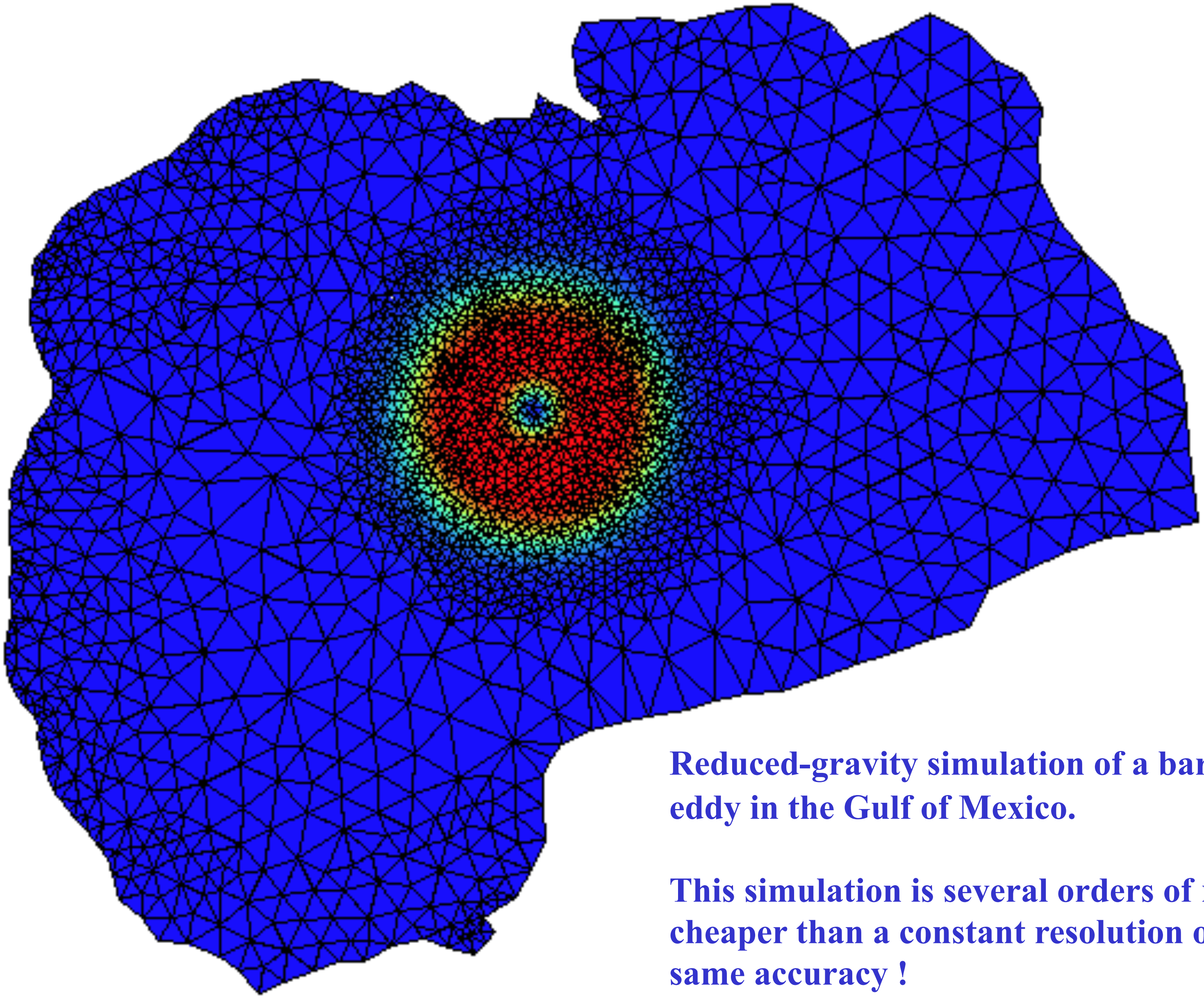
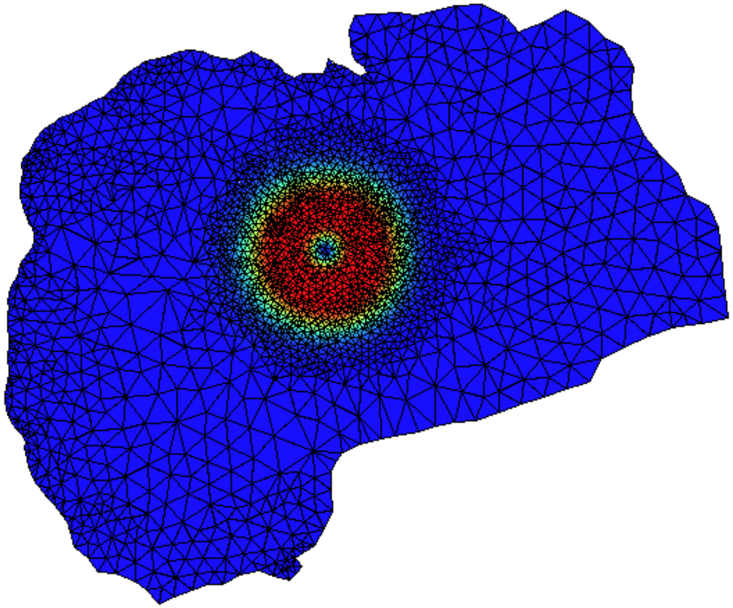


Reduced-gravity simulation of a baroclinic eddy in the Gulf of Mexico.

This simulation is several orders of magnitude cheaper than a constant resolution one of the same accuracy !

The Finite Element Method



is a way of computing approximate solutions to a mathematical model describing a physical process.

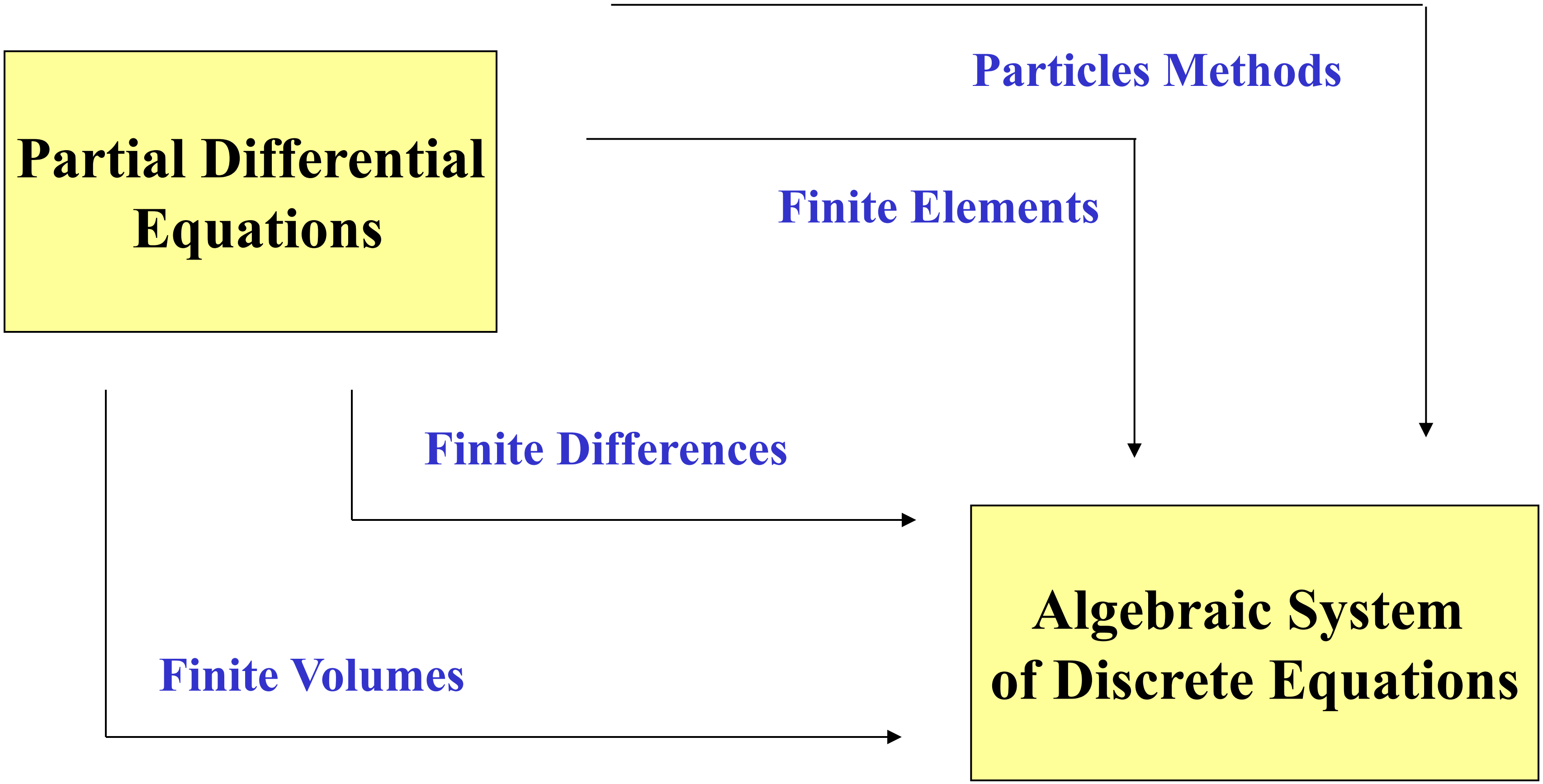
What is a mathematical model ?
A boundary value problem.

What is a boundary value problem ?
A set of partial differential equations with boundary and initial conditions.

Typical applications

- Deformable solids mechanics
- Fluid dynamics (CFD)
- Electromagnetism
- Transport phenomena
- Climatology

Finite Elements, Finite Differences, Finite Volumes etc.



Partial Differential Equations

Particles Methods

Finite Elements

Finite Differences

Finite Volumes

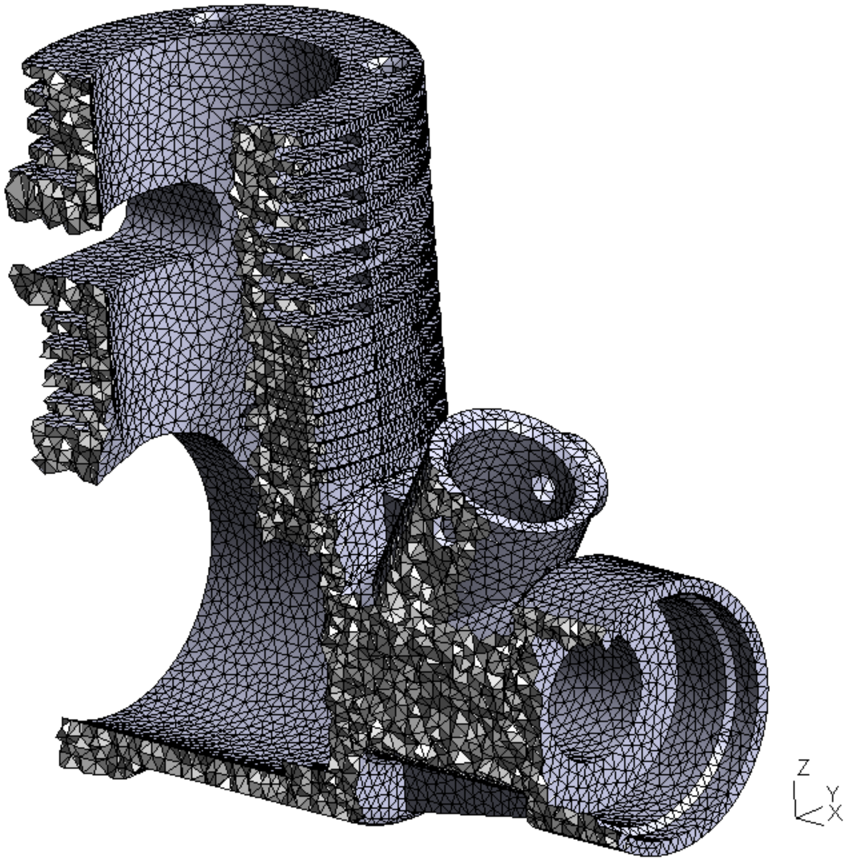
Algebraic System of Discrete Equations

The Finite Elements Method is a discretization method

The problem geometry is divided in small finite elements.

On each element, the solution is approximated by means of unknown nodal values and given polynomials

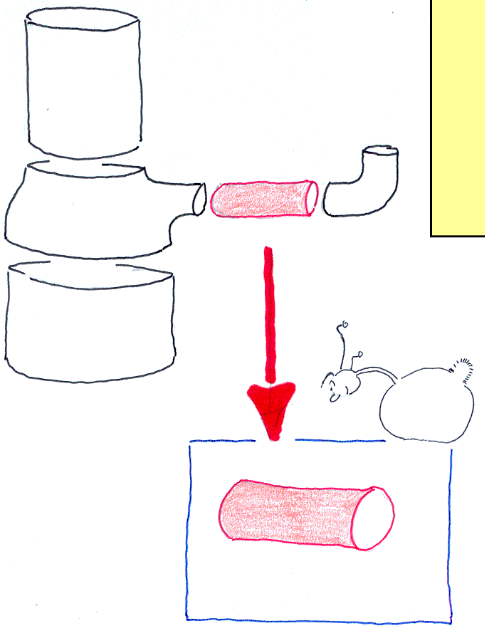
$$u(\mathbf{x}) \approx u^h(\mathbf{x}) = \sum_{j=1}^{n} U_j \tau_j(\mathbf{x})$$



Classical Engineering Analysis

Analysis through simple geometries and a limited combination of approximate models :

- Lubrication theory
- Bars
- Beams
- Plates and shells



Exact solution
to approximate problems

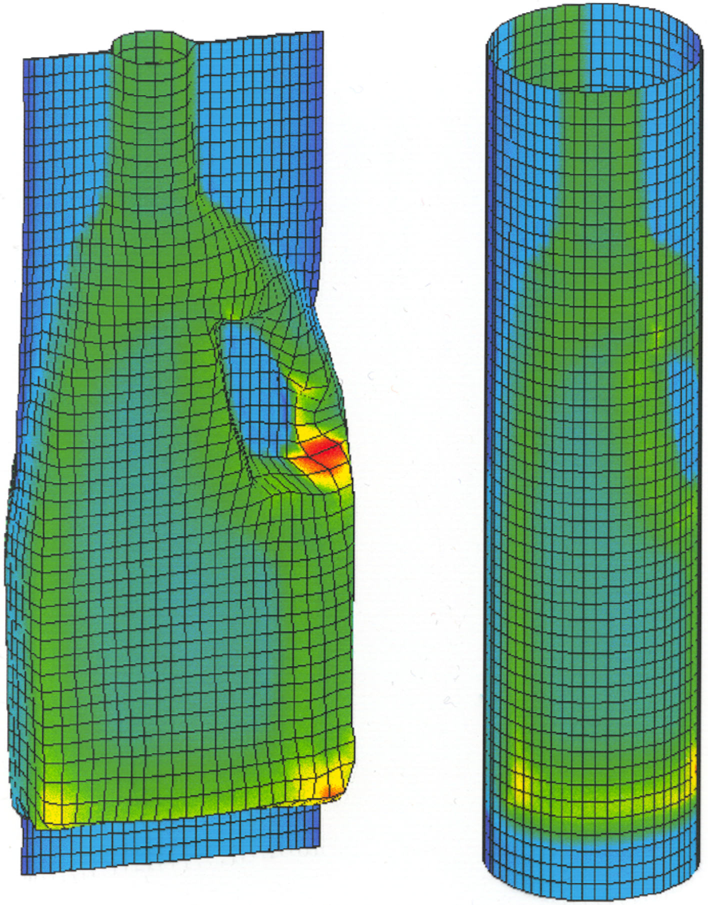
Low computer's cost
Good physical understanding

Simplicity of models
Complex geometries and loads cannot be handled
Complex materials cannot be analyzed

Computer Aided Engineering Analysis

The Finite Element Method provides approximate solutions to more realistic problems

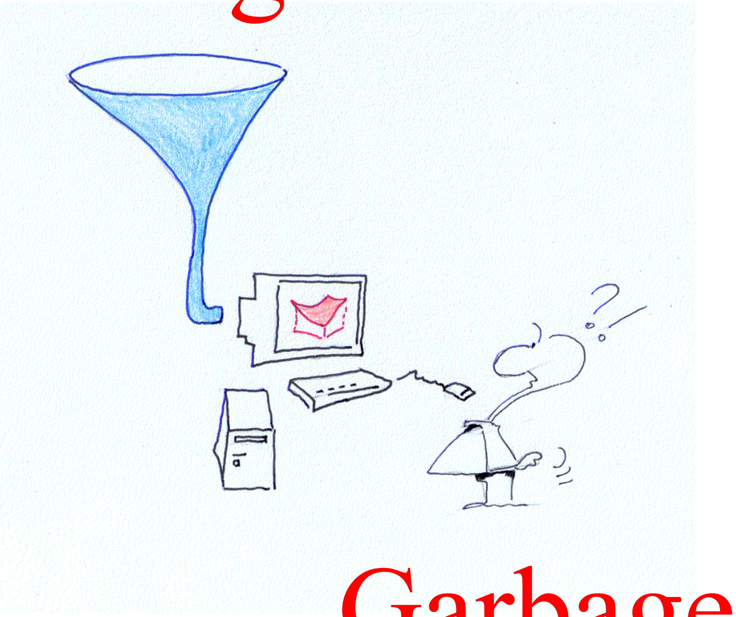
FEM developed in the sixties for linear elasticity and generalized to many other applications…



Powerful and flexible
High (cheap) computer's cost
Low (expensive) engineer's cost
Complex processes can be analyzed
Complex material laws can be included

However...

Garbage in



Garbage out

Illusion of non-qualified users to be able to analyze everything

New modeling issues requires higher qualifications…

How to define complex problems in an accurate and efficient way for the computer software ?

A l'issue de ce cours, vous serez capables de…

- **Comprendre la méthode des éléments finis**
- **Réaliser un petit programme en C**
- **Certifier et valider une simulation**
- **Choisir la voie numérique la plus efficace**
- **Estimer la précision d'un résultat**
- **Découvrir les joies et les aléas du numérique**

Non, non : ceci on ne fera pas !

- **Apprendre le génie logiciel de l'orienté-objet**
- **Utiliser des logiciels commerciaux**
- **Faire de l'analyse numérique théorique**
- **Faire du calcul parallèle**
- **Résoudre les équations de Navier-Stokes**
- **Créer automatiquement des maillages**

Learning Outcomes

Objectifs du projet

Réaliser	Créer une application pour prédire un tsunami.
Certifier	Tester et valider le travail de votre groupe.
Expliquer	Expliquer de manière efficace et rapide à l'enseignant et aux autres étudiants ce que vous avez réalisé.
Comprendre	Comprendre ce que vous avez réalisé. Comprendre ce que d'autres groupes ont réalisé.

Exercices : 10 petits problèmes

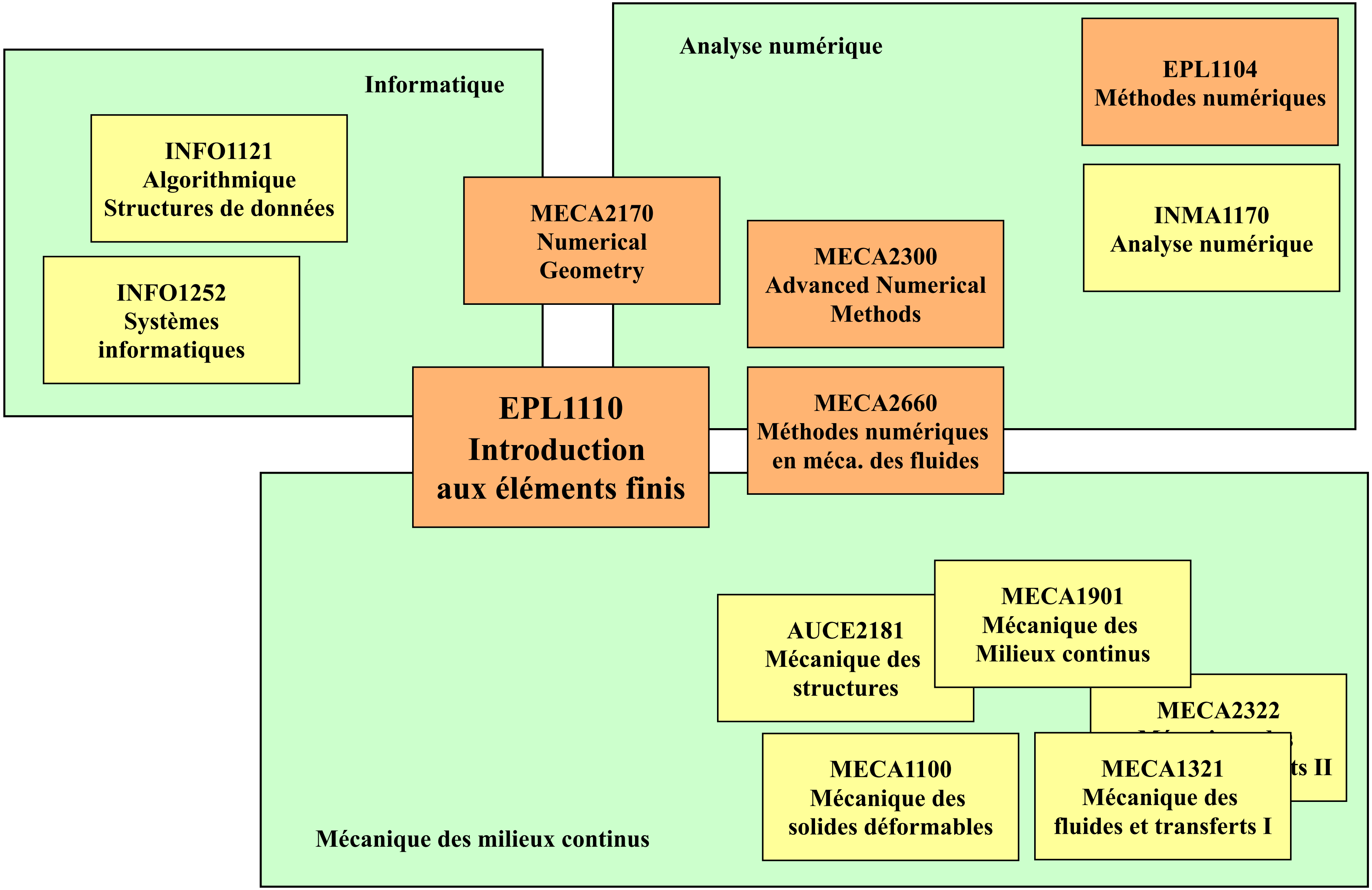
Quelques petits problèmes élémentaires pour apprivoiser le C

Projet en C :

Une petite application efficace pour prédire un tsunami…

The Practice of Programming :
Simplicity, Clarity, Generality. (B.K Kernighan & R. Pike 99)

Et les autres cours….



Informatique

INFO1121
Algorithmique
Structures de données

INFO1252
Systèmes
informatiques

MECA2170
Numerical
Geometry

Analyse numérique

EPL1104
Méthodes numériques

INMA1170
Analyse numérique

MECA2300
Advanced Numerical
Methods

MECA2660
Méthodes numériques
en méca. des fluides

EPL1110
Introduction
aux éléments finis

AUCE2181
Mécanique des
structures

MECA1901
Mécanique des
Milieux continus

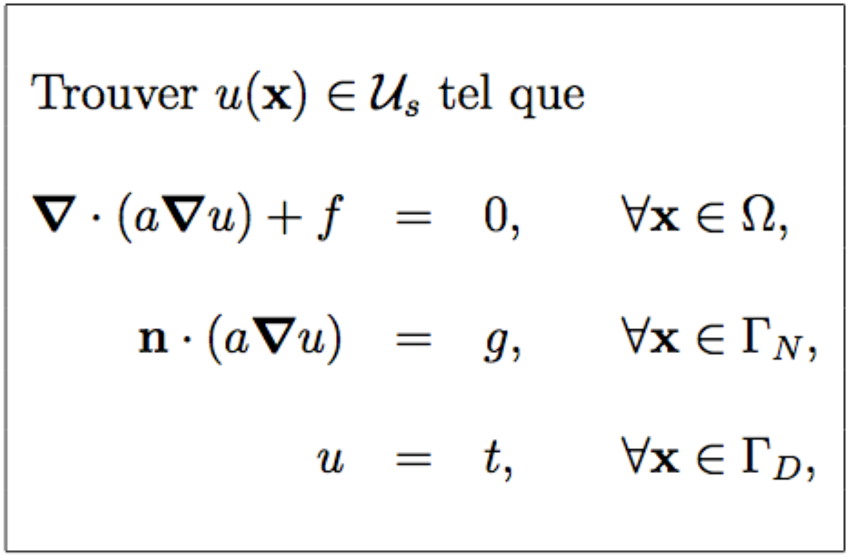
MECA2322
ts II

MECA1100
Mécanique des
solides déformables

MECA1321
Mécanique des
fluides et transferts I

Mécanique des milieux continus

Typical Elliptic BoundaryValue Problem



Trouver $u(\mathbf{x}) \in \mathcal{U}_s$ tel que

$$\begin{array}{rcll} \boldsymbol{\nabla} \cdot (a \boldsymbol{\nabla} u) + f & = & 0, & \forall \mathbf{x} \in \Omega, \\ \mathbf{n} \cdot (a \boldsymbol{\nabla} u) & = & g, & \forall \mathbf{x} \in \Gamma_N, \\ u & = & t, & \forall \mathbf{x} \in \Gamma_D, \end{array}$$

En général, la fonction n'est pas connue…
Mais, c'est la solution d'une équation aux dérivées partielles !

Commençons par une équation de Poisson

Conditions essentielles
Conditions de Dirichlet

Conditions naturelles
Conditions de Neumann

? $u(x)$ tel que

SOURCE DE CHALEUR

$$k \frac{d^2u}{dx^2} + f = 0 \quad \text{SUR } \underbrace{]0,1[}_{\Omega}$$

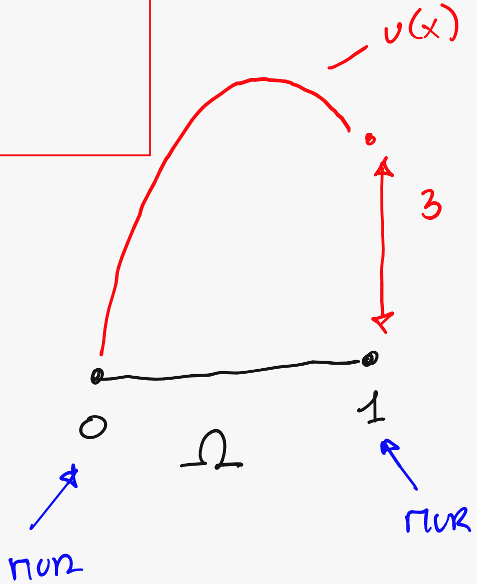
$= 1$

$u(0) = 0$

$u(1) = 3$

TEMPERATURE

CONDUCTIBILITE THERMIQUE



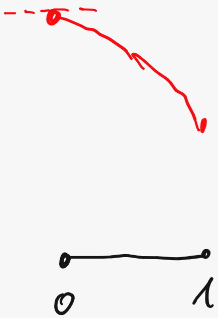
NATUREL

NEUMANN → $u'(0) = 0$

$u(1) = 3$

DIRICHLET

ESSENTIEL



Some nice spaces and notations…

Ne faisons pas comme les mathématiciens…
On ne va pas être rigoureux maintenant !
On fera cela plus tard…

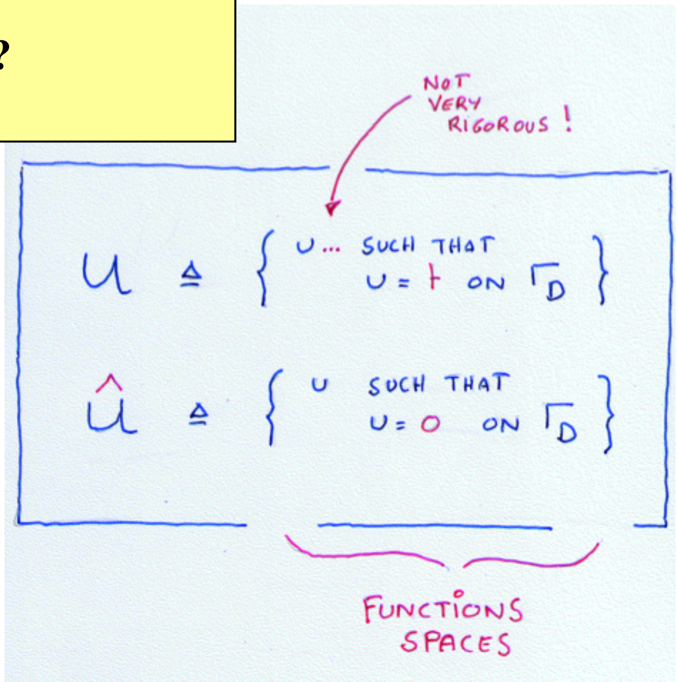
D’ailleurs, est-ce que cela est utile ?
Et pourtant, oui !

$$< f\ g > \;=\; \int_\Omega fg \; d\Omega,$$

$$\ll f\ g \gg \;=\; \int_{\partial\Omega} fg \; ds,$$

$$\ll f\ g \gg_N \;=\; \int_{\Gamma_N} fg \; ds,$$

$$\ll f\ g \gg_D \;=\; \int_{\Gamma_D} fg \; ds.$$



NOT VERY RIGOROUS !

$$\mathcal{U} \triangleq \left\{ u \ldots \text{ SUCH THAT } u = t \text{ ON } \Gamma_D \right\}$$

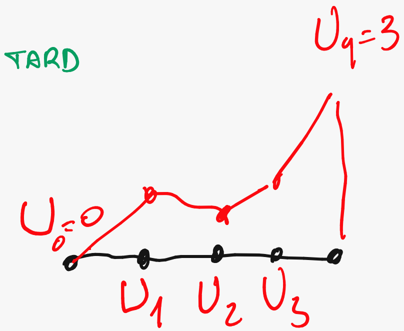
$$\hat{\mathcal{U}} \triangleq \left\{ u \text{ SUCH THAT } u = 0 \text{ ON } \Gamma_D \right\}$$

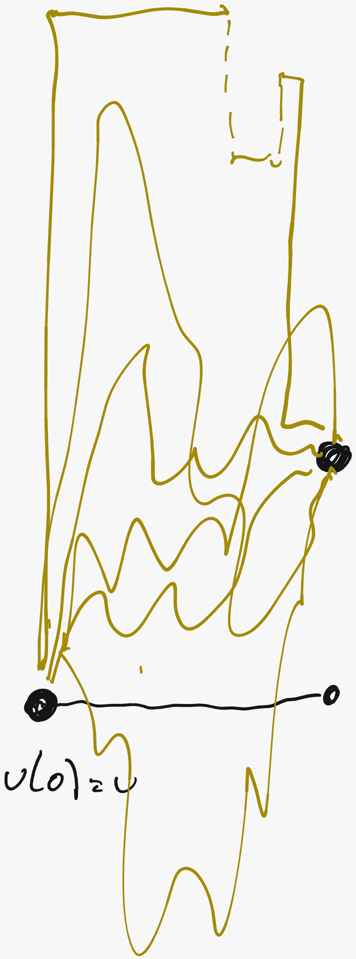
FUNCTIONS SPACES

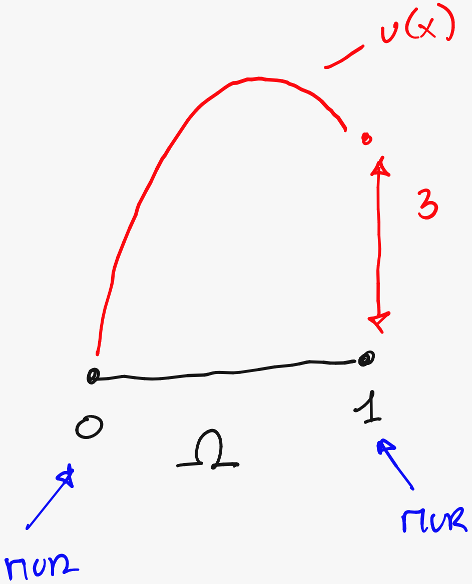
… to do calculus !

$$\mathcal{U} \triangleq \underbrace{\left\{ u(x) : \overset{]0,1[}{\Omega} \longrightarrow \mathbb{R} \quad \ldots\ldots \quad \text{tel que} \quad \begin{array}{l} u(0) = 0 \\ u(1) = 3 \end{array} \right\}}_{\text{ESPACE DES CANDIDATS SOLUTIONS}}$$

DO NOT WORRY! ON VERRA CELA PLUS TARD







$$\dim(\mathcal{U}) = \infty$$

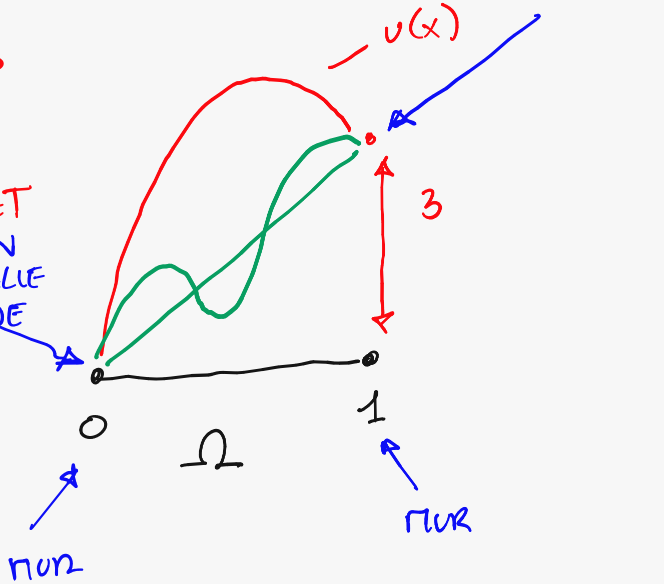
$$\mathcal{U}^h \subset \mathcal{U}$$

$\dim(\mathcal{U}^h) = n$

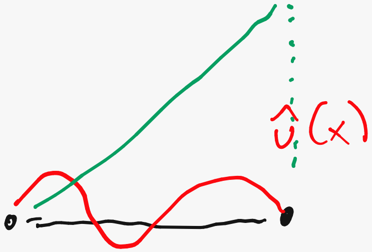
$\dim(\mathcal{U}) = \infty$

$$\mathcal{U} \triangleq \underbrace{\left\{ \begin{array}{l} v(x) : \overset{]0,1[}{\Omega} \to \mathbb{R} \quad \ldots\ldots \\ \text{tel que} \quad \begin{array}{l} v(0) = 0 \\ v(1) = 3 \end{array} \end{array} \right\}}_{\text{ESPACE DES CANDIDATS SOLUTIONS}}$$

DO NOT WORRY ! ON VERRA CELA PLUS TARD



$$\hat{\mathcal{U}} = \left\{ \begin{array}{l} \hat{v}(x) \quad \ldots\ldots \\ \text{tel que} \quad \begin{array}{l} v(0) = 0 \\ v(1) = 0 \end{array} \end{array} \right\}$$



$\mathcal{U}$ EST UNE VARIETE LINEAIRE DE $\hat{\mathcal{U}}$

TIP #1

$(fg)' = fg' + f'g$

INTEGRAL BY PARTS

$$\langle \hat{u}\ \underset{\sim}{\nabla}\cdot(a\underset{\sim}{\nabla} u)\rangle = \underbrace{\langle \underset{\sim}{\nabla}\cdot(\hat{u}\, a\underset{\sim}{\nabla} u)\rangle} - \langle(\underset{\sim}{\nabla}\hat{u})\cdot(a\underset{\sim}{\nabla} u)\rangle$$

$\in \hat{\mathcal{U}}$

$\in \mathcal{U}$

TIP #2

DIVERGENCE'S THEOREM

$\langle \underset{\sim}{\nabla}\cdot\underset{\sim}{a}\rangle = \langle\!\langle \underset{\sim}{n}\cdot\underset{\sim}{a}\rangle\!\rangle$

$$= \langle\!\langle \underset{\sim}{n}\cdot(a\underset{\sim}{\nabla} u)\ \hat{u}\rangle\!\rangle$$

TIP #3

BY DEFINITION OF $\hat{\mathcal{U}}$

$$= \langle\!\langle \underbrace{\underset{\sim}{n}\cdot(a\underset{\sim}{\nabla} u)}_{g \text{ ON } \Gamma_N\ !}\ \hat{u}\rangle\!\rangle_N$$

STRONG FORMULATION

? u SUCH THAT

$$\underset{\sim}{\nabla}\cdot(a\underset{\sim}{\nabla}u)+f=0 \quad \text{IN } \Omega$$

$$\underset{\sim}{n}\cdot(a\underset{\sim}{\nabla}u)=g \quad \text{ON } \Gamma_N$$

$$u=t \quad \text{ON } \Gamma_D$$

$\Downarrow \quad \not\Uparrow$ ⚠

WEAK FORMULATION

? $u\in\mathcal{U}$ SUCH THAT

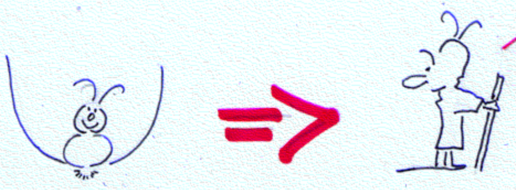
$$\underbrace{\langle\underset{\sim}{\nabla}\hat{u}\cdot a\underset{\sim}{\nabla}u\rangle}_{a(\hat{u},u)} = \underbrace{\langle f\hat{u}\rangle+\langle\!\langle g\hat{u}\rangle\!\rangle_N}_{b(\hat{u})} \quad \forall\hat{u}\in\hat{\mathcal{U}}$$

$\Updownarrow$

MINIMIZATION PROBLEM

? u SUCH THAT $J(u)=\min\limits_{v\in\mathcal{U}}\underbrace{\left(\frac{1}{2}a(v,v)-b(v)\right)}_{J(v)}$

VARIATIONAL CALCULUS



WEAK
OF VARIATIONAL
FORMULATION

u ∈ 𝒰
AND MINIMIZES
J

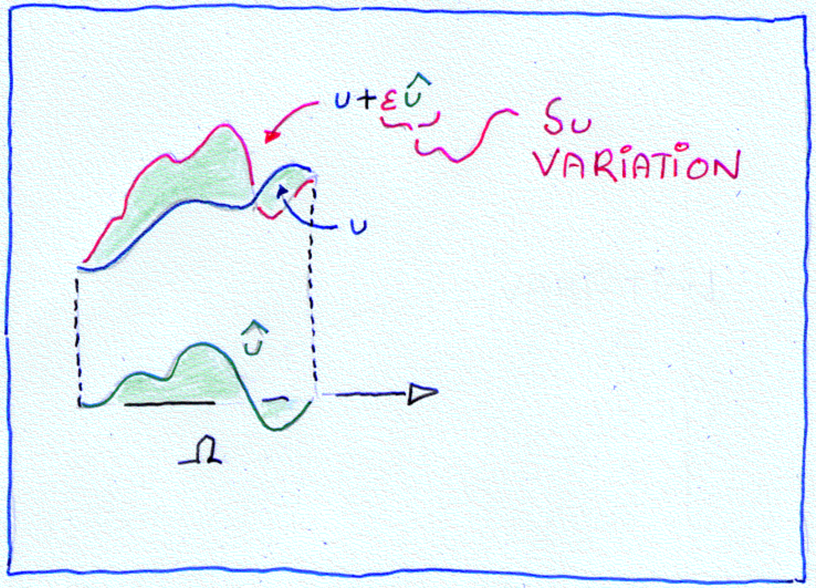
∈ ℝ

VARIATION
∈ $\hat{\mathcal{U}}$

FUNCTIONAL

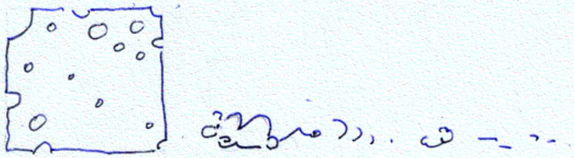
$$\underbrace{\left.\frac{dJ(u+\varepsilon\hat{u})}{d\varepsilon}\right|_{\varepsilon=0}}_{} = 0 \qquad \forall\, \hat{u} \in \hat{\mathcal{U}}$$

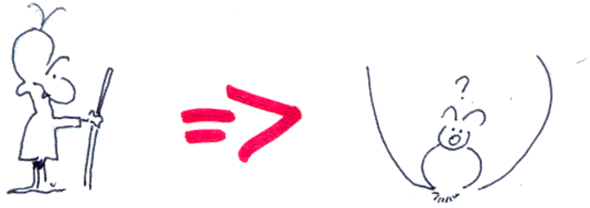
$$\delta J(u) = 0$$



$$J(u+\varepsilon\hat{u}) = \underbrace{\begin{array}{l} \frac{1}{2}\langle \underset{\sim}{\nabla} u \cdot a \underset{\sim}{\nabla} u \rangle \\ - \langle f u \rangle \\ - \langle\!\langle g u \rangle\!\rangle_N \end{array}}_{J(u)} \begin{array}{l} + \varepsilon \langle \underset{\sim}{\nabla} u \cdot a \underset{\sim}{\nabla} \hat{u} \rangle + \frac{\varepsilon^2}{2} \langle \underset{\sim}{\nabla} \hat{u} \cdot a \underset{\sim}{\nabla} \hat{u} \rangle \\ - \varepsilon \langle f \hat{u} \rangle \\ - \varepsilon \langle\!\langle g \hat{u} \rangle\!\rangle_N \end{array}$$

$$\underbrace{\left. \frac{d \overbrace{J(u+\varepsilon\hat{u})}^{g(\varepsilon)}}{d\varepsilon} \right|_{\varepsilon=0}}_{\begin{array}{c} = 0 \\ \forall \hat{u} \in \hat{\mathcal{U}} \end{array}} = \begin{array}{l} \langle \underset{\sim}{\nabla} u \cdot a \underset{\sim}{\nabla} \hat{u} \rangle - \langle f \hat{u} \rangle \\ \qquad - \langle\!\langle g \hat{u} \rangle\!\rangle_N \end{array}$$





$$J(\underset{\substack{\uparrow\\ \text{SOLUTION OF THE WEAK PROBLEM}}}{u} + \underset{\substack{\uparrow\\ \in \hat{\mathcal{U}}}}{\hat{u}}) = \underbrace{\frac{1}{2}\langle \underset{\sim}{\nabla} u . a \underset{\sim}{\nabla} u \rangle - \langle f u \rangle - \langle\!\langle g u \rangle\!\rangle_N}_{J(u)} + \underbrace{\langle \underset{\sim}{\nabla} u . a \underset{\sim}{\nabla} \hat{u} \rangle - \langle f \hat{u} \rangle - \langle\!\langle g \hat{u} \rangle\!\rangle_N}_{\substack{= 0\\ u \text{ SOLUTION OF THE WEAK PROBLEM}}} + \underbrace{\frac{1}{2}\langle \underset{\sim}{\nabla} \hat{u} . a \underset{\sim}{\nabla} \hat{u} \rangle}_{\geq 0}$$

$$\boxed{\text{IF } a > 0 \,!}$$

$$\geq J(u)$$

FINITE ELEMENT = VARIATIONAL METHOD

? $u^h \in \mathcal{U}^h$

$$a(u^h\ \hat{u}^h) = b(\hat{u}^h) \quad \forall \hat{u} \in \hat{\mathcal{U}}^h$$

GALERKIN METHOD

$$\Updownarrow$$

? $u^h \in \mathcal{U}^h$

$$J(u^h) = \min_{v^h \in \mathcal{U}^h} \underbrace{\frac{1}{2} a(v^h, v^h) - b(v^h)}_{J(v^h)}$$

RITZ METHOD

DISCRETE FORM.

DISCRETE FORMULATION = LINEAR SYSTEM !

$$J(u^h) = \frac{1}{2} < \underbrace{\sum_i U_i \underset{\sim}{\nabla} \tau_i}_{\underset{\sim}{\nabla} u^h} \cdot \sum_j U_j \underset{\sim}{\nabla} \tau_j > - < f \sum_i U_i \tau_i >$$

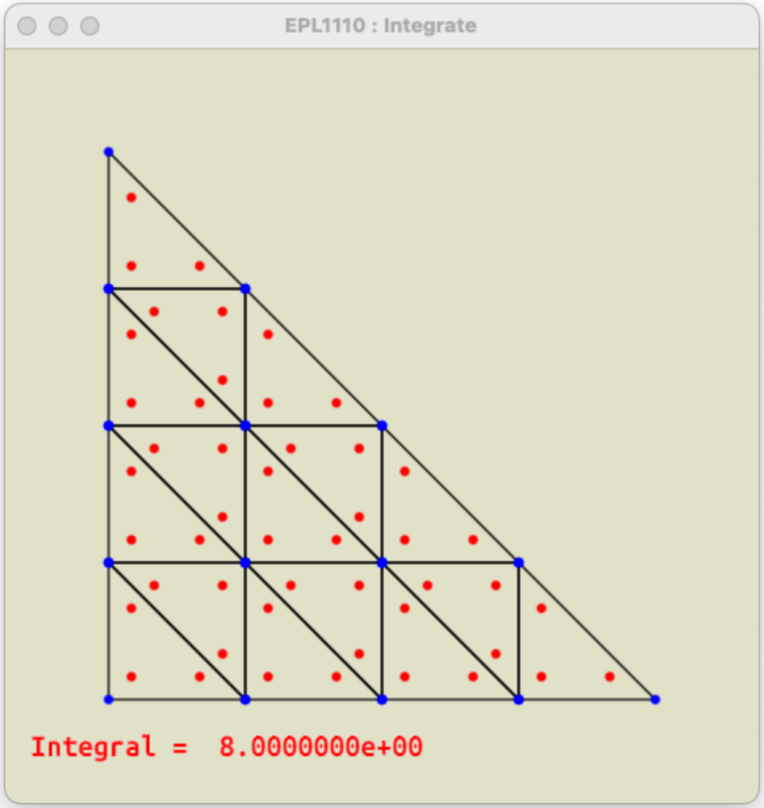
$$= \frac{1}{2} \sum_i \sum_j U_i U_j \underbrace{< \underset{\sim}{\nabla} \tau_i \cdot \underset{\sim}{\nabla} \tau_j >}_{A_{ij}} - \sum_i U_i \underbrace{< f \tau_i >}_{B_i}$$

$$0 = \underbrace{\frac{\partial J(u^h)}{\partial U_i}} \longrightarrow \boxed{\sum_{j=1}^{n} A_{ij} U_j = B_i}$$

ASSUME
$a = 1$
$g = 0$

Homework 1

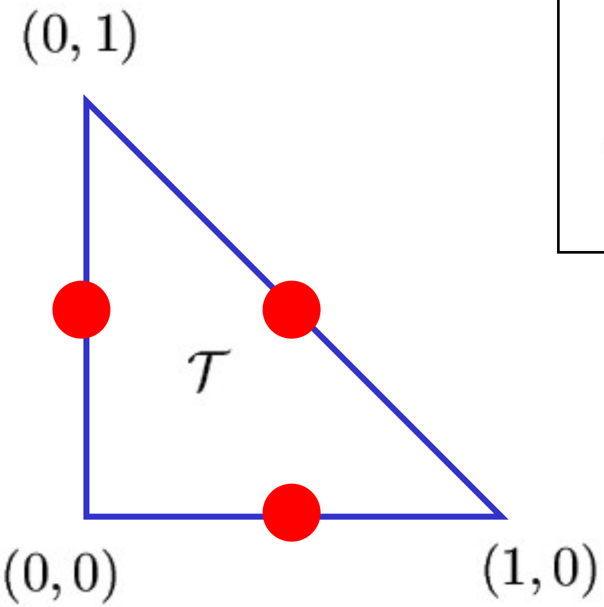
$$\underbrace{\int_{\widehat{\Omega}} f(x,y)\;dx\;dy}_{I} \approx \underbrace{\sum_{k=1}^{3} w_k\;f(x_k,y_k)}_{I_h}$$



Integral = 8.0000000e+00

Ecrire la règle de Hammer

Intégration sur un triangle : Règle de Hammer à 3 points



$$\underbrace{\int_{\mathcal{T}} f(x,y)\ dx\ dy}_{I} \approx \underbrace{\sum_{k=1}^{3} w_k f(X_k, Y_k)}_{I^h}$$

	X_k	Y_k	w_k
1	0.5	0.0	1/6
2	0.5	0.5	1/6
3	0.0	0.5	1/6

Démontrer que la formule de Hammer à trois points permet d'intégrer exactement n'importe quel polynôme à deux variables de degré deux : $a + bx + cy + dx^2 + ey^2 + fxy$

Question

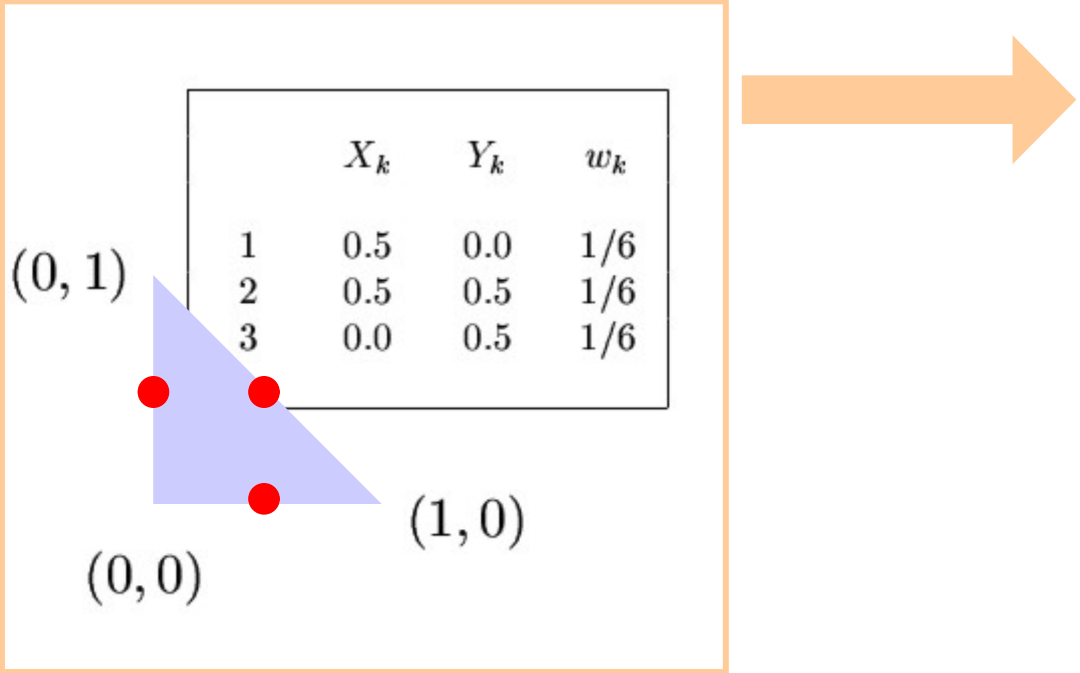
$$\begin{aligned}
I &= \frac{a}{2} + b\int_0^1 x \int_0^{1-x} dy\ dx + c\int_0^1 y \int_0^{1-y} dx\ dy \\
&\quad + d\int_0^1 x^2 \int_0^{1-x} dy\ dx + e\int_0^1 y^2 \int_0^{1-y} dx\ dy + f\int_0^1 x \int_0^{1-x} y\ dy\ dx \\
&= \frac{a}{2} + b\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 + c\left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1 \\
&\quad + d\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 + e\left[\frac{y^3}{3} - \frac{y^4}{4}\right]_0^1 + f\left[\frac{x^2}{4} - \frac{x^3}{3} + \frac{x^4}{8}\right]_0^1 \\
&= \frac{a}{2} + \frac{b}{6} + \frac{c}{6} + \frac{d}{12} + \frac{e}{12} + \frac{f}{24} \\
&= I^h
\end{aligned}$$

□

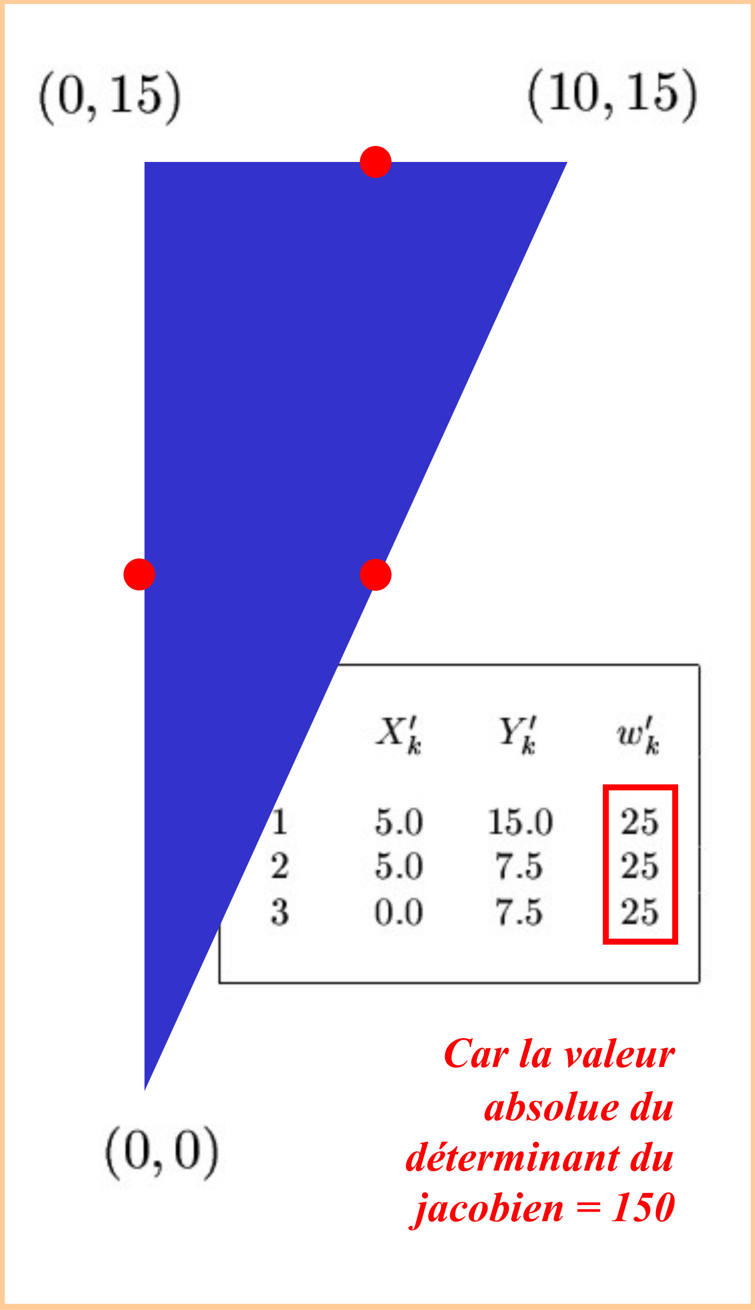
Degré de précision

Et un autre triangle ?

$$\begin{aligned} x' &= 10x \\ y' &= 15 - 15y \end{aligned}$$



	X_k	Y_k	w_k
1	0.5	0.0	1/6
2	0.5	0.5	1/6
3	0.0	0.5	1/6



	X'_k	Y'_k	w'_k
1	5.0	15.0	25
2	5.0	7.5	25
3	0.0	7.5	25

Car la valeur absolue du déterminant du jacobien = 150

Application to Finite Elements

Each triangle can be transformed in the parent element through a linear trasnformation.