import CubicSpline as spline
from matplotlib import pyplot as plt

X = arange(-55,70,10)
U = [3.25, 3.37, 3.35, 3.20, 3.12, 3.02, 3.02,
           3.07, 3.17, 3.32, 3.30, 3.20, 3.10]
x = linspace(X[0],X[-1],100)
uhLag = polyval(polyfit(X,U,len(X)-1),x)
uhSpl = spline(X,U)

plt.plot(x,uhLag,'--r',x,uhSpl(x),'-b')
plt.plot(X,U,'or')
```

Exemple



A quoi servent les méthodes numériques ?

End of the Millenium Question

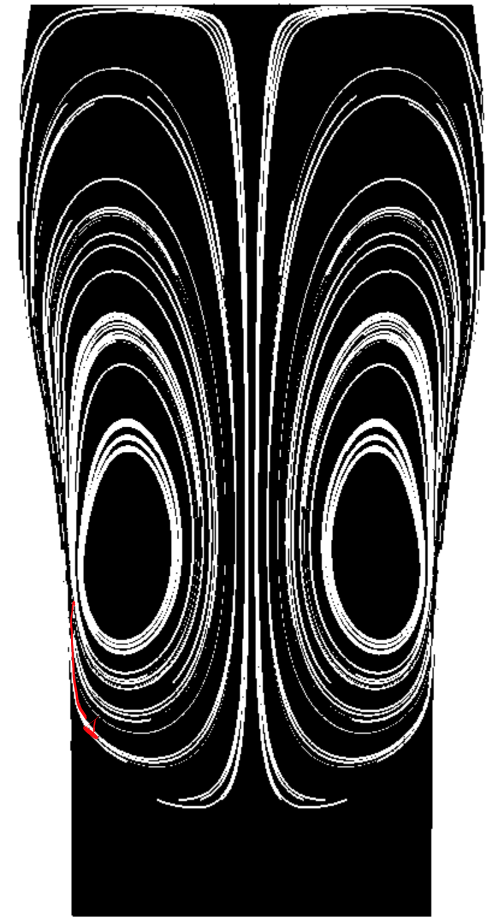
Do the Bubbles in a Glass of Guinness Beer Go Up or Down?

Since time immemorial, mankind has been troubled by natural phenomena that cannot be easily explained. Why do the bubbles in a glass of that venerable dark brew called Guinness appear to travel downward in the glass?

This goes against what we know about the physics of bubbles, or does it?

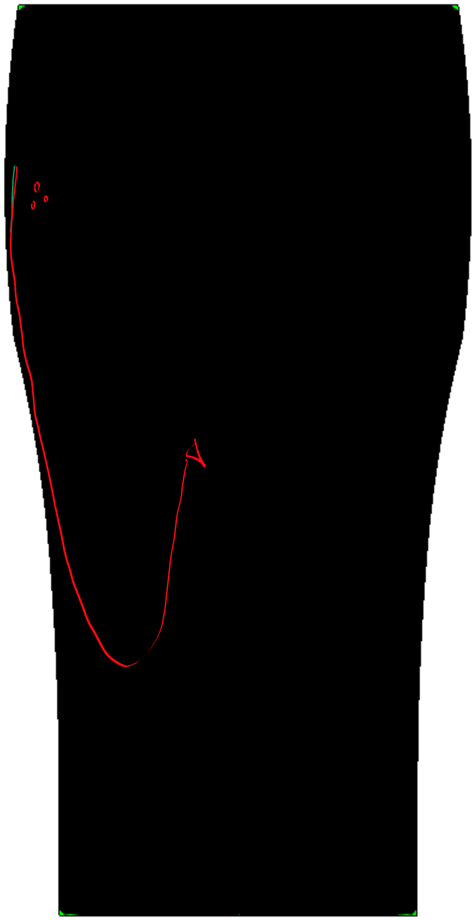
The answer to this most fundamental question has been locked away for centuries, resisting the direct assault of the most determined philosophers and theoreticians without compassion.

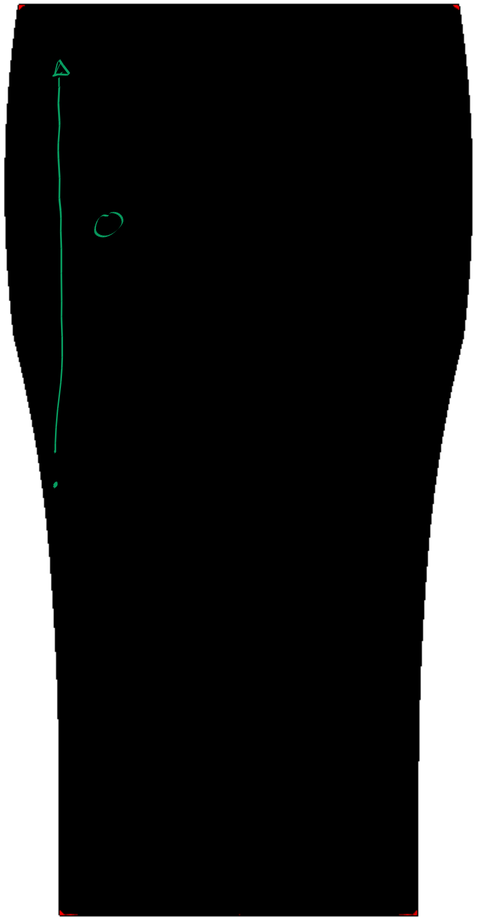
But, with the help of Computational Fluid Dynamics, we can provide a categorical and definitive answer: the bubbles go both up and down.



Fletcher et al. (1999)
CANCES-UNSW *Australia*
Fluent *United States*

La réponse des scientifiques…





Bubble tracks show that the 1 mm bubbles move steadily upwards while the 60 micron bubbles are dragged downwards near the side of the glass.

Python for dummies !

Exécution et soumission d'un programme sur le serveur...

Deadline : February 15 2023 23:59:59.
Now : February 14 2023 22:28:36.

10 pt

```
from numpy import *
from numpy.linalg import solve
#
# TOUTE AUTRE INSTRUCTION CONTENANT import / from SERA AUTOMATIQUEMENT SUPPRIMEE
#

def padeInterpolationCompute(X,U):

#
# A COMPLETER / MODIFIER
```

Position: Ln 1, Ch 1 | Total: Ln 25, Ch 305

Soumettre le programme | Voir le diagnostic

Valider son programme

Date limite pour le problème : 15/Fev/2023 23:59:59
et nous sommes aujourd'hui : 15/Fev/2023 10:30:00...