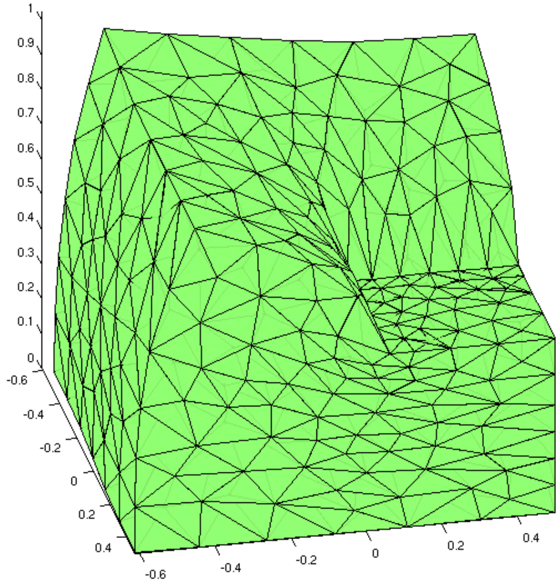
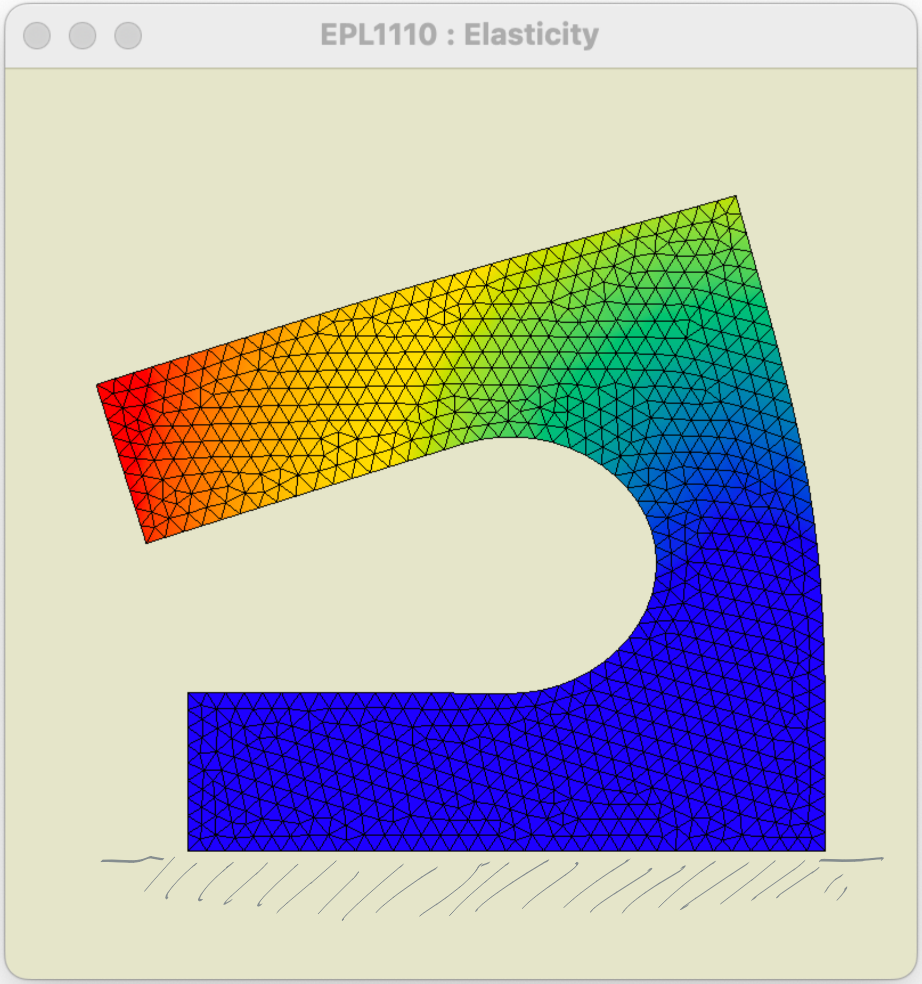
Elasticité linéaire

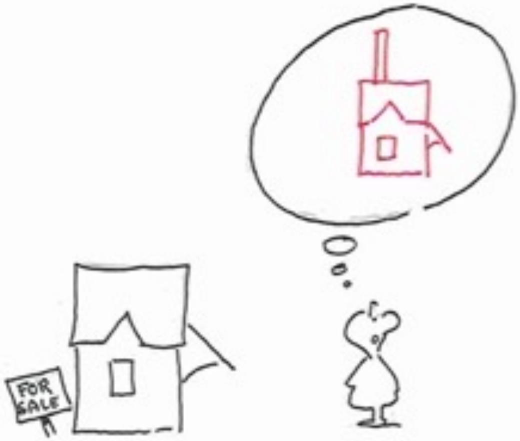
Y AURA DES
NOTES SUR LE COURS
.... CE SOIR
C'EST PROMIS



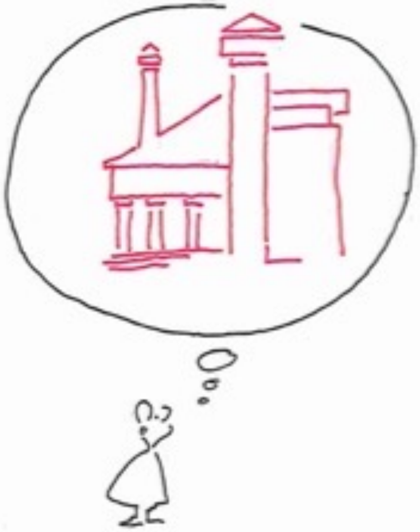


Le projet

J - 15



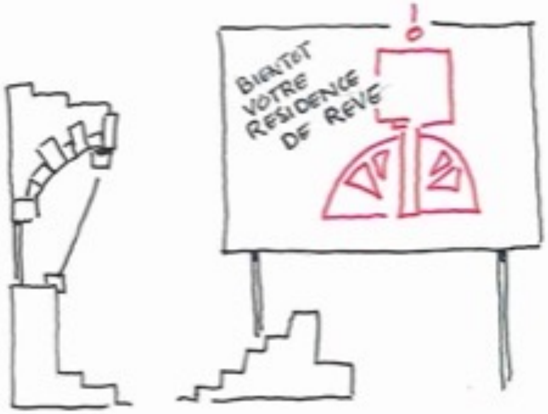
J-14



J-12

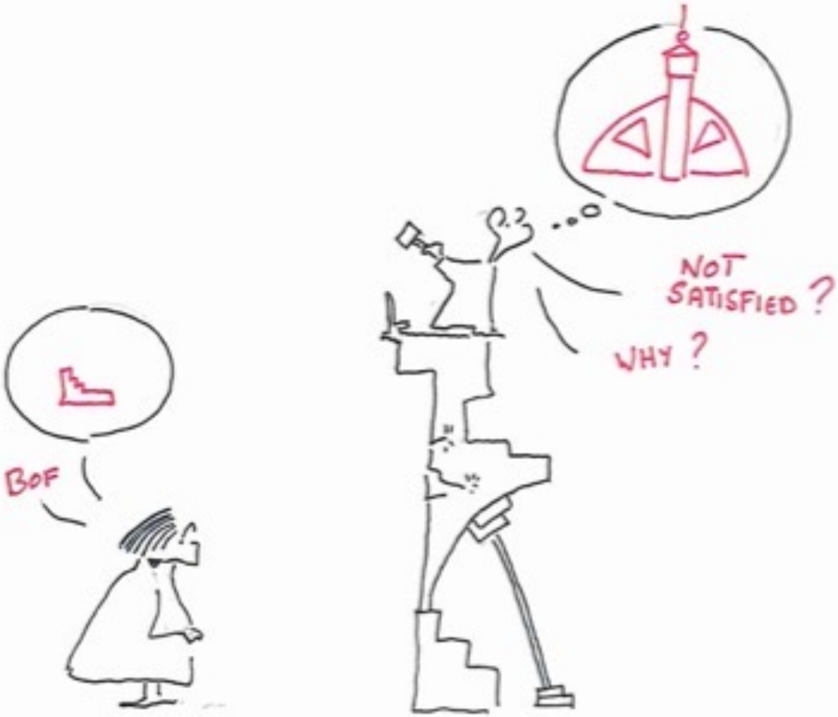
J-10

J-5



J-4

J-1



J

PROBLÈME ELLIPTIQUE

ELEMENTS FINIS *

⚠

* "USUAL FEM"
CLASSICAL
GALERKIN
FORMULATION

ABSTRACT GENERIC ELLIPTIC PROBLEM

HEAT CONDUCTION
LINEAR ELASTICITY
SIMPLIFIED MODELS OF LINEAR ELASTICITY
    BEAM / SHELLS
    ROPE / MENBRANE
STOKES PROBLEM

NOW,
WE CONSIDER
A VECTORIAL UNKNOWN FIELD !

WEAK FORMULATION

$$? \underset{\sim}{u} \in \mathcal{U} \quad \text{SUCH THAT}$$

$$\underbrace{\underbrace{a(\underset{\sim}{u}\ \hat{\underset{\sim}{u}})}_{\text{BILINEAR CONTINUOUS SYMMETRIC COERCIVE FORM}} = \underbrace{b(\hat{\underset{\sim}{u}})}_{\text{CONTINUOUS LINEAR FORM}}}_{\text{ASSUMPTIONS REQUIRED TO HAVE AN ABSTRACT MINIMIZATION PROBLEM}} \quad \forall \hat{\underset{\sim}{u}} \in \hat{\mathcal{U}}$$

$$\Updownarrow$$

$$? \underset{\sim}{u} \in \mathcal{U} \quad \text{SUCH THAT}$$

$$J(\underset{\sim}{u}) = \min_{\underset{\sim}{v} \in \mathcal{U}} \underbrace{\frac{1}{2} a(\underset{\sim}{v}\ \underset{\sim}{v}) - b(\underset{\sim}{v})}_{J(\underset{\sim}{v})}$$

MINIMIZATION PROBLEM

HEAT CONDUCTION

CONSTITUTIVE LAW

FOURIER

$$q = -k \underset{\sim}{\nabla} u$$

TEMPERATURE

CONSERVATION LAW

$$-\underset{\sim}{\nabla} \cdot q + f = 0$$

HEAT FLOW

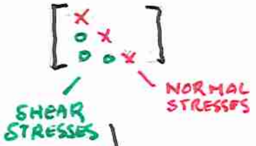
?u SUCH THAT

$$\underset{\sim}{\nabla} \cdot \left(\overbrace{k \underset{\sim}{\nabla} u}^{-q(u)} \right) + f = 0 \quad \text{IN } \Omega$$

$$-q \cdot \underset{\sim}{n} = g \quad \text{ON } \Gamma_N$$

$$u = 0 \quad \text{ON } \Gamma_D$$

3D LINEAR ISOTROPIC ELASTICITY



HOOKE ELASTIC BODY
SMALL DEF.

CONSTITUTIVE LAW

$$\underline{\underline{\sigma}} = \underbrace{\frac{E}{(1+\nu)}}_{2\mu}\underline{\underline{\varepsilon}} + \underbrace{\frac{E\nu}{(1+\nu)(1-2\nu)}}_{\lambda}\operatorname{tr}(\underline{\underline{\varepsilon}})\,\underline{\underline{S}}$$

DEFORMATION TENSOR

$$\underline{\underline{\varepsilon}} \triangleq \frac{1}{2}\left(\underline{\nabla}\,\underline{u} + (\underline{\nabla}\,\underline{u})^T\right)$$

CONSERVATION LAWS

$$\underline{\nabla}\cdot\underline{\underline{\sigma}} + \underline{f} = 0$$

$$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T$$

? $\underline{u}$

$$\underline{\nabla}\cdot\underline{\underline{\sigma}}(\underline{u}) + \underline{f} = 0 \quad \text{in } \Omega$$

$$\underline{\underline{\sigma}}\cdot\underline{n} = \underline{g} \quad \text{on } \Gamma_N$$

$$\underline{u} = 0 \quad \text{on } \Gamma_D$$

STOKES PROBLEM

CONSERVATION LAWS

$$\underbrace{\cancel{(\underset{\sim}{v}\cdot\underset{\sim}{\nabla})}\,\underset{\sim}{v}} = \underset{\sim}{\nabla}\cdot\underset{\approx}{\sigma} + \underset{\sim}{f}$$

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{v} = 0$$

CREEPING FLOW

$Re \simeq 0$

INCOMPRESSIBLE FLOW

CONSTITUTIVE LAW

NEWTONIAN FLUID

$$\underset{\approx}{\sigma} = 2\mu\,\underset{\approx}{d} - p\,\underset{\approx}{I}$$

RATE OF DEFORMATION TENSOR

$$\underset{\approx}{d} \triangleq \frac{1}{2}\left(\underset{\sim}{\nabla}\underset{\sim}{v} + (\underset{\sim}{\nabla}\underset{\sim}{v})^T\right)$$

$$?(\underset{\sim}{v}, p)$$

$$\begin{cases} \underset{\sim}{\nabla}\cdot\left(\underset{\approx}{\sigma}(\underset{\sim}{v}, p)\right) + \underset{\sim}{f} = 0 \\ \underset{\sim}{\nabla}\cdot\underset{\sim}{v} = 0 \end{cases} \quad \text{in } \Omega$$

$$\underset{\approx}{\sigma}\cdot\underset{\sim}{n} = g \quad \text{on } \Gamma_N$$

$$\underset{\sim}{v} = 0 \quad \text{on } \Gamma_D$$

CALCULUS

$\underset{\sim}{\nabla} \cdot \underset{\sim}{v} = 0$ IN Ω $\iff$ $\exists \, \underset{\sim}{\psi}$ SUCH THAT $\underset{\sim}{v} = \underset{\sim}{\nabla} \times \underset{\sim}{\psi}$

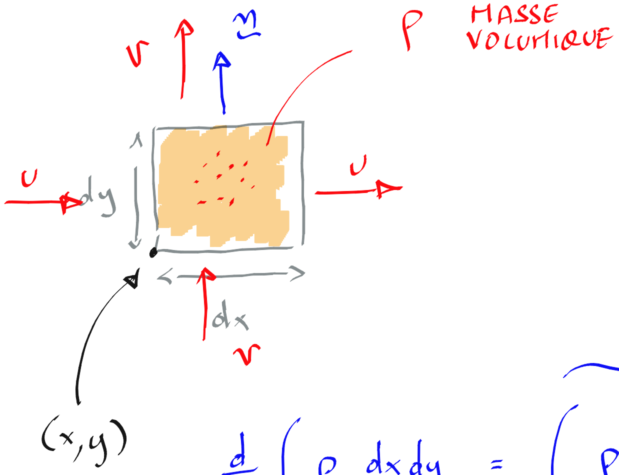
STREAM FUNCTION

? $\underset{\sim}{\psi}$

$$\underset{\sim}{\nabla}^4 \underset{\sim}{\psi} + \underset{\sim}{\nabla} \times \underset{\sim}{f} = 0 \quad \text{in } \Omega$$

+ SUITABLE BOUNDARY CONDITIONS

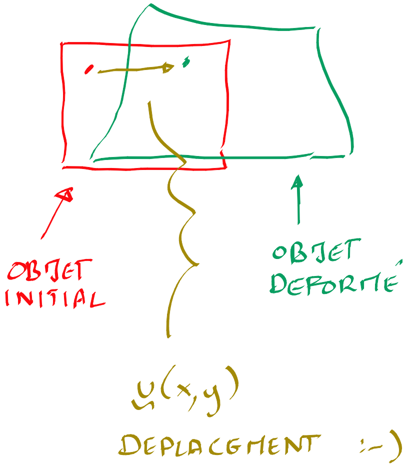
Bilan de masse…



$$\int_\Omega \underline{\nabla}\cdot(\rho\,\underline{v})$$

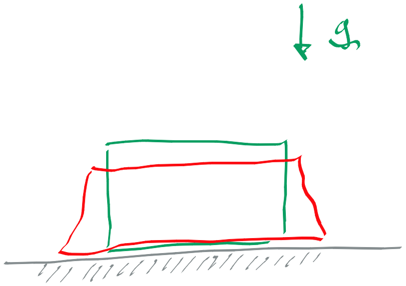
$$\frac{d}{dt}\int_\Omega \rho\; dx\,dy = \int_{\partial\Omega} \rho\; \underset{\text{VITESSE}}{\underline{v}}\cdot\underline{n}\;\; d$$

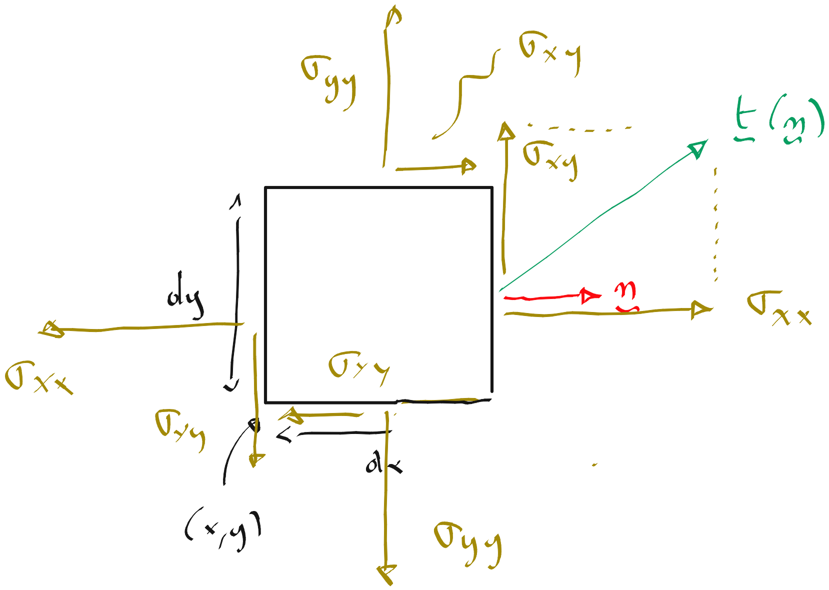
$$\frac{\partial\rho}{\partial t} = \underline{\nabla}\cdot(\rho\,\underline{v})$$



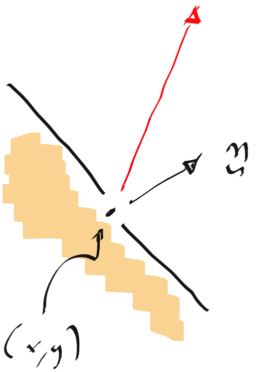
$$dx\,dy\,\frac{\partial\rho}{\partial t} = \underbrace{\left[\rho\, u(x,y) - \rho\, u(x+dx,y)\right] dy}_{} + \underbrace{\left[v(x,y) - \rho\, v(x,y+dy)\right] dx}_{}$$

$$\frac{\partial\rho}{\partial t} = \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v)$$



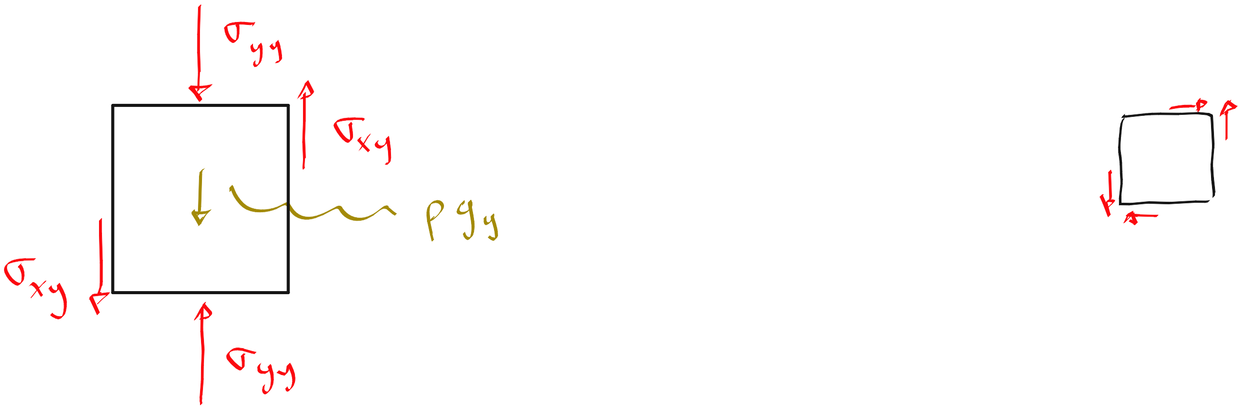


$$\underset{\sim}{t}(\underset{\sim}{n}) = \underset{=}{\sigma} \cdot \underset{\sim}{n}$$



$$\underset{\sim}{t}(\underset{\sim}{n}) = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix} \cdot \begin{bmatrix} n_x \\ n_y \end{bmatrix}$$

Tenseur de contraintes



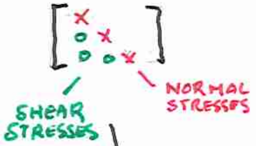
$$\rho g_y + \frac{\partial}{\partial x}(\sigma_{xy}) + \frac{\partial}{\partial y}(\sigma_{yy}) = 0$$

$$\frac{\partial}{\partial x}(\sigma_{xx}) + \frac{\partial}{\partial y}(\sigma_{yy}) = 0$$

$$\underset{\sim}{\nabla} \cdot \underline{\underline{\sigma}} + \rho \underline{g} = 0$$

Bilan de quantité de mouvement

3D LINEAR ISOTROPIC ELASTICITY



HOOKE ELASTIC BODY
SMALL DEF.

CONSTITUTIVE LAW

$$\underline{\underline{\sigma}} = \underbrace{\frac{E}{(1+\nu)}}_{2\mu} \underline{\underline{\varepsilon}} + \underbrace{\frac{E\nu}{(1+\nu)(1-2\nu)}}_{\lambda} \mathrm{tr}(\underline{\underline{\varepsilon}})\, \underline{\underline{\delta}}$$

DEFORMATION TENSOR

$$\underline{\underline{\varepsilon}} \triangleq \frac{1}{2}\left(\underline{\nabla}\,\underline{u} + (\underline{\nabla}\,\underline{u})^T\right)$$

CONSERVATION LAWS

$$\underline{\nabla} \cdot \underline{\underline{\sigma}} + \underline{f} = 0$$

$$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T$$

$? \, \underline{u}$

$$\underline{\nabla} \cdot \underline{\underline{\sigma}}(\underline{u}) + \underline{f} = 0 \quad \text{in } \Omega$$

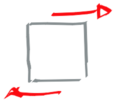
$$\underline{\underline{\sigma}} \cdot \underline{n} = \underline{g} \quad \text{on } \Gamma_N$$

$$\underline{u} = 0 \quad \text{on } \Gamma_D$$

$$\begin{bmatrix} \sigma_{xx} & \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix}$$

CONTRAINTE
DE CISAILLEMENT

CONTRAINTES
NORMALES



$$tr(\underline{\underline{\varepsilon}}) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \nabla \cdot \underline{u}$$

$$\begin{bmatrix} \varepsilon_{xx} & \\ \varepsilon_{xy} & \varepsilon_{yy} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2\,\partial u/\partial x & \\ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} & 2\,\partial v/\partial y \end{bmatrix}$$

$$\underline{\underline{\sigma}} = \underbrace{\frac{E}{(1+\nu)}}_{2\mu} \underline{\underline{\varepsilon}} + \underbrace{\frac{E\nu}{(1+\nu)(1-2\nu)}}_{\lambda} tr(\underline{\underline{\varepsilon}})\, \underline{\underline{S}}$$

SHEAR
MODULUS

$\lambda \rightarrow \infty$
IF $\nu \rightarrow 1/2$

POISSON'S
COEFFICIENT

$\nu \in]-1 \quad 1/2[$

INCOMPRESSIBLE
MATERIAL

STRANGE
MATERIALS

USUAL
BEHAVIOUR

ABSTRACT GENERIC DISCRETE FORMULATION

$\begin{bmatrix} u \\ v \end{bmatrix}$ 2D

$\begin{bmatrix} u \\ v \\ w \end{bmatrix}$ 3D

$$\underbrace{\underset{\sim}{u}(\underset{\sim}{x})}_{\in \mathcal{U}} \simeq \underbrace{\underset{\sim}{u}^h(\underset{\sim}{x})}_{\in \mathcal{U}^h \subset \mathcal{U}} = \sum_{i=1}^{n} \underset{\sim}{U}_i \, \tau_i(\underset{\sim}{x})$$

$\dim(\mathcal{U}^h) = 2n$ ← 2D PROBLEMS

$\ldots = 3n$ ← 3D PROBLEMS

$$\mathcal{U}^h = \text{SPAN} \left\{ \underbrace{\begin{bmatrix} \tau_1 \\ 0 \end{bmatrix} \begin{bmatrix} \tau_2 \\ 0 \end{bmatrix} \cdots \begin{bmatrix} \tau_n \\ 0 \end{bmatrix}}_{n} \underbrace{\begin{bmatrix} 0 \\ \tau_1 \end{bmatrix} \cdots \begin{bmatrix} 0 \\ \tau_n \end{bmatrix}}_{n} \right\}$$

? $\underset{\sim}{u}^h$ SUCH THAT

$$a(\underset{\sim}{u}^h \ \hat{\underset{\sim}{u}}^h] = b(\hat{\underset{\sim}{u}}^h)$$

$$\forall \hat{\underset{\sim}{u}}^h \in \hat{\mathcal{U}}^h$$

DISCRETE FORMULATION

$$\Big\langle \underbrace{\hat{\underset{\sim}{u}}}_{\in \hat{\mathcal{U}}} \cdot \Big(\underset{\sim}{\nabla} \cdot \underset{\approx}{\sigma}\big(\underbrace{\underset{\sim}{u}}_{\in \mathcal{U}}\big)\Big) \Big\rangle \overset{\boxed{\text{TIP \#1}}}{\underset{}{=}} \underbrace{\Big\langle \underset{\sim}{\nabla} \cdot \Big(\hat{\underset{\sim}{u}} \cdot \underset{\approx}{\sigma}(\underset{\sim}{u})\Big) \Big\rangle} - \overbrace{\Big\langle \underset{\sim}{\nabla} \hat{\underset{\sim}{u}} : \underset{\approx}{\sigma}(\underset{\sim}{u}) \Big\rangle}^{\triangleq\, a(\hat{\underset{\sim}{u}}, \underset{\sim}{u})}$$

$$\boxed{\text{TIP \#2}} \rightarrow \Big\downarrow$$

$$= \ll \underset{\sim}{n} \cdot \hat{\underset{\sim}{u}} \cdot \underset{\approx}{\sigma}(\underset{\sim}{u}) \gg$$

$$\Big\downarrow \leftarrow \boxed{\text{TIP \#3}}$$

$$= \ll \hat{\underset{\sim}{u}} \cdot \Big(\underbrace{\underset{\sim}{n} \cdot \underset{\approx}{\sigma}(\underset{\sim}{u})}_{g \text{ on } \Gamma_N}\Big) \gg_N$$

USUAL CALCULUS !

IS $a(\underset{\sim}{\hat{u}}\ \underset{\sim}{u})$ SYMMETRIC ?

$\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}$

$$\langle \underset{\sim}{\nabla}\underset{\sim}{\hat{u}} : \underline{\underline{\sigma}}(\underset{\sim}{u}) \rangle = \langle \left(\underbrace{\overbrace{\frac{\underset{\sim}{\nabla}\underset{\sim}{\hat{u}} + (\underset{\sim}{\nabla}\underset{\sim}{\hat{u}})^T}{2}}^{\underline{\underline{\varepsilon}}(\underset{\sim}{\hat{u}})}}_{\text{SYM}} + \underbrace{\frac{\underset{\sim}{\nabla}\underset{\sim}{\hat{u}} - (\underset{\sim}{\nabla}\underset{\sim}{\hat{u}})^T}{2}}_{\text{ANTI SYM.}} \right) : \underbrace{\underline{\underline{\sigma}}(\underset{\sim}{u})}_{\text{SYM.}} \rangle$$

$$\Big\downarrow$$

$$= \langle \underline{\underline{\varepsilon}}(\underset{\sim}{\hat{u}}) : \underbrace{\underline{\underline{\sigma}}(\underset{\sim}{u})}_{\boxed{\underline{\underline{\underline{\underline{C}}}}\, :\, \underline{\underline{\varepsilon}}(\underset{\sim}{u})}} \rangle$$

GENERALIZED HOOKE'S LAW

$$\Big\downarrow$$

$$= \langle \underline{\underline{\varepsilon}}(\underset{\sim}{\hat{u}}) : \underline{\underline{\underline{\underline{C}}}} : \underline{\underline{\varepsilon}}(\underset{\sim}{u}) \rangle$$

WEAK FORMULATION

? $\underset{\sim}{u} \in \mathcal{U}$ SUCH THAT

$$\underbrace{\langle \underset{=}{\varepsilon}(\hat{\underset{\sim}{u}}) : \underset{\equiv}{C} : \underset{=}{\varepsilon}(\underset{\sim}{u}) \rangle}_{a(\hat{\underset{\sim}{u}}\ \underset{\sim}{u})} = \underbrace{\langle \hat{\underset{\sim}{u}} \cdot \underset{\sim}{f} \rangle + \langle\!\langle \hat{\underset{\sim}{u}} \cdot \underset{\sim}{g} \rangle\!\rangle_N}_{b(\hat{\underset{\sim}{u}})}$$

$\forall \hat{u} \in \hat{\mathcal{U}}$

$$\Updownarrow$$

FOR SUITABLE $\underset{\equiv}{C}$ ONLY !

? $\underset{\sim}{u} \in \mathcal{U}$ SUCH THAT

$$J(\underset{\sim}{u}) = \min_{v \in \mathcal{U}} \underbrace{\frac{1}{2} a(\underset{\sim}{v}\ \underset{\sim}{v}) - b(\underset{\sim}{v})}_{J(\underset{\sim}{v})}$$

MINIMIZATION PROBLEM

$$\underbrace{\frac{1}{2}\,a(\underset{\sim}{v},\underset{\sim}{v})}_{\text{ENERGY OF DEFORMATION}} = \underbrace{\left\langle \frac{1}{2}\,\lambda\,\underline{\underline{\varepsilon}}(\underset{\sim}{v}) : \underline{\underline{I}}\;\;\overbrace{\underline{\underline{I}} : \underline{\underline{\varepsilon}}(\underset{\sim}{v})}^{tr(\underline{\underline{\varepsilon}}(\underset{\sim}{v}))} + \mu\,\underline{\underline{\varepsilon}}(\underset{\sim}{v}) : \underline{\underline{\varepsilon}}(\underset{\sim}{v}) \right\rangle}_{\text{MUST BE A QUADRATIC POSITIVE FORM IN ORDER TO OBTAIN A MINIMIZATION PROBLEM}}$$

$$\downarrow$$

$$\boxed{\begin{aligned} \mu &> 0 \\ \frac{3}{2}\lambda + \mu &> 0 \end{aligned}}\qquad \begin{aligned} &(\varepsilon_{12} \neq 0) \\ &(\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} \neq 0) \end{aligned}$$

ADMISSIBLE VALUES FOR LAMÉ COEFFICIENTS

"DEFORMATION ENERGY"

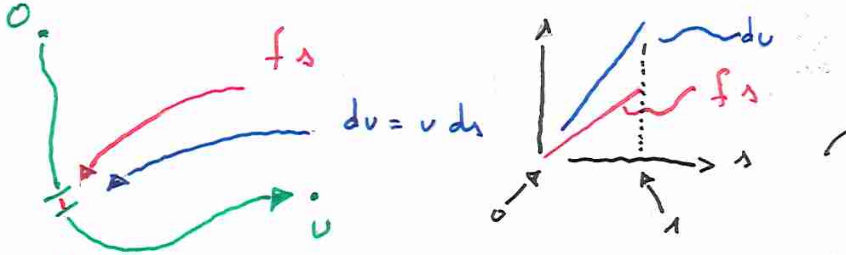
WHAT IS IT ?

ENERGY OF DEFORMATION

= WORK PERFORMED TO DEFORM THE STRUCTURE AND IS STORED INSIDE.

SUCH AN ENERGY IS RECOVERED WHEN EXTERNAL FORCES ARE REMOVED

$$\text{ENERGY} = \underbrace{\text{FORCE} \cdot \text{DISPLACEMENT}}_{\text{WORK PERFORMED}}$$



$$\text{WORK PERFORMED} = \int_0^u s f \, du$$

$$= \int_0^1 f u \, s \, ds$$

$$= f u \underbrace{\int_0^1 s \, ds}_{1/2}$$

$$= \frac{1}{2} f u \quad !$$

VERY VERY SLOW LOADING

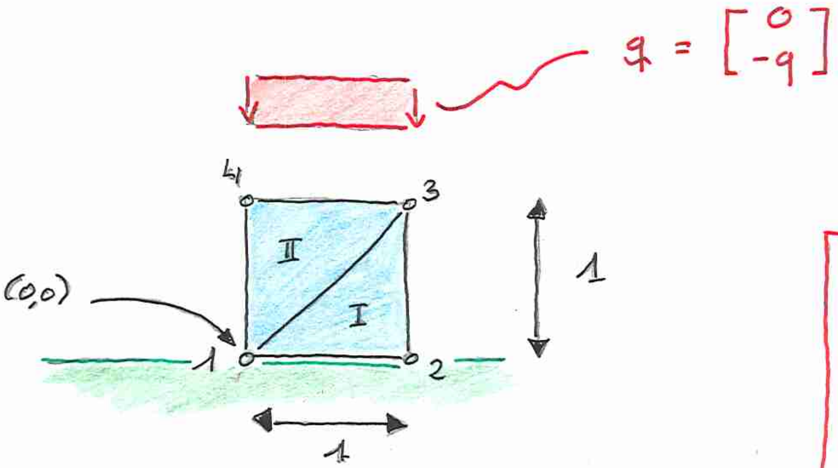
→ QUASI-STATIC APPROACH

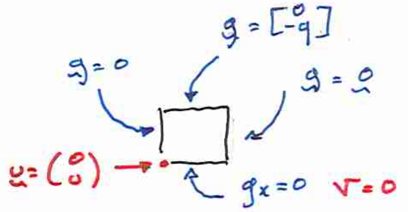
- IRREVERSIBILITY
- DISSIPATION

} ARE NEGLECTED !

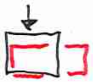
NUMERICAL EXAMPLE

$$\underset{\sim}{u} = \begin{bmatrix} u \\ v \end{bmatrix}$$





5 UNKNOWN DISPLACEMENTS



$$\begin{bmatrix} U_i \\ V_i \end{bmatrix}$$

$$? \; \underset{\sim}{u}^h = \sum_{i=1}^{n} \underset{\sim}{U}_i \, \tau_i$$

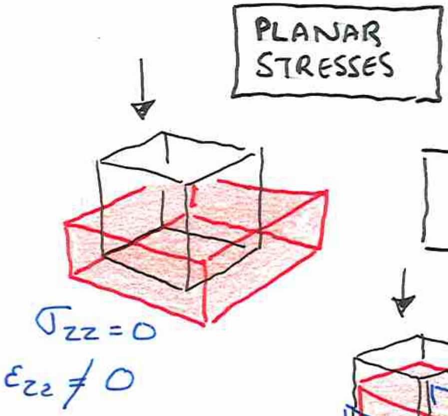
$$\langle \underset{\approx}{\varepsilon}(\underset{\sim}{\hat{u}}^h) : \underset{\approx}{\sigma}(\underset{\sim}{u}^h) \rangle = \langle \underset{\sim}{f} \cdot \underset{\sim}{\hat{u}}^h \rangle + \langle\!\langle \underset{\sim}{g} \cdot \underset{\sim}{\hat{u}}^h \rangle\!\rangle_N$$

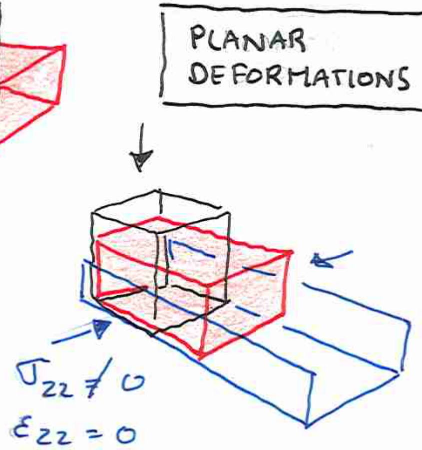
$$\forall \underset{\sim}{\hat{u}}^h$$

$$\begin{bmatrix} \tau_i \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ \tau_i \end{bmatrix}$$

$2n$ TEST FUNCTIONS

ANALYTICAL SOLUTION





PLANAR STRESSES

$$\underset{\sim}{u} = \frac{q}{E}\begin{bmatrix} \nu x \\ -y \end{bmatrix}$$

$$\underset{\approx}{\varepsilon} = \frac{q}{E}\begin{bmatrix} \nu & 0 \\ 0 & -1 \end{bmatrix}$$

$$\underbrace{\underset{\sim}{\nabla} \cdot \underset{=}{\sigma}}_{=0\,!} = 0$$

$$\sigma_{xx} = \frac{E}{(1-\nu^2)}\left(\underbrace{\varepsilon_{xx}}_{\frac{q\nu}{E}} + \underbrace{\nu\,\varepsilon_{yy}}_{-\frac{q\nu}{E}}\right) = 0$$

$$\sigma_{yy} = \frac{E}{(1-\nu^2)}\left(\underbrace{\nu\,\varepsilon_{xx} - \varepsilon_{yy}}_{\frac{q}{E}(\nu^2-1)}\right) = -q \; !$$

$$\begin{bmatrix}\tau_i\\0\end{bmatrix} \text{ OR } \begin{bmatrix}0\\\tau_i\end{bmatrix}$$

$$\boxed{\langle \underbrace{\underline{\underline{\varepsilon}}(\hat{\underline{u}}^h)} : \overbrace{\underline{\underline{\sigma}}(\underline{u}^h)} \rangle \quad \forall \hat{\underline{u}}^h}$$

$$\sum_j U_j \, \underline{\underline{\sigma}}\begin{pmatrix}\tau_j\\0\end{pmatrix} + V_j \, \underline{\underline{\sigma}}\begin{pmatrix}0\\\tau_j\end{pmatrix}$$

$$\sum_j \begin{bmatrix}U_j\\V_j\end{bmatrix}\tau_j = \sum U_j \begin{bmatrix}\tau_j\\0\end{bmatrix} + V_j \begin{bmatrix}0\\\tau_j\end{bmatrix}$$

$$\boxed{\sum_j \underline{\underline{A}}_{ij} \cdot \underline{U}_j}$$

$$\sum_j \begin{bmatrix} \langle \underline{\underline{\varepsilon}}\begin{pmatrix}\tau_i\\0\end{pmatrix} : \underline{\underline{\sigma}}\begin{pmatrix}\tau_j\\0\end{pmatrix} \rangle & \langle \underline{\underline{\varepsilon}}\begin{pmatrix}\tau_i\\0\end{pmatrix} : \underline{\underline{\sigma}}\begin{pmatrix}0\\\tau_j\end{pmatrix} \rangle \\ \langle \underline{\underline{\varepsilon}}\begin{pmatrix}0\\\tau_i\end{pmatrix} : \underline{\underline{\sigma}}\begin{pmatrix}\tau_j\\0\end{pmatrix} \rangle & \langle \underline{\underline{\varepsilon}}\begin{pmatrix}0\\\tau_i\end{pmatrix} : \underline{\underline{\sigma}}\begin{pmatrix}0\\\tau_j\end{pmatrix} \rangle \end{bmatrix} \cdot \begin{bmatrix}U_j\\V_j\end{bmatrix}$$

$$\forall i$$

DISCRETE OPERATOR

$$\underline{\underline{\varepsilon}}\begin{pmatrix}\tau_i\\ 0\end{pmatrix} = \begin{bmatrix}\tau_{i,x} & \tau_{i,y}/2\\ \tau_{i,y}/2 & 0\end{bmatrix}$$

$$\underline{\underline{\varepsilon}}\begin{pmatrix}0\\ \tau_i\end{pmatrix} = \begin{bmatrix}0 & \tau_{i,x}/2\\ \tau_{i,x}/2 & \tau_{i,y}\end{bmatrix}$$

$$\underline{\underline{\sigma}}\left(\underline{\underline{\varepsilon}}\right) = \begin{bmatrix}A\,\varepsilon_{xx} + B\,\varepsilon_{yy} & 2C\,\varepsilon_{xy}\\ 2C\,\varepsilon_{xy} & A\,\varepsilon_{yy} + B\,\varepsilon_{xx}\end{bmatrix}$$

HOW TO CALCULATE IT ?

	PLANAR DEFORMATIONS	PLANAR STRESSES
A	$\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$	$\frac{E}{(1-\nu^2)}$
B	$\frac{E\nu}{(1+\nu)(1-2\nu)}$	$\frac{E\nu}{(1-\nu^2)}$
C	$\frac{E}{2(1+\nu)}$	

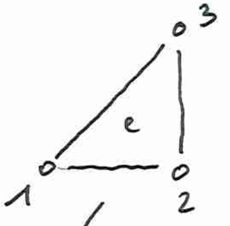
$$\underline{\underline{\varepsilon}} : \underline{\underline{\sigma}} = \varepsilon_{xx}\left(A\,\varepsilon_{xx} + B\,\varepsilon_{yy}\right) + 4C\,\varepsilon_{xy}\,\varepsilon_{xy} + \varepsilon_{yy}\left(A\,\varepsilon_{yy} + B\,\varepsilon_{xx}\right)$$

$$\underline{\underline{A}}_{ij} = \begin{bmatrix} \langle \tau_{i,x}\, A\, \tau_{j,x} \rangle + \langle \tau_{i,y}\, C\, \tau_{j,y} \rangle & \langle \tau_{i,x}\, B\, \tau_{j,y} \rangle + \langle \tau_{i,y}\, C\, \tau_{j,x} \rangle \\ \underbrace{\langle \tau_{i,y}\, B\, \tau_{j,x} \rangle + \langle \tau_{i,x}\, C\, \tau_{j,y} \rangle}_{\underline{\underline{\varepsilon}}\begin{pmatrix} 0 \\ \tau_i \end{pmatrix} : \underline{\underline{\sigma}}\left(\underline{\underline{\varepsilon}}\begin{pmatrix} \tau_j \\ 0 \end{pmatrix}\right)} & \langle \tau_{i,y}\, A\, \tau_{j,y} \rangle + \langle \tau_{i,x}\, C\, \tau_{j,x} \rangle \end{bmatrix}$$

LOCAL
ELASTICITY
MATRIX

COMPUTING A LOCAL ELASTICITY MATRIX

$$\left[\begin{array}{c|c} A\langle \tau_{i,x}\,\tau_{j,x}\rangle + C\langle \tau_{i,y}\,\tau_{j,y}\rangle & B\langle \tau_{i,x}\,\tau_{j,y}\rangle + C\langle \tau_{i,y}\,\tau_{j,x}\rangle \\ \hline B\langle \tau_{i,y}\,\tau_{j,x}\rangle + C\langle \tau_{i,x}\,\tau_{j,y}\rangle & A\langle \tau_{i,y}\,\tau_{j,y}\rangle + C\langle \tau_{i,x}\,\tau_{j,x}\rangle \end{array}\right]$$



	$\tau_{i,x}$	$\tau_{i,y}$
1	-1	0
2	1	-1
3	0	1

$\underline{\underline{A}}^e_{11}$

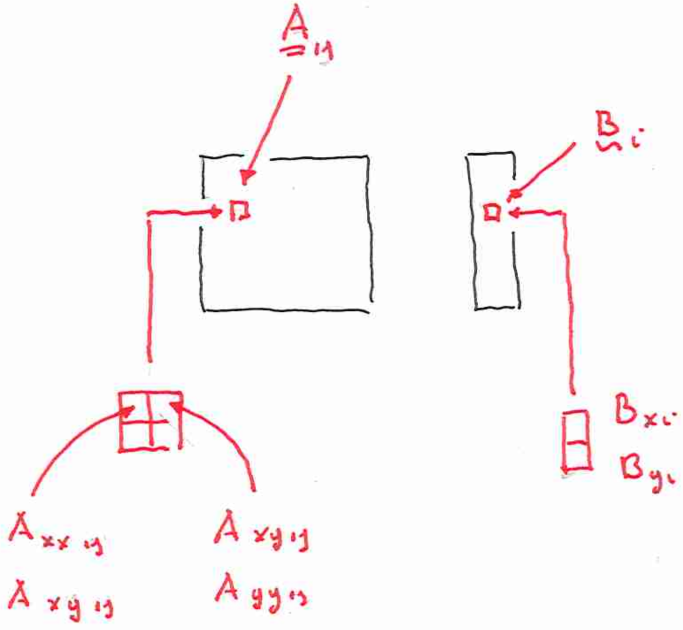
$A_{xx11} = A$
$A_{yy11} = C$
$A_{xy11} = 0$
$A_{yx11} = 0$

$$\underline{\underline{A}}^e_y = \left[\begin{array}{cc|cc|cc} \boxed{\begin{array}{cc} A & 0 \\ 0 & C \end{array}} & & -A & B & 0 & -B \\ & & C & -C & -C & 0 \\ \hline & & A+C & -B-C & -C & B \\ & & -B-C & A+C & C & -A \\ \hline & & & & C & 0 \\ & & & & 0 & A \end{array}\right] \frac{1}{2}$$

$$\underset{\sim}{B}_i = \langle\langle g \cdot \hat{\underset{\sim}{u}}^h \rangle\rangle_N$$

$$\begin{bmatrix} g_x \\ g_y \end{bmatrix} \qquad \begin{bmatrix} \tau_i \\ 0 \end{bmatrix} \text{ OR } \begin{bmatrix} 0 \\ \tau_i \end{bmatrix}$$

$$\boxed{\begin{bmatrix} \langle\langle g_x \ \tau_i \rangle\rangle \\ \langle\langle g_y \ \tau_i \rangle\rangle \end{bmatrix}}$$

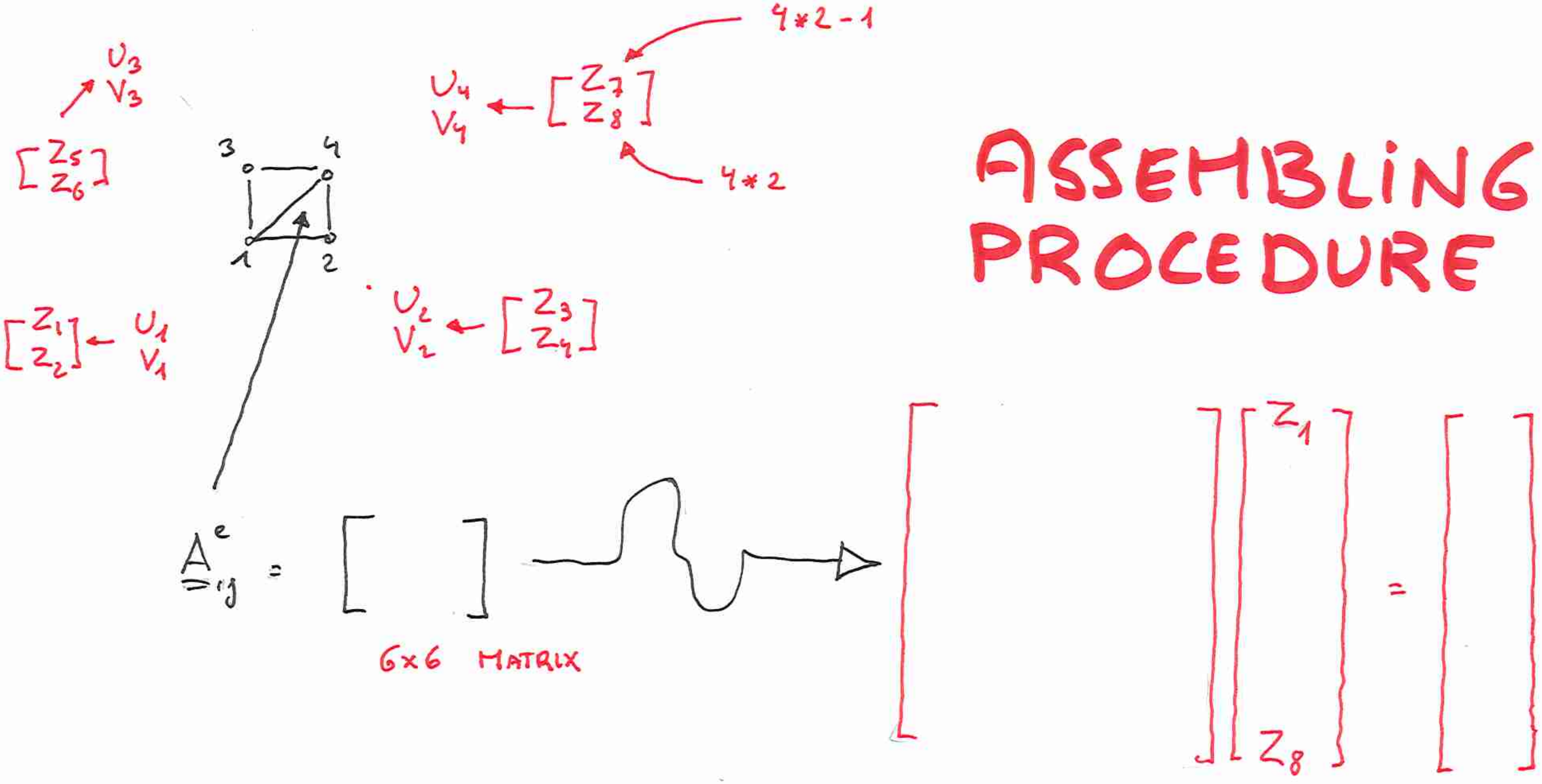


AND

HYPER

VECTOR !

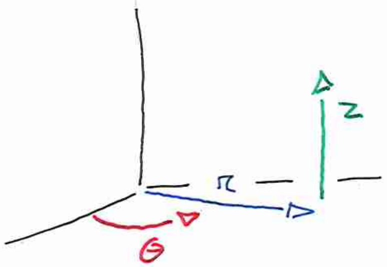
ASSEMBLING PROCEDURE



$$\underbrace{\text{GLOBAL HORIZONTAL INDEX} = 2 * \text{GLOBAL SCALAR INDEX}}_{\text{GLOBAL}} - 1$$

$$\begin{bmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \\ U_3 \\ V_3 \\ U_4 \\ V_3 \end{bmatrix}$$

AXISYMMETRIC PROBLEMS



AXISYMMETRY !

$u_\theta = 0$

$\frac{\partial}{\partial \theta} = 0$

$$\underset{\sim}{u} = \begin{bmatrix} u_r \\ u_\theta \\ u_z \end{bmatrix} = \begin{bmatrix} u(r,z) \\ 0 \\ v(r,z) \end{bmatrix}$$

$$\underset{\approx}{\varepsilon} = \begin{bmatrix} u_{r,r} & 0 & (u_{r,z} + u_{z,r})/2 \\ & u_r/r & 0 \\ & & u_{z,z} \end{bmatrix}$$

A, B, C

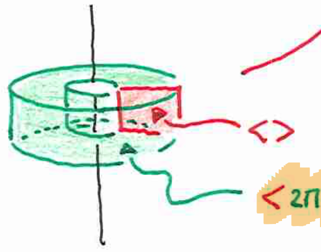
OF PLANAR DEFORMATIONS !

$$\underset{\approx}{\sigma} = \begin{bmatrix} A\varepsilon_{rr} + B(\varepsilon_{\theta\theta} + \varepsilon_{zz}) & 0 & 2C\varepsilon_{rz} \\ & A\varepsilon_{\theta\theta} + B(\varepsilon_{rr} + \varepsilon_{zz}) & 0 \\ & & A\varepsilon_{zz} + B(\varepsilon_{rr} + \varepsilon_{\theta\theta}) \end{bmatrix}$$

$$\underline{\underline{\varepsilon}}\left([\tau_i \;\; 0 \;\; 0]\right) = \begin{bmatrix} \tau_{i,r} & 0 & \tau_{i,z}/2 \\ & \tau_i/r & 0 \\ & & 0 \end{bmatrix}$$

$$\underline{\underline{\varepsilon}}\left([0 \;\; 0 \;\; \tau_i]\right) = \begin{bmatrix} 0 & 0 & \tau_{i,r}/2 \\ & 0 & 0 \\ & & \tau_{i,z} \end{bmatrix}$$

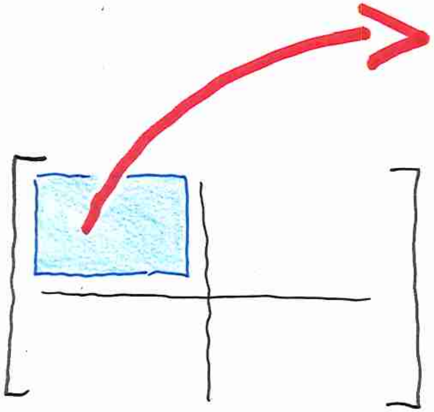
$$\underbrace{2\pi \left[\begin{array}{c|c} \left\langle r \; \underline{\underline{\varepsilon}}\begin{pmatrix}\tau_i\\0\\0\end{pmatrix} : \underline{\underline{\sigma}}\begin{pmatrix}\tau_j\\0\\0\end{pmatrix}\right\rangle & \left\langle r \; \underline{\underline{\varepsilon}}\begin{pmatrix}\tau_i\\0\\0\end{pmatrix} : \underline{\underline{\sigma}}\begin{pmatrix}0\\0\\\tau_j\end{pmatrix}\right\rangle \\ \hline \left\langle r \; \underline{\underline{\varepsilon}}\begin{pmatrix}0\\0\\\tau_i\end{pmatrix} : \underline{\underline{\sigma}}\begin{pmatrix}\tau_j\\0\\0\end{pmatrix}\right\rangle & \left\langle r \; \underline{\underline{\varepsilon}}\begin{pmatrix}0\\0\\\tau_i\end{pmatrix} : \underline{\underline{\sigma}}\begin{pmatrix}0\\0\\\tau_j\end{pmatrix}\right\rangle \end{array}\right]}_{\underline{\underline{A}}_{ij}}$$



$\langle 2\pi r \rangle = \langle \quad \rangle_{Ax_i}$!

$$\underline{\underline{\varepsilon}}\begin{pmatrix}\tau_i\\0\\0\end{pmatrix} = \begin{bmatrix}\tau_{i,r} & 0 & \tau_{i,z}/2\\ & \tau_i/r & 0\\ & & 0\end{bmatrix}$$

$$\underline{\underline{\sigma}}\begin{pmatrix}\tau_j\\0\\0\end{pmatrix} = \begin{bmatrix}A\tau_{j,r} + B\tau_j/r & 0 & C\tau_{j,z}\\ & A\tau_j/r + B\tau_{j,r} & 0\\ & & B\tau_j/r + B\tau_{j,r}\end{bmatrix}$$



$$\underline{\underline{\varepsilon}}\begin{pmatrix}\tau_i\\0\\0\end{pmatrix} : \underline{\underline{\sigma}}\begin{pmatrix}\tau_j\\0\\0\end{pmatrix} = \langle \tau_{i,r}\left(A\tau_{j,r} + B\tau_j/r\right) r \rangle + \langle \tau_{i,z}\, C\, \tau_{j,z}\, r \rangle + \langle \frac{\tau_i}{r}\left(A\frac{\tau_j}{r} + B\tau_{j,r}\right) r \rangle$$

$$\underline{\underline{A}}_{ij}$$

IT IS ALMOST LIKE A 2D PROBLEM !

$$
\left[
\begin{array}{c|c}
\begin{array}{l}
A \langle \tau_{i,r} \; \tau_{j,r} \; r \rangle \\
\quad + C \langle \tau_{i,z} \; \tau_{j,z} \; r \rangle \\
\qquad + B \langle \tau_{i,r} \; \tau_j \rangle \\
\qquad + \langle \underbrace{\tau_i}_{\varepsilon_{\theta\theta}/r} \underbrace{\left(B \, \tau_{j,r} + A \frac{\tau_j}{r} \right.}_{\sigma_{\theta\theta}} \rangle
\end{array}
&
\begin{array}{l}
B \langle \tau_{i,r} \; \tau_{j,z} \; r \rangle \\
\quad + C \langle \tau_{i,z} \; \tau_{j,r} \; r \rangle \\
\qquad + B \langle \tau_i \; \tau_{j,z} \rangle
\end{array}
\\ \hline
\begin{array}{l}
B \langle \tau_{i,z} \; \tau_{j,r} \; r \rangle \\
\quad + C \langle \tau_{i,r} \; \tau_{j,z} \; r \rangle \\
\qquad + B \langle \tau_{i,z} \; \tau_j \rangle
\end{array}
&
\begin{array}{l}
A \langle \tau_{i,z} \; \tau_{j,z} \; r \rangle \\
\quad + C \langle \tau_{i,r} \; \tau_{j,r} \; r \rangle
\end{array}
\end{array}
\right]
$$

$J(\underset{\sim}{u}^h) =$ MINIMUM OF THE POTENTIAL ENERGY

$$\underbrace{\text{OF THE POTENTIAL ENERGY}}_{\text{DEFORMATION ENERGY} \; - \; \text{WORK OF EXTERNAL FORCES}}$$

$\underset{\sim}{u}^h$ MINIMIZES THE ENERGY

DISPLACEMENTS FORMULATION

DISCONTINUOUS ! → SEVERAL VALUES AT EACH VERTEX !

TYPICALLY CONTINUOUS

LINEAR OR QUADRATIC

BUT $\underline{\underline{\sigma}}(\underset{\sim}{u}^h)$

MUST BE VIEWED AS A LEAST-SQUARE FIT OF THE EXACT SOLUTION $\underline{\underline{\sigma}}(\underset{\sim}{u})$

→ ALLOWS US TO USE THE BEST STRATEGY FOR THE INTERPRETATION OF $\underline{\underline{\sigma}}(\underset{\sim}{u}^h)$

$\underline{\underline{\sigma}}(\underset{\sim}{u}^h)$ = SOLUTION OF A LEAST-SQUARES FIT PROBLEM

$$a(\hat{\underset{\sim}{u}}, \underset{\sim}{u}) = b(\hat{\underset{\sim}{u}})$$
$$\forall \hat{v} \in \hat{\mathcal{U}}$$

$$a(\hat{\underset{\sim}{u}}^h, \underset{\sim}{u}^h) = b(\hat{\underset{\sim}{u}}^h)$$
$$\forall \underset{\sim}{u}^h \in \hat{\mathcal{U}}^h$$

? $\underset{\sim}{u}^h$

$$a(\hat{\underset{\sim}{u}}^h, \underset{\sim}{u}^h - \underset{\sim}{u}) = 0$$
$$\forall \hat{\underset{\sim}{u}}^h \in \hat{\mathcal{U}}^h$$

(CFR $\hat{\mathcal{U}}^h \subset \hat{\mathcal{U}}$!

CFR LINEARITY !

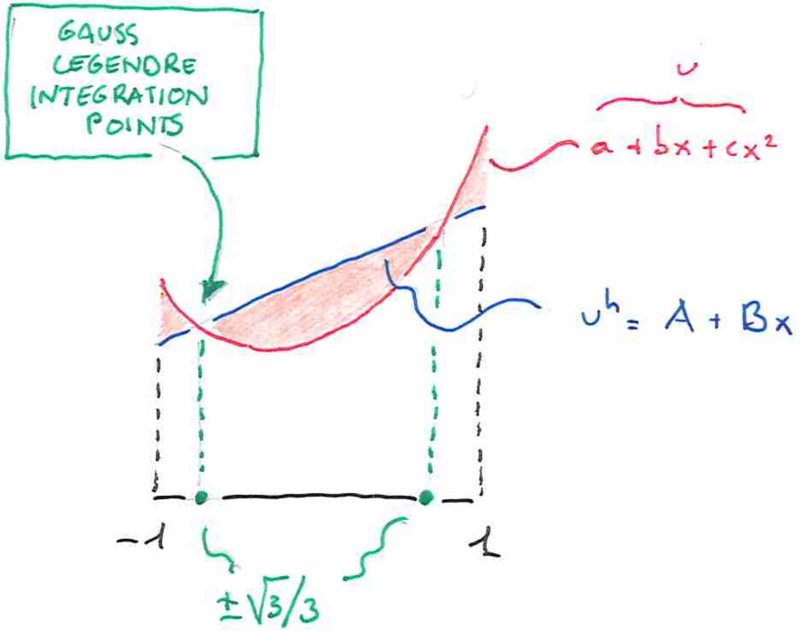
$\Updownarrow$

$$\overset{\diamond}{b}(\hat{\underset{\sim}{u}}^h) = a(\hat{\underset{\sim}{u}}^h, \underset{\sim}{u})$$
$$\overset{\diamond}{a}(\hat{\underset{\sim}{u}}^h, \underset{\sim}{u}^h) = a(\hat{\underset{\sim}{u}}^h, \underset{\sim}{u}^h)$$

? $\underset{\sim}{u}^h$

$$G(\underset{\sim}{u}^h) = \min_{\underset{\sim}{v}^h \in \mathcal{U}^h} \underbrace{\frac{1}{2} \left\langle \left(\underset{\approx}{\varepsilon}(\underset{\sim}{v}^h) - \underset{\approx}{\varepsilon}(\underset{\sim}{u})\right) : \underset{\approx}{\underset{\approx}{C}} : \left(\underset{\approx}{\varepsilon}(\underset{\sim}{v}^h) - \underset{\approx}{\varepsilon}(\underset{\sim}{u})\right) \right\rangle}_{G(\underset{\sim}{v}^h)}$$

WHAT ARE THE "BEST VALUES" OF A LEAST-SQUARES POLYNOMIAL FIT ?



$$\frac{c}{3} - cx^2 = 0 \rightarrow x = \pm\frac{\sqrt{3}}{3}$$

? A, B SUCH THAT

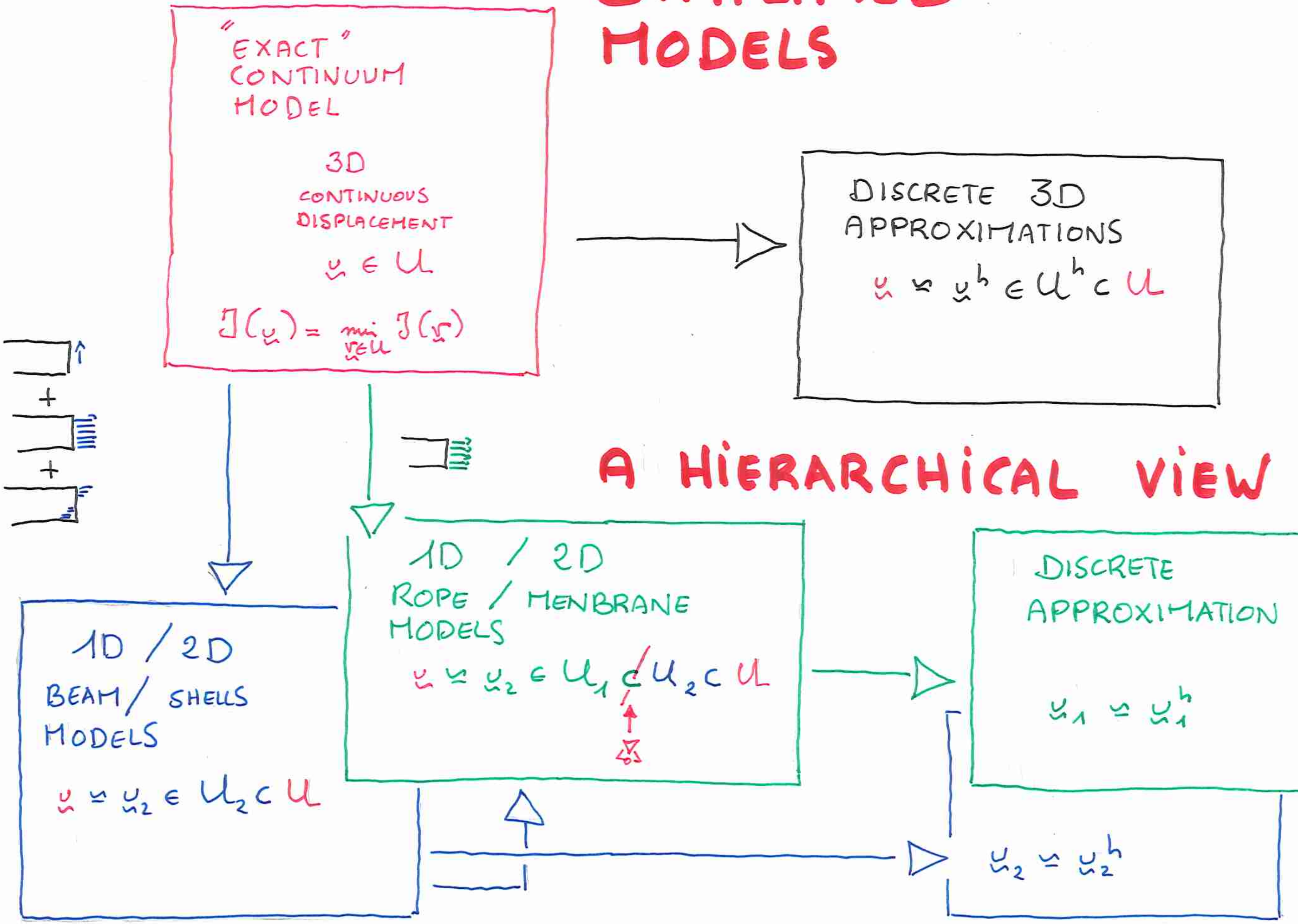
$$G(A,B) = \min_{\check{A}\check{B}} \int_{-1}^{1} (\check{A} + \check{B}x - a - bx - cx^2)^2 \, dx$$

$$\longrightarrow \begin{cases} B \int_{-1}^{1} (A + Bx - a - bx - cx^2)\, x \, dx = 0 \\ A \int_{-1}^{1} (A + Bx - a - bx - cx^2) \, dx = 0 \end{cases}$$

$$\begin{cases} B^2 \frac{2}{3} - Bb\frac{2}{3} = 0 \longrightarrow B = b \\ 2A^2 - 2Aa - Ac\frac{2}{3} = 0 \longrightarrow A = a + \frac{c}{3} \end{cases}$$

SIMPLIFIED MODELS

A HIERARCHICAL VIEW



"EXACT"
CONTINUUM
MODEL

3D
CONTINUOUS
DISPLACEMENT

$\underset{\sim}{u} \in \mathcal{U}$

$J(\underset{\sim}{u}) = \min_{\underset{\sim}{v} \in \mathcal{U}} J(\underset{\sim}{v})$

DISCRETE 3D
APPROXIMATIONS

$\underset{\sim}{u} \simeq \underset{\sim}{u}^h \in \mathcal{U}^h \subset \mathcal{U}$

1D / 2D
BEAM / SHELLS
MODELS

$\underset{\sim}{u} \simeq \underset{\sim}{u}_2 \in \mathcal{U}_2 \subset \mathcal{U}$

1D / 2D
ROPE / MENBRANE
MODELS

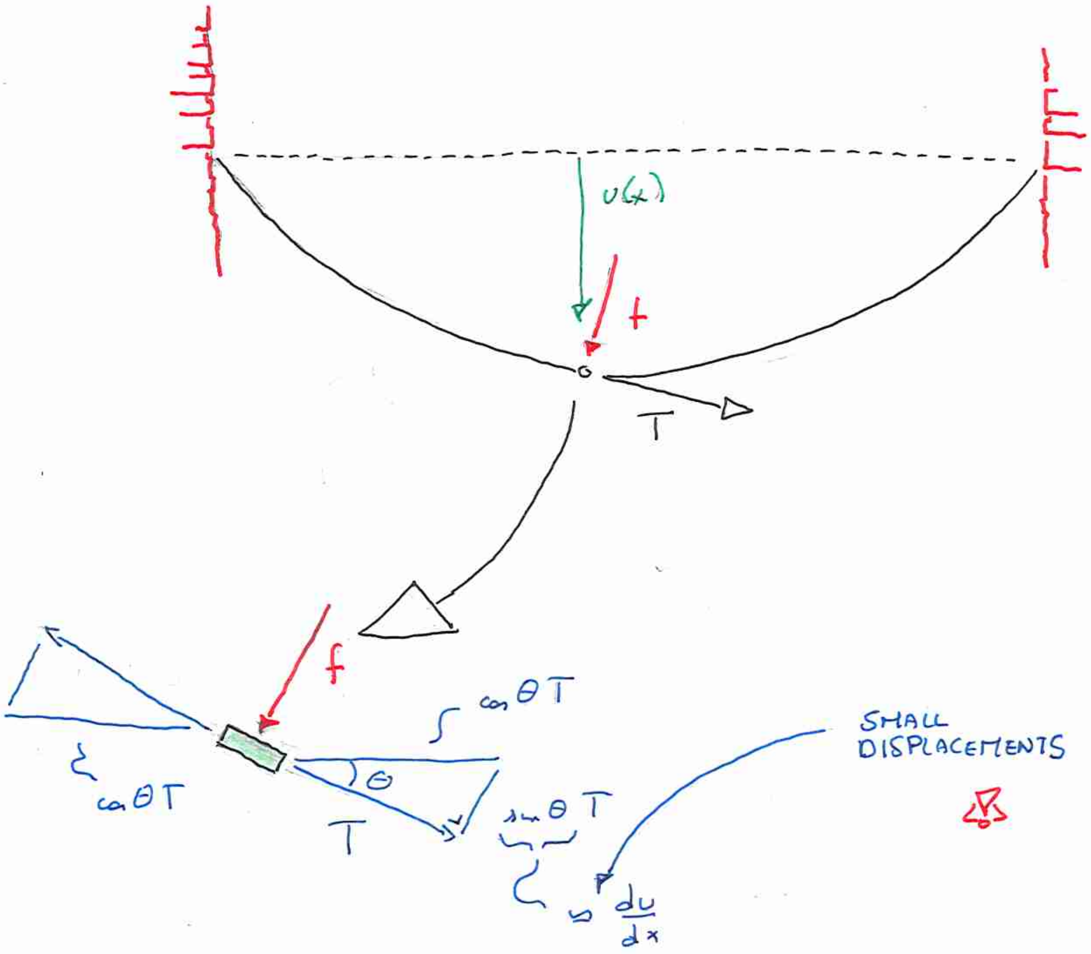
$\underset{\sim}{u} \simeq \underset{\sim}{u}_2 \in \mathcal{U}_1 \cancel{\subset} \mathcal{U}_2 \subset \mathcal{U}$

DISCRETE
APPROXIMATION

$\underset{\sim}{u}_1 \simeq \underset{\sim}{u}_1^h$

$\underset{\sim}{u}_2 \simeq \underset{\sim}{u}_2^h$

ROPE MODEL

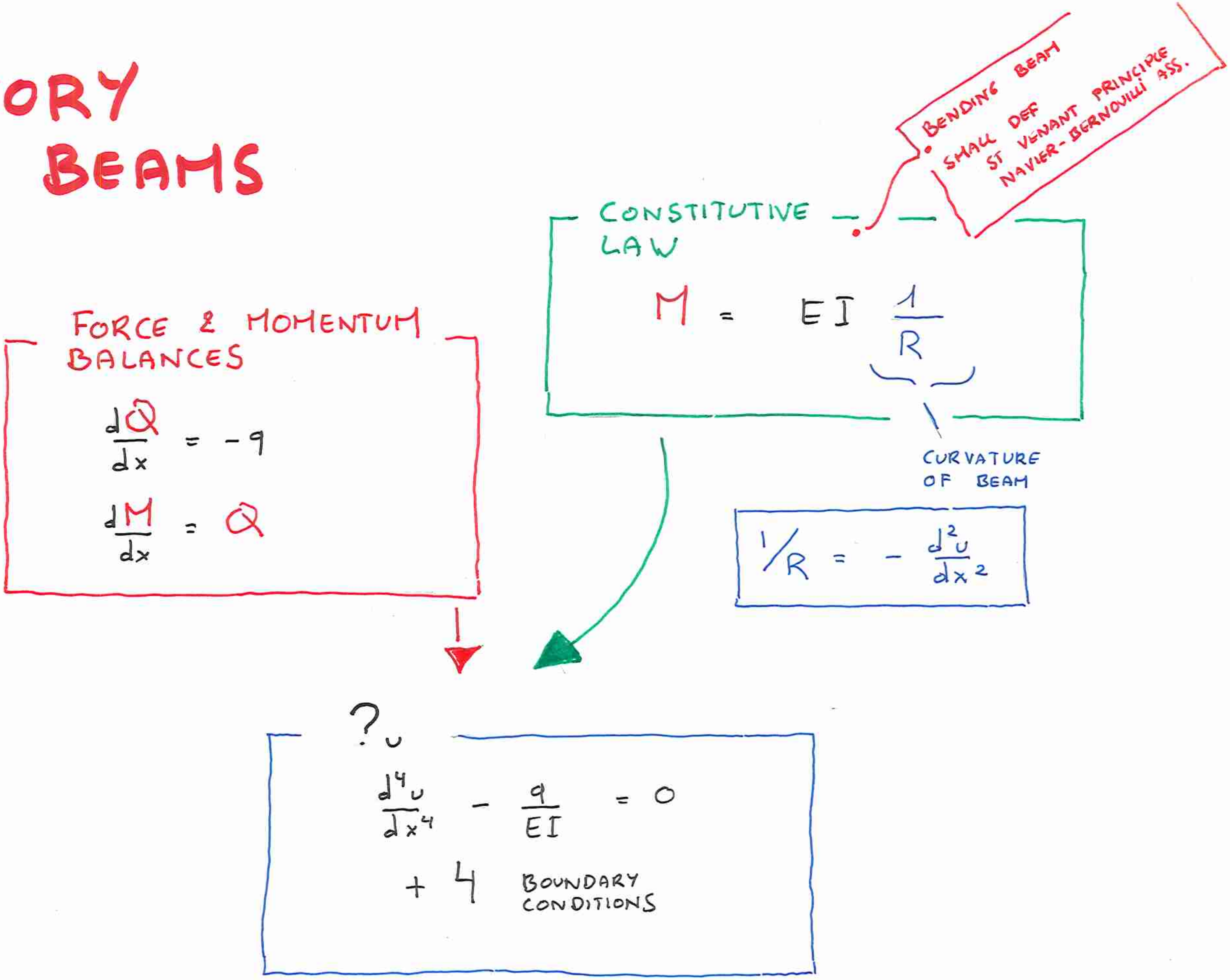


VERTICAL FORCE BALANCE

$$\frac{d}{dx}\left(T\frac{du}{dx}\right) = -f$$

$$T\frac{d^2u}{dx^2} + f = 0$$

THEORY OF BEAMS



FORCE & MOMENTUM BALANCES

$$\frac{dQ}{dx} = -q$$

$$\frac{dM}{dx} = Q$$

CONSTITUTIVE LAW

$$M = EI \underbrace{\frac{1}{R}}$$

- BENDING BEAM
- SMALL DEF
- ST VENANT PRINCIPLE
- NAVIER-BERNOVILLI ASS.

CURVATURE OF BEAM

$$1/R = -\frac{d^2u}{dx^2}$$

? u

$$\frac{d^4u}{dx^4} - \frac{q}{EI} = 0$$

$+ 4$ BOUNDARY CONDITIONS

$$\left\langle EI \frac{d^4u}{dx^4} \hat{u} \right\rangle = \left\langle q \hat{u} \right\rangle$$

$$\Big\downarrow$$

$$-\left\langle EI \frac{d^3u}{dx^3} \frac{d\hat{u}}{dx} \right\rangle = \left\langle q \hat{u} \right\rangle - \left[\overbrace{EI \frac{d^3u}{dx^3}}^{Q} \hat{u} \right]$$

$$\Big\downarrow$$

$$\underbrace{\left\langle EI \frac{d^2u}{dx^2} \frac{d^2\hat{u}}{dx^2} \right\rangle}_{a(u,\hat{u})} = \underbrace{\left\langle q \hat{u} \right\rangle - \left[Q \hat{u} \right] + \left[\underbrace{EI \frac{d^2u}{dx^2}}_{M} \frac{d\hat{u}}{dx} \right]}_{b(\hat{u})}$$

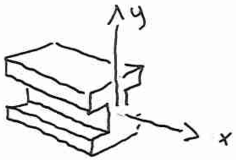
IMPOSING u OR Q

$\frac{du}{dx}$ OR M

AT BOTH ENDS IS EASY !

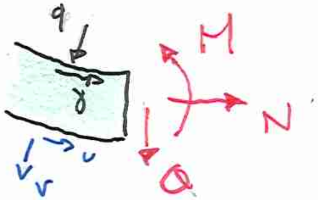
IT IS A MINIMIZATION PROBLEM...

$$J(u) = \underbrace{\frac{1}{2}\left\langle \underbrace{EI\frac{d^2u}{dx^2}}_{\substack{M(u)\\ \text{MOMENTUM}}}\ \underbrace{\frac{d^2u}{dx^2}}_{\substack{\chi(u)\\ \text{CURVATURE}}} \right\rangle}_{\text{DEFORMATION ENERGY}} \underbrace{- \langle q u \rangle + [Q u] - \left[M\frac{du}{dx}\right]}_{\text{WORK OF EXTERNAL FORCES}}$$



$$\frac{1}{2}\int_0^L\int_S \underbrace{\sigma}_{E y \frac{d^2u}{dx^2}}\ \underbrace{\varepsilon}_{y\frac{d^2u}{dx^2}}\ dS\, dx$$

TIMOSHENKO BEAM THEORY



CONSERVATION BALANCES

$$\frac{dN}{dx} = -\gamma$$

$$\frac{dQ}{dx} = -q$$

$$\frac{dM}{dx} = Q$$

CONSTITUTIVE LAW

$$M = -EI\,\frac{d^2v}{dx^2}$$

$$N = ES\,\frac{du}{dx}$$

$$EI\,\frac{d^4v}{dx^2} = q$$

$$ES\,\frac{d^2u}{dx^2} = -\gamma$$

BENDING OF A BEAM

HORIZONTAL FORCE OBSERVED

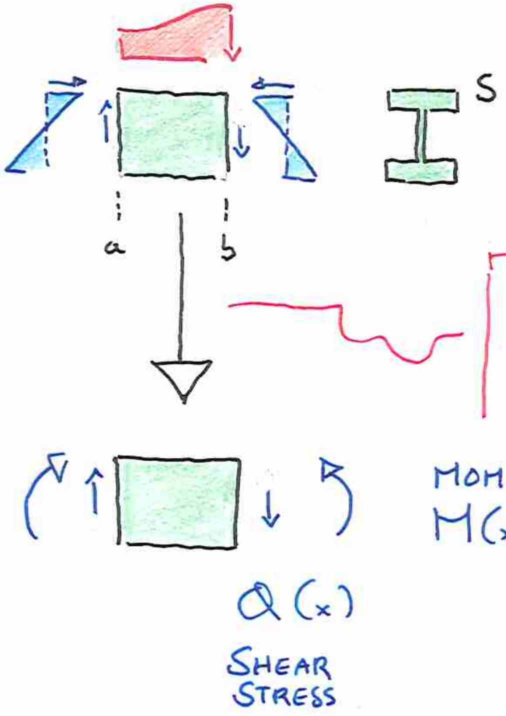
$$\overbrace{\int_S (\sigma)\, dS}^{=0} = 0$$

MOMENTUM OBSERVED

$$\underbrace{\int_S (\sigma)\,\eta\, dS}_{M} \underset{\underset{\text{HOOKE}}{\uparrow}}{=} \int_S \frac{E\eta^2}{R}\, dS$$

$$= \frac{E}{R}\underbrace{\int_S \eta^2\, dS}_{I = \frac{bh^3}{12}}$$

$$= EI \underbrace{\frac{1}{R}}_{-\frac{d^2u}{dx^2}}$$



THE PATTERN OF INTERNAL STRESS PRODUCES A TWISTING FORCE, A MOMENT ON THE MATERIAL

MOMENTUM
$M(x)$

FORCE BALANCE

$$\int_a^b q\,dx - Q_a + Q_b = 0$$

$$\downarrow$$

$$\int_a^b q + \frac{dQ}{dx}\,dx = 0$$

$$\boxed{\frac{dQ}{dx} = -q}$$

$$\boxed{\frac{dM}{dx} = Q}$$

MOMENTUM BALANCE

$$M_b - M_a - \int_a^b (x-a)\,q\,dx - (b-a)\,Q_b = 0$$

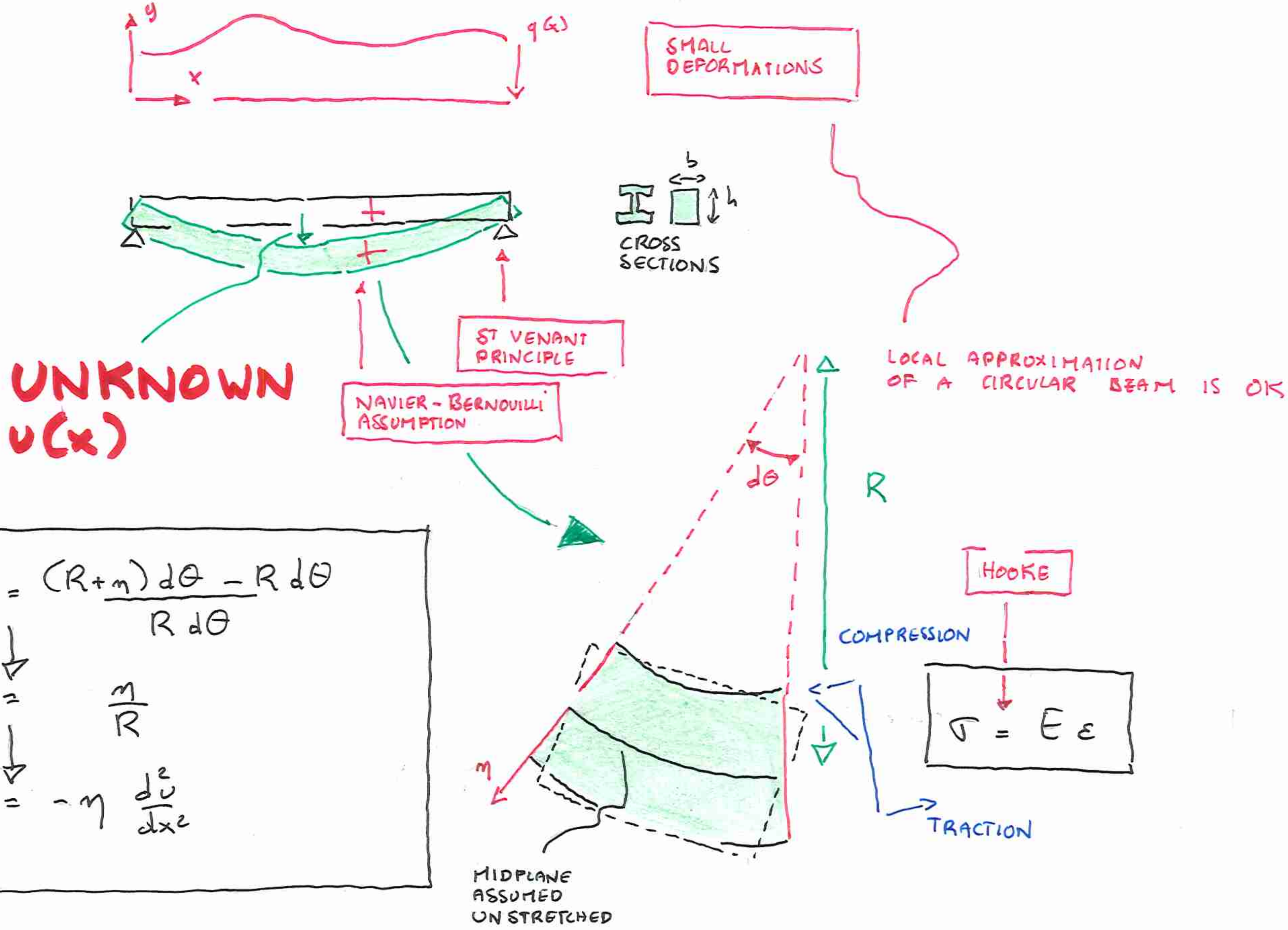
$$\downarrow$$

$$\int_a^b \left(\frac{dM}{dx} + (x-a)\,\frac{dQ}{dx} \right) dx - (b-a)\,Q_b = 0$$

$$\downarrow$$

$$\int_a^b \left(\frac{dM}{dx} - Q \right) dx - \cancel{(b-a)\,Q_b} = 0$$

$$+ \cancel{\Big[(x-a)\,Q \Big]_a^b}$$



$$\varepsilon = \frac{(R+\eta)\,d\theta - R\,d\theta}{R\,d\theta}$$

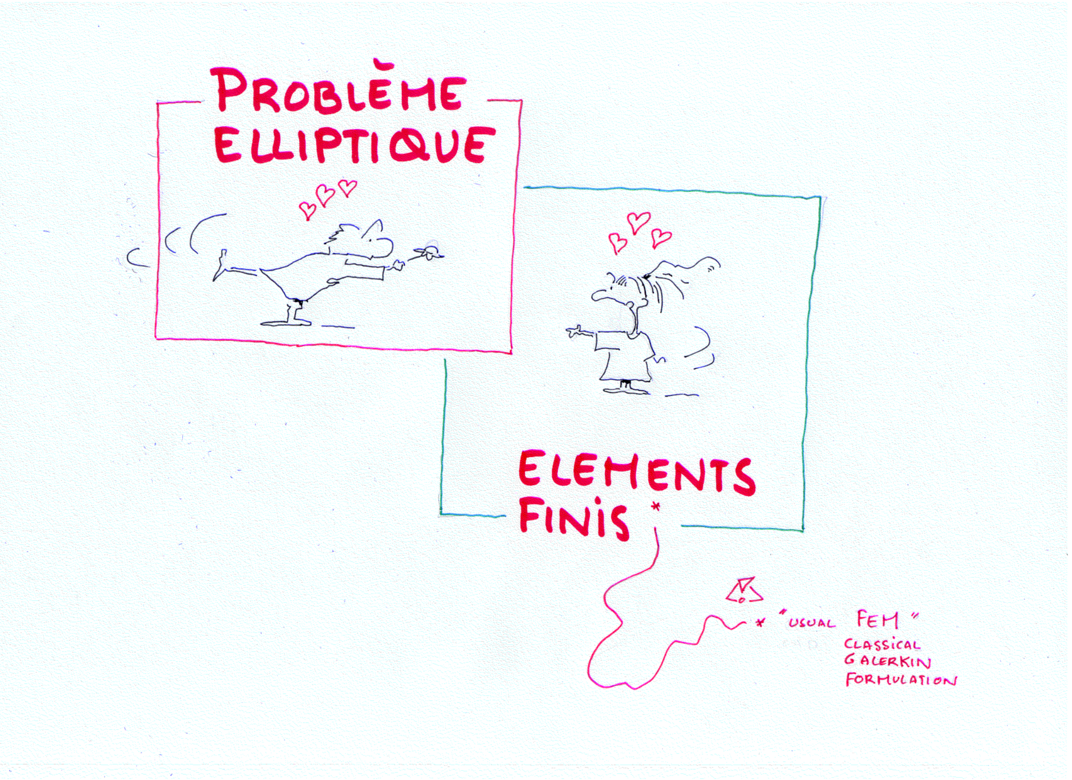
$$\downarrow$$

$$= \frac{\eta}{R}$$

$$\downarrow$$

$$= -\eta \frac{d^2 v}{dx^2}$$

$$\sigma = E\,\varepsilon$$



Galerkin, c’est donc optimal
pour des équations elliptiques