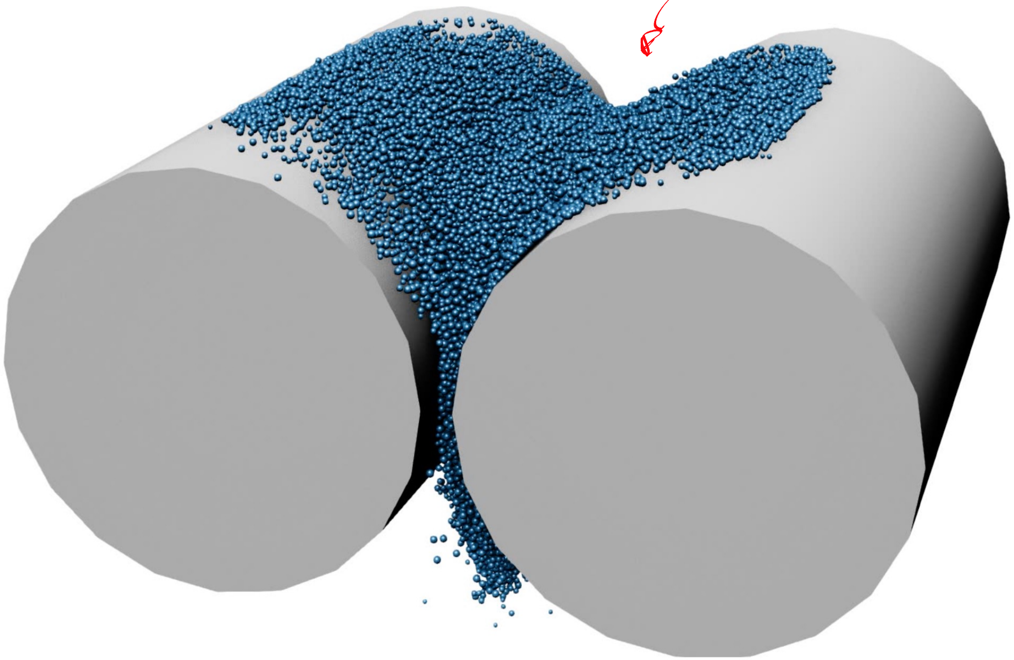
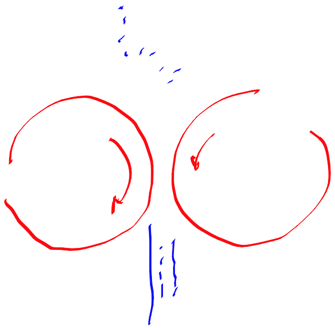
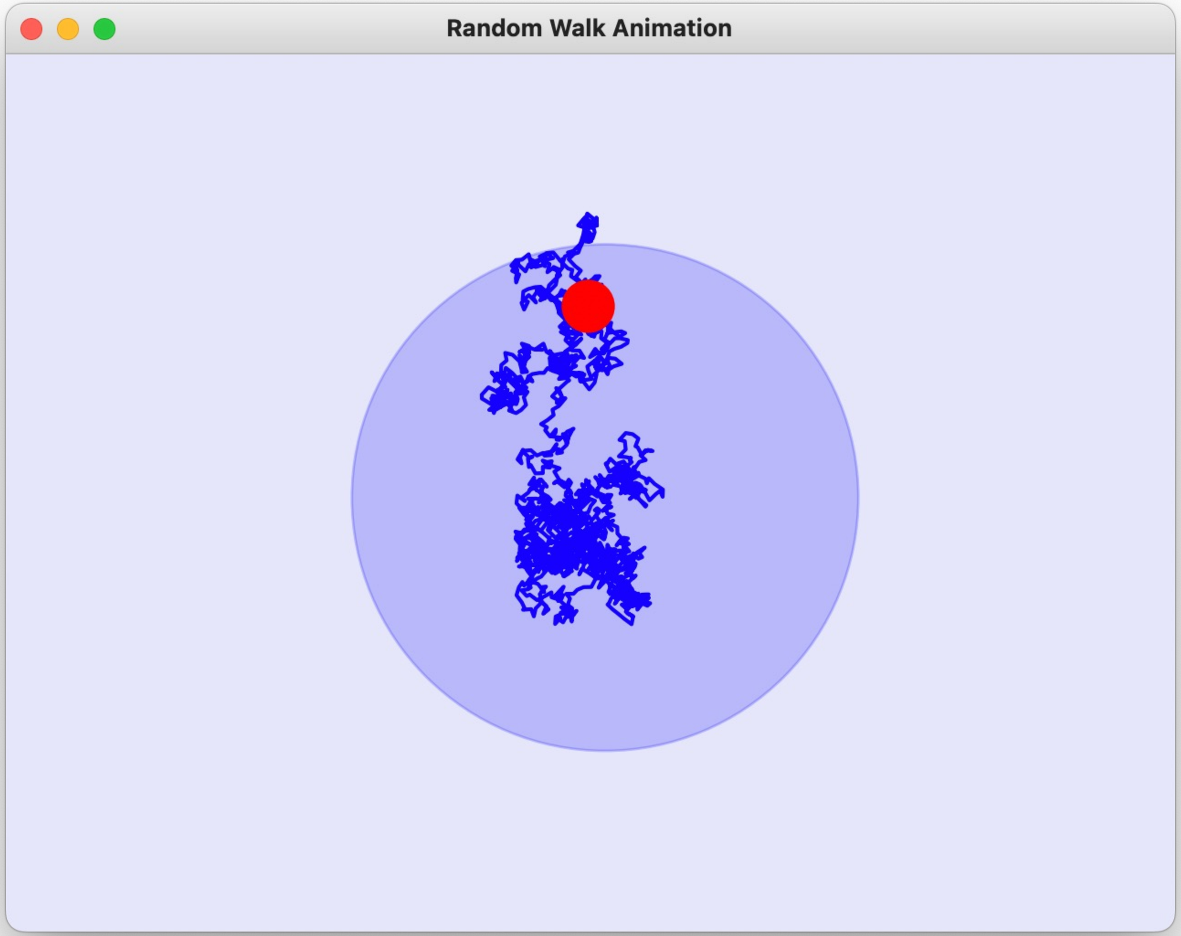
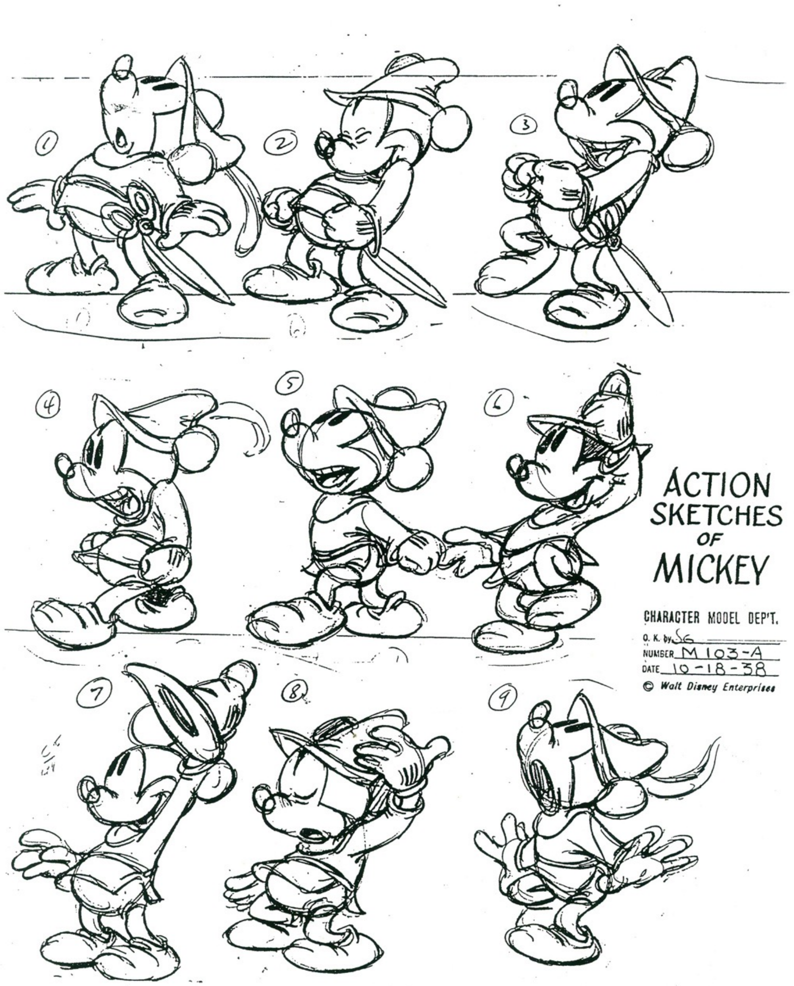
A quoi servent les méthodes numériques ?

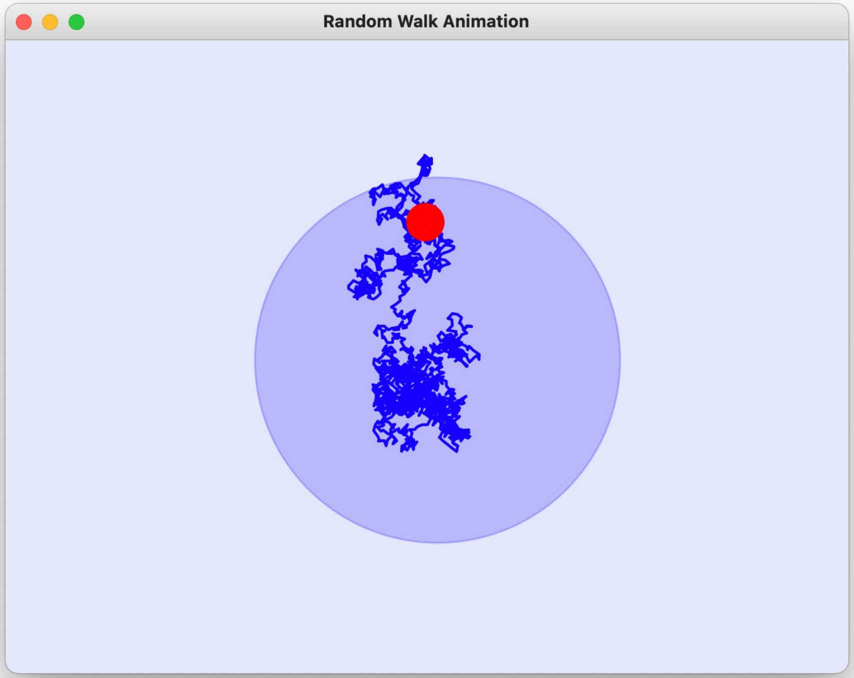




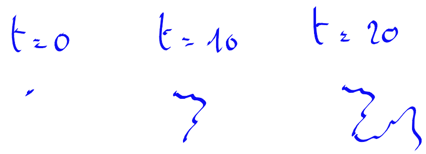


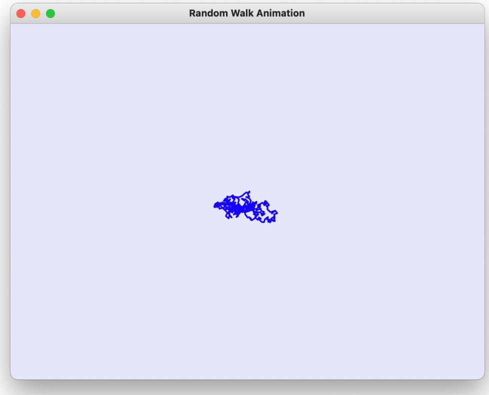
Faire une animation !

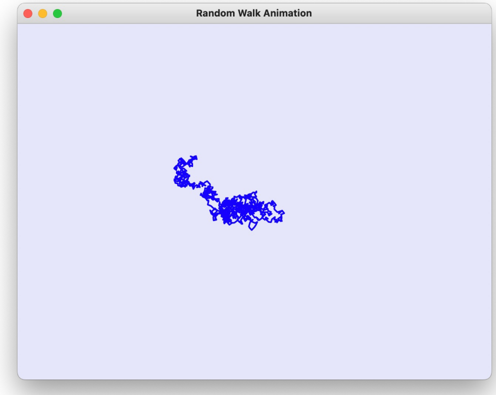




Comme Walt Disney !







Définir le dessin de chaque frame !

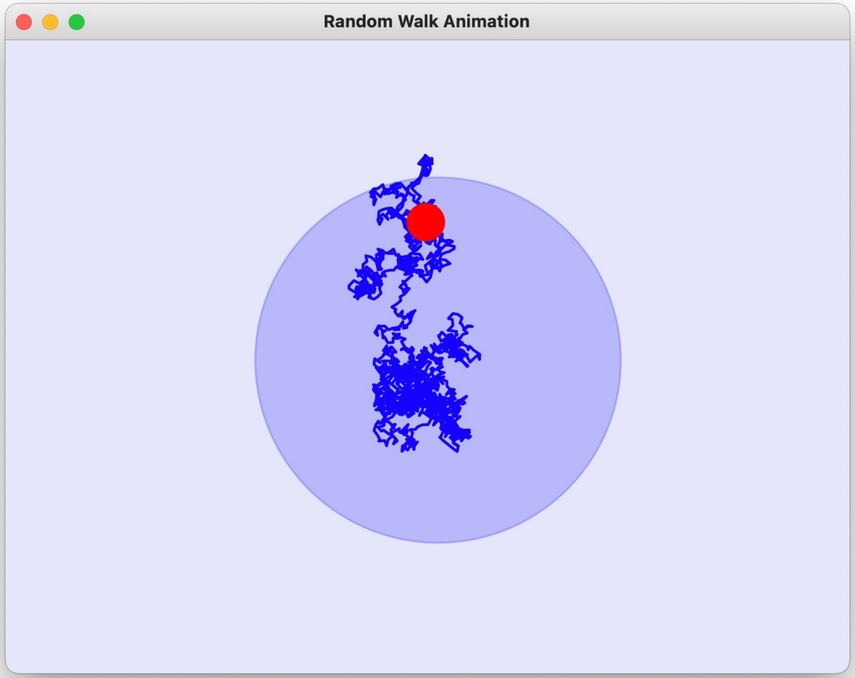
```
import numpy as np
import matplotlib.pyplot as plt
import matplotlib.animation as ani

x = np.zeros(2001)*np.nan; x[0] = 0
y = np.zeros(2001)*np.nan; y[0] = 0

def animate(frame):
  x[frame+1] = x[frame] + 0.1*(np.random.rand()-0.5);
  y[frame+1] = y[frame] + 0.1*(np.random.rand()-0.5);
  plt.cla()
  plt.plot(x,y,'-',color='blue')

fig = plt.figure("Random Walk Animation")
ani.FuncAnimation(fig,animate,frames=2000,interval=50,repeat=False)
plt.show()
```

Et le dessin complet…

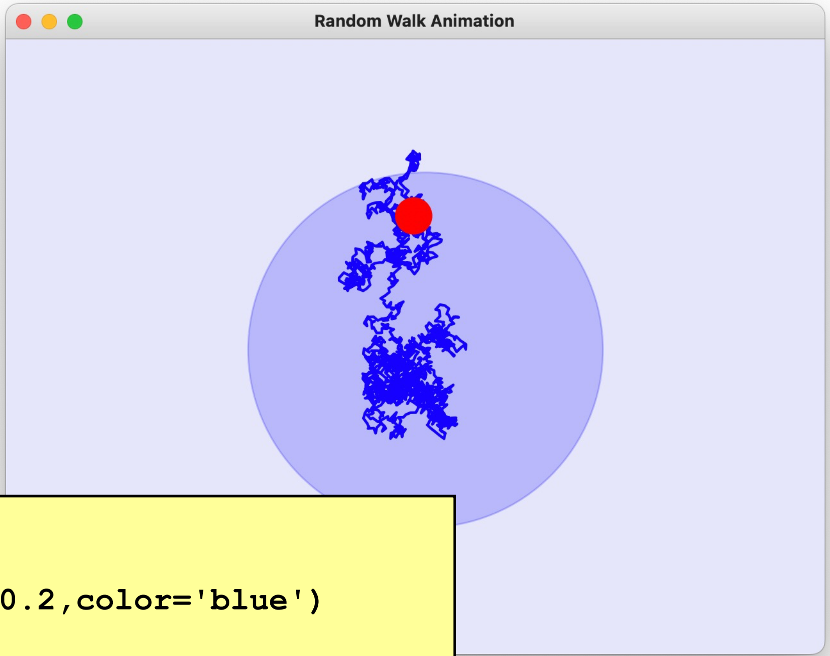



```
def animate(frame):
  xnew = x[frame] + 0.1*(np.random.rand()-0.5); x[frame+1] = xnew
  ynew = y[frame] + 0.1*(np.random.rand()-0.5); y[frame+1] = ynew

  plt.cla()
  ax = plt.gca()
  ax.set_xlim(-2,2); ax.set_ylim(-2,2)
  ax.set_aspect('equal’); plt.axis('off’)

  circle = plt.Circle((0,0),1.5,fill=True,alpha=0.2,color='blue')
  plt.plot(x, y,'-',color='blue')
  plt.plot(x[frame+1],y[frame+1],'o',markersize=20, color='red')
  ax.add_artist(circle)
```

Et un truc encore mieux !




```
fig = plt.figure("Random Walk Animation")
ax = plt.gca()
circle = plt.Circle((0,0),1.5,fill=True,alpha=0.2,color='blue')
line1, = plt.plot([],[],'-',color='blue')
line2, = plt.plot([],[],'o',markersize=20,color='red')
ax.add_artist(circle)
x = np.zeros(2001) * np.nan; x[0] = 0
y = np.zeros(2001) * np.nan; y[0] = 0

def animate(frame):
   x[frame+1] = x[frame] + 0.1*(np.random.rand()-0.5)
   y[frame+1] = y[frame] + 0.1*(np.random.rand()-0.5)
   line1.set_data(x,y)
   line2.set_data([x[frame+1]],[y[frame+1]])
   return line1,line2,

ani.FuncAnimation(fig,animate,frames=2000,interval=50,repeat=False)
plt.show()
```

Python
for dummies !

Exécution et soumission d'un programme sur le serveur...

Deadline : February 21 2024 23:59:59.
Now : February 13 2024 22:57:55.

10 pt

```
import numpy as np

#
# TOUTE AUTRE INSTRUCTION CONTENANT import / from SERA AUTOMATIQUEMENT SUPPRIMEE
#

# A MODIFIER / COMPLETER

def circlesCreate(radius,n) :

```

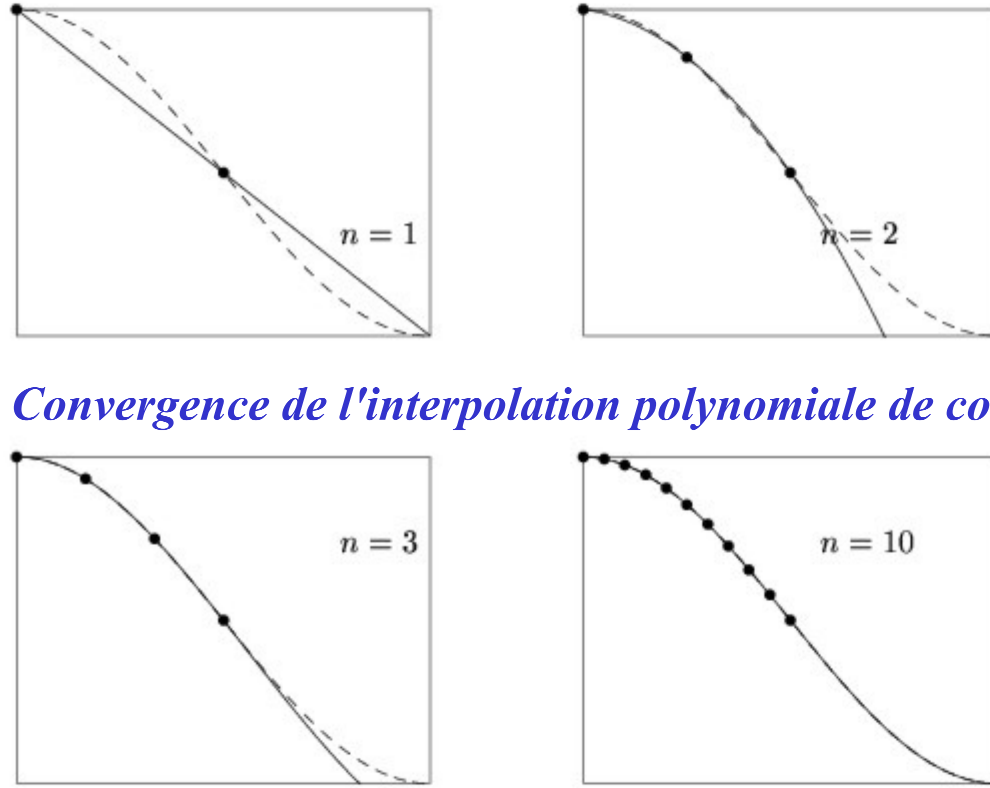
Position: Ln 1, Ch 1 | Total: Ln 25, Ch 584

Soumettre le programme | Voir le diagnostic

Valider son programme

Date limite pour le problème : 21/Fev/2024 23:59:59
et nous sommes aujourd'hui : 14/Fev/2024 10:30:00...

Convergence



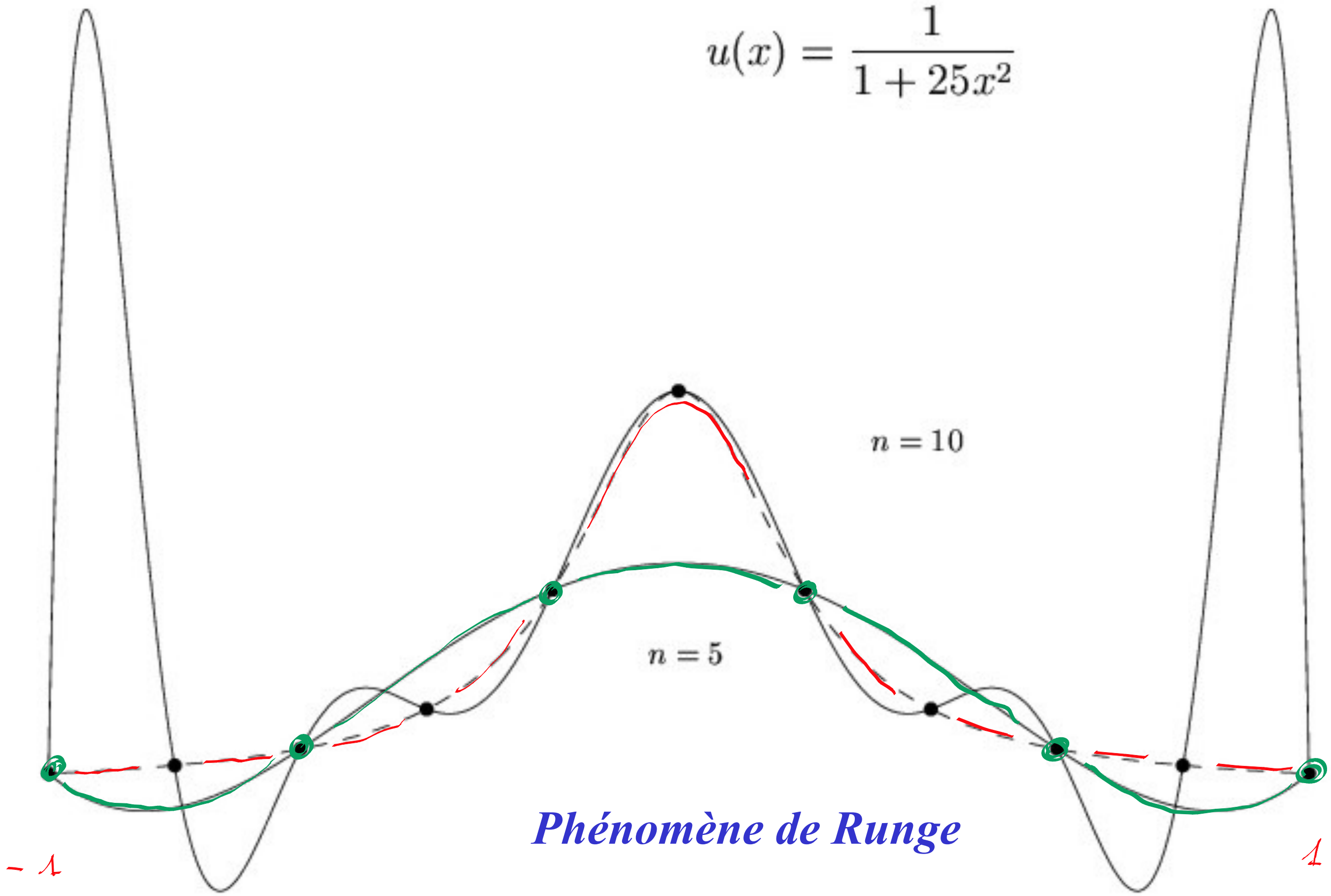
Convergence de l'interpolation polynomiale de cos(x)

$$e^h(x) = u(x) - u^h(x)$$

Définition 1.3. *Une interpolation est dite* convergente *si l'erreur d'interpolation tend vers zéro lorsque le nombre de degrés de liberté, c'est-à-dire n tend vers l'infini :*

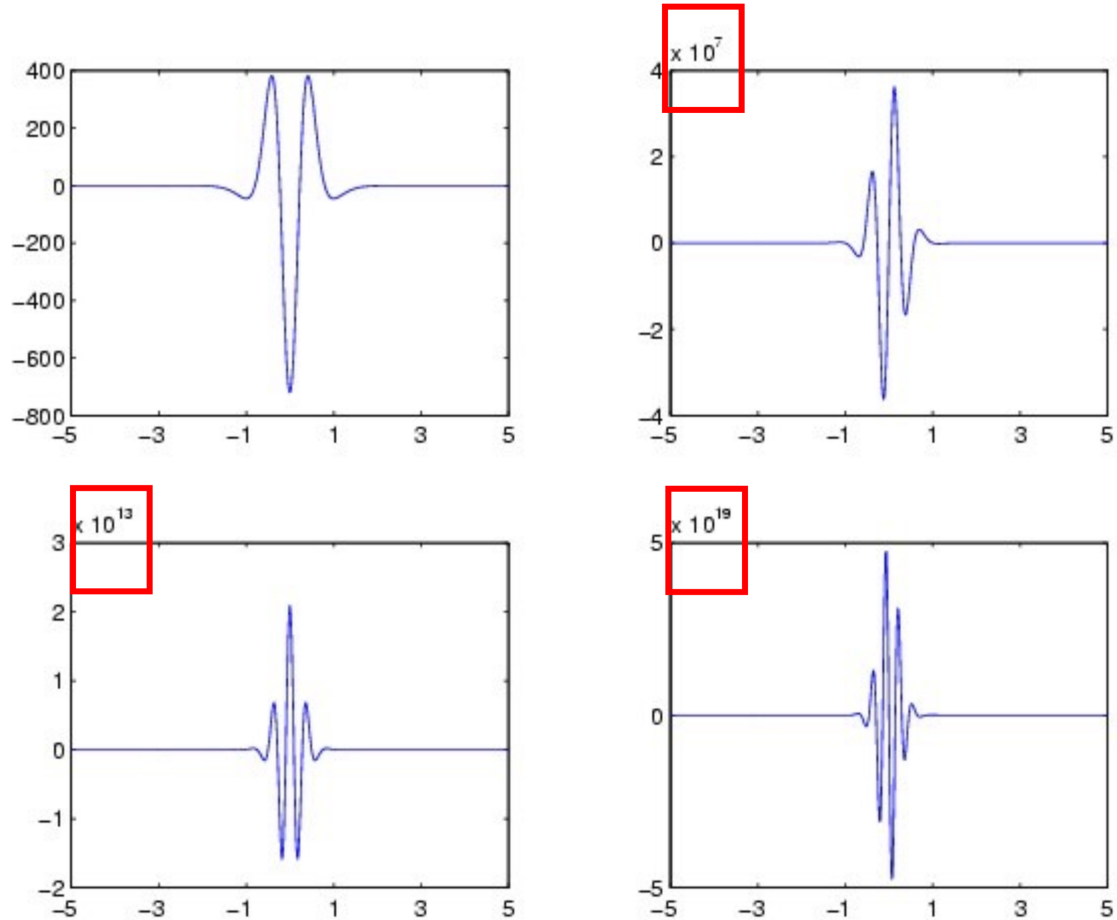
$$\lim_{n\to\infty} e^h(x) = 0 \quad \textit{pour } x \in [X_0, X_n].$$

L'interpolation polynomiale, parfois cela ne converge pas…



Phénomène de Runge

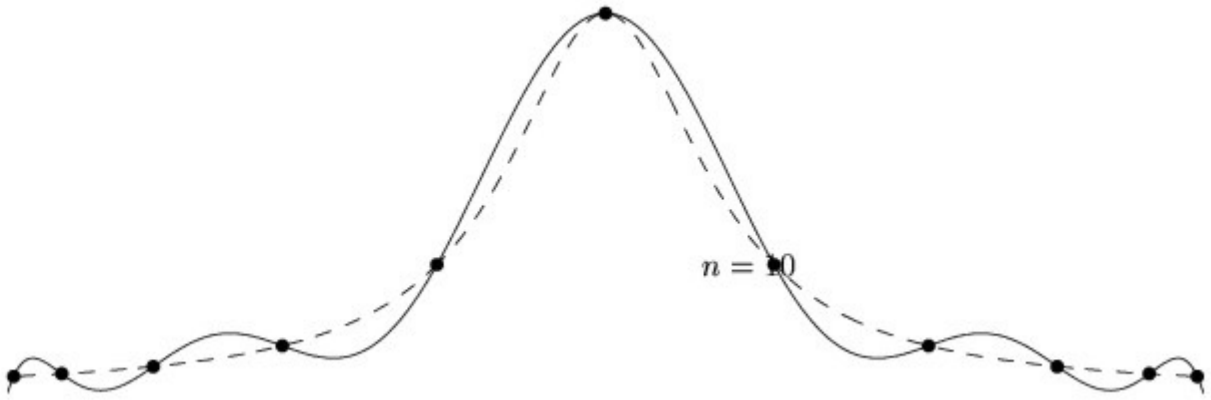
Why does it not work ?



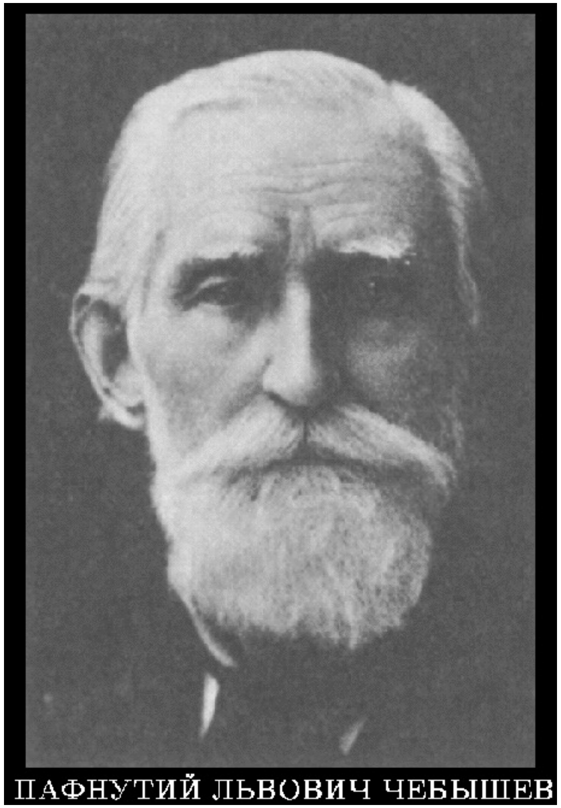
Dérivées d'ordre 6, 11, 16 et 21 de la fonction de Runge

$$e^h(x) = \boxed{\frac{u^{(n+1)}(\xi(x))}{(n+1)!}} (x - X_0)(x - X_1)(x - X_2) \cdots (x - X_n).$$

Parfois, on peut sauver la mise…



Abscisses de Chebyshev

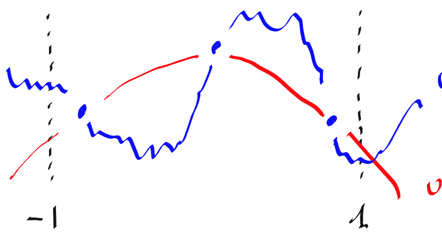


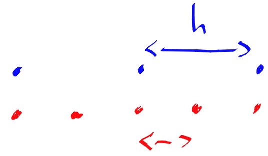
Pafnuty Lvovitch Chebyshev (1821-1894)

Abscisses de Chebychev

$$\underbrace{\left| u(x) - u^h(x) \right|}_{e^h(x)} \leq \left| \frac{u^{(n+1)}(\xi(x))}{(n+1)!} \right| \left| (x - X_0)(x - X_1) \dots (x - X_n) \right|$$

$$\leq \frac{n!\, h^{n+1}}{4}$$





$$\boxed{u^{(n+1)}(\xi(x))}$$

$\nearrow n$ $\nearrow$

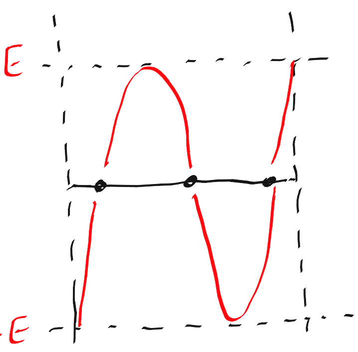
$$\boxed{\frac{h^{n+1}}{4(n+1)}}$$

$h \searrow$ $\searrow$

$n \nearrow$

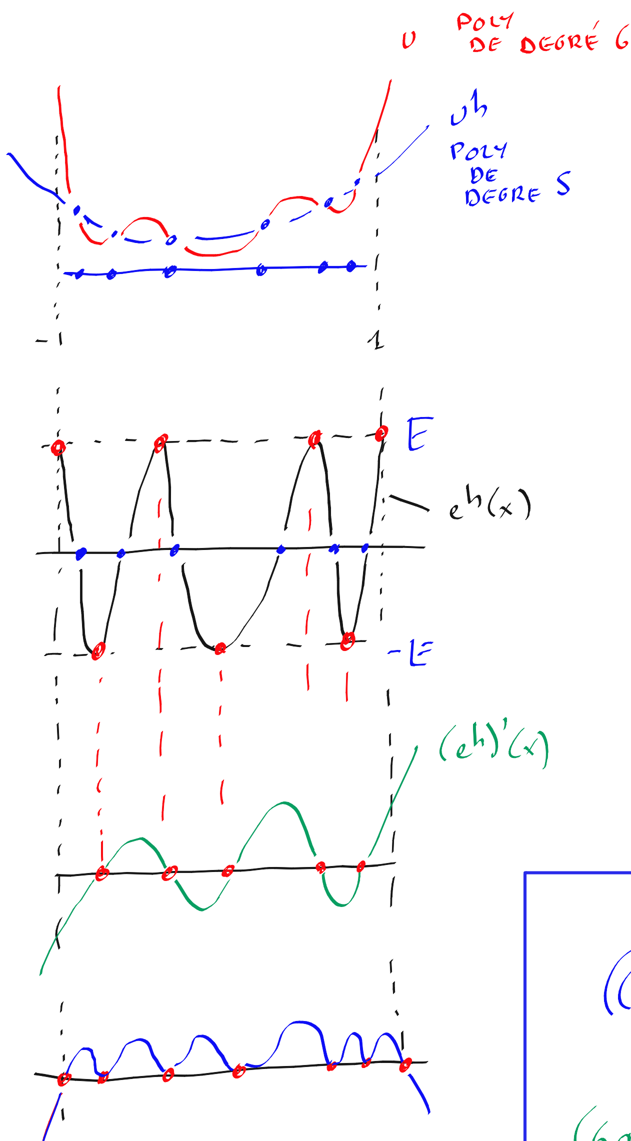
DIMINUER CECI LE PLUS EFFICACEMENT POSSIBLE

$$\boxed{X_i = \cos\left[\frac{(2i+1)\pi}{2(n+1)}\right]}$$



ESTIMATION ASYMPTOTIQUE DE L'ERREUR

$h \to 0$



$a_6x^6 + a_5x^5 \;\ldots\; a_0$ $\qquad [a_6\ a_5 \ldots a_0] = pv$

$b_5x^5 \;\ldots\; b_0$ $\qquad [b_5 \ldots b_0] = pvh$

$$peh = \underbrace{pv}_{[a_6 \ldots a_0]} - [0\ \underbrace{*pvh}_{b_5\ b_4 \ldots b_0}]$$

$[0\ b_5 \ldots b_0]$

$$pdeh = peh * [6\ 5\ 4\ 3\ 2\ 1\ 0]$$

$$pdeh = pdeh[0:\underbrace{6}_{-1\ :-)}]$$

$$ax^2 + bx + c = \underline{\underline{a}}(x-x')(x-x'')$$

$$\underbrace{((e^h)'(x))^2}_{(6a_6x^5\ldots)^2}(1-x^2) = (n+1)^2\Big(E^2 - \underbrace{(e^h(x))^2}_{(a_6x^6\ldots)^2}\Big)$$

$6^2 a_6\, x^{12} \ldots$

2 racines simples

5 racines doubles

$$\left((e^h)'(x)\right)^2(1-x^2) = (n+1)^2\left(E^2 - \left(e^h(x)\right)^2\right)$$

$$e'\sqrt{1-x^2} = \pm(n+1)\sqrt{E^2-e^2}$$

$$\frac{e'}{E}\frac{1}{\sqrt{1-\left(\frac{e}{E}\right)^2}} = \pm(n+1)\frac{1}{\sqrt{1-x^2}}$$

$$\left(\arccos\left(\frac{e}{E}\right)\right)' = \pm(n+1)\left(\arccos(x)\right)'$$

$$\downarrow$$

$$\arccos\left(\frac{e}{E}\right) = \pm(n+1)\arccos(x) + C$$

$$e^h(x) = \underbrace{E\cos\left[(n+1)\underbrace{\arccos(x)}_{\theta}\right]}_{T_{n+1}(x)}$$

X_i ZEROS DE T_{n+1}

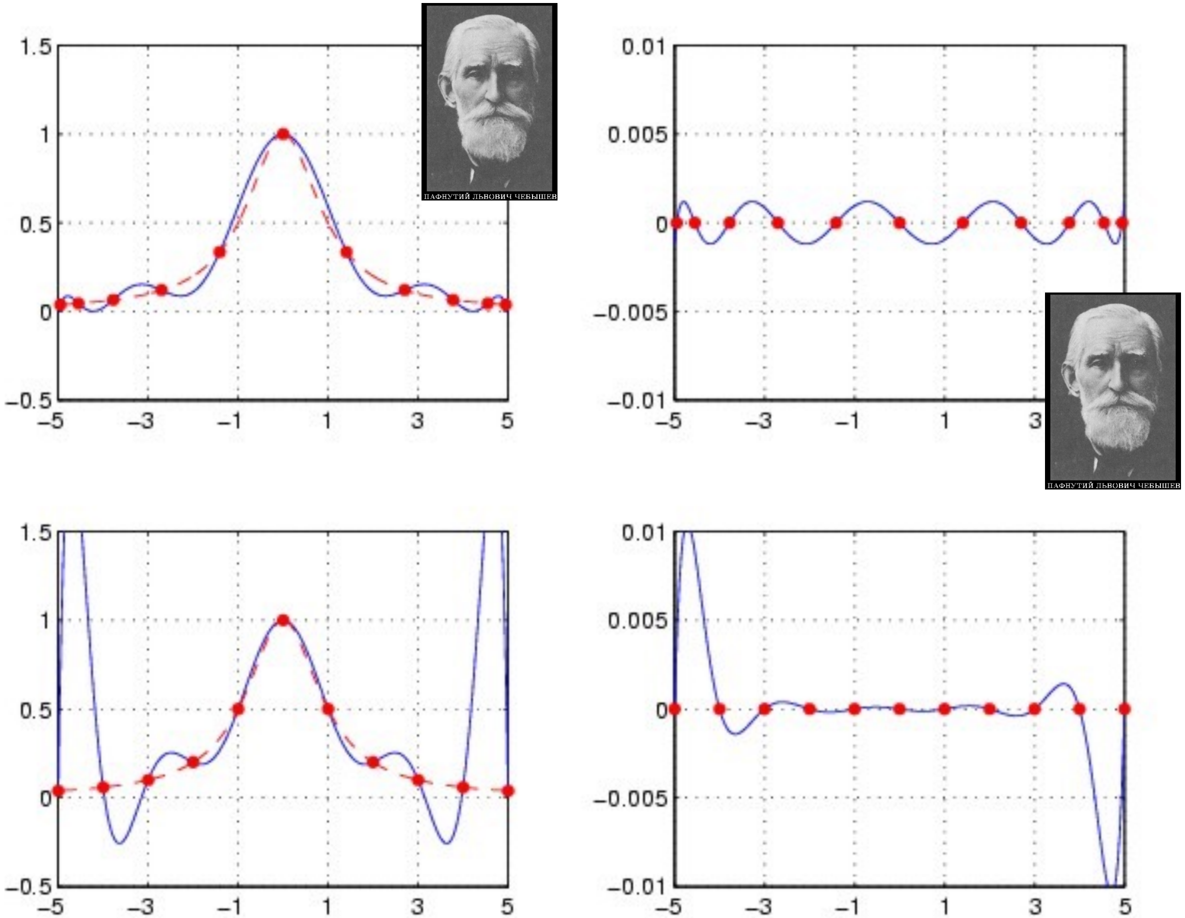
$$0 = \cos\left[(n+1)\arccos(X_i)\right]$$

$$\pi/2 + i\pi = (n+1)\arccos(X_i)$$

$$X_i = \cos\left[\frac{(2i+1)\pi}{2(n+1)}\right]$$

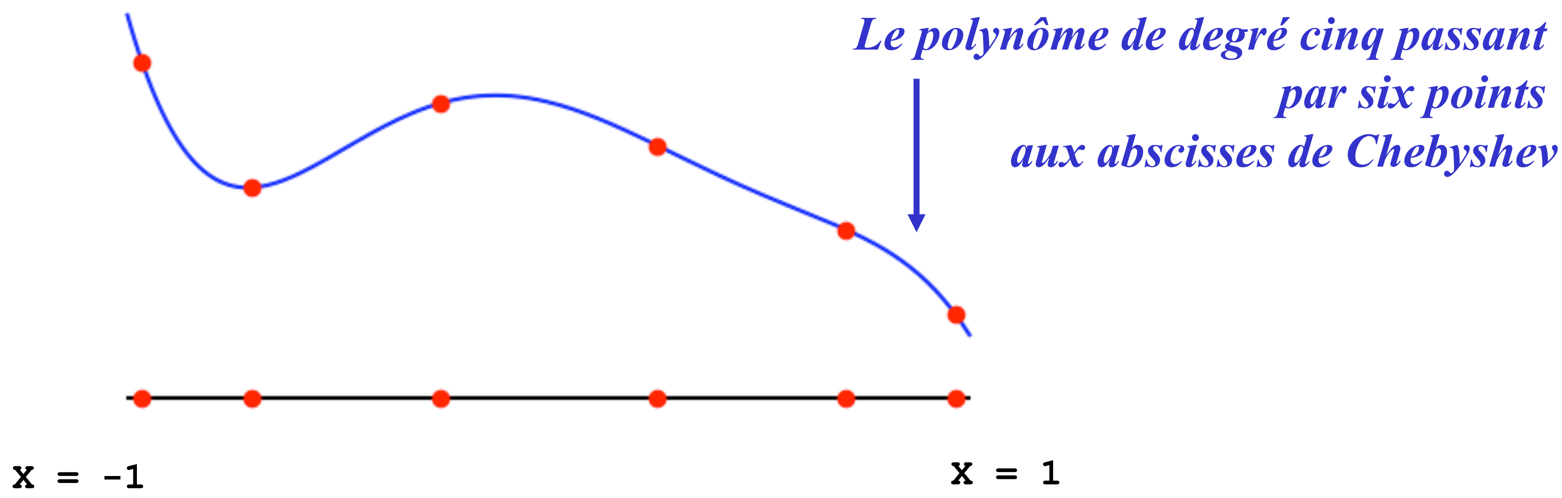
Polynômes de Chebychev

Why does it work ?



$$e^h(x) = \frac{u^{(n+1)}(\xi(x))}{(n+1)!} \boxed{(x - X_0)(x - X_1)(x - X_2)\cdots(x - X_n).}$$

Six points aux abscisses de Chebyshev….



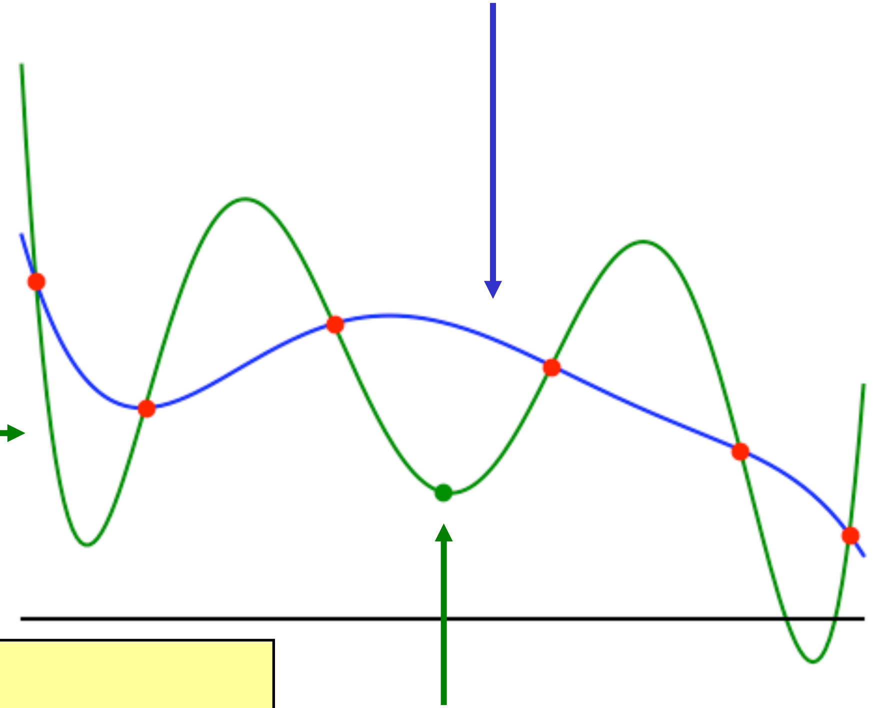

```
n = 5
X = cos(pi * (2*arange(0,n+1) + 1)/((n+1) * 2))
U = [0.2,0.4,0.6,0.7,0.5,0.8]

puh = polyfit(X,U,5)
x = linspace(-1,1,200)
uh = polyval(puh,x)
plt.plot(x,uh,'-b')
plt.plot(X,U,'or')
```

Ajoutons un septième point !

Le polynôme de degré cinq passant par six points aux abscisses de Chebyshev

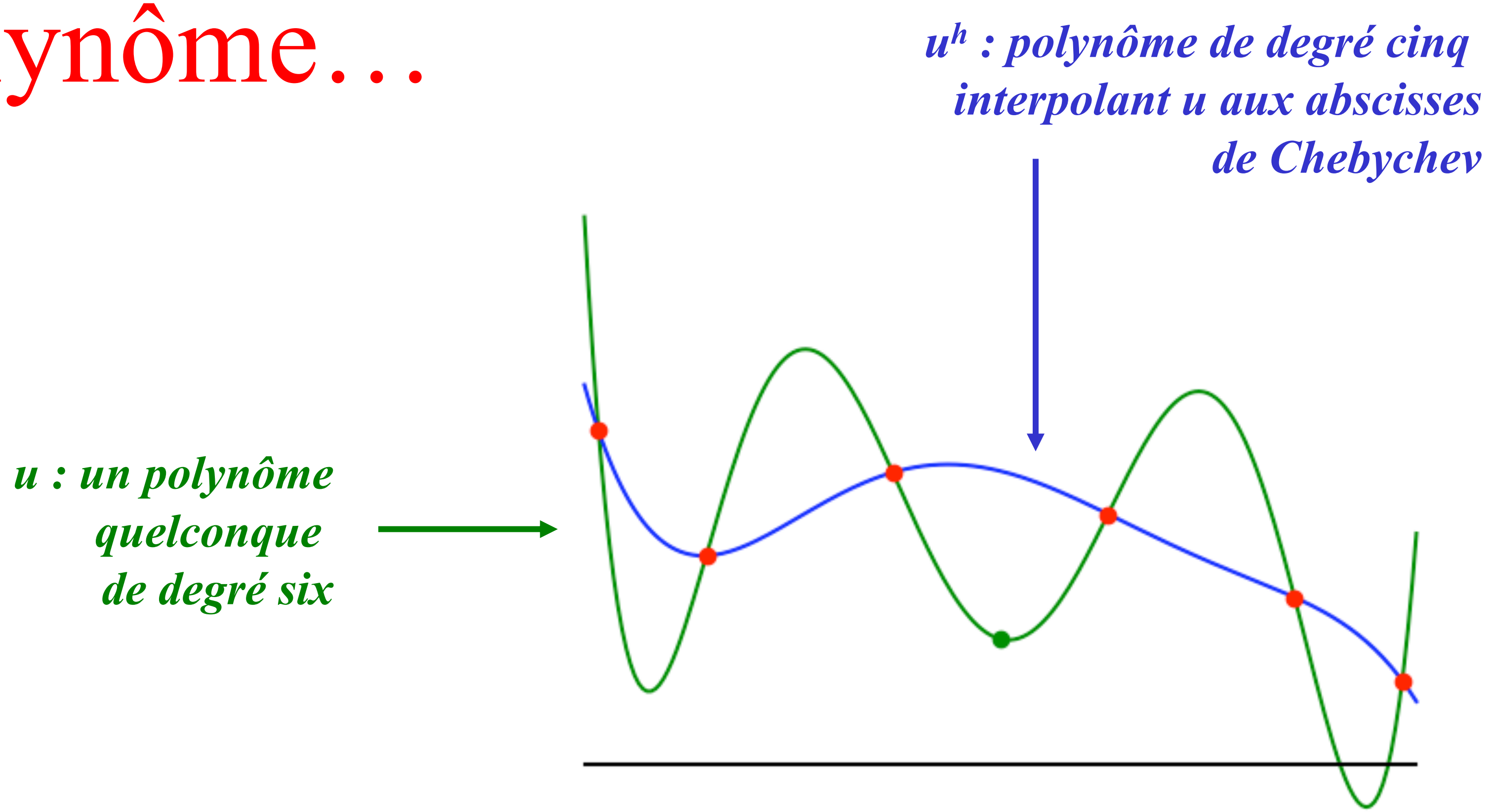
Le polynôme de degré six passant par six points aux abscisses de Chebyshev et par le septième point



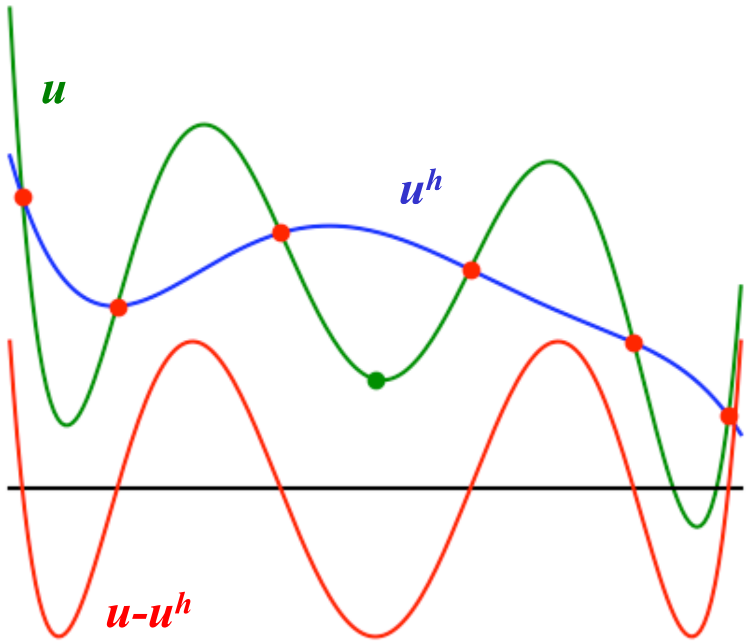
Le septième point

```
pu = polyfit([*X,0],[*U,0.3],6)
u = polyval(pu,x)
plt.plot(x,u,'-g')
plt.plot([0],[0.3],'og')
```

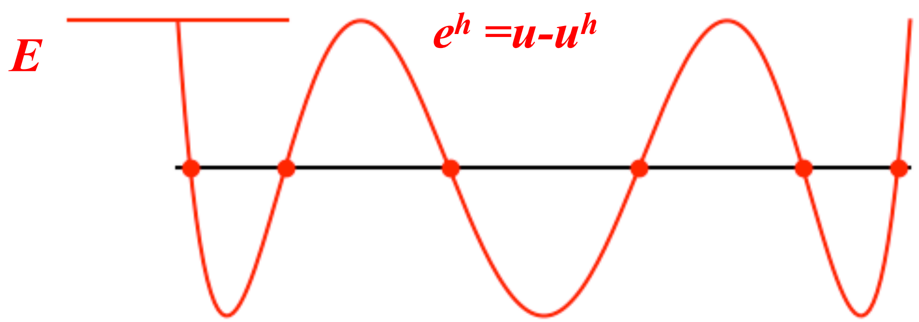
Interpolons un polynôme par un polynôme…



…c'est bête, je sais :-)

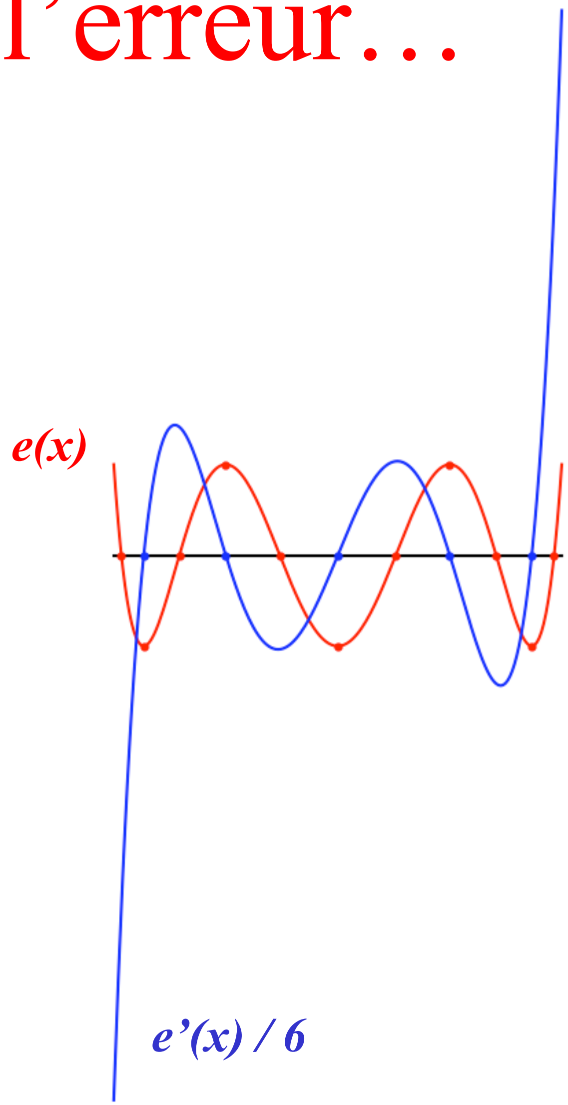



```
error = u - uh
E = error[0]
plt.plot(x,error,'-r');
```



Erreur d’interpolation

Dérivons et divisons l'erreur…

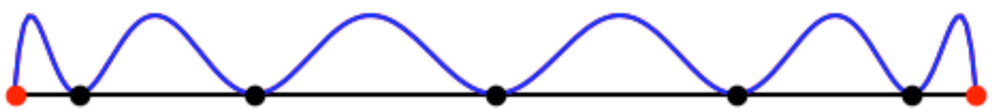



```
peh = pu - [0,*puh]
eh = polyval(peh, x)
plt.plot(x,eh,'-r')

pdeh = peh * [6,5,4,3,2,1,0]
pdeh = pdeh[0:-1]
deh = polyval(pdeh,x)/6
plt.plot(x,deh,'-b')
```

Et encore quelques petites manipulations…

$$\left(e'(x)\right)^2 (1 - x^2) \quad = \quad (n+1)^2 \left(E^2 - e^2(x)\right)$$



```
Xd = roots(pdeh)
Ed = polyval(peh,Xd)
E = Ed[0]

plt.plot(x,E**2-eh**2,'-r')
plt.plot(x,(1-x**2)*(deh**2),'-b')
plt.plot(Xd,zeros(size(Xd)),'ok')
plt.plot([-1,1],[0,0],'or')
```

Une solution analytique d'une équation différentielle ?

$$\left(e'(x)\right)^2 (1 - x^2) = (n+1)^2 \left(E^2 - e^2(x)\right)$$

$$\downarrow$$

$$\frac{\frac{e'(x)}{E}}{\sqrt{1 - \left(\frac{e}{E}\right)^2}} = \pm(n+1)\frac{1}{\sqrt{1-x^2}}$$

```
>>> from sympy import *
>>> x = symbols('x')
>>> f = 1/sqrt(1-x**2)
>>> integrate(f)
asin(x)
```

Et si on dérive la primitive…

```
>>> from sympy import *
>>> x = symbols('x')
>>> f = 1/sqrt(1-x**2)
>>> integrate(f)
asin(x)
>>> diff(asin(x))
1/sqrt(-x**2 + 1)
>>> diff(acos(x))
-1/sqrt(-x**2 + 1)
```

Une petite solution analytique comme le faisaient les anciens…

$$\left(e'(x)\right)^2\left(1-x^2\right) = (n+1)^2\left(E^2-e^2(x)\right)$$

$$\downarrow$$

$$\frac{\frac{e'(x)}{E}}{\sqrt{1-\left(\frac{e}{E}\right)^2}} = \pm(n+1)\frac{1}{\sqrt{1-x^2}}$$

$$\downarrow$$

$$\arccos\left(\frac{e(x)}{E}\right) = \pm\left((n+1)\arccos(x)+C\right)$$

$$\downarrow \quad \text{En vertu de la parité du cosinus !}$$

$$e(x) = E\cos\left((n+1)\arccos(x)+C\right)$$

$$\downarrow \quad \text{En imposant que e(1) = E}$$

$$e(x) = E\,\boxed{\underbrace{\cos\left((n+1)\arccos(x)\right)}_{T_{n+1}(x)}}$$

Polynôme de Chebyshev de degré n+1

Drôle d'expression pour un polynôme, non ?

Théorème 1.2. *Les polynômes de Chebyshev* $T_{n+1}(x) = \cos((n+1)\arccos(x))$ *définis sur l'intervalle* $[-1,1]$ *satisfont la relation de récurrence*

$$T_{i+1}(x) = 2x\, T_i(x) - T_{i-1}(x), \quad i = 1, 2, 3, \ldots,$$

avec $T_0(x) = 1$ *et* $T_1(x) = x$.

Calcul des polynômes de Chebyshev : formule de récurrence

Démonstration : Définissons $\theta = \arccos(x)$ et écrivons :

$$\begin{aligned} T_{i+1}(x) &= \cos((i+1)\theta) \\ &= \cos(\theta)\cos(i\theta) - \sin(\theta)\sin(i\theta) \\[1em] T_{i-1}(x) &= \cos((i-1)\theta) \\ &= \cos(\theta)\cos(i\theta) + \sin(\theta)\sin(i\theta) \end{aligned}$$

$$\begin{aligned} T_{i+1}(x) + T_{i-1}(x) &= 2\cos(\theta)\cos(i\theta) \\ &= 2xT_i(x) \end{aligned}$$

$\square$

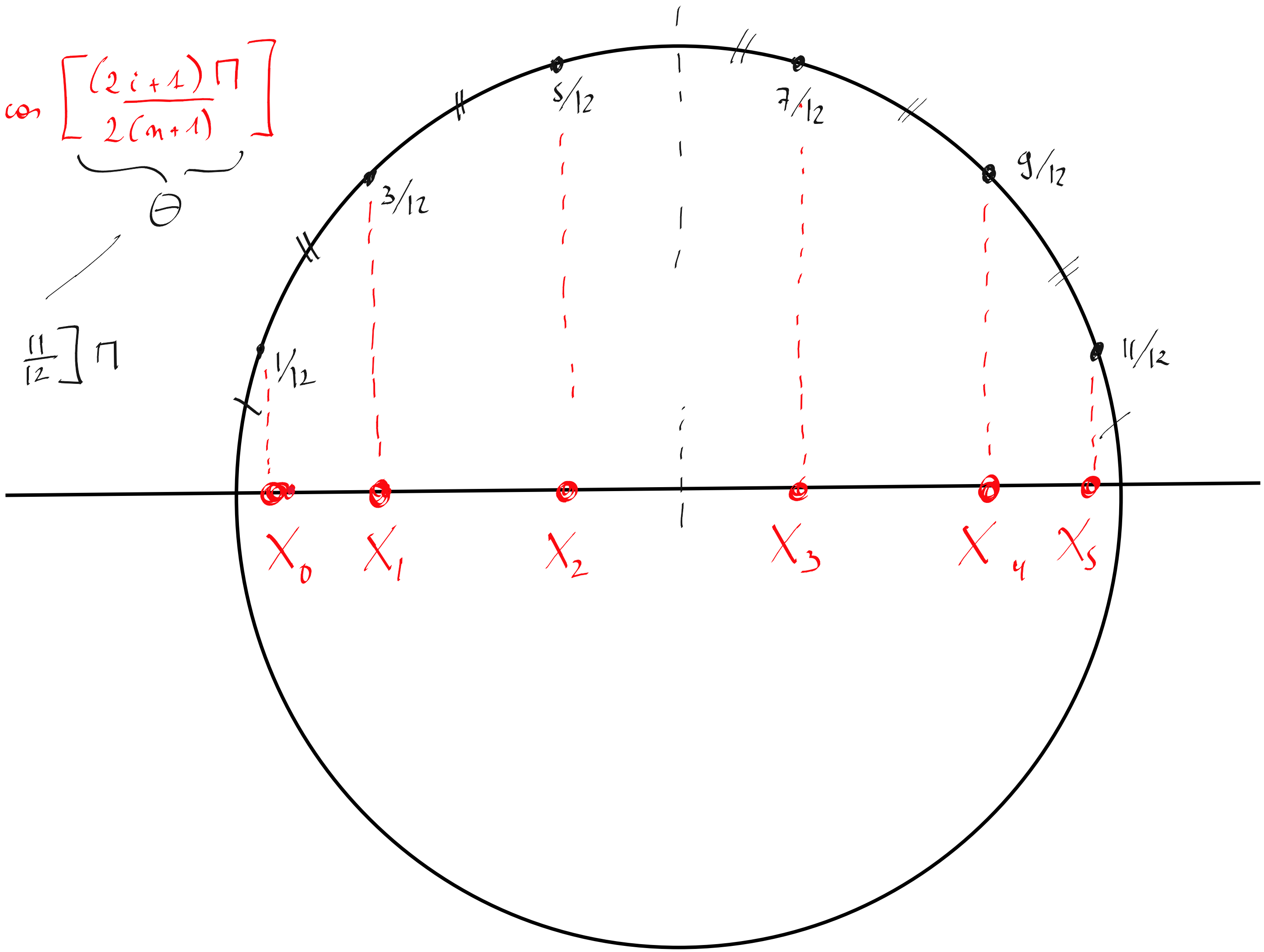
Abscisses de Chebyshev

$$
\begin{array}{rcll}
0 & = & \overbrace{\cos\left((n+1)\arccos(X_i)\right)}^{T_{n+1}(X_i)} & i = 0, \ldots, n \\
 & \Big\downarrow & & \\
\dfrac{\pi/2 + i\pi}{(n+1)} & = & \arccos(X_i) & i = 0, \ldots, n \\
 & \Big\downarrow & & \\
\cos\left(\dfrac{(2i+1)\pi}{2(n+1)}\right) & = & X_i & i = 0, \ldots, n
\end{array}
$$

$$X_i = \cos\left[\underbrace{\frac{(2i+1)\pi}{2(n+1)}}_{\theta}\right]$$

$n = 5$

$$\left[\frac{1}{12}\ \frac{3}{12} \ldots \frac{11}{12}\right]\pi$$



Abscisses
de Chebychev

Interpolation polynomiale : bilan

- **Pour une fonction u(x) très régulière : fonction cosinus**

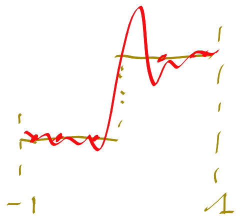
  Convergence de l'interpolation polynomiale

- **Pour une fonction u(x) suffisamment régulière : fonction de Runge**

  Divergence pour des abscisses équidistantes
  Convergence pour les abscisses de Chebyshev

- **Pour une fonction u(x) peu régulière : fonction échelon**

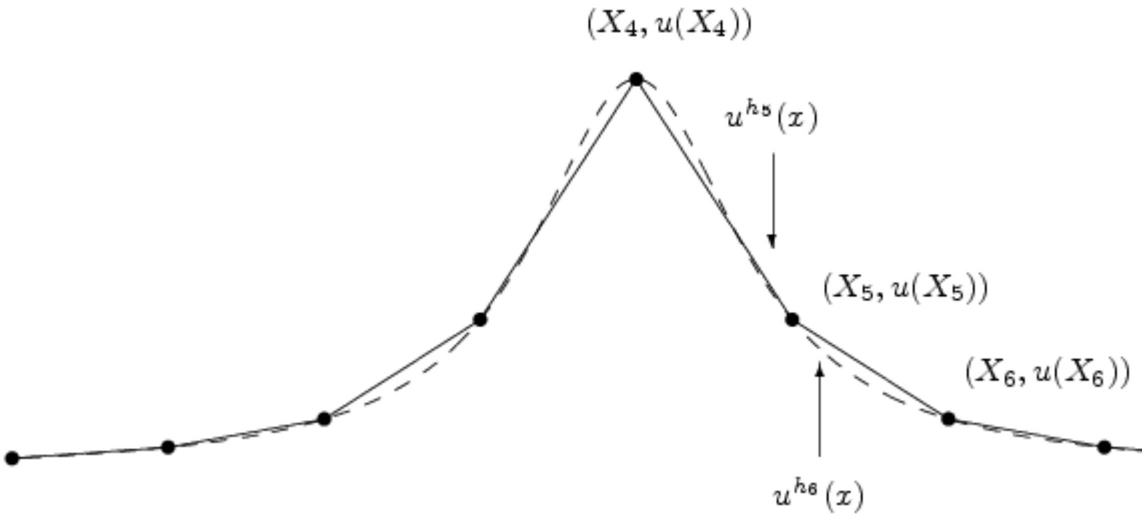
  Divergence !
  Eviter l'interpolation polynomiale de degré élevé



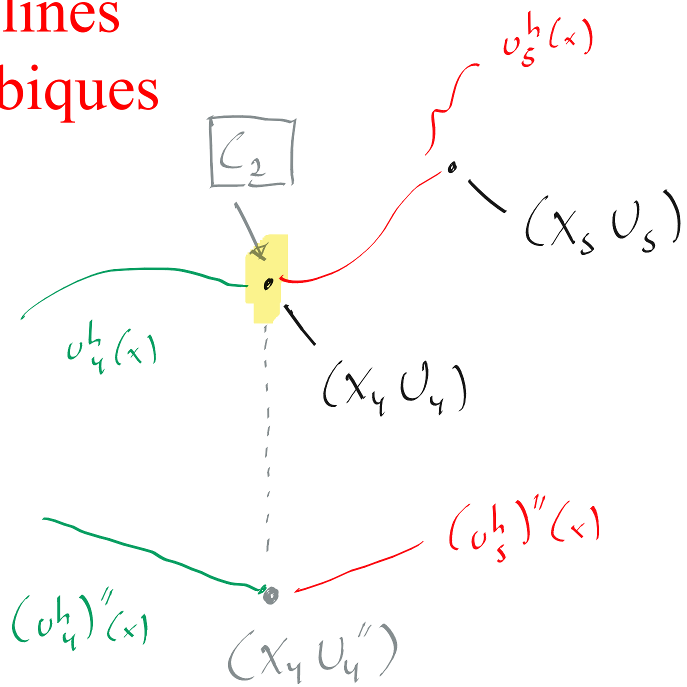
Idée :

Utiliser une interpolation par morceaux composée des polynômes de degré bas !

Interpolation linéaire par morceaux



Splines cubiques



BE CAREFUL

$U_4'' \neq u''(X_4)$

$u_5^h = a_3 x^3 + a_2 x^2 + a_1 x + a_0$

$n+1$ POINTS

n FONCTIONS

$4n$ COEFICIENTS

$$u_4^h(X_4) = U_4 = u_5^h(X_4)$$

$$(u_4^h)'(X_4) = (u_5^h)'(X_4) \neq u'(X_4)$$

$$(u_4^h)''(X_4) = (u_5^h)''(X_4) \neq u''(X_4)$$

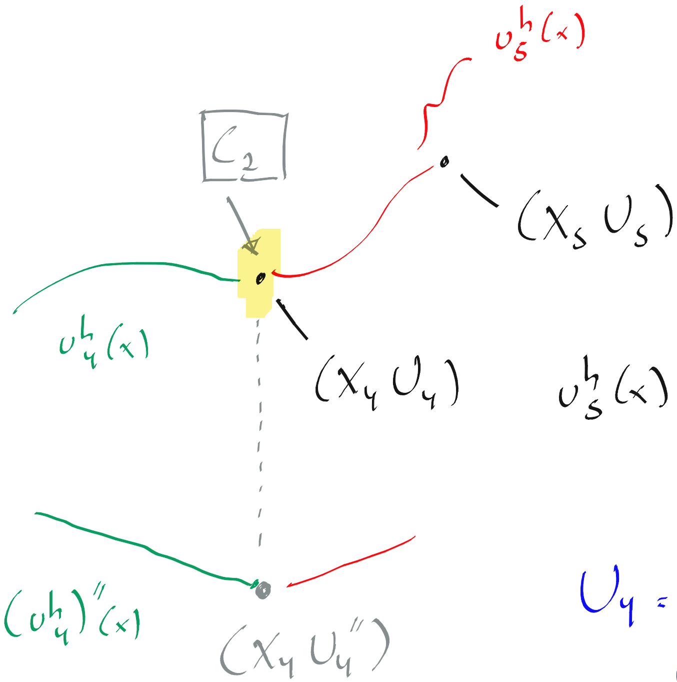
$$2(n-1) + 2 + 2(n-1)$$

IL Y A $4n-2$ CONDITIONS

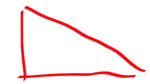
IL MANQUE 2 CONDITIONS

$$u_h''(X_0) = u_h''(X_n) = 0$$

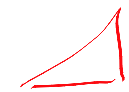
Etape 1
Dérivée seconde ok



$$(u_5^h)''(x) = U_4''\,\boxed{\frac{(x-X_5)}{(X_4-X_5)}} + U_5''\,\boxed{\frac{(x-X_4)}{(X_5-X_4)}}$$



$\frac{X_5-x}{h}$



$\frac{x-X_4}{h}$

$$u_5^h(x) = U_4''\frac{(X_5-x)^3}{6h} + U_5''\frac{(x-X_4)^3}{6h} + \underbrace{A\,\frac{(X_5-x)}{h} + B\,\frac{(x-X_4)}{h}}_{Cx+D}$$

$$\underbrace{U_4 = U_4''\frac{h^2}{6} + A}$$

$$\boxed{A = U_4 - U_4''\frac{h^2}{6}}$$

$$\boxed{B = U_5 - U_5''\frac{h^2}{6}}$$

Etape 2
Interpolation ok

Etape 3
Dérivée première ok

$$u_5^h(x) = U_4'' \frac{(X_5 - x)^3}{6h} + U_5'' \frac{(x - X_4)^3}{6h} + A \frac{(X_5 - x)}{h} + B \frac{(x - X_4)}{h}$$

$$(u_4^h)'(X_4) = (u_5^h)'(X_4)$$

$$= U_4'' \underbrace{\frac{-3h^2}{6h}}_{\hookrightarrow} - \frac{A}{h} + \frac{B}{h}$$

$$\left[\frac{(X_5 - x)^3}{6h}\right]'_{X_4} = \frac{-3(X_5 - x)^2}{6h}\Bigg|_{X_4}$$

$$\frac{h\,U_4''}{2} + \left[\frac{U_4 - U_3}{h}\right] - \left[\frac{U_4'' - U_3''}{h}\right]\frac{h^2}{6} = \frac{-h\,U_4''}{2} + \left[\frac{U_5 - U_4}{h}\right] - \left[\frac{U_5'' - U_4''}{h}\right]\frac{h^2}{6}$$

$$\left[\frac{U_3 - 2U_4 + U_5}{h}\right] = \frac{h}{6}\left[U_3'' + 4U_4'' + U_5''\right]$$

$$\boxed{A = U_4 - U_4''\frac{h^2}{6}}$$

$$\boxed{B = U_5 - U_5''\frac{h^2}{6}}$$

Et le système
à résoudre est finalement…

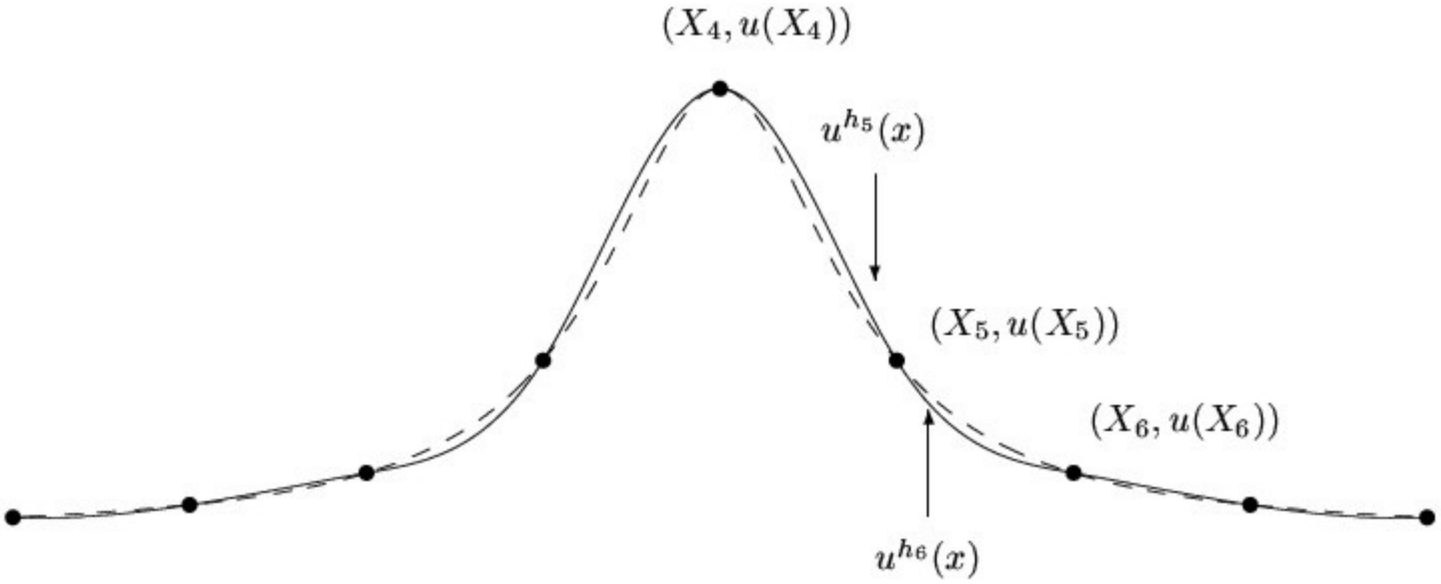
$$\left[\frac{U_3 - 2U_4 + U_5}{h} \right] = \frac{h}{6}\left[U_3'' + 4U_4'' + U_5'' \right]$$

$$\frac{h^2}{6}\begin{bmatrix} 1 & & & & & \\ 1 & 4 & 1 & & & \\ & 1 & 4 & 1 & & \\ & & & \ddots & & \\ & & & 1 & 4 & 1 \\ & & & & & 1 \end{bmatrix}\begin{bmatrix} U_0'' \\ \\ U_4'' \\ \\ U_M'' \end{bmatrix} = \begin{bmatrix} 0 \\ \\ U_3 - 2U_4 + U_5 \\ \\ 0 \end{bmatrix}$$

Interpolation par splines cubiques

$$u^{h_i}(x) = a_i + b_i x + c_i x^2 + d_i x^3 \qquad i = 1, 2, \ldots, n$$

4n coefficients inconnus



Comment trouver les coefficients ?

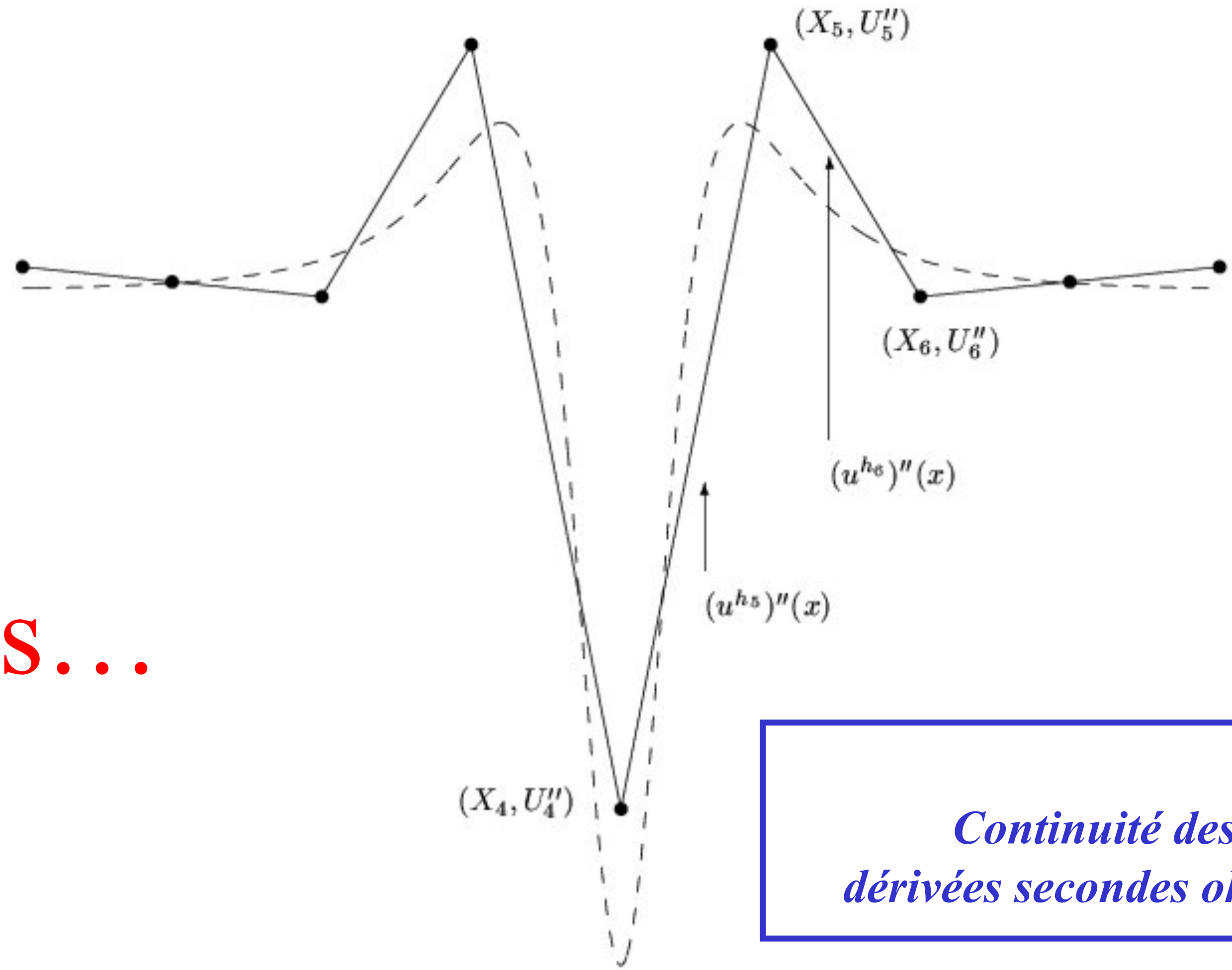
$$\begin{aligned} u^{h_1}(X_0) &= U_0 \\ u^{h_n}(X_n) &= U_n \end{aligned}$$

$$\begin{aligned} u^{h_i}(X_i) &= U_i & i = 1, \ldots, n-1 \\ u^{h_{i+1}}(X_i) &= U_i & i = 1, \ldots, n-1 \end{aligned}$$

$$\begin{aligned} (u^{h_i})'(X_i) &= (u^{h_{i+1}})'(X_i) & i = 1, \ldots, n-1 \\ (u^{h_i})''(X_i) &= (u^{h_{i+1}})''(X_i) & i = 1, \ldots, n-1 \end{aligned}$$

4n-2 conditions

$$(u^{h_i})'' = U''_{i-1}\frac{(x - X_i)}{(X_{i-1} - X_i)} + U''_i\frac{(x - X_{i-1})}{(X_i - X_{i-1})}$$



Ecrivons…

Continuité des dérivées secondes ok

Intégrons…

$$(u^{h_i})'' = U''_{i-1}\frac{(X_i - x)}{h_i} + U''_i\frac{(x - X_{i-1})}{h_i}$$

En intégrant deux fois,

$$(u^{h_i}) = U''_{i-1}\frac{(X_i - x)^3}{6h_i} + U''_i\frac{(x - X_{i-1})^3}{6h_i} + A_i\frac{(X_i - x)}{h_i} + B_i\frac{(x - X_{i-1})}{h_i}$$

$$U_{i-1} = U''_{i-1}\frac{h_i^3}{6h_i} + A_i\frac{h_i}{h_i} \quad \text{et} \quad U_i = U''_i\frac{h_i^3}{6h_i} + B_i\frac{h_i}{h_i}$$

$$A_i = U_{i-1} - \frac{U''_{i-1}\,h_i^2}{6} \qquad B_i = U_i - \frac{U''_i\,h_i^2}{6}$$

Continuité de la fonction ok

Calculons…

Continuité des dérivées premières ok

$$(u^{h_i})'(X_i) = (u^{h_{i+1}})'(X_i)$$

$$\frac{U_i'' h_i}{2} + \frac{(U_i - U_{i-1})}{h_i} - \frac{(U_i'' - U_{i-1}'')h_i}{6} = -\frac{U_i'' h_{i+1}}{2} + \frac{(U_{i+1} - U_i)}{h_{i+1}} - \frac{(U_{i+1}'' - U_i'')h_{i+1}}{6}$$

$$\frac{(2U_i'' + U_{i-1}'')h_i}{6} + \frac{(U_i - U_{i-1})}{h_i} = \frac{(U_{i+1} - U_i)}{h_{i+1}} - \frac{(U_{i+1}'' + 2U_i'')h_{i+1}}{6}$$

$$\frac{h_i}{6}\, U_{i-1}'' + \frac{2(h_i + h_{i+1})}{6}\, U_i'' + \frac{h_{i+1}}{6}\, U_{i+1}'' = \frac{(U_{i+1} - U_i)}{h_{i+1}} - \frac{(U_i - U_{i-1})}{h_i}$$

$$i = 1, \ldots, n-1$$

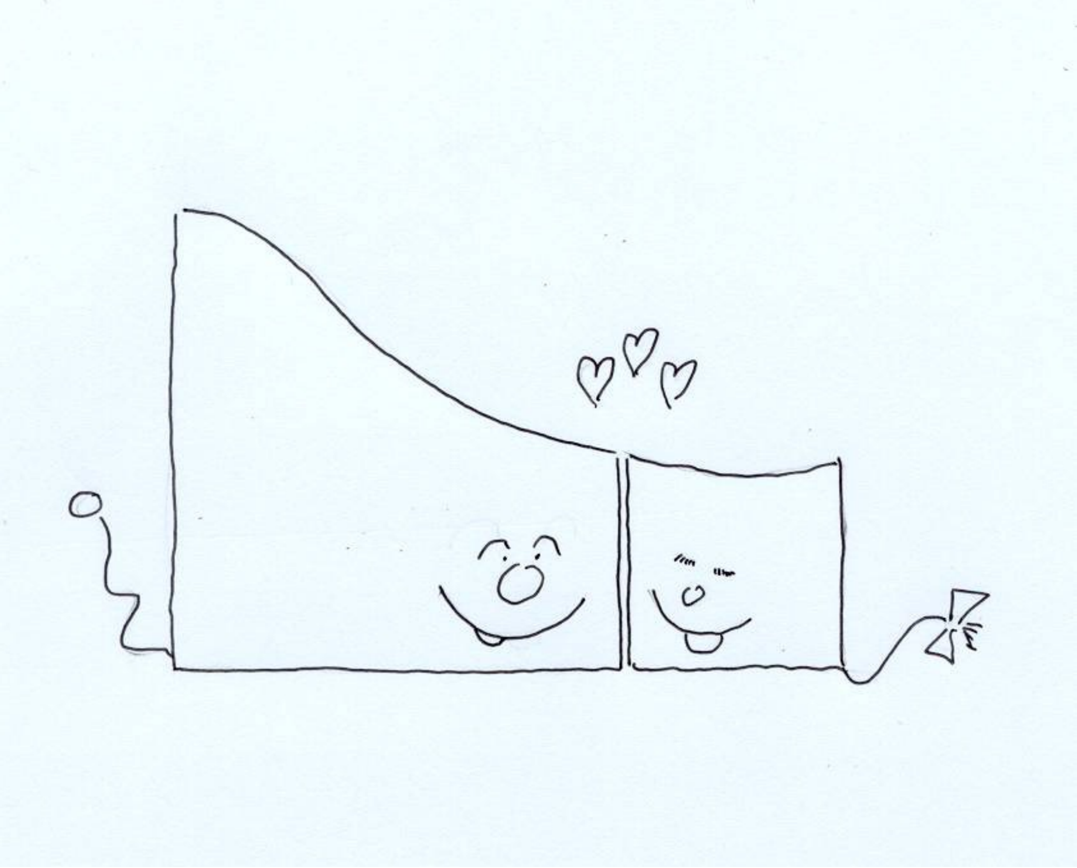
Abscisses équidistantes

$U_0'' = 0 \quad \text{et} \quad U_n'' = 0$

2 conditions supplémentaires
Courbe spline naturelle

$$\frac{h^2}{6}\begin{bmatrix} 1 & 0 & & & & & & & \\ 1 & 4 & 1 & & & & & & \\ & 1 & 4 & 1 & & & & & \\ & & 1 & 4 & 1 & & & & \\ & & & 1 & 4 & 1 & & & \\ & & & & \cdot & \cdot & \cdot & & \\ & & & & & 1 & 4 & 1 & \\ & & & & & & 1 & 4 & 1 \\ & & & & & & & 0 & 1 \end{bmatrix}\begin{bmatrix} U_0'' \\ U_1'' \\ U_2'' \\ U_3'' \\ U_4'' \\ \vdots \\ U_{n-2}'' \\ U_{n-1}'' \\ U_n'' \end{bmatrix} = \begin{bmatrix} 0 \\ U_0 - 2U_1 + U_2 \\ U_1 - 2U_2 + U_3 \\ U_2 - 2U_3 + U_4 \\ U_3 - 2U_4 + U_5 \\ \vdots \\ \vdots \\ U_{n-2} - 2U_{n-1} + U_n \\ 0 \end{bmatrix}$$

Ou de manière plus poétique…



```
from numpy import *
from scipy.interpolate import CubicSpline as spline
from matplotlib import pyplot as plt

X = arange(-55,70,10)
U = [3.25, 3.37, 3.35, 3.20, 3.12, 3.02, 3.02,
           3.07, 3.17, 3.32, 3.30, 3.20, 3.10]
x = linspace(X[0],X[-1],100)
uhLag = polyval(polyfit(X,U,len(X)-1),x)
uhSpl = spline(X,U)

plt.plot(x,uhLag,'--r',x,uhSpl(x),'-b')
plt.plot(X,U,'or')
```

Exemple