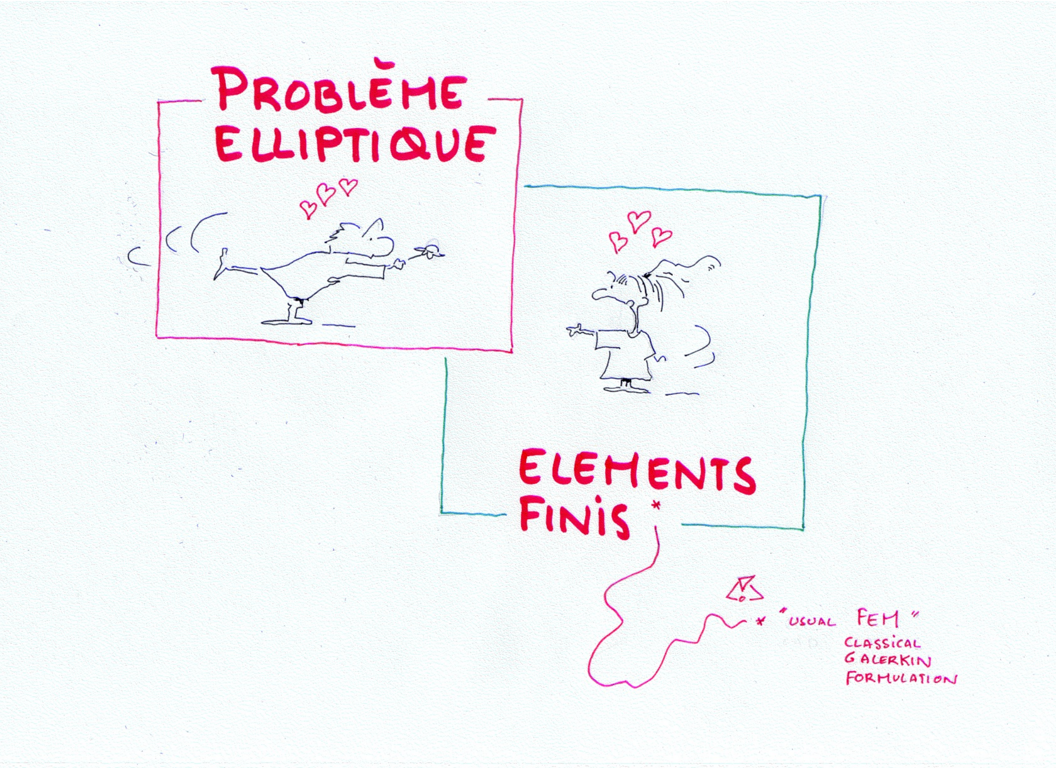
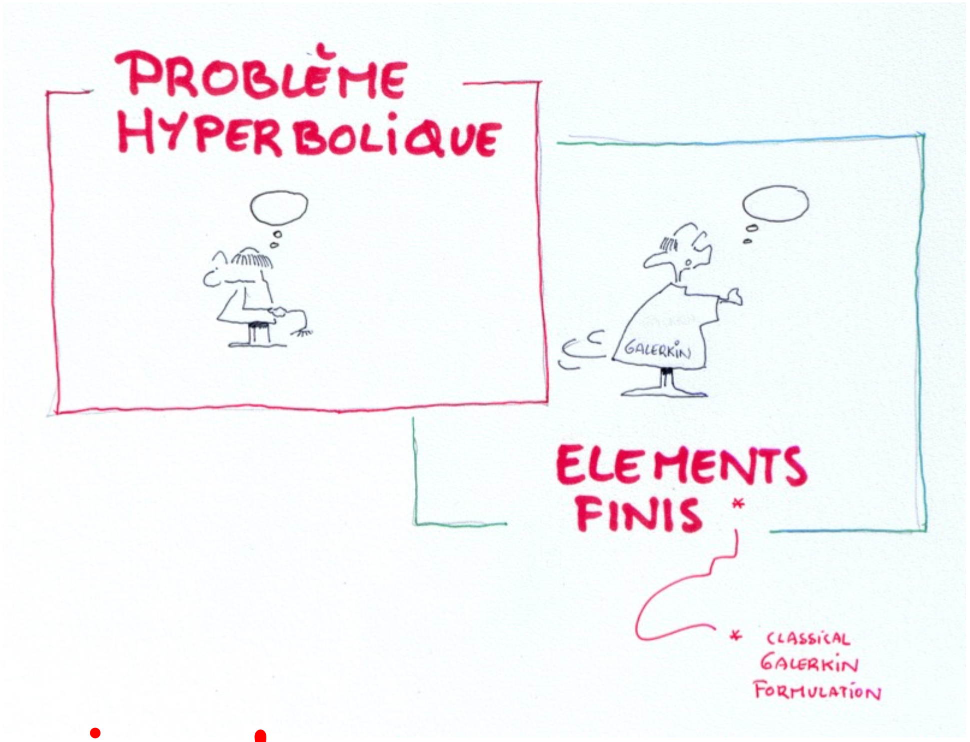
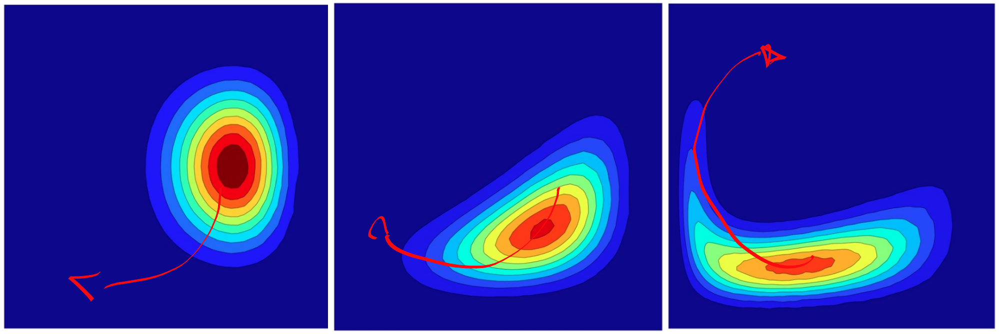


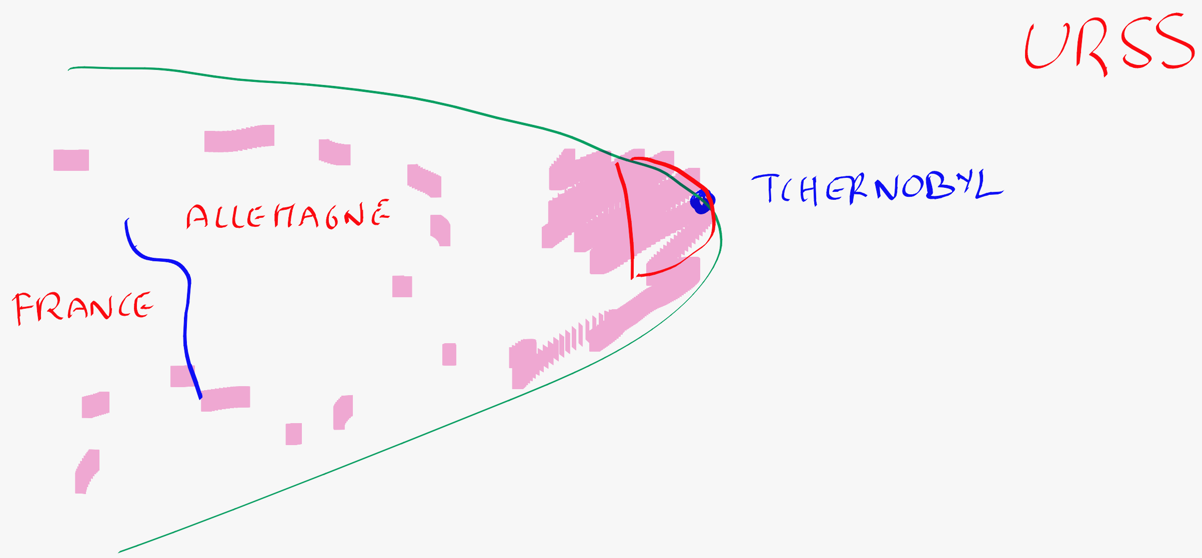
Galerkin, c’était donc optimal pour des équations elliptiques

Mais,
plus pour des
équations
d'advection diffusion !

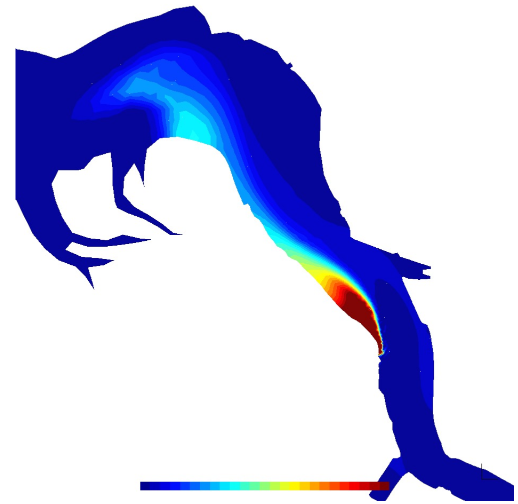


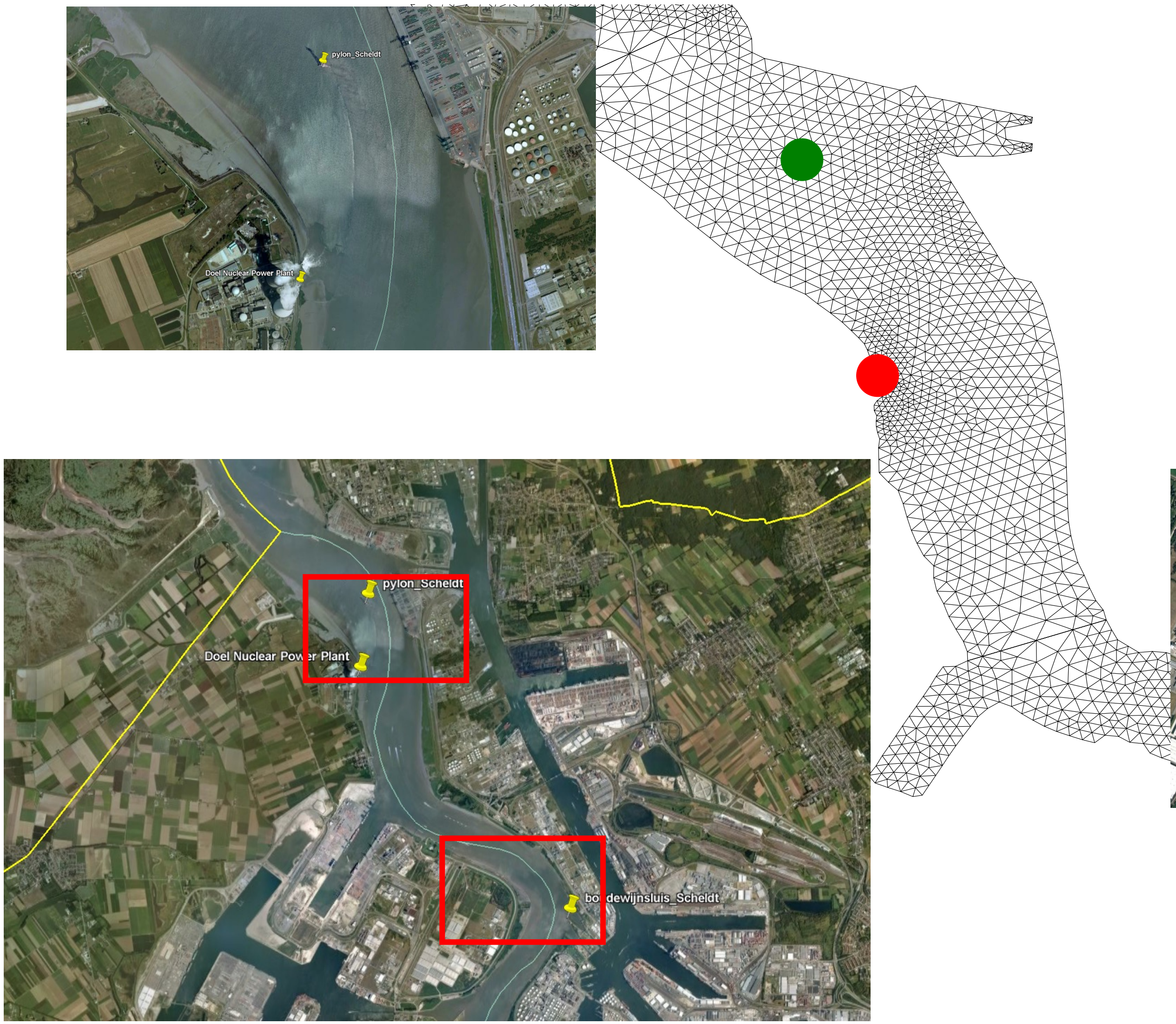


https://perso.uclouvain.be/vincent.legat/documents/epl1110/slides-2122/epl1110-cours7-advdiff.pdf



Un petit exemple concret





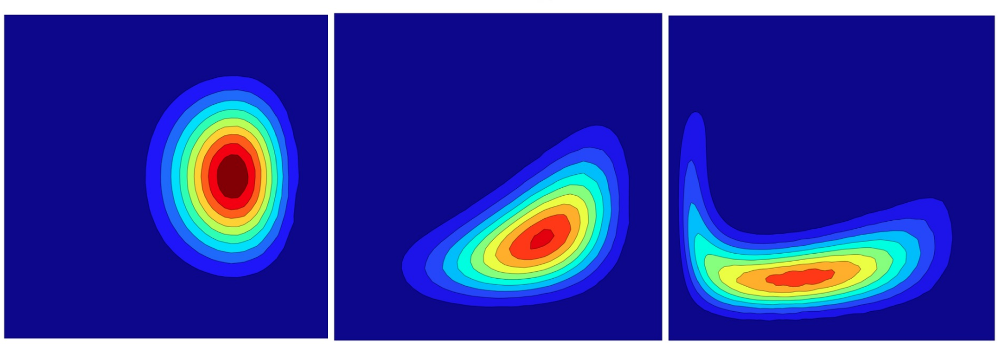
Diffusion et transport d'un traceur passif

MOYENNE DE LA CONCENTRATION SUR LA PROFONDEUR

COEFFICIENT DE DIFFUSION

$$\frac{\partial c}{\partial t} + \underbrace{\mathbf{v} \cdot \nabla c}_{\text{TERME DE TRANSPORT}} = \underbrace{\nabla \cdot (D \, \nabla c)}_{\text{TERME DE DIFFUSION}} + S$$

$\mathbf{v}$ = VITESSE HORIZONTALE

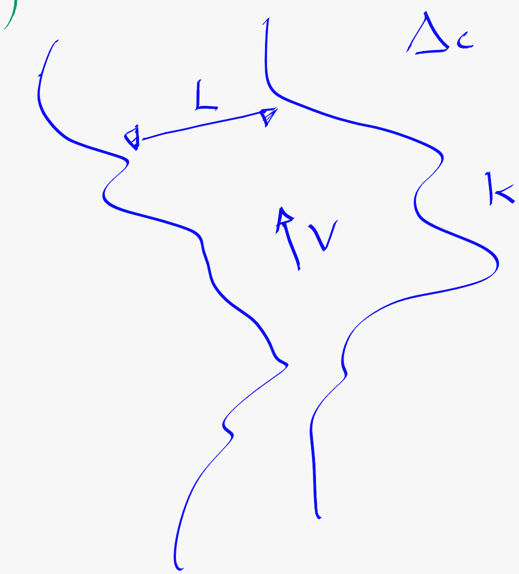


C'est une équation parabolique du second ordre !

Ce n'est pas elliptique !

$$\overbrace{\frac{\partial c}{\partial t} + \boxed{u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y}}}^{\frac{Dc}{Dt}} = \boxed{k\left[\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2}\right]} + f$$

$$\partial\left(\frac{V\Delta c}{L}\right) \qquad \partial\left(\frac{k\Delta c}{L^2}\right)$$



$$Pe = \qquad = \frac{V\cancel{\Delta c}}{\cancel{L}}\,\frac{L^{\cancel{2}}}{\cancel{\Delta c}\,k} = \underbrace{\frac{VL}{k}}_{\text{PECLET}}$$

$$\frac{\partial c}{\partial t} + v\frac{\partial c}{\partial x} = k\frac{\partial^2 c}{\partial x^2}$$

$$Pe = \frac{VL}{k}$$

	STATIONNAIRE	INSTATIONNAIRE
Pe PETIT	$\boxed{0 = k\frac{\partial^2 c}{\partial x^2}}$ ELLIPT [2]	$\boxed{\frac{\partial c}{\partial t} = k\frac{\partial^2 c}{\partial x^2}}$ PARA [2]
	ELLIPT [2] $\boxed{v\frac{\partial c}{\partial x} = k\frac{\partial^2 c}{\partial x^2}}$	$\boxed{\frac{\partial c}{\partial t} + v\frac{\partial c}{\partial x} = k\frac{\partial^2 c}{\partial x^2}}$ PARA [2]
Pe GRAND	$\boxed{v\frac{\partial c}{\partial x} = 0}$ HYP [1]	$\boxed{\frac{\partial c}{\partial t} + v\frac{\partial c}{\partial x} = 0}$ HYP [1]

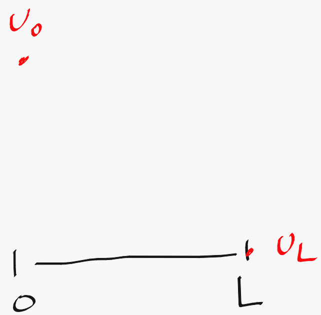
$$\frac{\partial c}{\partial t} + v\,\frac{\partial c}{\partial x} = k\,\frac{\partial^2 c}{\partial x^2}$$

	STATIONNAIRE		INSTATIONNAIRE	
P_e PETIT	$0 = k\,\frac{\partial^2 c}{\partial x^2}$	ELL 2	$\frac{\partial c}{\partial t} = k\,\frac{\partial^2 c}{\partial x^2}$	PARA 2
	$v\,\frac{\partial c}{\partial x} = k\,\frac{\partial^2 c}{\partial x^2}$	ELL 2	$\frac{\partial c}{\partial t} + v\,\frac{\partial c}{\partial x} = k\,\frac{\partial^2 c}{\partial x^2}$	PARA 2
P_e GRAND	$v\,\frac{\partial c}{\partial x} = 0$	HYP 1	$\frac{\partial c}{\partial t} + v\,\frac{\partial c}{\partial x} = 0$	HYP 1

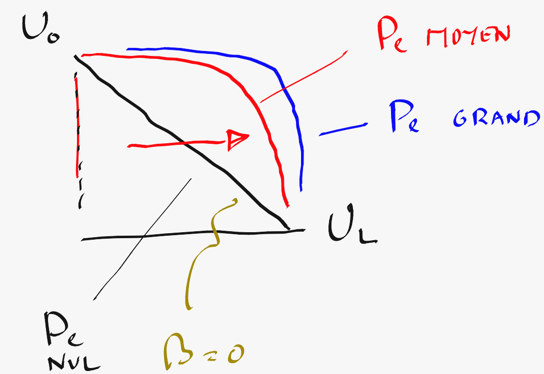
Le nombre de Péclet permet d’estimer l’importance du terme de transport par rapport à celui de la diffusion !

$$\boxed{P_e = \frac{vL}{k}}$$

$$\beta \frac{du}{dx} - \varepsilon \frac{d^2u}{dx^2} = 0 \qquad P_e = \frac{\beta L}{\varepsilon} \qquad \beta > 0$$



$$u(x) = C \exp\left[\frac{\beta x}{\varepsilon}\right]$$

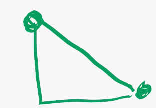


$$\varepsilon u'' = \frac{\beta^2}{\varepsilon^2} \varepsilon \exp[\,]\, C$$

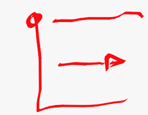
$$\beta u' = \beta \frac{\beta}{\varepsilon} \exp[\,]\, C$$

$$\boxed{\frac{u(x) - U_0}{U_L - U_0} = \frac{\exp\left[\frac{\beta x}{\varepsilon}\right] - 1}{\exp\left[\frac{\beta L}{\varepsilon}\right] - 1}}$$

$$\varepsilon \frac{d^2u}{dx^2} = 0$$



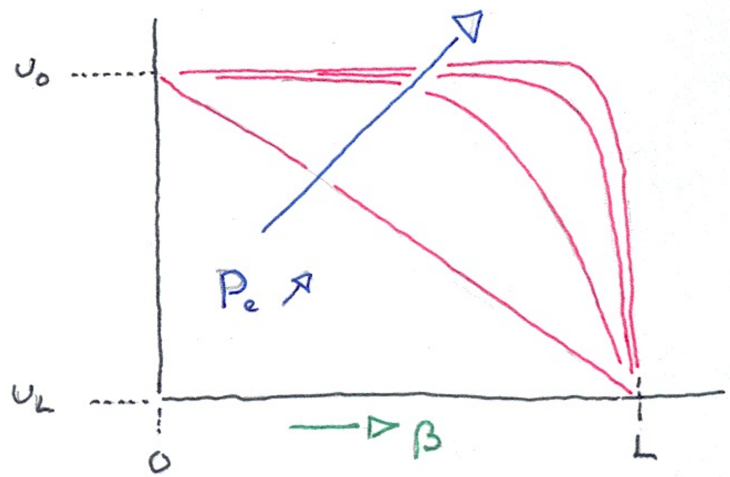
$$\beta \frac{du}{dx} = 0$$



EQUATION D'ADVECTION - DIFFUSION

$$\boxed{\begin{array}{l} \beta \dfrac{du}{dx} - \varepsilon \dfrac{d^2u}{dx^2} = 0 \\[2ex] u(0) = u_0 \\ u(L) = u_L \end{array}}$$

$$\boxed{Pe = \frac{\beta L}{\varepsilon}}$$



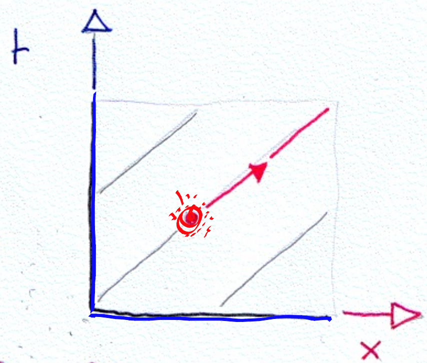
$$\frac{u - u_0}{u_L - u_0} = \frac{\exp\left(\overbrace{\beta x/\varepsilon}^{Pe\, x/L}\right) - 1}{\exp\left(\underbrace{\beta L/\varepsilon}_{Pe}\right) - 1}$$

PROBLEME BIEN POSE

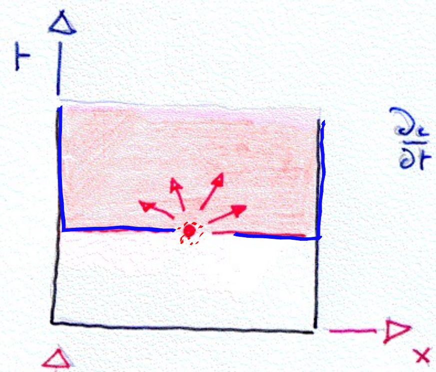
$$\cancel{\frac{\partial^2 c}{\partial t^2} + \frac{\partial^2 c}{\partial x^2} = f}$$

$$\frac{\partial^2 c}{\partial t^2} = a^2 \frac{\partial^2 c}{\partial x^2}$$

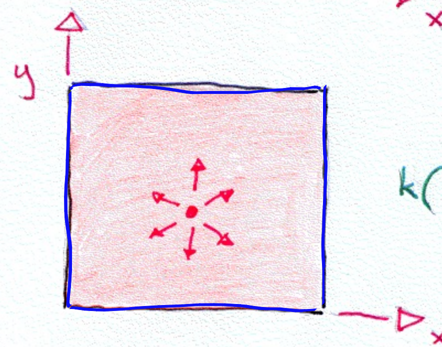
→ CONDITIONS AUX LIMITES



$$\frac{\partial c}{\partial t} + v \frac{\partial c}{\partial x} = 0$$

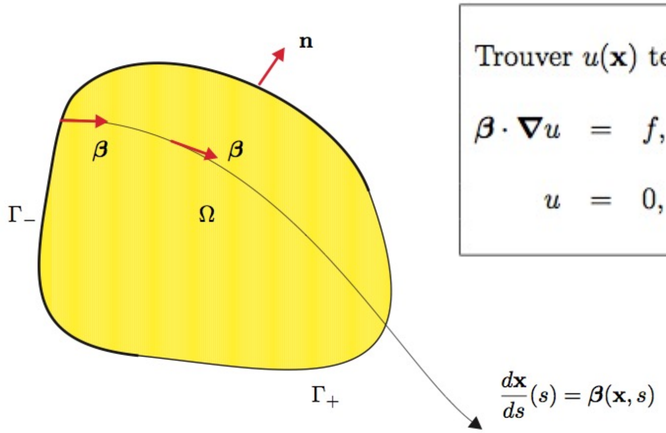


$$\frac{\partial c}{\partial t} = k \frac{\partial^2 c}{\partial x^2}$$



$$k\left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2}\right) = 0$$

Advection pure

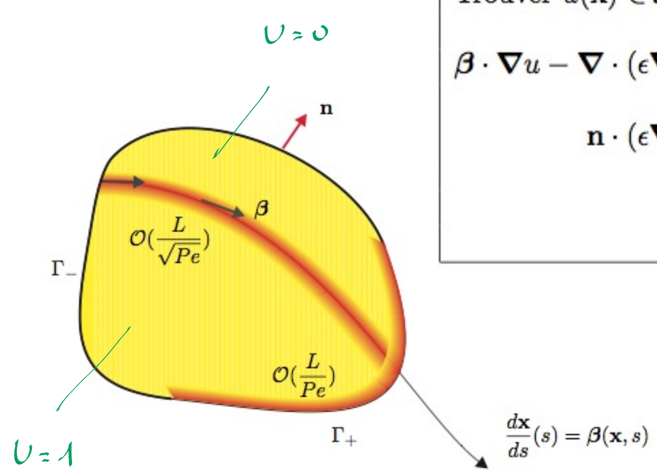


Trouver $u(\mathbf{x})$ tel que

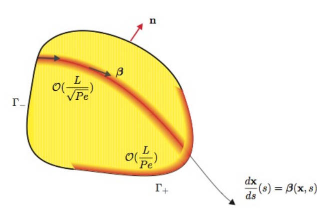
$$
\begin{aligned}
\boldsymbol{\beta} \cdot \boldsymbol{\nabla} u &= f, && \forall\, x \in \Omega, \\
u &= 0, && \forall\, x \in \Gamma_-
\end{aligned}
$$

Trouver $u(\mathbf{x}) \in \mathcal{U}_s$ tel que

$$
\begin{aligned}
\boldsymbol{\beta} \cdot \boldsymbol{\nabla} u - \boldsymbol{\nabla} \cdot (\epsilon \boldsymbol{\nabla} u) &= f, && \forall \mathbf{x} \in \Omega, \\
\mathbf{n} \cdot (\epsilon \boldsymbol{\nabla} u) &= g, && \forall \mathbf{x} \in \Gamma_N, \\
u &= t, && \forall \mathbf{x} \in \Gamma_D,
\end{aligned}
$$



Advection-diffusion



Trouver $u(\mathbf{x}) \in \mathcal{U}_s$ tel que

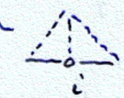
$$
\begin{aligned}
\boldsymbol{\beta} \cdot \boldsymbol{\nabla} u - \boldsymbol{\nabla} \cdot (\epsilon \boldsymbol{\nabla} u) &= f, && \forall \mathbf{x} \in \Omega, \\
\mathbf{n} \cdot (\epsilon \boldsymbol{\nabla} u) &= g, && \forall \mathbf{x} \in \Gamma_N, \\
u &= t, && \forall \mathbf{x} \in \Gamma_D,
\end{aligned}
$$

Advection-diffusion

EQUATION DE TRANSPORT

$$\boxed{\begin{aligned} \frac{du}{dx} &= f \\ u(0) &= u_0 \end{aligned}}$$

$$u \simeq u^h = \sum_i U_i \, \underbrace{\tau_i(x)}$$



GALERKIN

? U_i TELS QUE

$$\langle \tau_i \; r_h \rangle = 0$$

$$\sum_j \overbrace{\langle \tau_i \; \tau_{j,x} \rangle}^{A_{ij}} U_j = \overbrace{\langle \tau_i \; f \rangle}^{B_i}$$

≠ MATRICE DEFINIE POSITIVE !

UN
PETIT
RAPPEL
SYMPATHIQUE

LES
ELEMENTS
FINIS SONT
UNE METHODE
VARIATIONNELLE

$$u'' + f = 0$$

$$u(0) = u(1) = 0$$

LES
ELEMENTS
FINIS SONT
UNE METHODE
DE RESIDUS
PONDERES

$$u \simeq u^h = \sum U_i \tau_i$$

$$\underbrace{(u^h)'' + f}_{r^h} \simeq 0 \quad \Rightarrow \quad \langle r^h \, \tau_i \rangle = 0 \quad \forall i$$

$$\langle \sum U_j \tau_j'' \, \tau_i \rangle + \langle f \, \tau_i \rangle = 0$$

$$\sum U_j \underbrace{\langle \tau_j'' \, \tau_i \rangle}_{\underbrace{\left[- \langle \tau_j' \, \tau_i' \rangle \right]}_{A_{ij}}} + \underbrace{\langle f \, \tau_i \rangle}_{B_i} = 0$$

$$\sum_j A_{ij} U_j = B_i$$

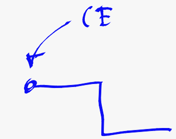
TRANSPORT
PUR

$$\sum U_j \langle \tau_j' \, \tau_i \rangle = \langle f \, \tau_i \rangle$$

N'EST
PAS DEFINIE
POSITIVE

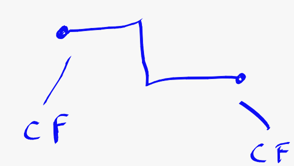
$$\frac{U_{j+1} - U_{j-1}}{2h} = F_j$$

$$\sum U_j \, \underbrace{\langle \tau_j' \; \tau_i' \rangle}_{\text{MATRICE DEFINIE POSITIVE}} = \langle f \; \tau_i' \rangle$$



$$\frac{du}{dx} = f$$

$$\frac{d^2u}{dx^2} = \frac{df}{dx}$$



$$\frac{U_{j-1} - 2U_j + U_{j+1}}{h^2} = \frac{F_j - F_{j-1}}{h}$$

LA BONNE SOLUTION

$$\frac{U_j - U_{j-1}}{h} = F_{j-1}$$

$$\boxed{\frac{du}{dx} = f}$$

GALERKIN

$$\boxed{\sum_j A_{ij} U_j = B_i}$$

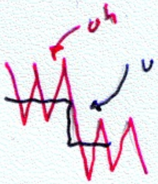
BUBNOV - GALERKIN

DIFFERENCES FINIES CENTREES $\quad \dfrac{U_{j+1} - U_{j-1}}{2\Delta x} = B_j$

$\not\Leftrightarrow$ **PROBLEME DE MINIMISATION**

$f(x) = \sin(x)$ ⟶ OK

$f(x) =$ ⟶ "Boum"



PETROV-GALERKIN

? U_j TELS QUE

$$\langle \tau_{j,x} \; r^h \rangle = 0$$

INTEGRATION LE LONG DES CARACTERISTIQUES

DIFFERENCES FINIES AMONT

$$\frac{U_{j+1} - U_j}{\Delta x} = B_j$$

$$\sum_j \underbrace{\langle \tau_{i,x} \; \tau_{j,x} \rangle}_{A_{ij}} U_j = \underbrace{\langle \tau_{i,x} \; f \rangle}_{B_i}$$

MATRICE DEFINIE POSITIVE

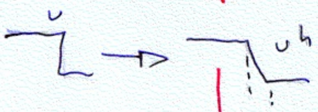
1 CONDITION AUX LIMITES !

$$\frac{du}{dx} = f$$

A ÉTÉ REMPLACÉ PAR $\frac{d^2u}{dx^2} = \frac{df}{dx}$

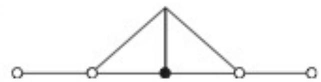
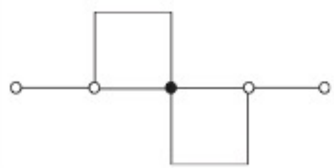
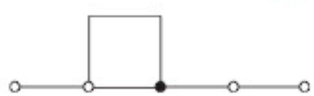
2 CONDITIONS AUX LIMITES !

HIC



En bref :-)

$$\boxed{\begin{aligned} \frac{du}{dx} &= f, \\ u(0) &= 0, \end{aligned}}$$

Galerkin $w_i = \tau_i$ 	*Différences finies centrées* Simple et donc tentant... Oscillations numériques si f n'est pas lisse ! $\frac{U_{i+1} - U_{i-1}}{2h} = \frac{F_{i+1} + 4F_i + F_{i-1}}{6},$
Petrov-Galerkin $w_i = \tau_{i,x}$ 	*Différences finies centrées d'ordre deux* Mathématiquement, tentant Condition frontière parasite ! $\frac{U_{i+1} - 2Ui + U_{i-1}}{h^2} = \frac{F_{i+1} - F_{i-1}}{2h},$
Petrov-Galerkin $w_i = \tau_{i-1}^{cst}$ 	*Différences finies amont* Quasiment optimal... Correspond à une intégration le long de la caractéristique, Pas d'oscillation numérique $\frac{U_i - U_{i-1}}{h} = \frac{F_{i+1} + F_{i-1}}{2},$

PETROV
GALERKIN

$$u' - \varepsilon u'' - f = 0 \qquad u^h = \sum U_i \tau_i$$

$$\langle (\tau_i + \zeta \tau_{i,x})(u^h_{,x} - \varepsilon u^h_{,xx} - f) \rangle = 0 \qquad \forall i$$

$$\sum U_j \Big[\underbrace{\zeta \langle \tau_{i,x} \tau_{j,x} \rangle + \varepsilon \langle \tau_{i,x} \tau_{j,x} \rangle}_{\text{MATRICE DEFINIE POSITIVE}} \; \underbrace{- \varepsilon \zeta \langle \tau_{i,x} \tau_{j,xx} \rangle + \langle \tau_i \tau_{j,x} \rangle}_{\text{LE COTE OBSCUR}} \Big] = \dots \qquad \forall i$$

$\zeta \nearrow$

COTE POSITIF $\nearrow$

COTE NEGATIF $\nearrow$

u^h EST LINEAIRE .... ALORS $\tau_{i,xx} = 0$

PETROV-GALERKIN

$$\langle (\underbrace{\tau_i + \zeta\, \tau_{i,x}}_{\hat{\tau}_i}) (\overbrace{u^h_{,x} - \varepsilon\, u^h_{,xx} - f}^{r^h}) \rangle = 0$$

HOW TO SELECT ζ ?

2D

$$\hat{\tau}_i = \tau_i + \zeta\, \underline{\beta} \cdot \underline{\nabla} \tau_i$$

STREAMLINE UPWINDING

$$\sum_j \Big(\zeta \langle \tau_{i,x}\, \tau_{j,x} \rangle \quad \varepsilon \langle \tau_{i,x}\, \tau_{j,x} \rangle \; - \; \varepsilon \zeta \langle \tau_{i,x}\, \tau_{j,xx} \rangle \; + \; \langle \tau_i\, \tau_{j,x} \rangle \Big) U_j = \dots$$

DIFF. CENTRÉES

$$\beta\,\frac{U_{i+1}-U_{i-1}}{2h} = \varepsilon\,\frac{U_{i+1}-2U_i+U_{i-1}}{h^2}$$

$U_0 = u_0$

$U_N = u_L$

EQUATION AUX RECURENCES !

$$U_i = A r^i + B$$

$$\cancel{A r^{i-1}}\,\underbrace{\frac{\beta h}{2\varepsilon}}\,(r^2-1) = \cancel{A r^{i-1}}\,(r^2-2r+1)$$

$$\triangleq \frac{Pe^h}{2}$$

PECLET DE MAILLE

$$0 = \left(\frac{1-Pe^h/2}{2}\right) r^2 - r + \left(\frac{1+Pe^h/2}{2}\right)$$

$$r = \frac{1 \pm \sqrt{1-(1+Pe^h/2)(1-Pe^h/2)}}{(1-Pe^h/2)}$$

$$r = \frac{1+Pe^h/2}{1-Pe^h/2}$$

REJECT $r = 1$ OF COURSE !

$$\frac{U_i - u_0}{u_L - u_0} = \frac{\left(\frac{1+P_e h/2}{1-P_e h/2}\right)^i - 1}{\underbrace{\left(\frac{1+P_e h/2}{1-P_e h/2}\right)^N} - 1}$$

$$\simeq \exp\left(P_e h\right)^N$$

$$\simeq \exp\left(\underbrace{\frac{\beta h N}{\varepsilon}}_{P_e}\right)$$

[SCHAUM p 551]

$$\frac{x}{e^x - 1} \simeq 1 - x/2$$

$$(1 + x/2) = e^x (1 - x/2)$$

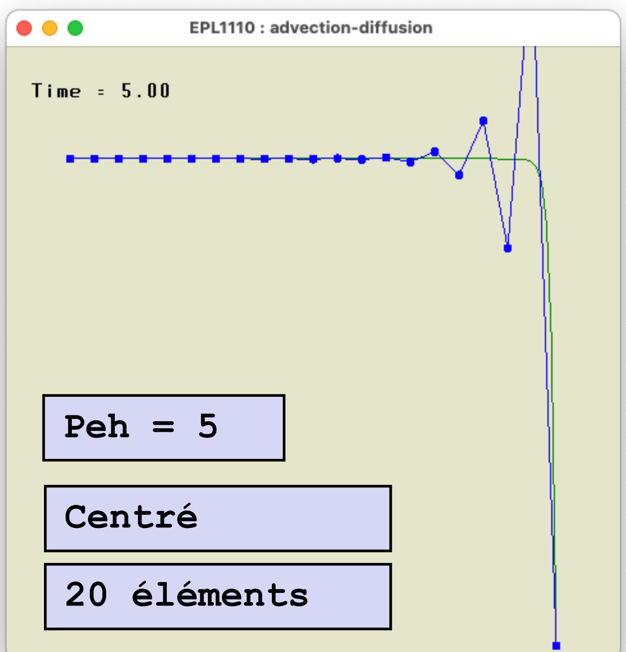
$$\frac{P_e h}{2} < 1$$

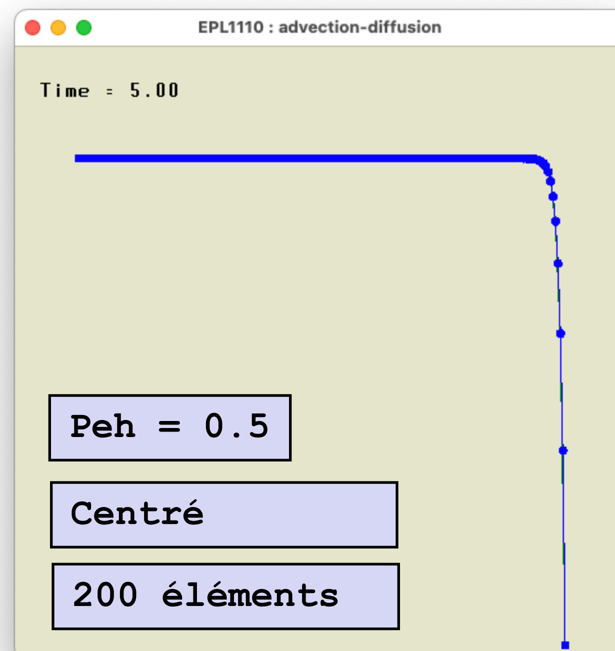
TO AVOID OSCILLATORY BEHAVIOUR

Galerkin converge si on raffine le maillage suffisament !

```
epsilon = 0.01;
beta = 1.0;
```

L'équation d'advection-diffusion est formellement une équation elliptique et donc c'était prévisible par la théorie !





UPWIND DIFF.

$$\boxed{\begin{array}{l} \beta \dfrac{U_i - U_{i-1}}{h} = \varepsilon \dfrac{U_{i+1} - 2U_i + U_{i-1}}{h^2} \\ U_0 = u_0 \\ U_N = u_L \end{array}}$$

$$\cancel{A r^{i-1}} \underbrace{\frac{\beta h}{\varepsilon}}_{\boxed{\triangleq Pe^h}} (r-1) = \cancel{A r^{i-1}} (r^2 - 2r + 1)$$

$$0 = \frac{r^2}{2} - (1 + Pe^h/2)\, r + \frac{(1 + Pe^h)}{2}$$

$$r = (1 + Pe^h/2) \pm \sqrt{\underbrace{(1 + Pe^h/2)^2 - (1 + Pe^h)}_{Pe^h/2}}$$

$$= 1 + Pe^h$$

$$\frac{U_i - U_0}{U_L - U_0} = \frac{(1+P_e^h)^i - 1}{(1+P_e^h)^N - 1}$$

$$\boxed{(1+P_e^h) > 0 \quad \forall h}$$

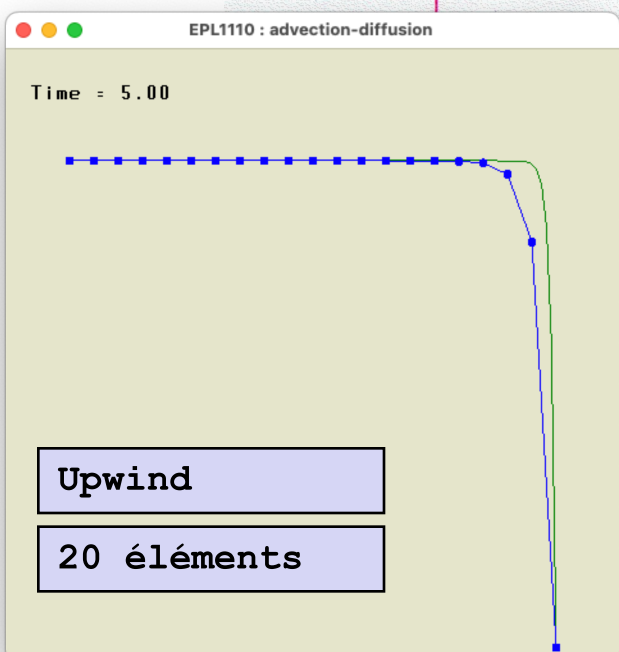
NO OSCILLATIONS

BUT...

NUMERICAL DIFFUSION

$$\beta \frac{U_i - U_{i-1}}{h} = \varepsilon \frac{U_{i+1} - 2U_i + U_{i-1}}{h^2}$$

$$\beta \frac{U_{i+1} - U_{i-1}}{2h} - \underbrace{\frac{\beta h}{2}}_{\text{NUMERICAL DIFFUSIVITY}} \frac{U_{i+1} - 2U_i + U_{i-1}}{h^2}$$



HYBRID SCHEME

$$\boxed{(1-\zeta)\beta\,\frac{U_{i+1}-U_{i-1}}{2h} + \zeta\beta\,\frac{U_i-U_{i-1}}{2h} = \varepsilon\,\frac{U_{i+1}-2U_i+U_{i-1}}{h^2}}$$

$$0 = \left(1-\frac{(1-\zeta)P_e^h/2}{2}\right)r^2 - \left(1+\zeta P_e^h/2\right)r + \left(1+\frac{\overbrace{(1-\zeta)P_e^h/2 + \zeta P_e^h}^{(1+\zeta)P_e^h/2}}{2}\right)$$

$$r = \frac{\left(1+\zeta P_e^h/2\right) \pm \sqrt{\left.\begin{matrix}(1+\zeta P_e^h/2)^2 \\ -(1-(1-\zeta)P_e^h/2)\cdot \\ (1+(1+\zeta)P_e^h/2)\end{matrix}\right\} = P_e^h/2}}{\left(1-(1-\zeta)P_e^h/2\right)}$$

$$\boxed{\pi = \frac{1 + (1+\zeta)\,Pe^h/2}{1 - (1-\zeta)\,Pe^h/2}}$$

$$\boxed{\text{HOW TO SELECT } \zeta\,?}$$

$$U_i = u(ih)$$

$$\downarrow$$

$$\underbrace{\left(\frac{1+(1+\zeta)\,Pe^h/2}{1-(1-\zeta)\,Pe^h/2}\right)^i}_{\pi^i} = \underbrace{\exp\left(\frac{\beta h}{\varepsilon}\, i\right)}_{\left(\exp\left(Pe^h\right)\right)^i}$$

$$\downarrow$$

$$\left(1 + Pe^h/2\right) - \zeta\, Pe^h/2 = \exp(Pe)\left(\left(1 - Pe^h/2\right) + \zeta\, Pe^h/2\right)$$

$$\zeta \, \frac{P_e^h}{2} \left(1 - \exp(P_e) \right) = \exp(P_e^h) \left(1 - \frac{P_e^h}{2} \right) - \left(1 + \frac{P_e^h}{2} \right)$$

$$\zeta = \frac{\exp(P_e^h) \left(\frac{2}{P_e^h} - 1 \right) - \left(\frac{2}{P_e^h} + 1 \right)}{\left(1 - \exp(P_e^h) \right)}$$

$$= - \underbrace{\frac{\left(1 + \exp(P_e^h) \right)}{\left(1 - \exp(P_e^h) \right)}}_{\coth(P_e^h/2)} - \frac{2}{P_e^h}$$

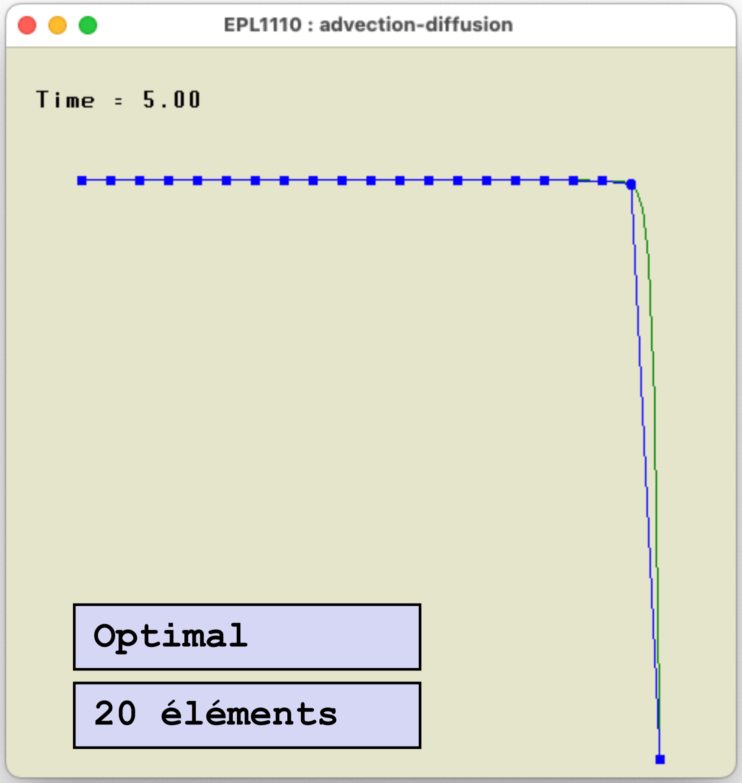
$$\boxed{\zeta = \coth\left(\frac{P_e^h}{2} \right) - \frac{2}{P_e^h}}$$

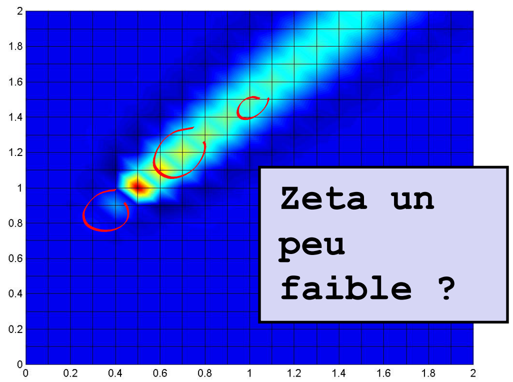
$\square$

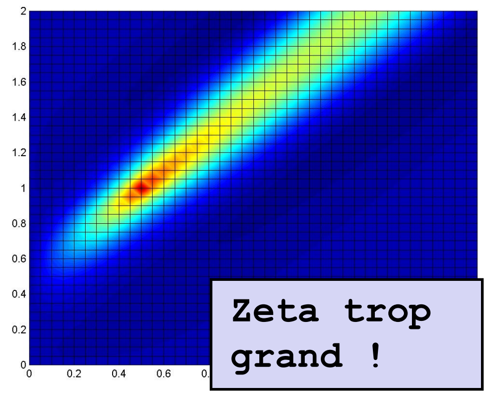
On a trouvé
la méthode parfaite
pour des équations
unidimensionnelles !

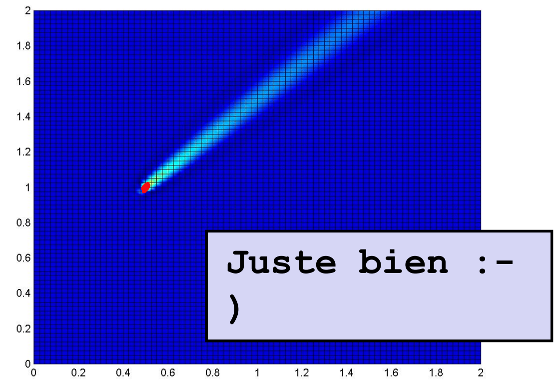
$$\begin{aligned}\beta\frac{du}{dx} - \epsilon\frac{d^2u}{dx^2} &= 0,\\ u(0) &= u_0,\\ u(L) &= u_L,\end{aligned}$$

$$\boxed{\zeta = \coth(\frac{Pe^h}{2}) - \frac{2}{Pe^h}}$$









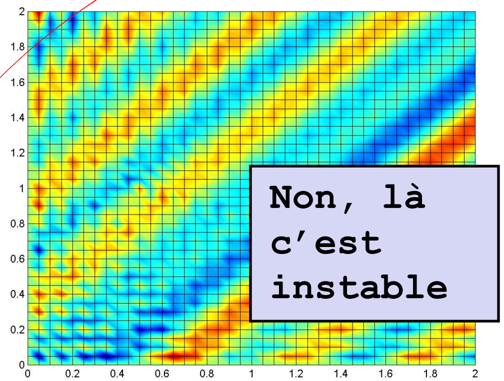
En extrapolant aux dimensions supérieures…

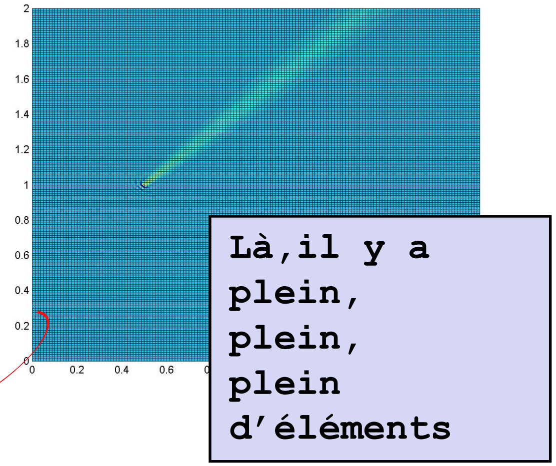
$$w_i = \tau_i + \zeta \boldsymbol{\beta} \cdot \nabla \tau_i$$

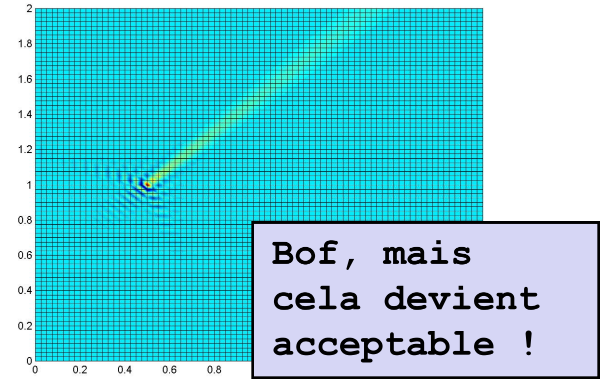
Facteur de stabilisation
Trop grand : diffusion numérique !
Trop petit : instable !

Trouver $U_j \in \mathbb{R}^n$ tel que

$$\sum_{j=1}^{n} \underbrace{< w_i \; \boldsymbol{\beta} \cdot \nabla \tau_j + \epsilon \nabla w_i \cdot \nabla \tau_j >}_{A_{ij}} U_j \quad = \quad \underbrace{< w_i f > + \ll w_i g \gg_N}_{B_i} , \qquad i = 1, \ldots, n,$$







Et en payant le prix, Galerkin fonctionne !

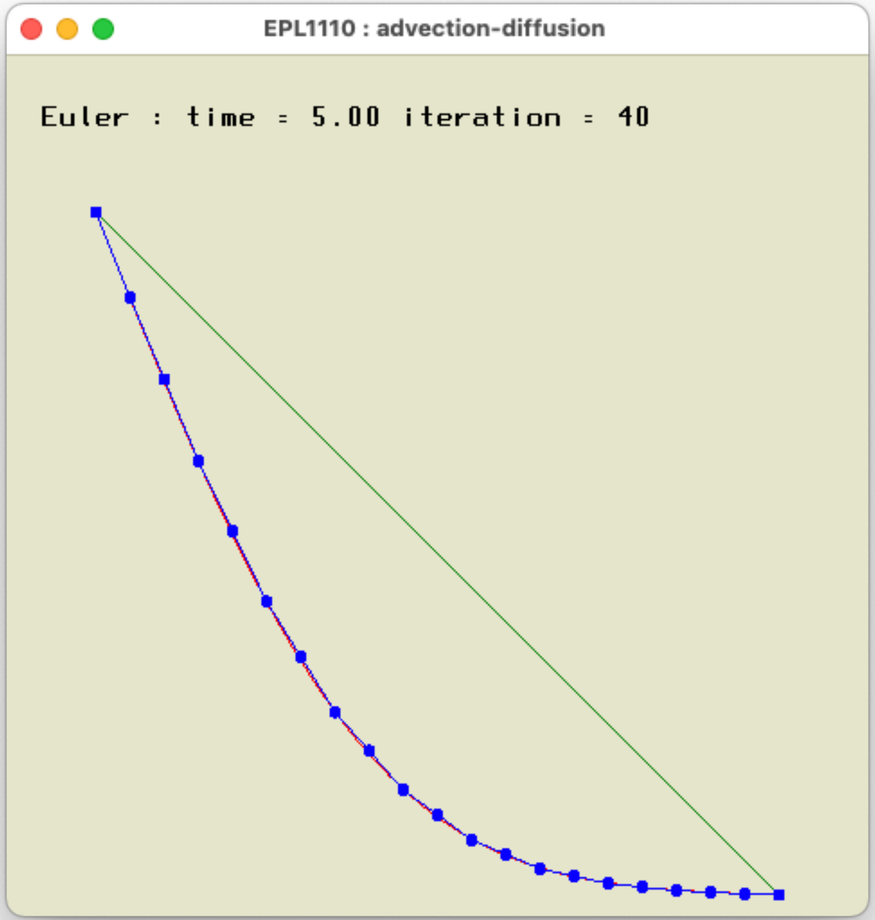
$$w_i = \tau_i + \zeta \boldsymbol{\beta} \cdot \boldsymbol{\nabla} \tau_i$$

Pas de stabilisation !
Zeta = 0 !

Trouver $U_j \in \mathbb{R}^n$ tel que

$$\sum_{j=1}^{n} \underbrace{< w_i \; \boldsymbol{\beta} \cdot \boldsymbol{\nabla} \tau_j + \epsilon \boldsymbol{\nabla} w_i \cdot \boldsymbol{\nabla} \tau_j >}_{A_{ij}} U_j \quad = \quad \underbrace{< w_i f > + \ll w_i g \gg_N}_{B_i} , \qquad i = 1, \ldots, n,$$

Et maintenant introduisons le temps…



$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \epsilon \frac{\partial^2 u}{\partial x^2} \\
u(0) &= 1 \\
u(1) &= 0
\end{aligned}
$$

```
epsilon = 0.01;
L = 1
```

Différences finies (espace) Euler explicite (temps)

$$\left(\frac{U_i^{n+1} - U_i^n}{\Delta t}\right) = \epsilon \left(\frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{(\Delta x)^2}\right)$$

En définissant $b = \dfrac{\epsilon \Delta t}{(\Delta x)^2}$,

$$\boxed{U_i^{n+1} = U_i^n + b\Big(U_{i+1}^n + U_{i-1}^n - 2U_i^n\Big)}$$

C'est une itération pour un vecteur qui doit converger vers la solution de régime
C'est quelque chose qu'on a déjà rencontré…

On intègre un système linéaire…

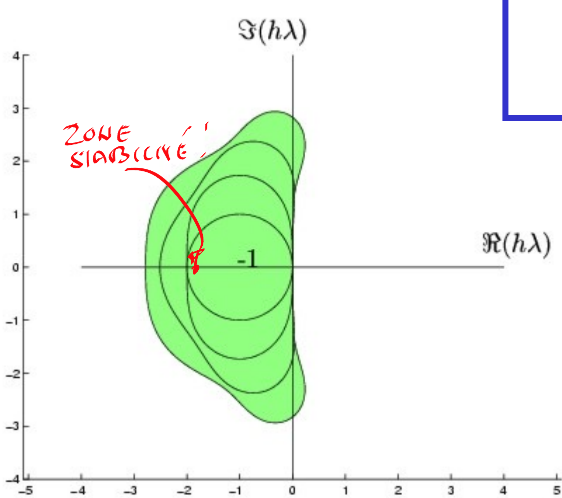
$$U_i^{n+1} = U_i^n + b\left(U_{i+1}^n + U_{i-1}^n - 2U_i^n\right)$$

En passant à une notation matricielle,

$$\begin{bmatrix} U_1^{n+1} \\ U_2^{n+1} \\ U_3^{n+1} \\ U_4^{n+1} \\ U_5^{n+1} \\ \\ \\ U_m^{n+1} \end{bmatrix} = \begin{bmatrix} U_1^{n} \\ U_2^{n} \\ U_3^{n} \\ U_4^{n} \\ U_5^{n} \\ \\ \\ U_m^{n} \end{bmatrix} + b \begin{bmatrix} -2 & 1 & & & & & & \\ 1 & -2 & 1 & & & & & \\ & 1 & -2 & 1 & & & & \\ & & 1 & -2 & 1 & & & \\ & & & 1 & -2 & 1 & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & 1 & -2 \end{bmatrix} \begin{bmatrix} U_1^{n} \\ U_2^{n} \\ U_3^{n} \\ U_4^{n} \\ U_5^{n} \\ \\ \\ U_m^{n} \end{bmatrix}$$

En définissant adéquatement u_n et A,

$$\mathbf{u}_{n+1} = \mathbf{u}_n + b\mathbf{A}\mathbf{u}_n$$



On résout le système linéaire défini par :

$$\mathbf{u}'(t) = \frac{\epsilon}{(\Delta x)^2}\mathbf{A}\mathbf{u}(t)$$

$$\Delta x = 0.1, \Delta t = 0.005$$

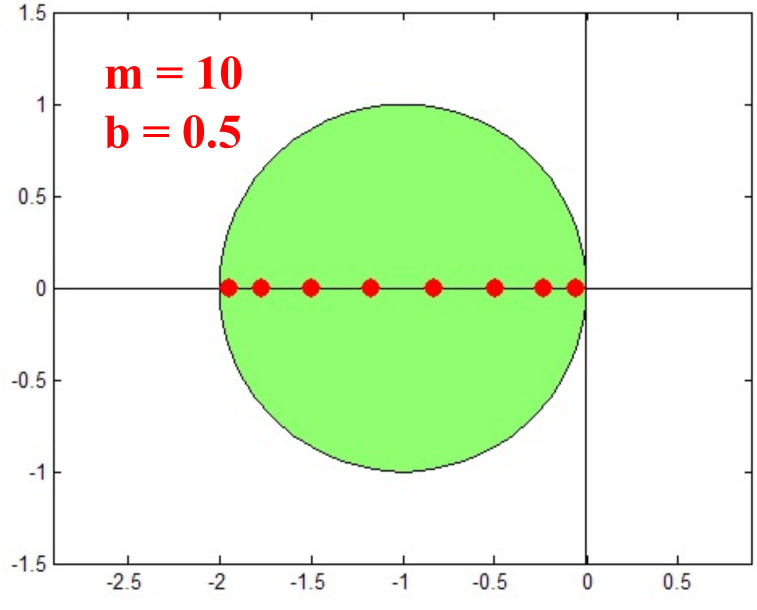
Euler explicite

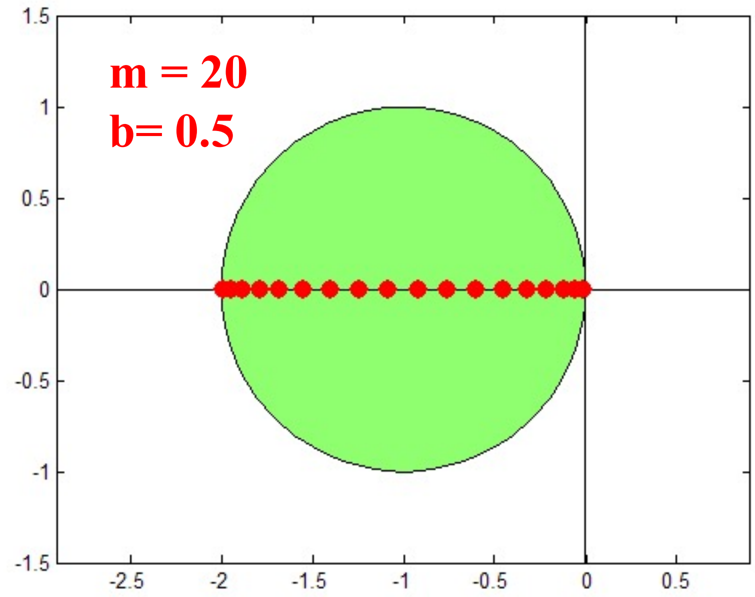
$$\mathbf{u}_{n+1} = \mathbf{u}_n + \underbrace{\frac{\epsilon \Delta t}{(\Delta x)^2}}_{b} \mathbf{A}\mathbf{u}_n$$

$$|1 + b\lambda_i| \leq 1$$

Valeurs propres de A

$$\Delta x = 0.05, \Delta t = 0.00125$$





Condition de stabilité pour la méthode d'Euler explicite….

$$\beta = \frac{\epsilon \Delta t}{(\Delta x)^2} \leq \frac{1}{2}$$

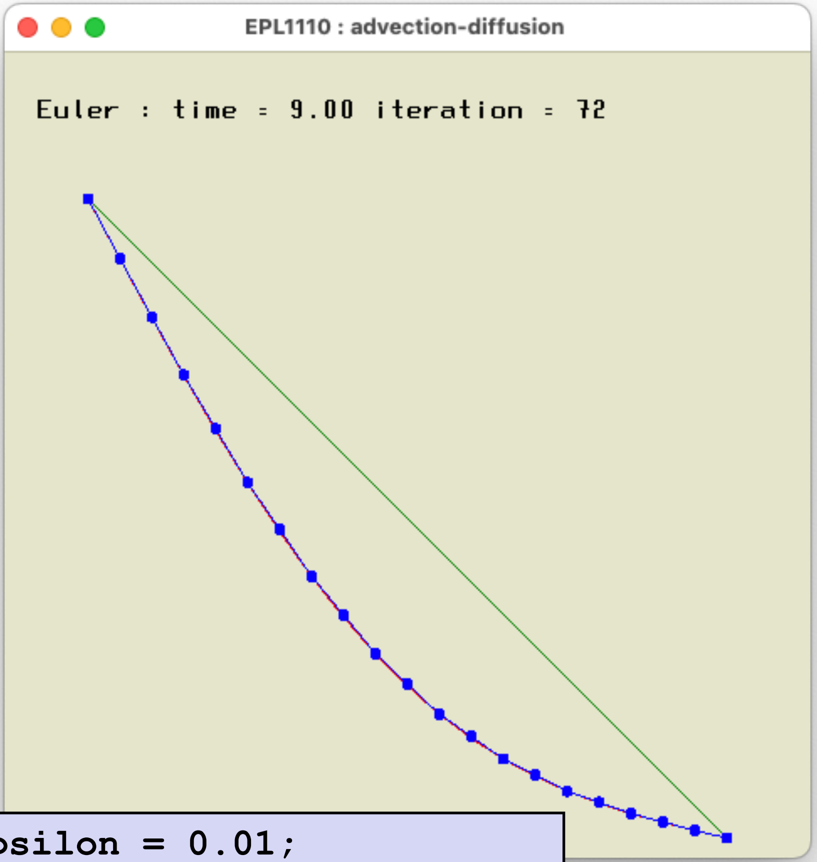
:-)

$$\boxed{\Delta t \leq \frac{(\Delta x)^2}{2\epsilon}}$$

Courant, Friedrichs et Lewy (1928)


```
epsilon = 0.01;
dt = h*h/(epsilon*1.94);
```

Condition de stabilité pour la méthode d'Euler explicite….



$$\beta \;=\; \frac{\epsilon \Delta t}{(\Delta x)^2} \;\leq\; \frac{1}{2}$$

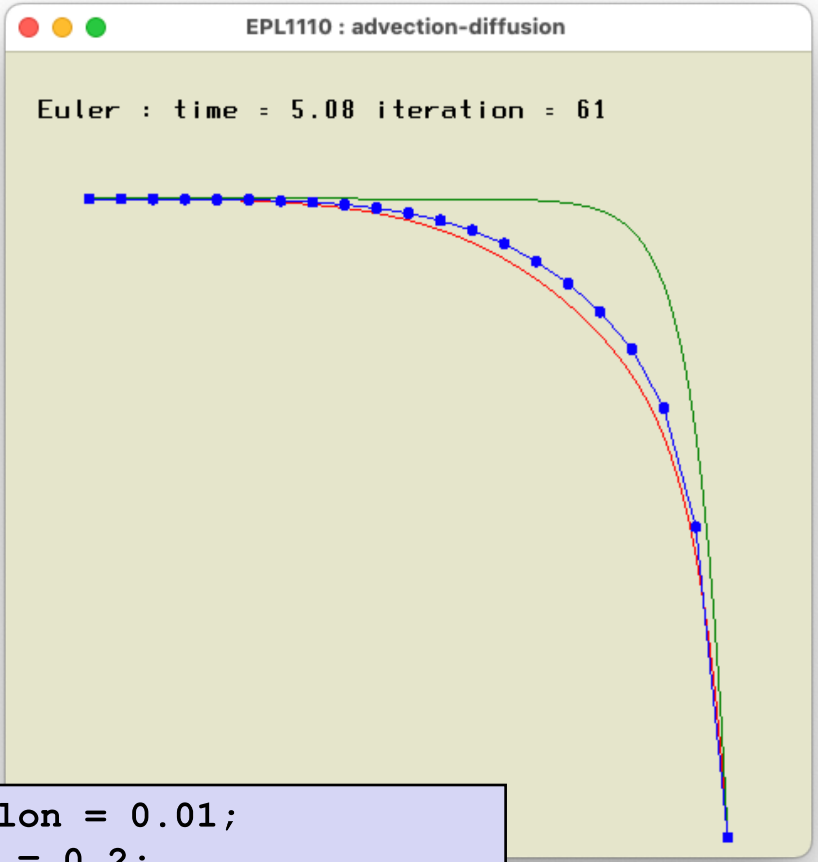
$$\boxed{\Delta t \;\leq\; \frac{(\Delta x)^2}{2\epsilon}}$$

Courant, Friedrichs et Lewy (1928)

```
epsilon = 0.01;
dt = h*h/(epsilon*2.0);
```

Et maintenant introduisons l'advection…

$$
\begin{aligned}
\frac{\partial u}{\partial t} + \beta \frac{\partial u}{\partial x} &= \epsilon \frac{\partial^2 u}{\partial x^2} \\
u(0) &= 1 \\
u(1) &= 0
\end{aligned}
$$

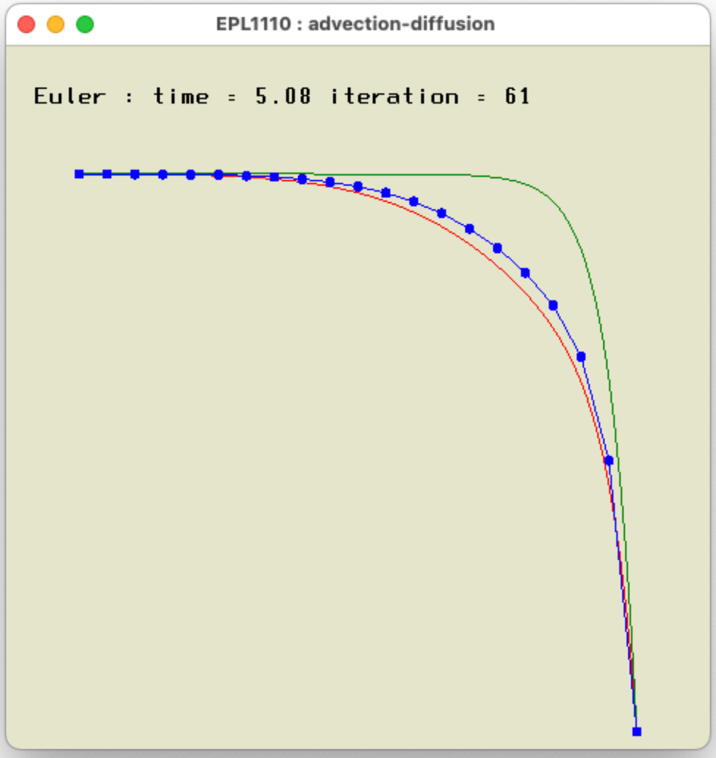



```
epsilon = 0.01;
beta = 0.2;
dt = h*h/(epsilon*3.0);
```

Et comment déduire le pas de temps ?

$$U_j^n = U^n e^{ikX_j}$$

Considérons une perturbation quelconque...



$$\begin{aligned}
U_j^{n+1} &= U_j^n + \Delta t \Bigg(\overbrace{\left((\zeta - 1)\frac{\beta}{2h} + \frac{\epsilon}{h^2} \right)}^{a} U_{j+1}^n + \Bigg(\overbrace{\left(-\zeta\frac{\beta}{h} - 2\frac{\epsilon}{h^2} \right)}^{b} U_j^n + \Bigg(\overbrace{\left((\zeta + 1)\frac{\beta}{2h} + \frac{\epsilon}{h^2} \right)}^{c} U_{j-1}^n \Bigg) \\
&= U_j^n \Big(1 + \Delta t \Big(a e^{ikh} + b + c e^{-ikh} \Big) \Big) \\
&= U_j^n \Big(1 + \Delta t \Big(\underbrace{(a + c)}_{-b} \cos kh + b + i \underbrace{(a - c)}_{-\beta/h} \sin kh \Big) \Big)
\end{aligned}$$

Il faut que le module du facteur d'amplification soit inférieur à l'unité :-)

$$\boxed{U = \left(1 + \Delta t\left(b - b\cos kh + b - i\frac{\beta}{h}\sin kh\right)\right)}$$

$$\begin{aligned}
\left|\left(1 + \Delta t b - \Delta t b\cos(kh)\right) - i\Delta t\left(\tfrac{\beta}{h}\sin(kh)\right)\right| &\leq 1 \\
&\downarrow \\
1 + \Delta t^2 b^2(1 - \cos(kh))^2 + 2b\Delta t(1 - \cos(kh)) + \Delta t^2\tfrac{\beta^2}{h^2}\sin(kh)^2) &\leq 1 \\
\Delta t^2 b^2(1 - \cos(kh))^2 + 2b\Delta t(1 - \cos(kh)) + \Delta t^2\tfrac{\beta^2}{h^2}(1 - \cos(kh)^2) &\leq 0 \\
\Delta t b^2(1 - \cos(kh)) + 2b + \Delta t\tfrac{\beta^2}{h^2}(1 + \cos(kh)) &\leq 0
\end{aligned}$$

On déduit finalement :

$$\begin{aligned}
\Delta t &\leq \frac{-2b}{(1 - \cos(kh))b^2 + (1 + \cos(kh))\frac{\beta^2}{h^2}} \\
\Delta t &\leq \frac{2(\zeta\beta h + 2\epsilon)h^2}{(\zeta h\beta + 2\epsilon)^2 + \beta^2 h^2 + \cos(kh)(\beta^2 h^2 - (\zeta h\beta + 2\epsilon)^2)}
\end{aligned}$$

On conclut donc : $\boxed{\Delta t \quad \leq \quad \min\left(\frac{\zeta h\beta + 2\epsilon}{\beta^2}, \frac{h^2}{\zeta h\beta + 2\epsilon}\right)}$

Notons que l'on obtient les résultats habituels $\Delta t \leq \frac{h}{\beta}$ pour $\epsilon = 0, \zeta = 1$ et $\Delta t \leq \frac{h^2}{2\epsilon}$ pour $\beta = 0$.

Pratiquement…