Un peu de physique !

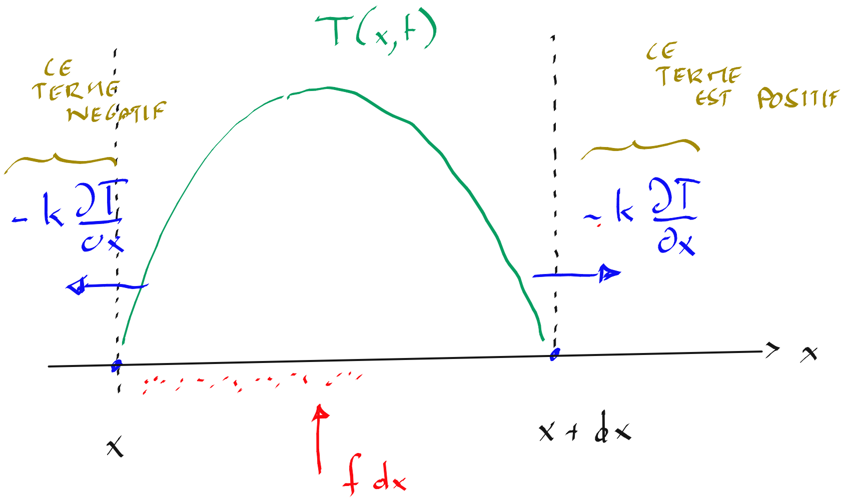
$$\rho c \frac{\partial T}{\partial t} dx = f\, dx + k \left.\frac{\partial T}{\partial x}\right|_{x+dx} - k \left.\frac{\partial T}{\partial x}\right|_{x}$$

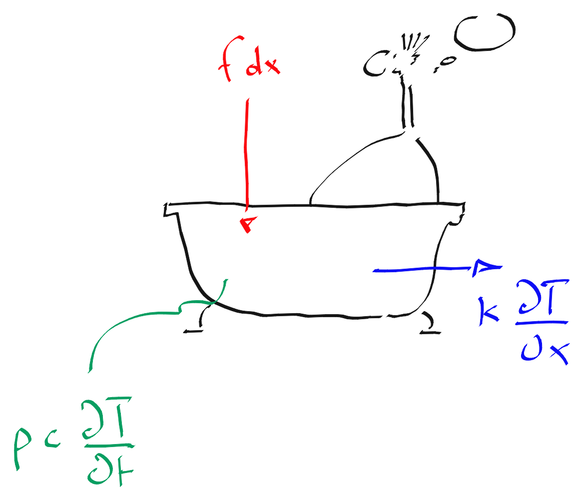
ρ : MASSE VOLUMIQUE

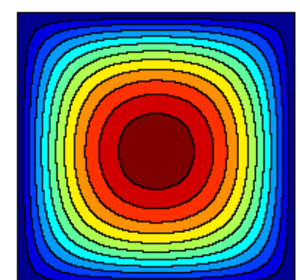
c : CAPACITE THERMIQUE

k : CONDUCTIBILITE THERMIQUE

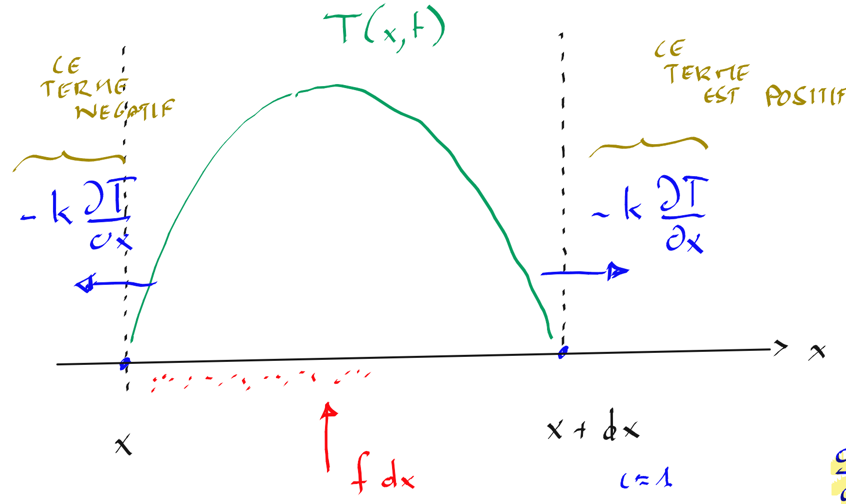
$$\boxed{\rho c \frac{\partial T}{\partial t} = f + k \frac{\partial^2 T}{\partial x^2}}$$







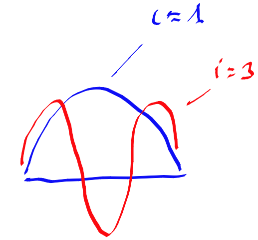
Solution analytique !



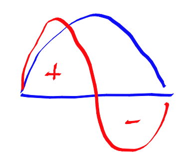
$$\frac{d^2u}{dx^2} + 1 = 0$$

$$u(-1) = 0$$
$$u(1) = 0$$

$$u(x) = \sum_{\substack{i=1\\ \text{impairs}}}^{\infty} C_i \sin\left(\frac{i(x+1)\pi}{2}\right)$$



$$1 - \sum C_i \sin[\ldots] \frac{i^2\pi^2}{4} = 0$$



$$\int_{-1}^{1} \sin[j\ldots] \sum C_i \sin[i\ldots] \frac{i^2\pi^2}{4} = \int_{-1}^{1} \sin[j\ldots]\, 1 \qquad \forall j$$

$$\sum C_i \frac{i^2\pi^2}{4} \int_{-1}^{1} \sin[j\ldots] \sin[i\ldots] = \int_{-1}^{1} \sin[j\ldots]$$

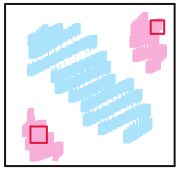
$= 0$ si $i \neq j$

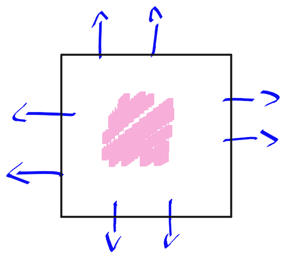
$\neq 0$ si $i = j$

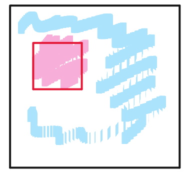
$$j^2 \frac{\pi^2}{4} C_j = \frac{\int \ldots}{\int \ldots} \Bigg\} \frac{4}{j\pi}$$

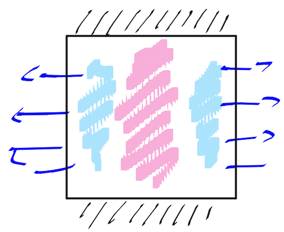
$$C_j = \frac{16}{j^3 \pi^3}$$

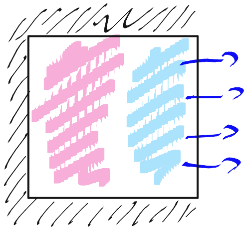
$$k\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right] = 1$$



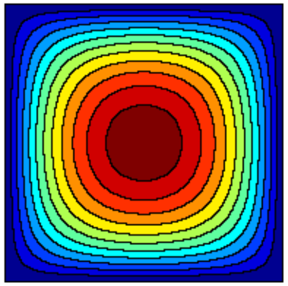




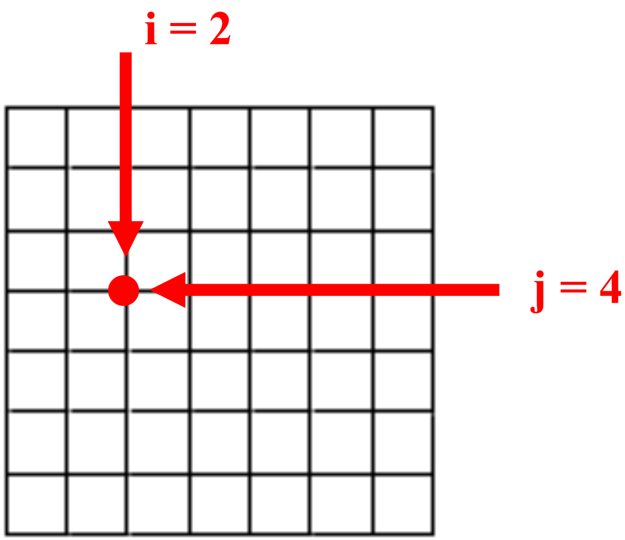


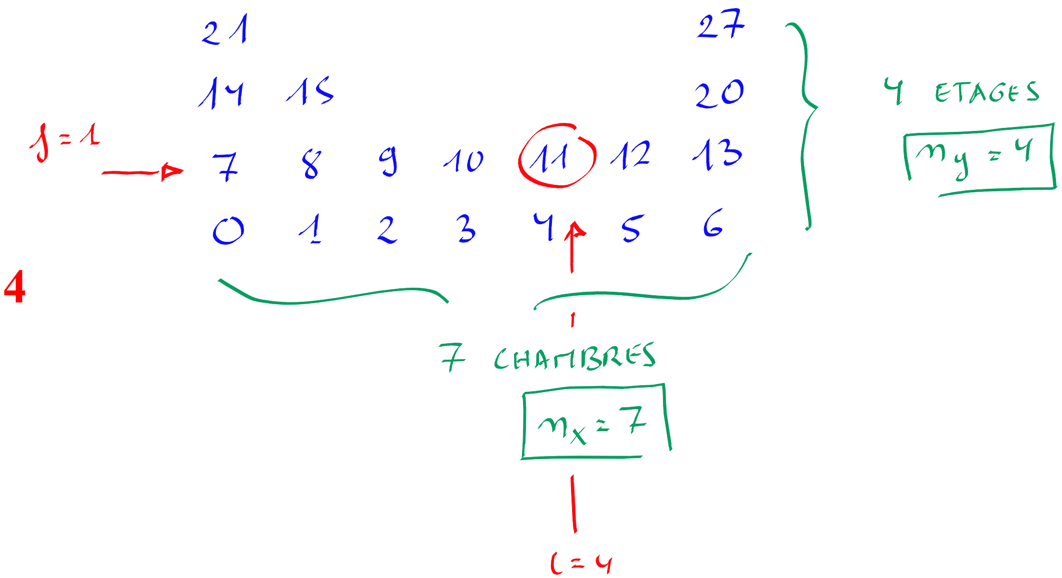


En deux dimension !

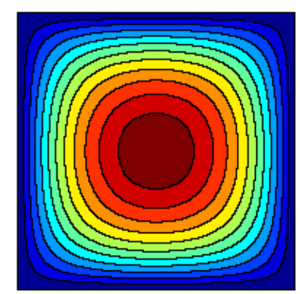


Un peu de numérotation

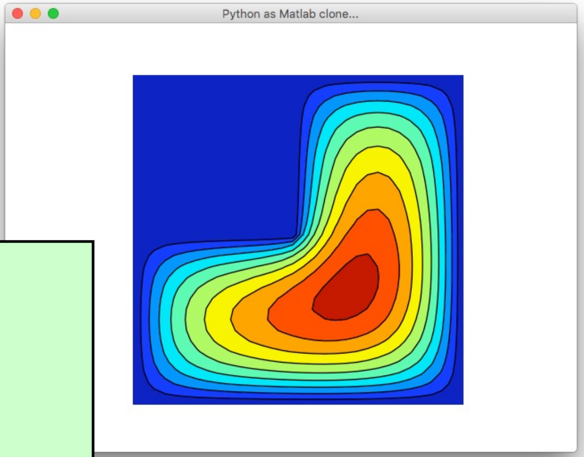




$$\text{INDEX} = \underbrace{4}_{=i} + \underbrace{1}_{=j} \times n_x = 11$$



Plan du cours de méthodes numériques



Comment résoudre numériquement un problème aux valeurs initiales ?

Comment interpoler une fonction ?

Comment dériver numériquement une fonction ?

Comment résoudre numériquement un problème aux conditions frontières ?

Comment approximer une fonction ?

Comment intégrer numériquement une fonction ?

Et les équations non-linéaires ?

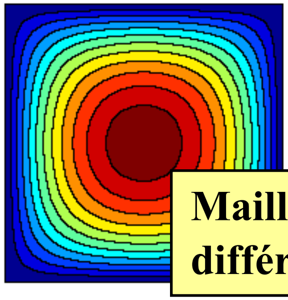
Et les méthodes itératives ?

Comment résoudre numériquement une équation différentielle ordinaire ?

Comment résoudre numériquement une équation aux dérivées partielles ?

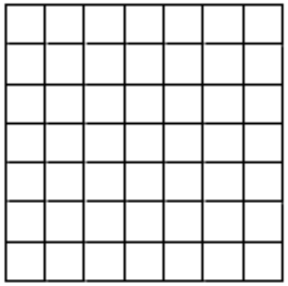
Comment résoudre numériquement une équation aux dérivées partielles ?

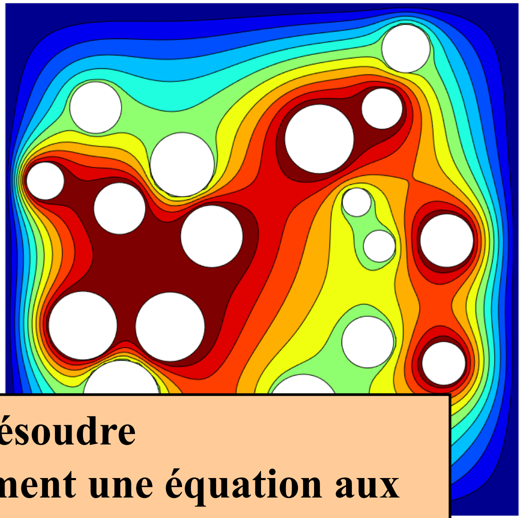
A quoi servent les méthodes numériques ?



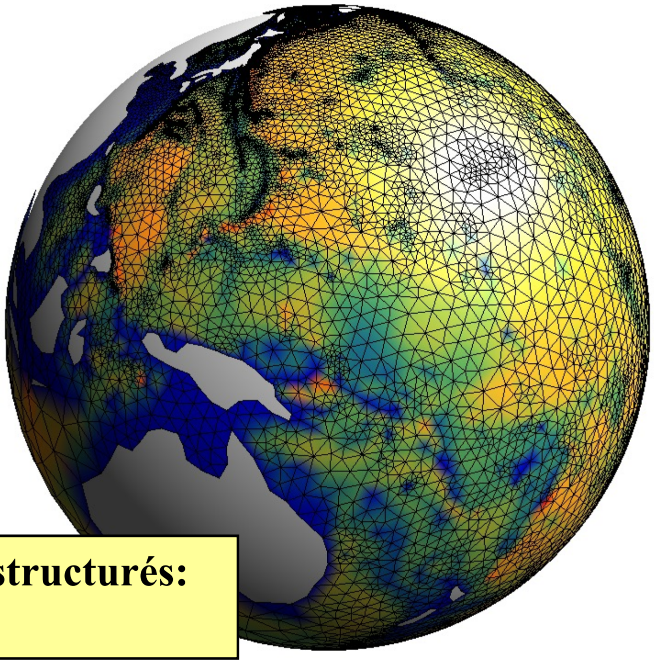
LEPL1104

Maillages structurés: différences finies





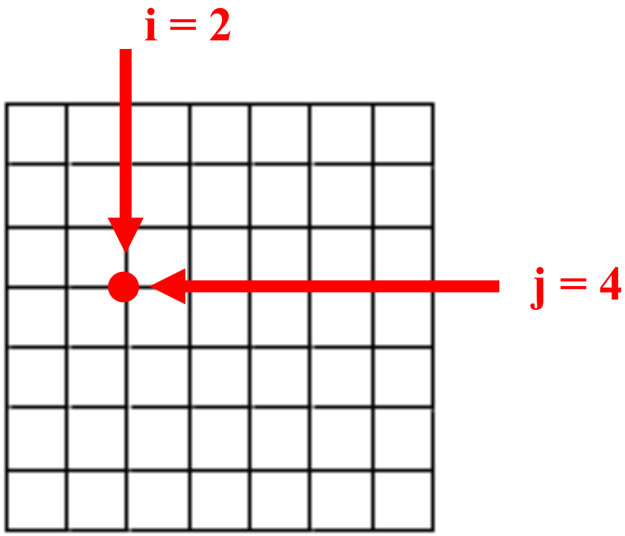
Comment résoudre numériquement une équation aux dérivées partielles avec des conditions aux limites ?



LEPL1110

Maillages non-structurés: éléments finis

Grille / Maillage



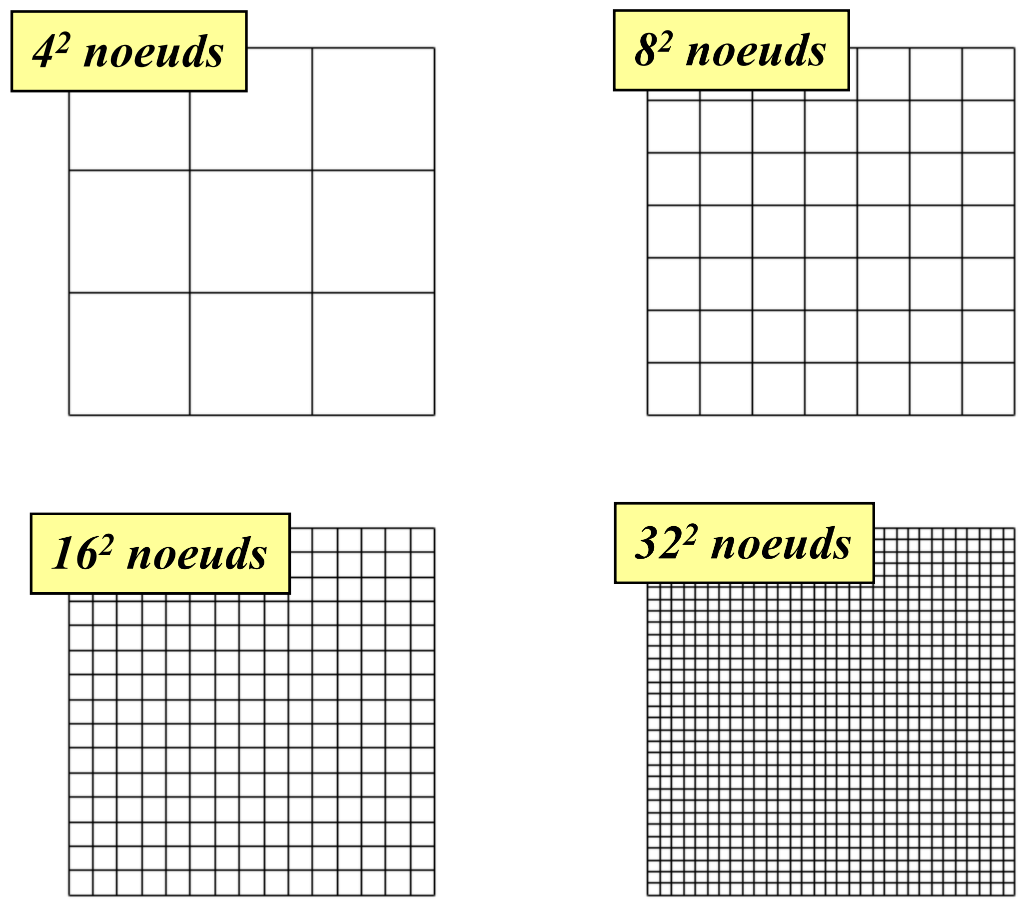
$$h = (b-a)/(m)$$

$$X_i = a + ih, \quad i = 0, \ldots, m$$

$$\mathbf{X}_{ij} = (X_i, Y_j) = (ih, jh)$$

$$U_{ij} = u^h(\mathbf{X}_{ij}) \approx u(\mathbf{X}_{ij})$$

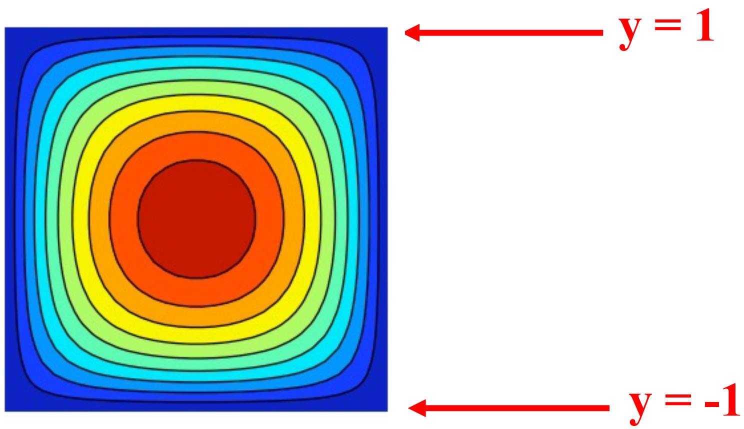
Méthode des différences finies



$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$

Exemple

$$\begin{aligned} \nabla^2 u(x,y) + 1 &= 0, \quad (x,y) \in \Omega, \\ u(x,y) &= 0, \quad (x,y) \in \partial\Omega, \end{aligned}$$



$$u(x,y) = \sum_{i,j\ impairs} C_{ij} \sin\left(\frac{i\pi(x+1)}{2}\right) \sin\left(\frac{j\pi(y+1)}{2}\right)$$

Comment trouver C_{ij} ?

$$\nabla^2 u(x,y) + 1 = 0,$$

$$\sum_{i,j \text{ impairs}} \frac{-\pi^2(i^2+j^2)}{4} C_{ij} \sin\left(\frac{i\pi(x+1)}{2}\right) \sin\left(\frac{j\pi(y+1)}{2}\right) + 1 = 0,$$

En vertu de l'orthogonalité des sinus,

$$\frac{\pi^2(i^2+j^2)}{4} C_{ij} - \underbrace{\int_\Omega \sin\left(\frac{i\pi(x+1)}{2}\right) \sin\left(\frac{j\pi(y+1)}{2}\right) d\Omega}_{16/(ij\pi^2)} = 0,$$

$$u(x,y) = \sum_{i,j \ impairs} \frac{64}{\pi^4(i^2+j^2)ij} \sin\left(\frac{i\pi(x+1)}{2}\right) \sin\left(\frac{j\pi(y+1)}{2}\right)$$

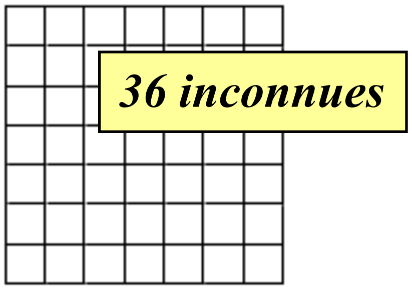
Différences finies

$$\left(\frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h^2} + \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{h^2}\right) + 1 \quad = \quad 0$$

$$\frac{U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} - 4U_{i,j}}{h^2} + 1 \quad = \quad 0$$

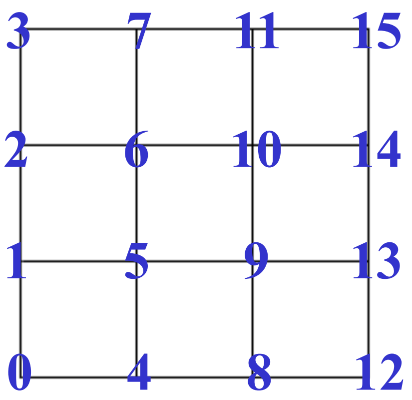
(n-2)(n-2) équations linéaires
(n-2)(n-2) inconnues

Il suffit donc de résoudre un grand système linéaire



Programme Python….

```
def poissonSolve(nx,ny):
  n = nx*ny; h = 2/(ny-1)
  A = eye(n); B = zeros(n)
  for i in range(1,nx-1):
    for j in range(1,ny-1):
      index = i + j*nx
      A[index,index]    =  4.0
      A[index,index-1]  = -1.0
      A[index,index+1]  = -1.0
      A[index,index-nx] = -1.0
      A[index,index+nx] = -1.0
      B[index] = 1
  return solve(A,B).reshape(ny,nx)
```



$$\frac{U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} - 4U_{i,j}}{h^2} + 1 \quad = \quad 0$$

Comment résoudre le système discret avec python...

~~`A = inv(A)`~~
~~`x = A @ b`~~

```
x = solve(A,b)
```

On résout un système linéaire, on ne l'inverse jamais…. (J. Meinguet)

A frequent misuse of inv arises when solving the system of linear equations. One way to solve this is with

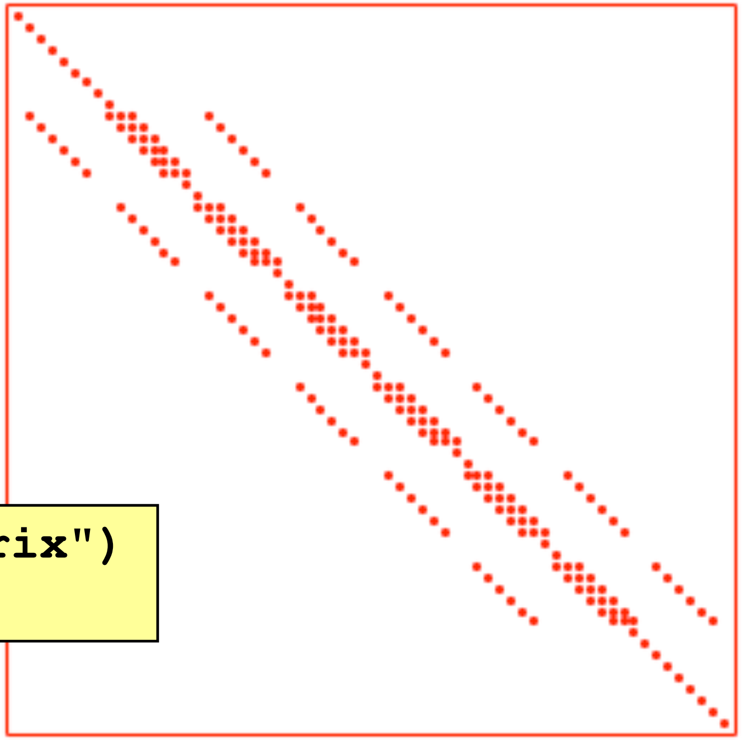
```
x = inv(A) @ b
```

A better way, from both an execution time and numerical accuracy standpoint, is to use the solve function

```
x = solve(A,b)
```

This produces the solution using Gaussian elimination, without forming the inverse.

Les différences finies produisent une matrice creuse…



```
plt.figure("Sparcity of the matrix")
plt.spy(A,marker='o',color='r')
```

Utiliser scipy.sparse !

```
from scipy.sparse import dok_matrix
from scipy.sparse.linalg import spsolve

n = nx*ny; h = 2/(ny-1)
A = dok_matrix((n,n),dtype=float32);
B = zeros(n)
for i in range(n):
  A[i,i] = 1.0
for i in range(1,nx-1):
  for j in range(1,ny-1):
    index = i + j*nx
    A[index,index]    =  4.0
    A[index,index-1]  = -1.0
    A[index,index+1]  = -1.0
    A[index,index-nx] = -1.0
    A[index,index+nx] = -1.0
    B[index] = 1
U = spsolve(A/(h*h).tocsr(),B).reshape(ny,nx)
```

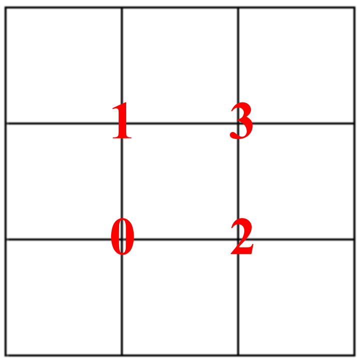
Pour les matrices de grande taille c’est plus rapide !

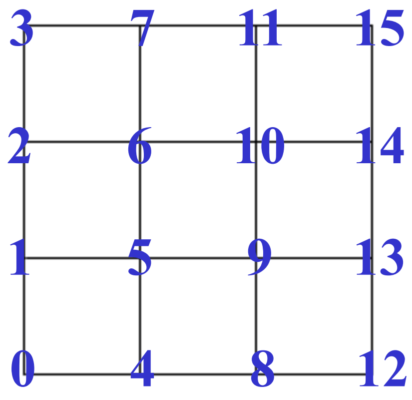
```
bash-3.2$ python poissonSimple.py
 === Considering nx=ny=150
 === Full solver    : elapsed time is 68.746733 seconds.
 === Sparse solver : elapsed time is 1.635676 seconds.
```

https://docs.scipy.org/doc/scipy-0.9.0/reference/sparse.html

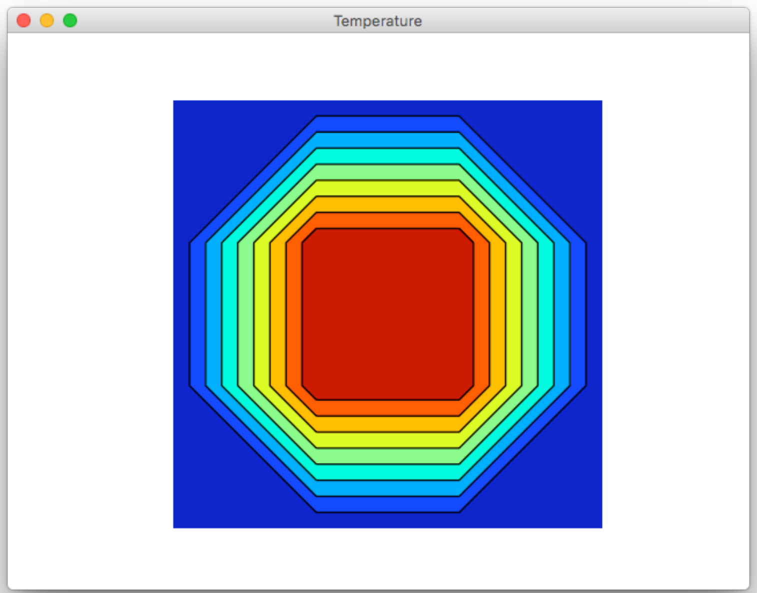
Petits problèmes de numérotation…

```
n = nx*ny; h = 2/(ny-1)
A = zeros((n,n))
B = ones((nx-2)*(ny-2))
map = zeros((nx-2)*(ny-2),dtype=int)
for i in range(1,nx-1):
  for j in range(1,ny-1):
    index = i + j*nx
    map[i-1 + (j-1)*(nx-2)] = index
    A[index,index]    =  4.0
    A[index,index-1]  = -1.0
    A[index,index+1]  = -1.0
    A[index,index-nx] = -1.0
    A[index,index+nx] = -1.0
A = A / (h*h)
V = solve(A[map,:][:,map],B)
```

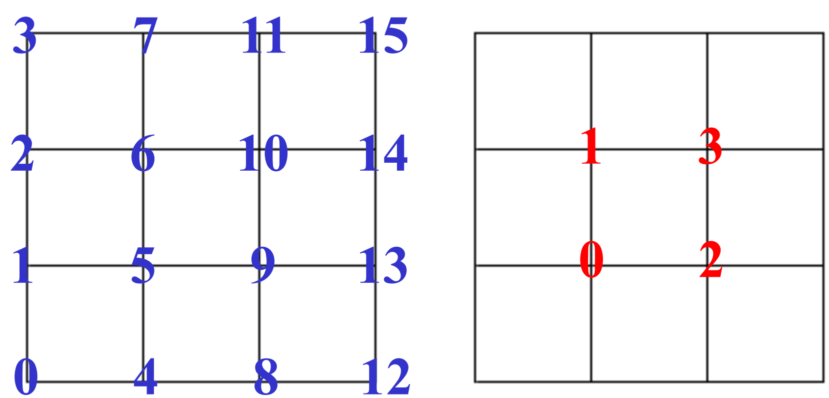




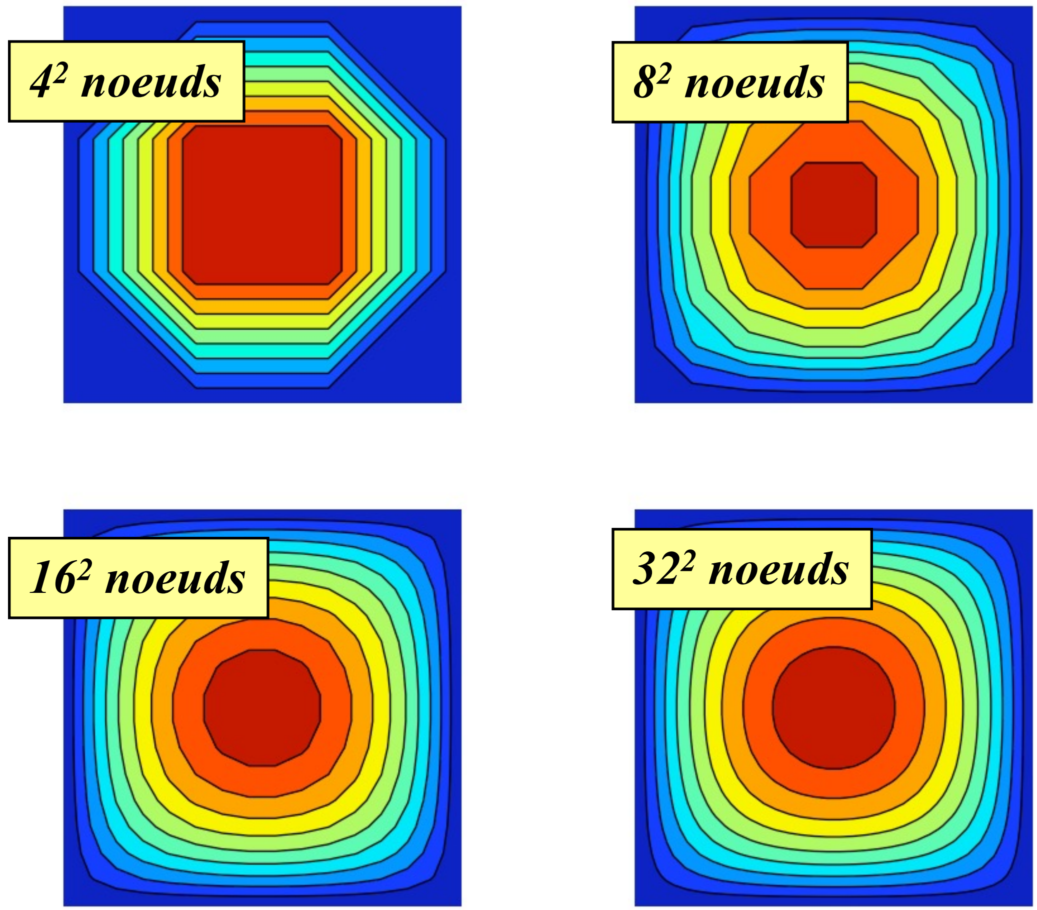
```
U = zeros((ny,nx))
U[1:ny-1,1:nx-1] = V.reshape(ny-2,nx-2)

X,Y = meshgrid(linspace(-1,1,nx),linspace(-1,1,ny))
myColorMap = matplotlib.cm.jet
plt.contour(X,Y,U,10,cmap=myColorMap)
plt.contour(X,Y,U,10,colors='k',linewidths=1)
plt.axis("equal"); plt.axis("off")
```



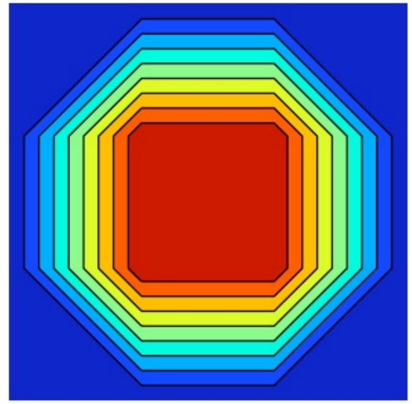
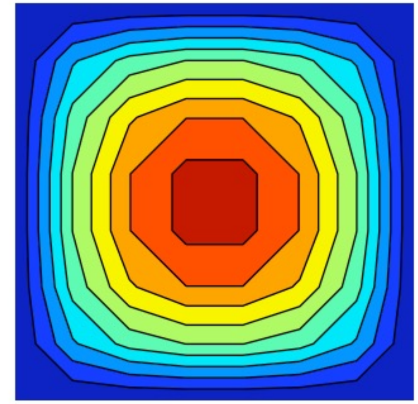
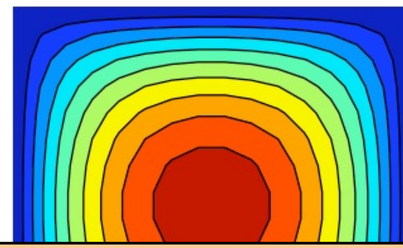
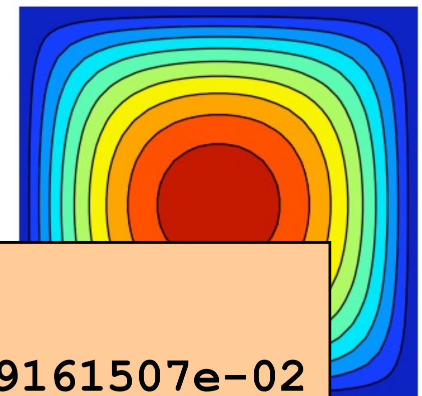
Et faire un joli plot



Est-ce que cela converge ?



Quelle est la précision ?

```
bash-3.2$ python poissonSuperBasic.py
 === Discretization nx =     4 : error   =    1.9161507e-02
 === Discretization nx =     8 : error   =    4.4728014e-03
 === Discretization nx =    16 : error   =    1.0188850e-03
 === Discretization nx =    32 : error   =    2.4094496e-04

 ===== Estimated order of convergence  =    2.1044545e+00
```

Théoriquement…

Théorème 6.1.

Soit $u(x,y)$ la solution exacte d'un problème de Poisson aux conditions aux limites sur un domaine $\Omega \subset \mathbb{R}^2$.
Si la fonction u et toutes ses dérivées partielles jusqu'au quatrième ordre sont continues sur le domaine fermé $\overline{\Omega}$, alors il existe une constante positive telle que :

$$\max |u(X_i, Y_j) - \underbrace{u^h(Xi, Y_j)}_{U_{ij}}| \leq C\ M\ h^2$$

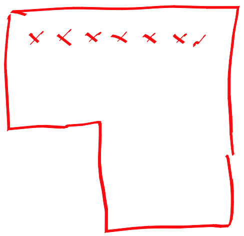
où M est la valeur maximale atteinte par une des dérivées quatrièmes de u sur $\overline{\Omega}$ et u^h est la solution discrète obtenue au moyen d'un schéma à 5 points basé sur des différences finies centrées du second ordre.

Ce résultat n'est pas évident a priori…
Démonstration :
Isaacson and Keller, Analysis of Numerical Methods (1966)

Plus compliqué :-)

```
m = 5
map = numgrid(2*m+1)
print(map)
```

```
[[ 0   0   0   0   0   0   0   0   0   0   0]
 [ 0   1   2   3   4   5   6   7   8   9   0]
 [ 0  10  11  12  13  14  15  16  17  18   0]
 [ 0  19  20  21  22  23  24  25  26  27   0]
 [ 0  28  29  30  31  32  33  34  35  36   0]
 [ 0   0   0   0   0   0  37  38  39  40   0]
 [ 0   0   0   0   0   0  41  42  43  44   0]
 [ 0   0   0   0   0   0  45  46  47  48   0]
 [ 0   0   0   0   0   0  49  50  51  52   0]
 [ 0   0   0   0   0   0  53  54  55  56   0]
 [ 0   0   0   0   0   0   0   0   0   0   0]]
```



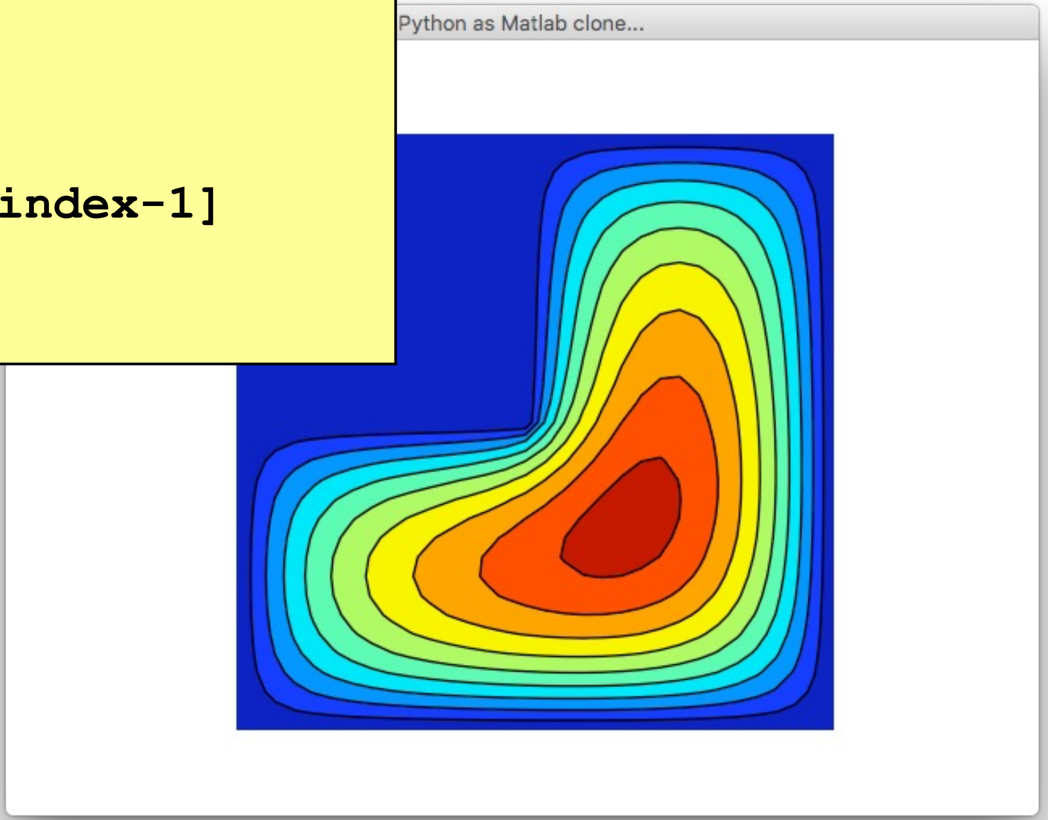
Et pourtant si simple…

```
m = 3;
map = numgrid(2*m+1)
A = delsq(map); print(A.toarray())
```

```
[[ 4. -1.  0.  0.  0. -1.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.]
 [-1.  4. -1.  0.  0.  0. -1.  0.  0.  0.  0.  0.  0.  0.  0.  0.]
 [ 0. -1.  4. -1.  0.  0.  0. -1.  0.  0.  0.  0.  0.  0.  0.  0.]
 [ 0.  0. -1.  4. -1.  0.  0.  0. -1.  0.  0.  0.  0.  0.  0.  0.]
 [ 0.  0.  0. -1.  4.  0.  0.  0.  0. -1.  0.  0.  0.  0.  0.  0.]
 [-1.  0.  0.  0.  0.  4. -1.  0.  0.  0.  0.  0.  0.  0.  0.  0.]
 [ 0. -1.  0.  0.  0. -1.  4. -1.  0.  0.  0.  0.  0.  0.  0.  0.]
 [ 0.  0. -1.  0.  0.  0. -1.  4. -1.  0.  0.  0.  0.  0.  0.  0.]
 [ 0.  0.  0. -1.  0.  0.  0. -1.  4. -1. -1.  0.  0.  0.  0.  0.]
 [ 0.  0.  0.  0. -1.  0.  0.  0. -1.  4.  0. -1.  0.  0.  0.  0.]
 [ 0.  0.  0.  0.  0.  0.  0.  0. -1.  0.  4. -1. -1.  0.  0.  0.]
 [ 0.  0.  0.  0.  0.  0.  0.  0.  0. -1. -1.  4.  0. -1.  0.  0.]
 [ 0.  0.  0.  0.  0.  0.  0.  0.  0.  0. -1.  0.  4. -1. -1.  0.]
 [ 0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0. -1. -1.  4.  0. -1.]
 [ 0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0. -1.  0.  4. -1.]
 [ 0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0. -1. -1.  4.]]
```

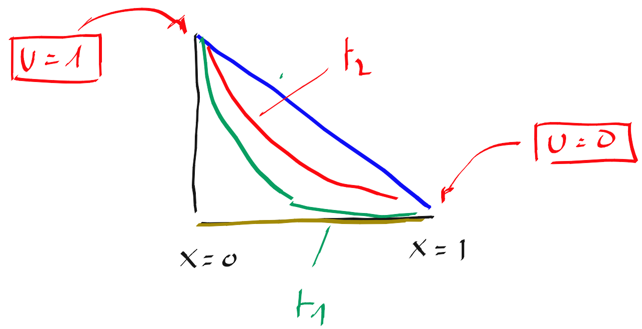
Et finalement…

```
map = numgrid(32)
index = map[map>0]

A = delsq(map)
B = ones(size(index))
U = zeros(shape(map))
U[map>0] = spsolve(A,B)[index-1]

plt.contourf(U,10)
```



Malheureusement,
les fonctions numgrid et delsq
de MATLAB ne font pas encore
partie de numpy et de scipy !

$$\rho c \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

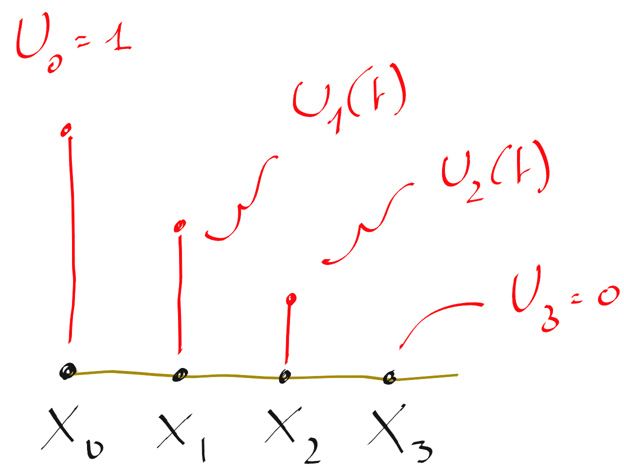


$$\frac{\partial u}{\partial t} = \underbrace{\frac{k}{\rho c}}_{\alpha} \frac{\partial^2 u}{\partial x^2}$$

$\frac{\partial u}{\partial t}$: $\left[\frac{°C}{s}\right]$

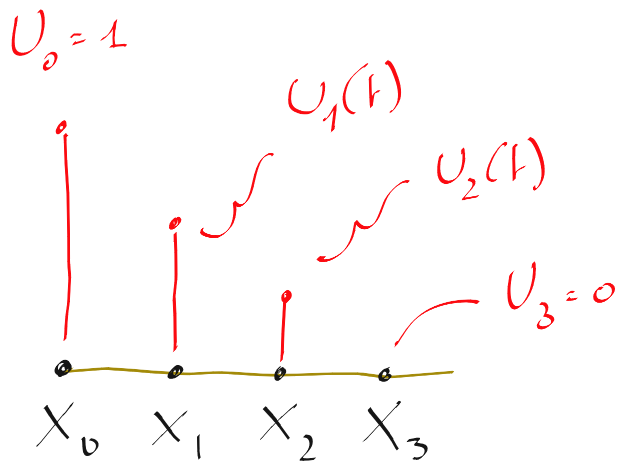
α : DIFFUSIVITE THERMIQUE $\left[\frac{m^2}{s}\right]$

$\frac{\partial^2 u}{\partial x^2}$: $\left[\frac{°C}{m^2}\right]$



Et faisons varier le temps

Exemple



$$\frac{dU_0}{dt} = 0$$

$$\frac{dU_1}{dt} = \alpha \left[\frac{U_0 - 2U_1 + U_2}{(\Delta x)^2} \right]$$

$$\frac{dU_2}{dt} = \alpha \left[\frac{U_1 - 2U_2 + U_3}{(\Delta x)^2} \right]$$

$$\frac{dU_3}{dt} = 0$$

$$\underbrace{\frac{d}{dt}\begin{bmatrix} U_1(t) \\ U_2(t) \\ U_3(t) \\ U_4(t) \end{bmatrix}}_{\underline{u}} = \underbrace{\frac{\alpha}{(\Delta x)^2}}_{\beta} \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & \\ & 1 & -2 & 1 \\ & & & \end{bmatrix}}_{\underline{\underline{A}}} \underbrace{\begin{bmatrix} U_0(t) \\ U_1(t) \\ U_2(t) \\ U_3(t) \end{bmatrix}}_{\underline{u}}$$

$$\underline{u}'(t) = \beta \, \underline{\underline{A}} \, \underline{u}(t)$$

Différences finies (espace)
Euler explicite (temps)

En définissant $\alpha = k/(\rho c)$

$$\left(\frac{U_i^{n+1} - U_i^n}{\Delta t}\right) = \alpha \left(\frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{(\Delta x)^2}\right)$$

En définissant $\beta = \frac{\alpha \Delta t}{(\Delta x)^2}$,

INDICE TEMPS

$$U_i^{n+1} = U_i^n + \beta \left(U_{i+1}^n + U_{i-1}^n - 2U_i^n\right)$$

INDICE SPATIAL

C'est une itération pour un vecteur qui doit converger vers la solution de régime

C'est quelque chose qu'on a déjà rencontré…

En fait, on intègre un système linéaire…

$$\mathbf{u}' = \frac{\alpha}{(\Delta x)^2}\mathbf{A}\mathbf{u}$$

$$\mathbf{u}_n = \begin{bmatrix} U_1^n \\ U_2^n \\ U_3^n \\ U_4^n \\ U_5^n \\ \\ \\ U_{n_x}^n \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} -2 & 1 & & & & & & \\ 1 & -2 & 1 & & & & & \\ & 1 & -2 & 1 & & & & \\ & & 1 & -2 & 1 & & & \\ & & & 1 & -2 & 1 & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & 1 & -2 \end{bmatrix}$$

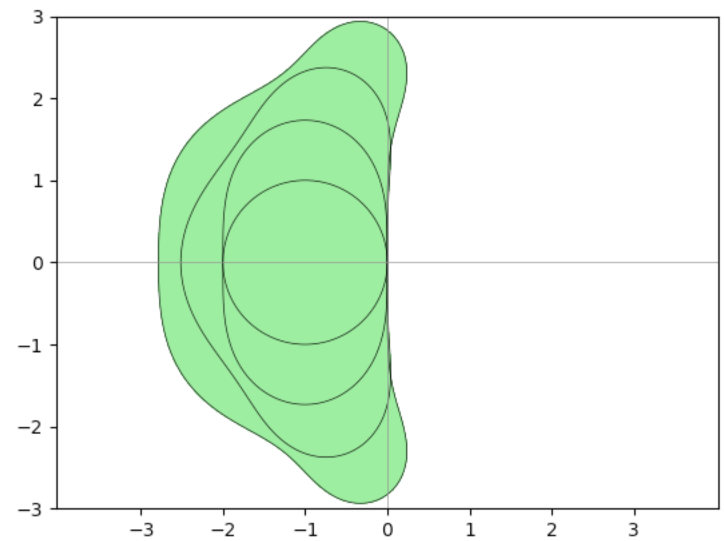
Stabilité des systèmes

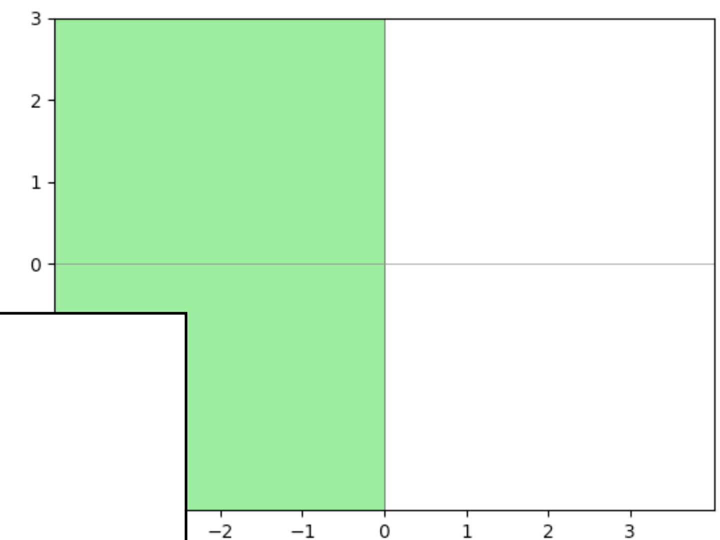
$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$$

$$\mathbf{v}(x) = \mathbf{P}^{-1}\mathbf{u}(x)$$

$$\mathbf{u}' = \mathbf{A}\mathbf{u}$$

$$\mathbf{v}' = \mathbf{D}\mathbf{v}$$





Le problème différentiel initial est équivalent à n équations scalaires

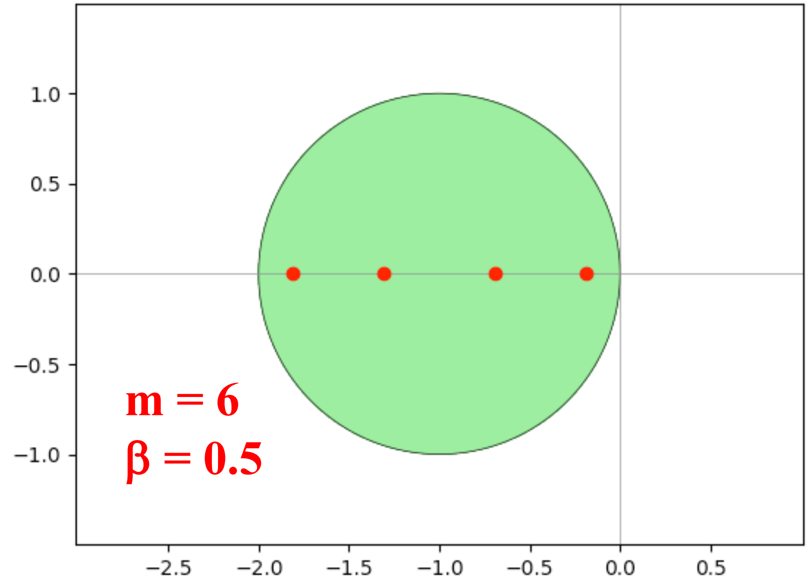
$$v_i'(x) = \lambda_i v_i(x), \qquad i = 1, \ldots, n$$

$$\mathbf{u}_{n+1} = \mathbf{u}_n + \beta \mathbf{A} \mathbf{u}_n$$

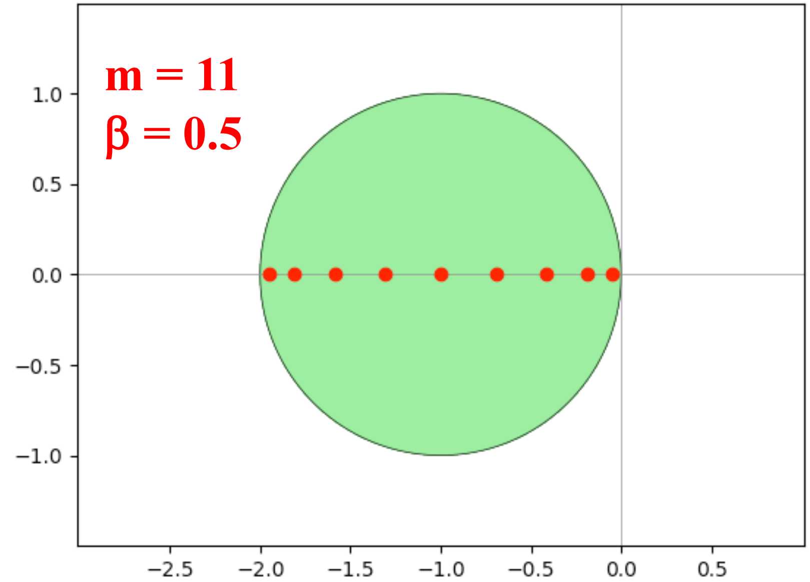
Euler explicite

$$|1 + \beta \lambda_i| < 1$$

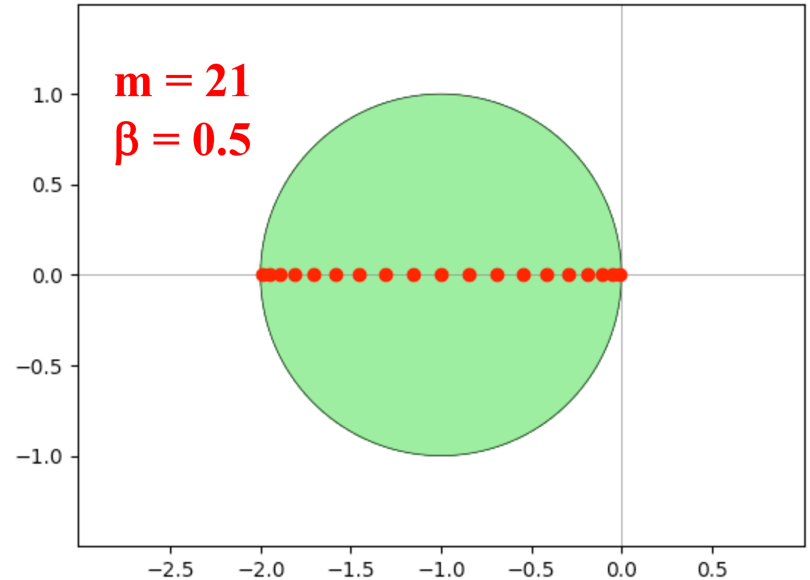
$$\Delta x = 0.2, \Delta t = 0.02$$



$$\Delta x = 0.1, \Delta t = 0.005$$



En raffinant le maillage…



$$\Delta x = 0.05, \Delta t = 0.00125$$

Adaptons…

$$\mathbf{u}_{n+1} = \underbrace{\left(\mathbf{I} + \beta \mathbf{A}\right)}_{\mathbf{M}} \mathbf{u}_n$$

Condition pour qu'une méthode itérative converge ?

$$\mathbf{x}_{i+1} = \mathbf{M}\mathbf{x}_i + \mathbf{c}$$

$$\underbrace{(\mathbf{x}_{i+1} - \mathbf{x})}_{\mathbf{e}_{i+1}} = \mathbf{M}\mathbf{x}_i + \mathbf{c} - \mathbf{M}\mathbf{x} - \mathbf{c}$$

$$= \mathbf{M}\,(\mathbf{x}_i - \mathbf{x})$$

En procédant de la même manière pour chaque étape,

$$= \mathbf{M}^{i+1}\,\underbrace{(\mathbf{x}_0 - \mathbf{x})}_{\mathbf{e}_0}$$

Il faut que…

$$\lim_{i \to \infty} \mathrm{M}^i\, \mathrm{e}_0 = 0$$

$$\mathbf{u}_n = \begin{bmatrix} U_1^n \\ U_2^n \\ U_3^n \\ U_4^n \\ U_5^n \\ \\ \\ U_{n_x}^n \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} -2 & 1 & & & & & \\ 1 & -2 & 1 & & & & \\ & 1 & -2 & 1 & & & \\ & & 1 & -2 & 1 & & \\ & & & 1 & -2 & 1 & \\ & & & & & & \\ & & & & & 1 & -2 \end{bmatrix}$$

$$\mathbf{u}_{n+1} = \mathbf{u}_n + \beta \mathbf{A} \mathbf{u}_n$$

$$|1 + \beta \lambda_i| < 1$$

Et toujours, les valeurs propres…

Ecrivons l'erreur comme une combili de vecteurs propres…

$$\mathbf{e}_0 = \sum_{j=1}^{n} \alpha_j \mathbf{v}_j$$

Puisque $\mathbf{M}\ \mathbf{v}_i = \lambda_i \mathbf{v}_i$,

$$\mathbf{e}_i = \sum_{j=1}^{n} \alpha_j \lambda_j^i \mathbf{v}_j$$

Pour obtenir…

$$\lim_{i \to \infty} \mathrm{M}^i\, \mathbf{e}_0 = 0$$

… on doit exiger que le rayon spectral de la matrice M soit inférieur à l'unité

$$|\lambda_i| < 1 \qquad \forall\ i$$

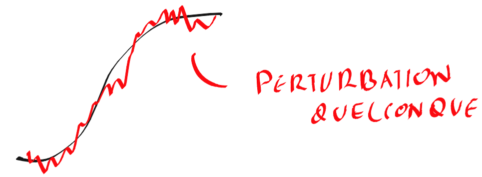
Mais, il faut calculer les valeurs propres d'un opérateur laplacien discret quelconque….

$$\beta = \frac{\alpha \Delta t}{(\Delta x)^2} \leq \frac{1}{2}$$

Courant, Friedrichs et Lewy (1928)

Eux, ils ne disposaient pas de Python….

Analyse de stabilité



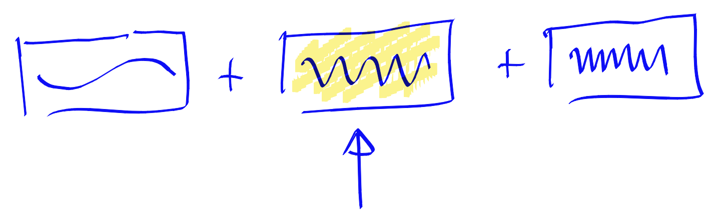
NBR COMPLEXE

$$U_i^n = \boxed{U^n} \exp\, i k X_i$$

AMPLITUDE

NBR ONDE QUELCONQUE

ON VA LA DECOMPOSER



ON CHOISIT ARBITRAIREMENT UNE FREQUENCE

Considérons une perturbation quelconque de la forme suivante et analysons son évolution….
On souhaite que son amplitude diminue.

Analyse de stabilité

Amplitude quelconque de la perturbation

Nombre imaginaire

$$U_i^n = \boxed{U^n} e^{ikX_i}$$

Indice spatial

k quelconque

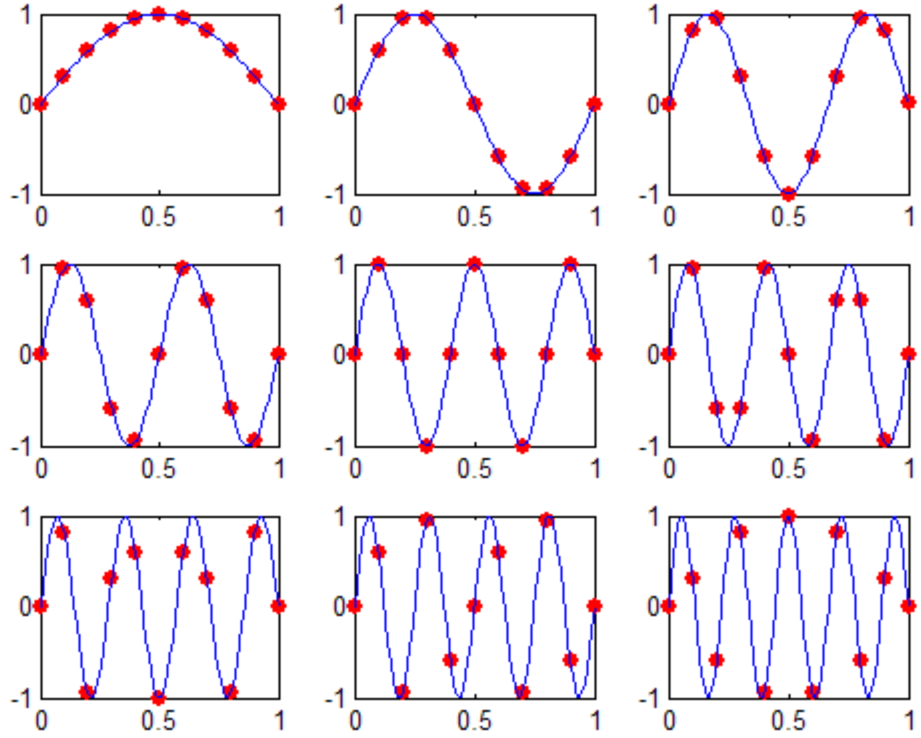
Indice spatial

Considérons une perturbation quelconque de la forme suivante et analysons son évolution….
On souhaite que son amplitude diminue.

Quelques k bien choisis…

m = 11
β = 0.5

$$U_i^n = U^n \sin(\frac{\hat{k}\pi X_i}{L})$$



$$\Delta x = 0.1$$

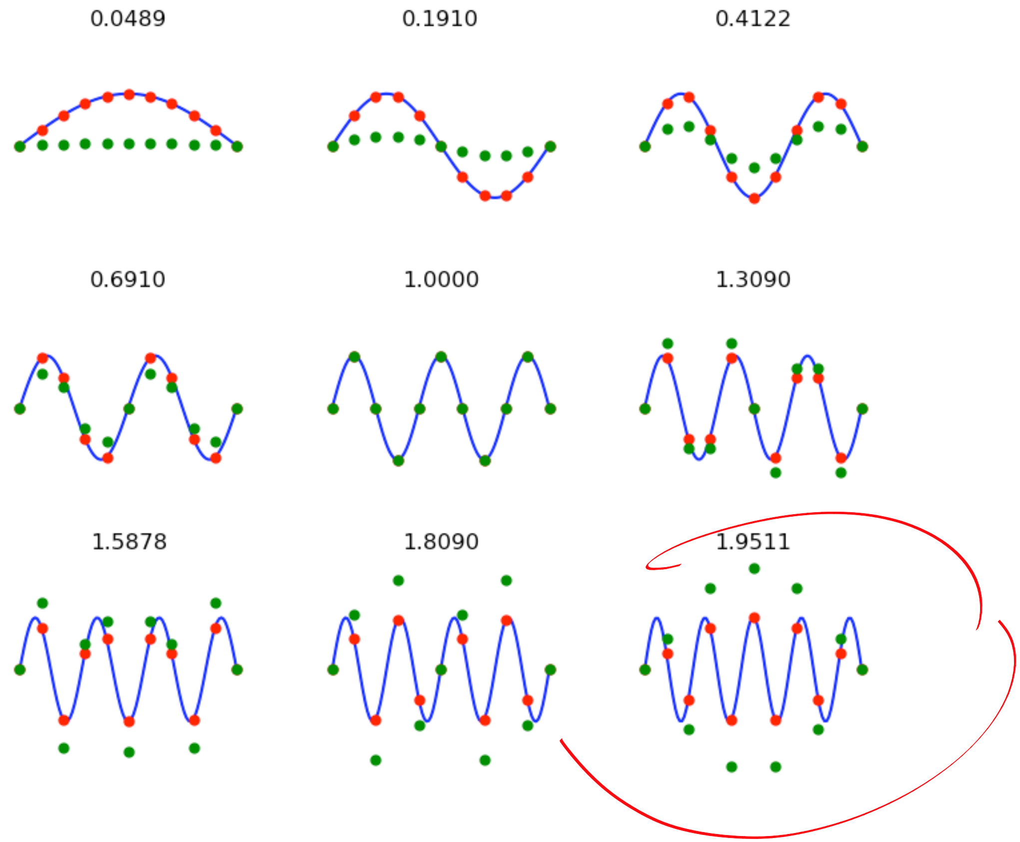
Des perturbations bien particulières et propres…

m = 11
β = 0.5

$$U_i^n = U^n \sin\left(\frac{\hat{k}\pi X_i}{L}\right)$$

$\mathbf{u}$

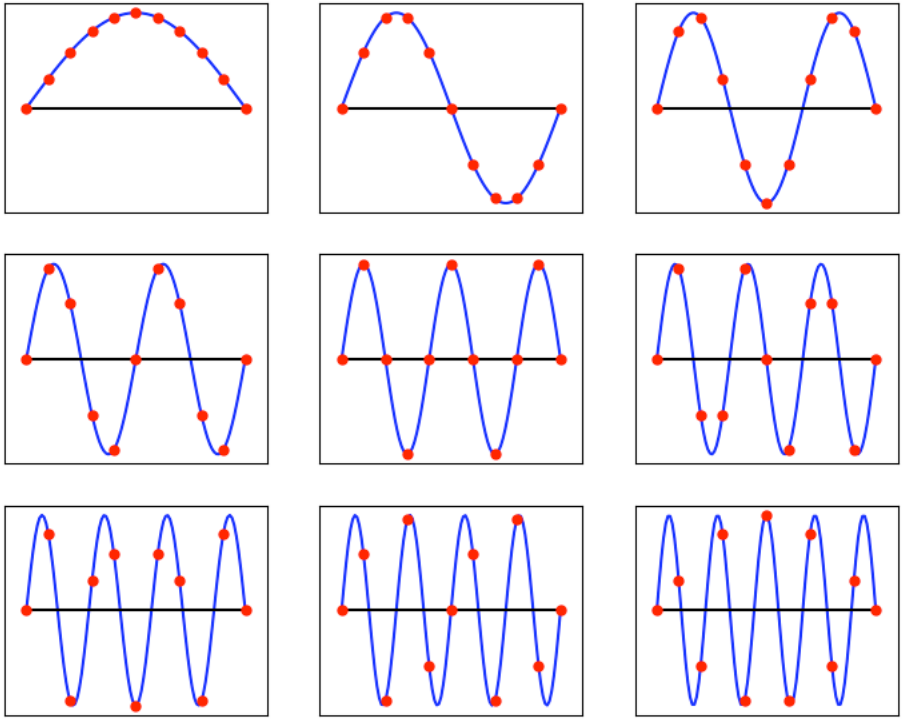
$-\beta \mathbf{A}\mathbf{u}$



C'est quoi ces perturbations ?

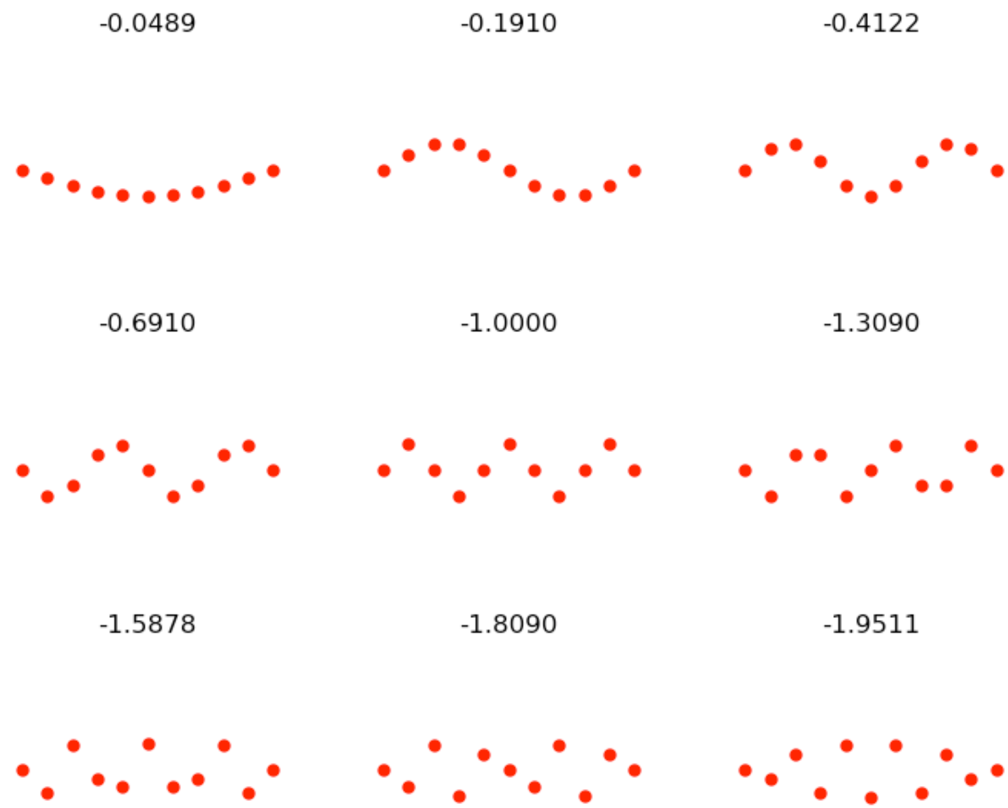
N'importe quelle perturbation peut être écrite comme une combinaison linéaire de ces vecteurs propres de notre opérateur….

m = 11
β = 0.5



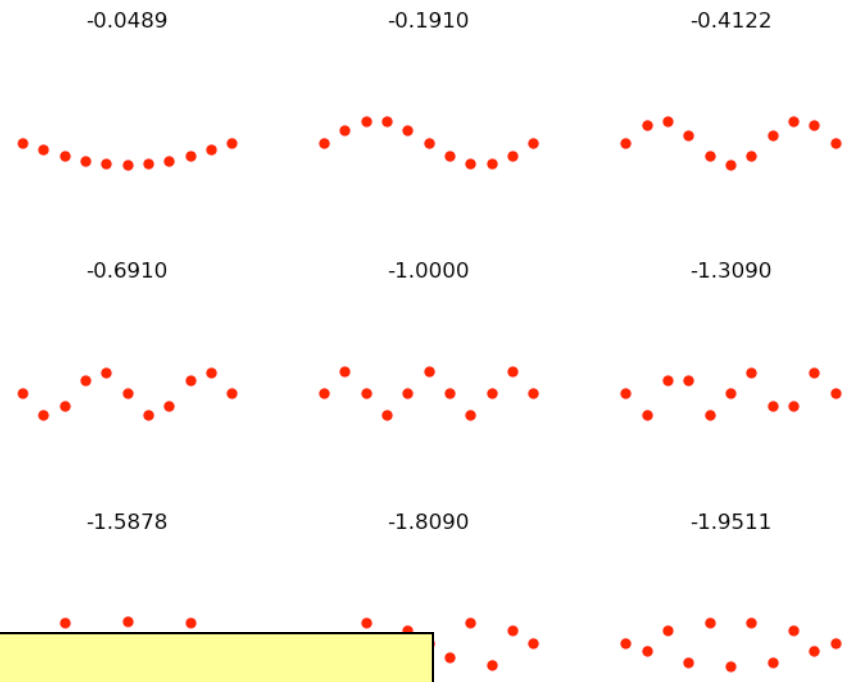
Dimension de l'espace discret = 9
Nombre de vecteurs de base = 9

Calculons les valeurs et les vecteurs propres…

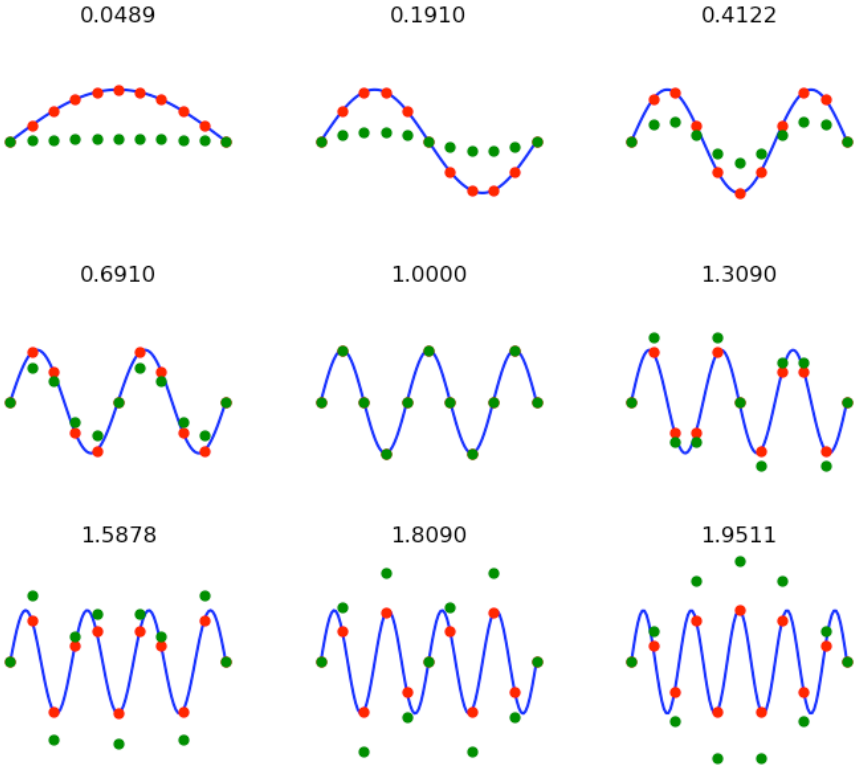



```
m = 11; dx = 1.0/(m-1); beta = (dx*dx)/2
e = array([0,*ones(m-2),0])
D = spdiags([e,-2*e,e],[-1,0,1],m,m)
D = D.T/(dx*dx)
lam,eigen = eig(beta * D.toarray()[1:-1,1:-1])
```

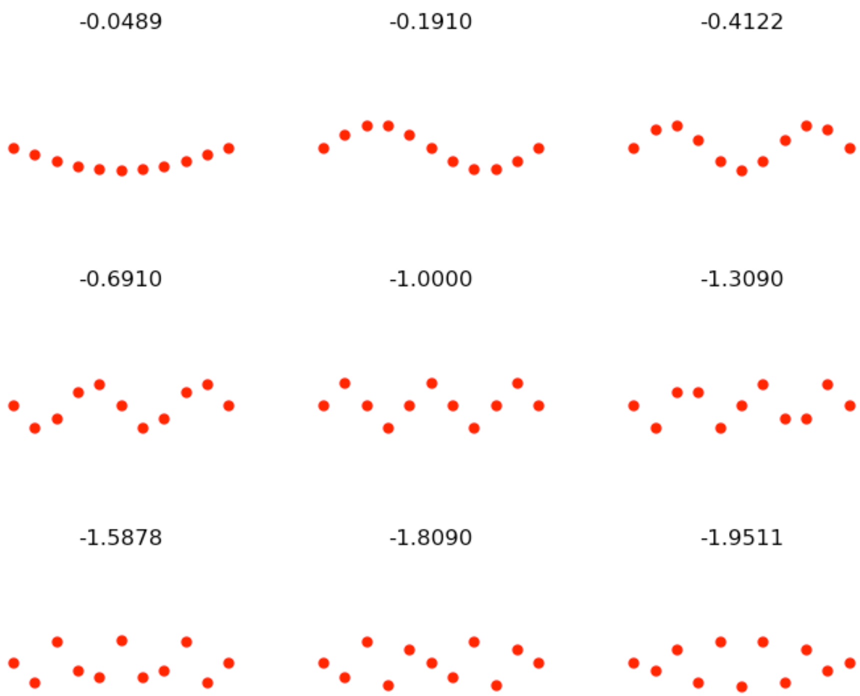
Trier et dessiner les vecteurs propres




```
order = argsort(-lam.real)
X = linspace(0,1,m); V = zeros(m)
for i in range(m-2):
  plt.subplot(msqrt,msqrt,i+1)
  V[1:-1] = eigen[:,order[i]]
  plt.plot(X,V,'.r',markersize=10)
  plt.title("%6.4f" % lam[order[i]].real)
  plt.xlim((-0.1,1.1))
  plt.ylim((-2.1,2.1))
  plt.axis('off')
```



$$U_i^n = U^n \sin(\frac{\hat{k}\pi X_i}{L})$$



Et c'est vraiment cela ?

$U_i^n = U^n \exp ikX_i$

Propagation des erreurs

$$U_i^{n+1} = U_i^n + \beta\left(U_{i-1}^n - 2U_i^n + U_{i+1}^n\right)$$

$$U^{n+1} e^{ikX_i} = U^n\left[e^{ikX_i} + \beta\left(e^{ikX_i} e^{-ik\Delta x} - 2e^{ikX_i} + e^{ikX_i} e^{ik\Delta x}\right)\right]$$

$$U^{n+1} = U^n\left[1 + \beta\left(e^{ik\Delta x} + e^{-ik\Delta x} - 2\right)\right]$$

$$2\cos(k\Delta x) = -2$$

JE CHOISIS LE CAS LE PLUS DEFAVORABLE

$$|U^{n+1}| < |U^n|\,|1 - 4\beta|$$

$$-1 < 1 - 4\beta < 1$$

$$4\beta < 2 \quad \rightarrow \quad \beta = 1/2$$

$$\left|1 + \beta\Big(2\cos(k\Delta x) - 2\Big)\right| \leq 1$$

Un peu d'algèbre …

En prenant le cas le plus défavorable où $k\Delta x = \pi$,

$$-1 \leq 1 - 4\beta$$

$$2\beta \leq 1$$

$$\beta = \frac{\alpha \Delta t}{(\Delta x)^2} \leq \frac{1}{2}$$

Condition de stabilité très pénalisante sur le pas de temps….

Courant, Friedrichs et Lewy (1928)

The so-called Stupid Single Student Behaviour Law