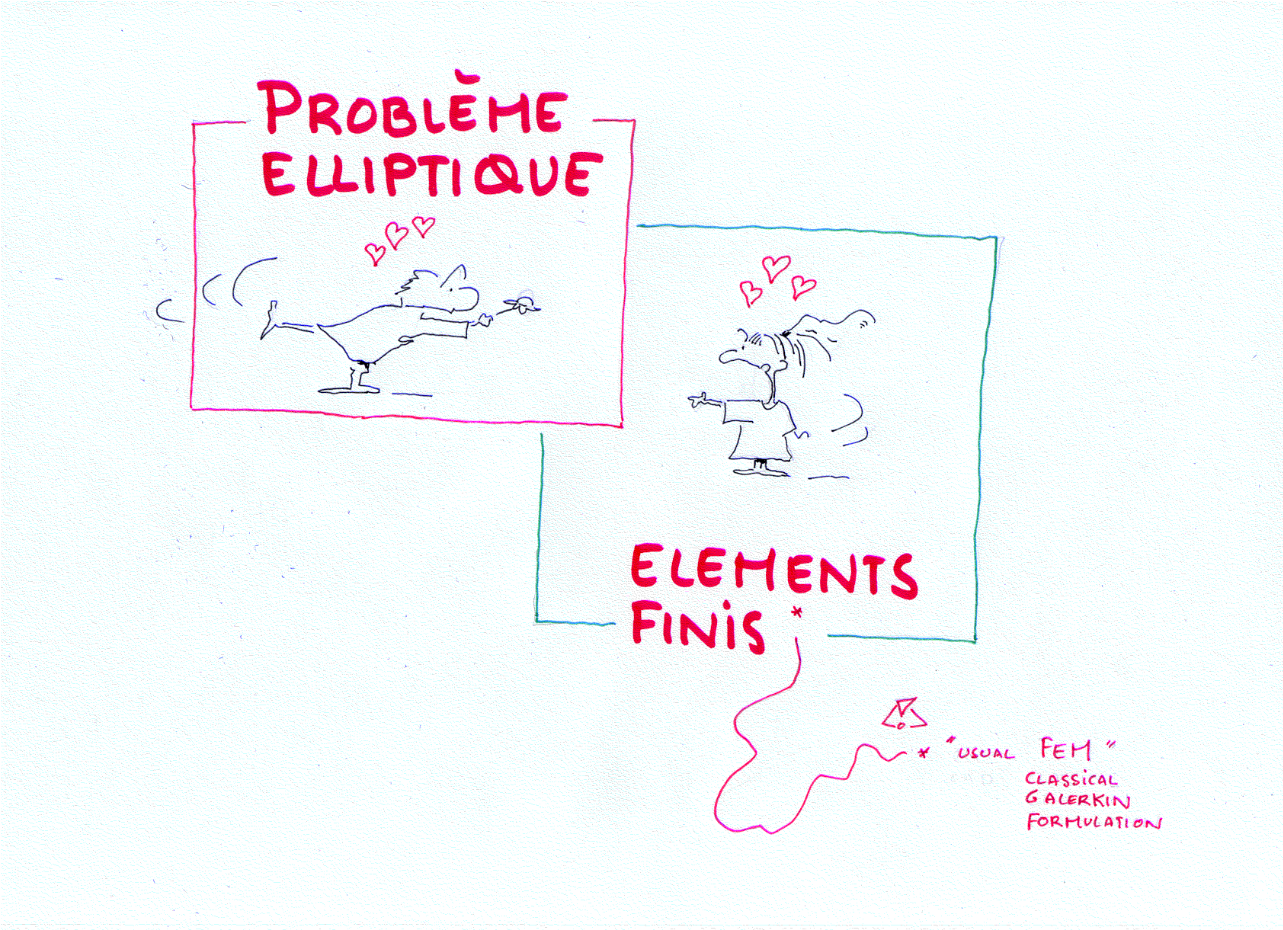
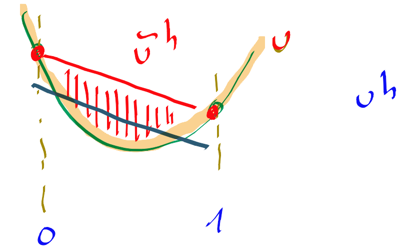


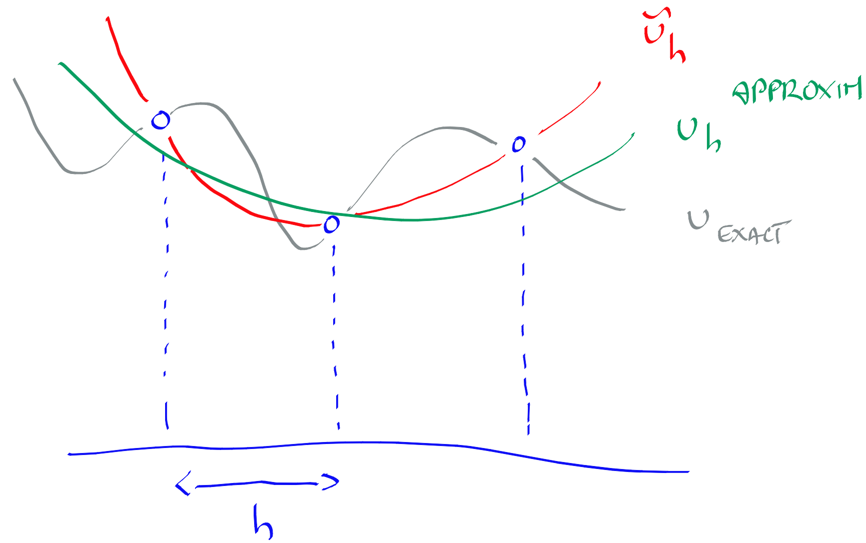
Un peu de mathématiques pour les éléments finis

Estimation a priori de l'erreur

$$\boxed{\int_0^1 (u-u^h)^2 \quad \text{MINIMISER}}$$



$$u^h = \sum U_i \tau_i \qquad \sum \langle \tau_i \tau_j \rangle U_j = \langle u \, \tau_i \rangle$$



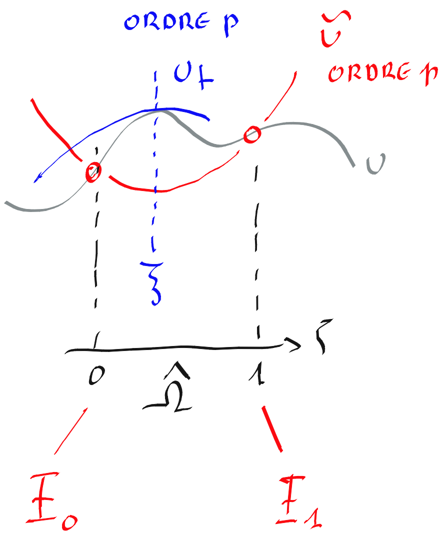
$$\boxed{\|e\|_m^2 \;\leq\; \tfrac{c}{\alpha}\, C^2\, h^{2(p+1-m)} \|u\|_{p+1}^2,}$$

Lemme de Bramble-Hilbert

Théorème de la meilleure approximation énergétique

Lemme de Cea

Erreur de l'interpolation
Lemme de Bramble-Hilbert



$$\frac{d^i}{d\xi^i}(\underbrace{u-\tilde{u}^h}_{\tilde{e}}) = \frac{d^i u}{d\xi^i} - \frac{d^i u^I}{d\xi^i} + \underbrace{\frac{d^i u^I}{d\xi^i}}_{\sum_{j=0}^{p} u^I(\Xi_j)\frac{d^i \phi_j}{d\xi^i}} - \sum_{j=0}^{p} u(\Xi_j)\frac{d^i \phi_j}{d\xi^i}$$

CAR $\tilde{u}^I = u^I$

$$\left|\frac{d^i \tilde{e}}{d\xi^i}\right| \leq \underbrace{\left|\frac{d^i u}{d\xi^i} - \frac{d^i u^I}{d\xi^i}\right|}_{\leq C \max\left|\frac{d u^{p+1}}{d\xi^{p+1}}\right|} + \sum_{j=0}^{p} \underbrace{\left|u^I(\Xi_j) - u(\Xi_j)\right|}_{\leq C \ldots} \underbrace{\left|\frac{d^i \phi_j}{d\xi^i}\right|}_{\leq C}$$

$$\leq C \max\left|\frac{d u^{p+1}}{d\xi^{p+1}}\right|$$

Lemme de Bramble-Hilbert

Théorème de la meilleure approximation énergétique

Lemme de Cea

Et on calcule finalement notre norme de Sobolev

$$\|\tilde{e}\|_m^2 = \sum_{e=1}^{n} \sum_{i=0}^{m} \underbrace{\int_{\Omega_e} \left(\frac{d^i \tilde{e}}{dx^i}\right)^2 dx}$$

$$= \left(\frac{h}{2}\right)^{1-2i} \underbrace{\int_{-1}^{1} \left(\frac{d^i \tilde{e}}{d\xi^i}\right)^2 d\xi}$$

$$\leq C^2 \underbrace{\int_{-1}^{1} \max \left|\frac{d^{p+1} u}{d\xi^{p+1}}\right|^2 d\xi}$$

$$\left(\frac{2}{h}\right)^{1-2(p+1)} \int_{\Omega_e} \max \left|\frac{d^{p+1}}{dx^{p+1}}\right|^2$$

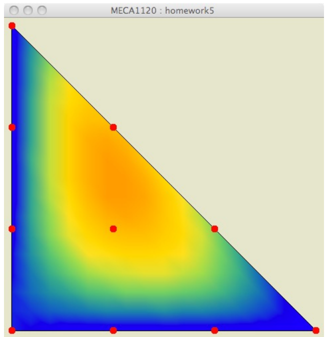
$$\leq C^2 \sum_{i=0}^{m} h^{2(p+1-i)} \|u\|_{p+1}^2$$

$$\boxed{\|\tilde{e}\|_m \leq C\, h^{p+1-m} \|u\|_{p+1}}$$

Lemme de Bramble-Hilbert

Théorème de la meilleure approximation énergétique

Lemme de Cea



Erreur de l'interpolation

$$\|\tilde{e}\|_m \leq C h^{p+1-m} \|u\|_{p+1},$$

lorsque h tend vers 0.

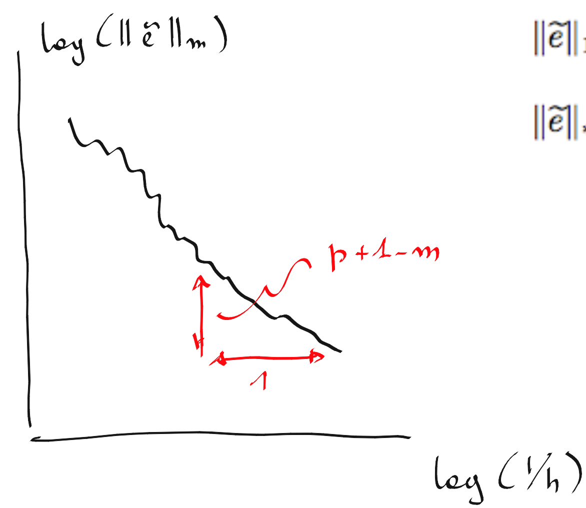
Lemme de Bramble Hilbert

Concrètement...

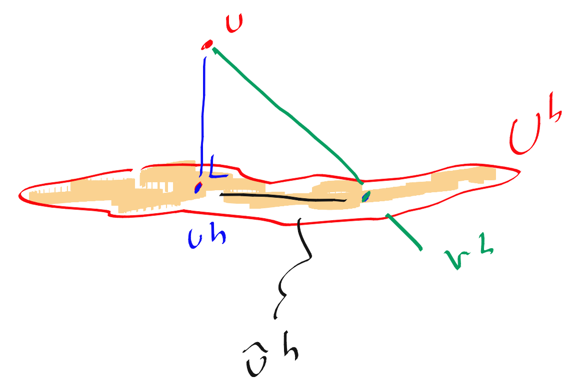
$$\|\tilde{e}\|_0 \leq C_0\, h^2\, \|u\|_2$$

$$\|\tilde{e}\|_1 \leq C_1\, h^1\, \|u\|_2$$

$$\|\tilde{e}\|_* \leq C_1\, h^1\, \|u\|_2$$



L'erreur énergétique des éléments finis est minimale !



$$a(u, \hat{v}) = b(\hat{v}) \qquad \forall \hat{v} \in \hat{\mathcal{U}}$$

$$a(u, \hat{v}^h) = b(\hat{v}^h) \qquad \forall \hat{v}^h \in \hat{\mathcal{U}}^h \subset \hat{\mathcal{U}}$$

$$a(u^h, \hat{v}^h) = b(\hat{v}^h) \qquad \forall \hat{v}^h \in \hat{\mathcal{U}}^h$$

$$a(\underbrace{u - u^h}_{e}, \hat{v}^h) = 0 \qquad \forall \hat{v}^h \in \hat{\mathcal{U}}^h$$

Lemme de Bramble-Hilbert

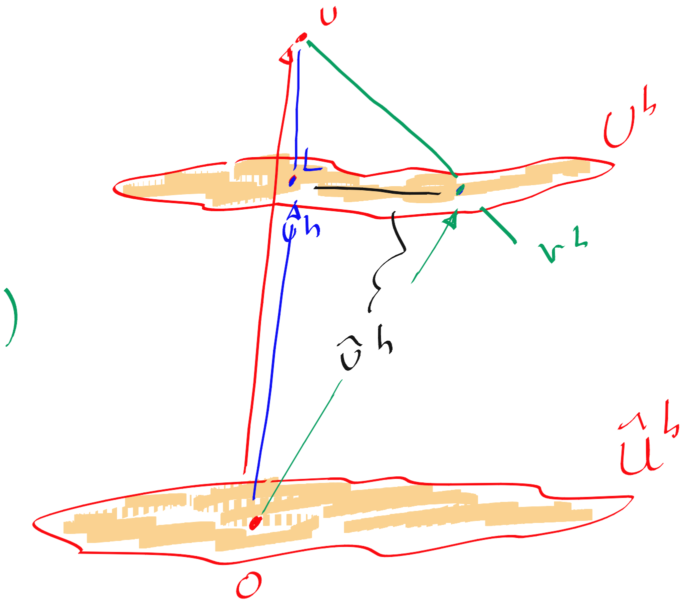
Théorème de la meilleure approximation énergétique

Lemme de Cea

$$a(u - v^h, u - v^h) =$$

$$a(\underbrace{u - u^h}_{e} + u^h - v^h, \; \ldots\ldots)$$

$$a(e, e) + \underbrace{a(\underbrace{u^h - v^h}_{\hat{v}^h}, u^h - v^h)}_{\geq 0} + 2\underbrace{a(e, \overbrace{u^h - v^h}^{\hat{v}^h})}_{= 0}$$



$$\boxed{\| u - v^h \|_* \geq \| e \|_*}$$

$$\| u - v^h \|_*^2 = \| e \|_*^2 + \| u^h - v^h \|_*^2$$

Si $U = \hat{U}$ $\quad \| u \|_*^2 = \| e \|_*^2 + \| u^h \|_*^2$

Les éléments finis,
c’est une projection orthogonale !

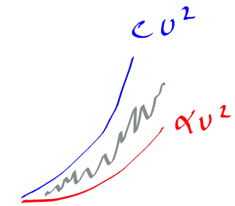
MEILLEURE APPROXIMATION ENERGETIQUE

CONTINUITE

$$\|e\|^2 \leq \frac{1}{\alpha}\|e\|_*^2 \leq \frac{1}{\alpha}\|\tilde{e}\|_*^2 \leq \frac{c}{\alpha}\underbrace{\|\tilde{e}\|^2}_{C^2\, h^{2(p+1-m)}\, \|u\|_{p+1}^2}$$

COERCIF !

Oui, mais c’est une norme bizarre, non ?



C’est le meilleure approximation mais dans une norme bizarre !

Erreur énergétique ?
Cela signifie quoi ?

Lemme de Bramble-Hilbert

Théorème de la meilleure approximation énergétique

Lemme de Cea

Lemme de Cea

$$\|e\|^2 \leq \frac{1}{\alpha}\|e\|_*^2 \leq \frac{1}{\alpha}\|\tilde{e}\|_*^2 \leq \frac{c}{\alpha}\|\tilde{e}\|^2,$$

En vertu de la continuité de a,

Car u^h est la meilleure approximation énergétique,

En vertu de la coercivité de a,

$$\boxed{\|e\|^2 \leq \frac{c}{\alpha}\|\tilde{e}\|^2}$$

Non, c'est aussi vrai dans toutes les normes de Sobolev !

Estimation de l'erreur

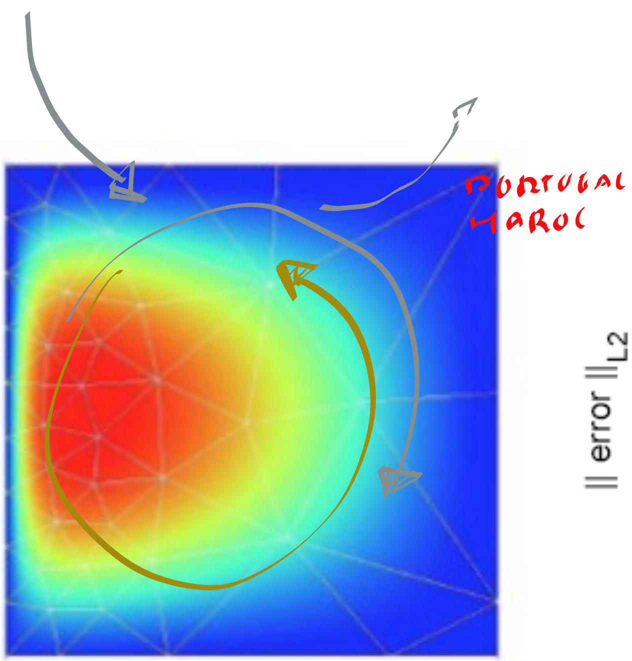
$$\|e\|_m^2 \leq \frac{c}{\alpha}\|\tilde{e}\|_m^2 \leq \frac{c}{\alpha} C^2\, h^{2(p+1-m)}\, \|u\|_{p+1}^2,$$

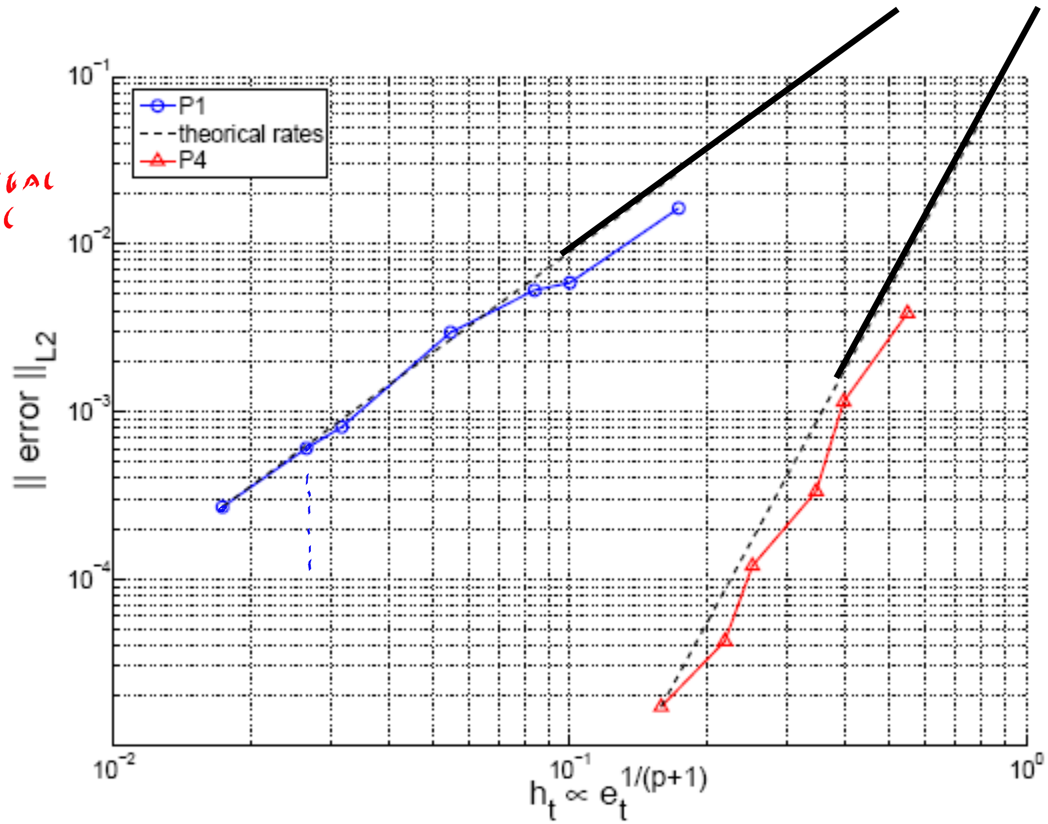
Estimation de l'erreur d'interpolation,

En vertu du lemme de Cea,

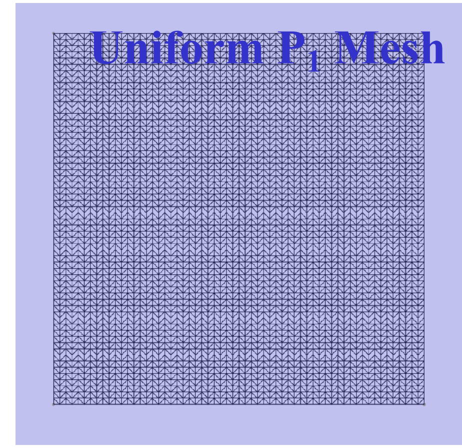
$$\boxed{\|e\|_m^2 \leq \frac{c}{\alpha} C^2\, h^{2(p+1-m)} \|u\|_{p+1}^2,}$$

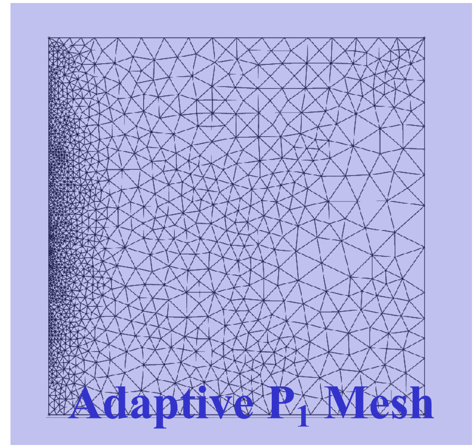
Theoretical rates of convergence are obtained for the analytical Stommel problem

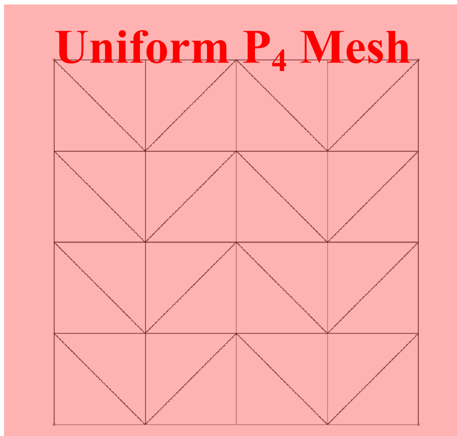


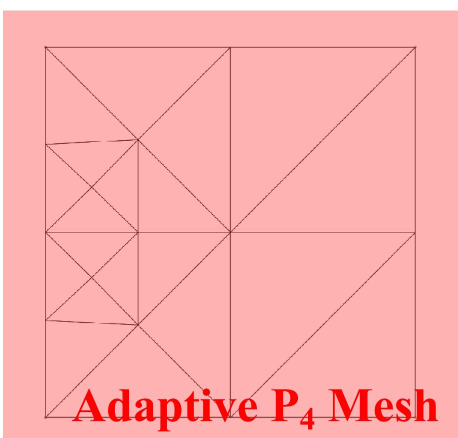


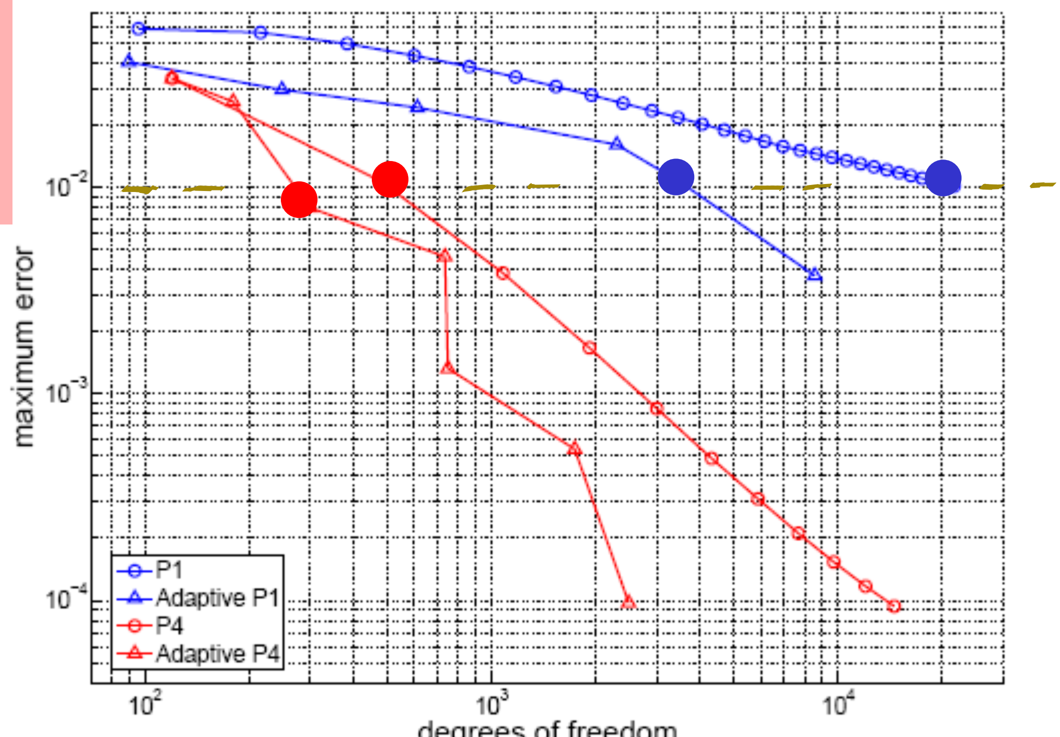
How does it converge ?











$$r^h = \nabla \cdot (a \nabla u^h) + f.$$

La technique de Galerkin consiste à annuler en moyenne le produit du résidu avec les fonctions de forme

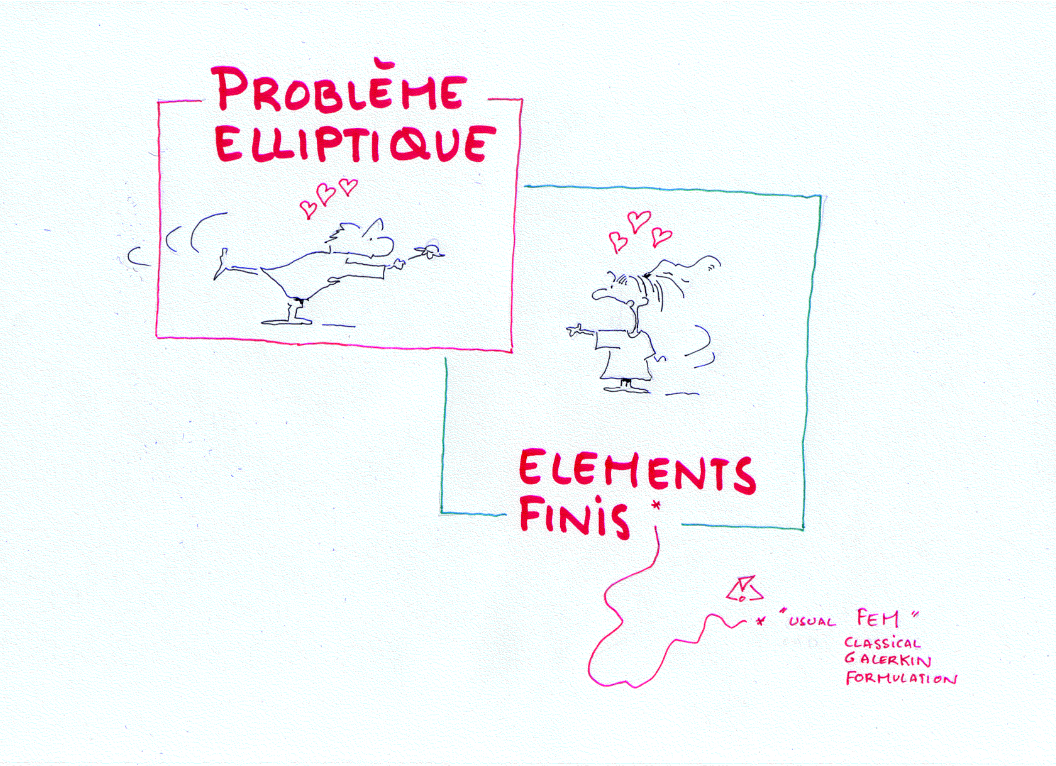
$u^h = \sum U_i \tau_i$

$$
\begin{aligned}
< \tau_i \, r^h > &= 0, \\
&\downarrow \\
< \tau_i \, (\nabla \cdot (a \nabla u^h)) > + < \tau_i \, f > &= 0, \\
&\downarrow \\
< (\nabla \tau_i) \cdot (a \nabla u^h) > &= < \tau_i f >, \\
&\downarrow \\
\sum_{j=1}^{n} \underbrace{< (\nabla \tau_i) \cdot (a \nabla \tau_j) >}_{A_{ij}} \, U_j &= \underbrace{< \tau_i f >}_{B_i}
\end{aligned}
$$

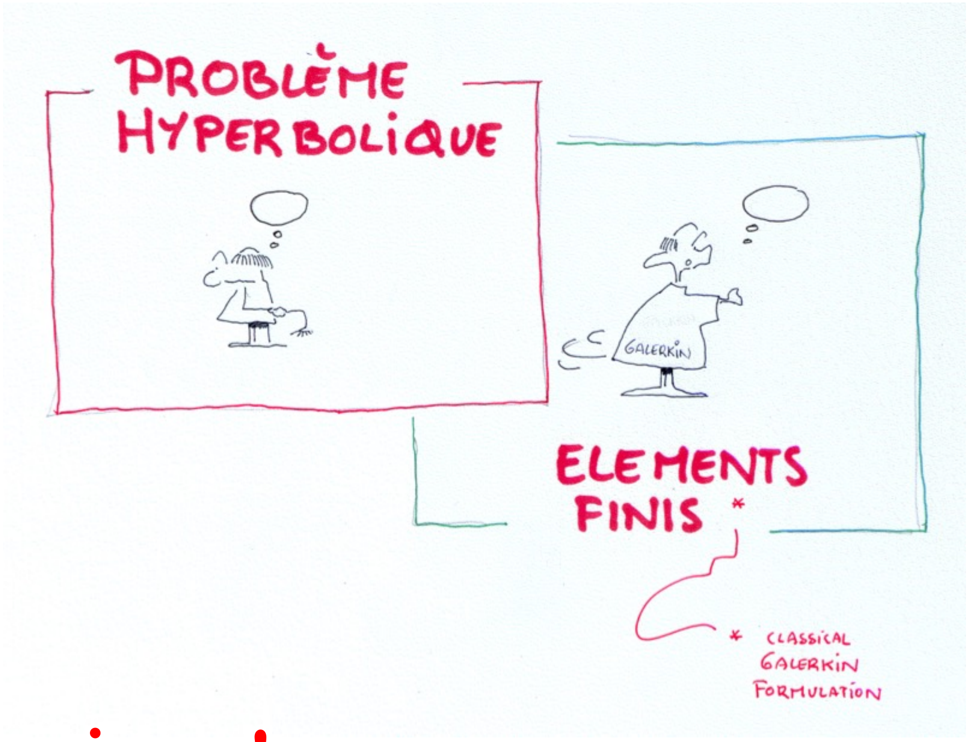
Et cette manière de procéder minimise l'erreur dans la norme L2 : c'est une formulation optimale en ce sens !

Galerkin 1871-1945

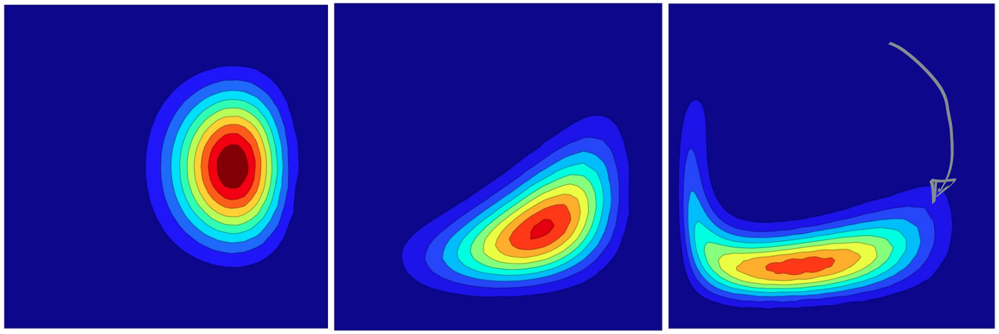
Les éléments finis sont une méthode de résidus pondérés

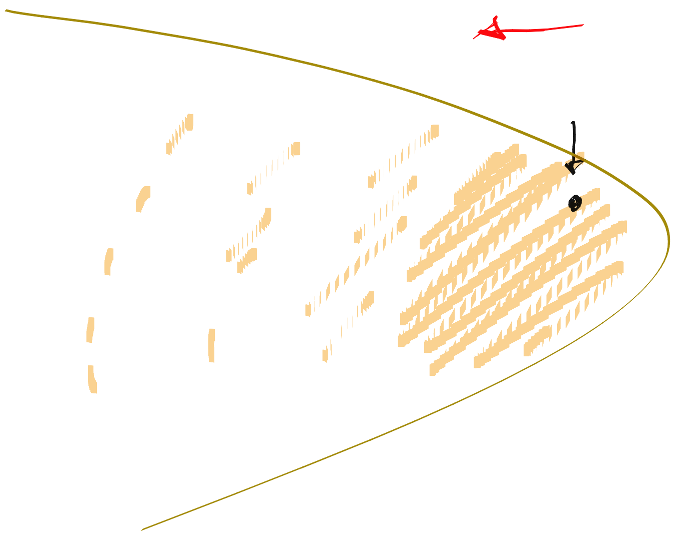


Galerkin, c’est donc optimal pour des équations elliptiques



Mais,
plus pour des
équations
d'advection diffusion !

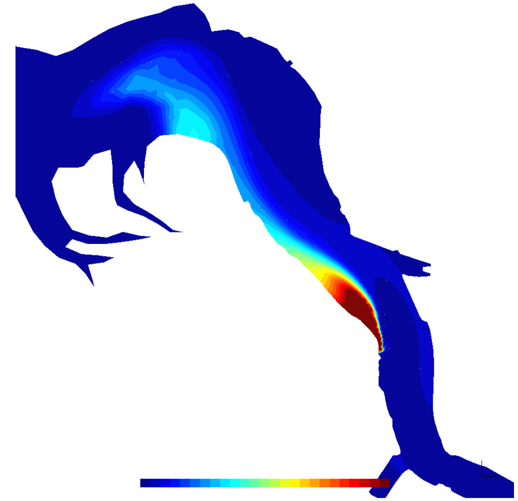


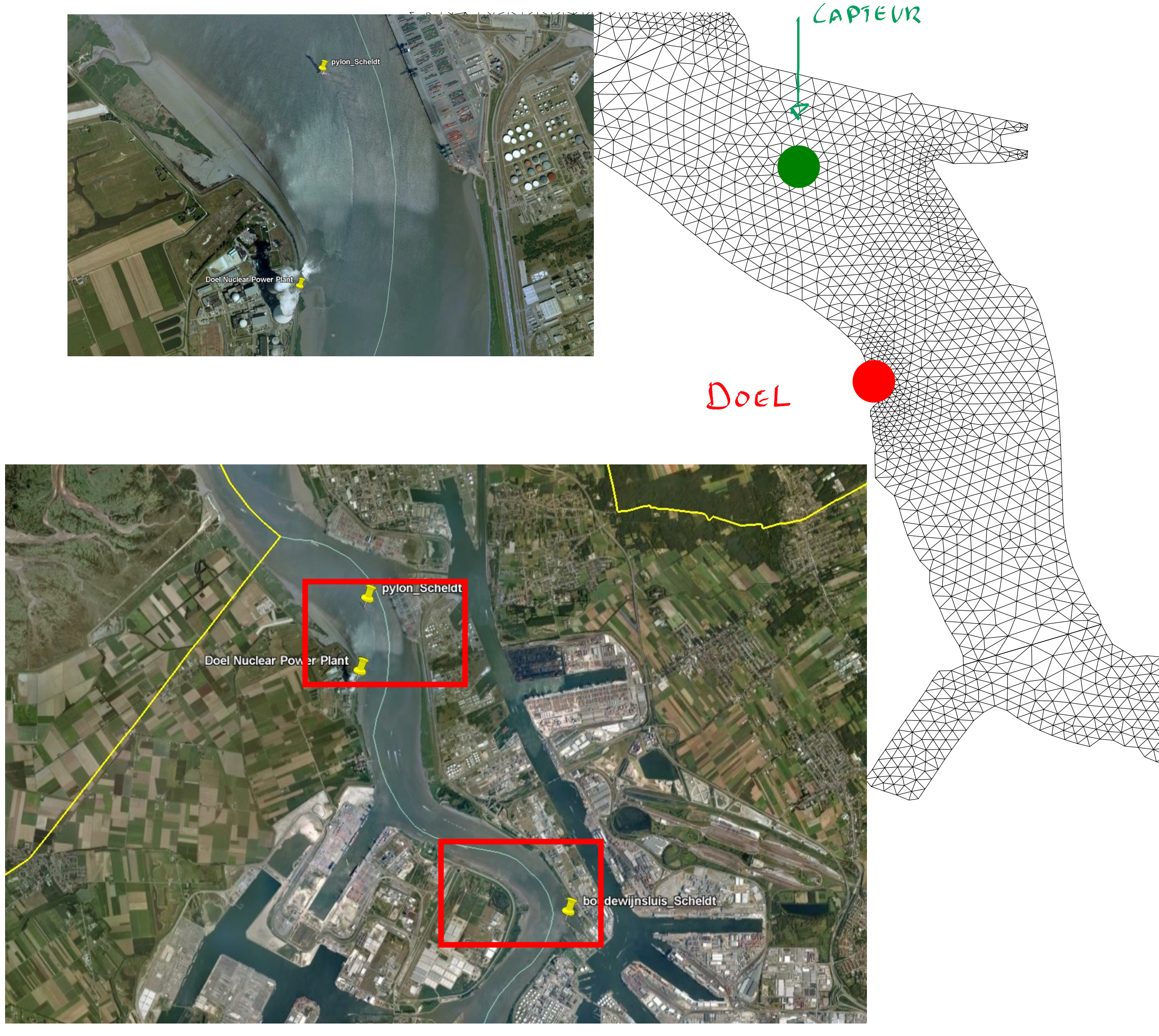


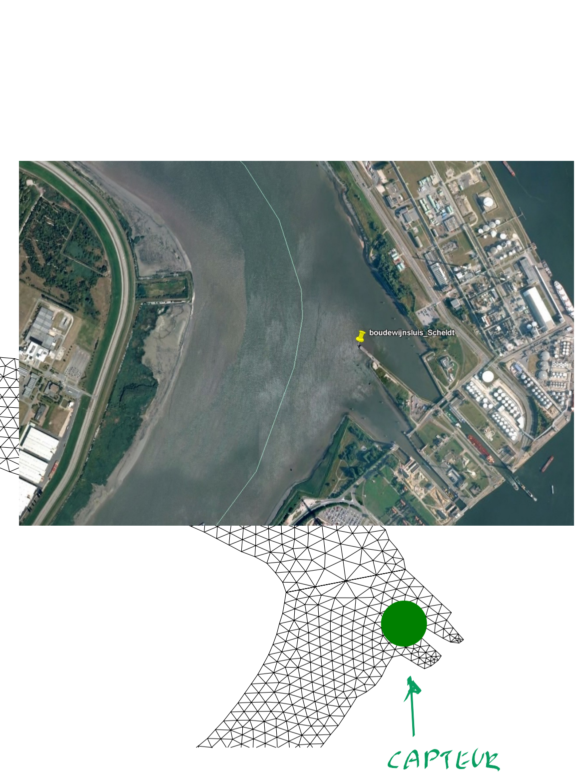

Smolensk
Minsk
Belarus
Mogilev
Krychaw
Cherykaw
Russia
Bryansk
Bykhov
Babruysk
Rahachow
Chachersk
Vetka
Gomel
Novozybkov
Luninets
Mazyr
Yelsk
Chernobyl Plant
Pershotravneve
Ovruch
Lypnyky
Korosten
Chernobyl
Chernihiv
Ukraine
Kiev
Zhytomyr
Confiscated/Closed Zone
Greater than 40 curies per square kilometer (Ci/km²) of Cesium-137
Permanent Control Zone
15 to 40 Ci/km² of Cesium-137
Periodic Control Zone
5 to 15 Ci/km² of Cesium-137
Unnamed zone
1 to 15 Ci/km² of Cesium-137

Tchernobyl…

Un petit exemple concret







Diffusion et transport d'un traceur passif

MOYENNE DE LA CONCENTRATION SUR LA PROFONDEUR

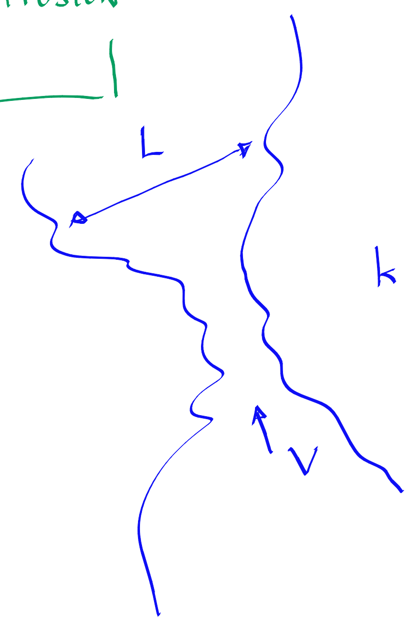
COEFFICIENT DE DIFFUSION

$$\overbrace{\overbrace{\frac{\partial \hat{c}}{\partial t}} + \boxed{\underbrace{\underline{v}\cdot\underline{\nabla} c}_{\text{TERME DE TRANSPORT}}}}^{\frac{Dc}{Dt}} = \boxed{\underbrace{\underline{\nabla}\cdot\left(k\, \underline{\nabla} c\right)}_{\text{TERME DE DIFFUSION}}} + S$$

$\underline{v}$ = VITESSE HORIZONTALE

$$\frac{\partial c}{\partial t} + \boxed{u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y}} = \boxed{k\left[\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2}\right]}$$

$$\partial\left(V\frac{\Delta c}{L}\right) \qquad \partial\left[\frac{k\,\Delta c}{L^2}\right]$$



$$P_e = \frac{\blacksquare}{\blacksquare} = \frac{V\,\cancel{\Delta c}}{L}\,\frac{L^2}{k\,\cancel{\Delta c}} = \frac{VL}{k}$$

Analysons les cas particuliers !

$$\frac{\partial c}{\partial t} + v\frac{\partial c}{\partial x} = k\frac{\partial^2 c}{\partial x^2}$$

$$P_e = \frac{VL}{k}$$

	STATIONNAIRE		INSTATIONNAIRE	
P_e PETIT	$0 = k\frac{\partial^2 c}{\partial x}$	ELLIPTIQUE 2	$\frac{\partial c}{\partial t} = k\frac{\partial^2 c}{\partial x}$	PARA 2
	$v\frac{\partial c}{\partial x} = k\frac{\partial^2 c}{\partial x^2}$	ELLIPTIQUE 2	$\frac{\partial c}{\partial t} + v\frac{\partial c}{\partial x} = k\frac{\partial^2 c}{\partial x^2}$	PARA 2
P_e GRAND	$v\frac{\partial c}{\partial x} = 0$	HYP 1	$\frac{\partial c}{\partial t} + v\frac{\partial c}{\partial x} = 0$	HYPERBOLIQUE 1

Le nombre de Péclet permet d'estimer l'importance du terme de transport par rapport à celui de la diffusion !

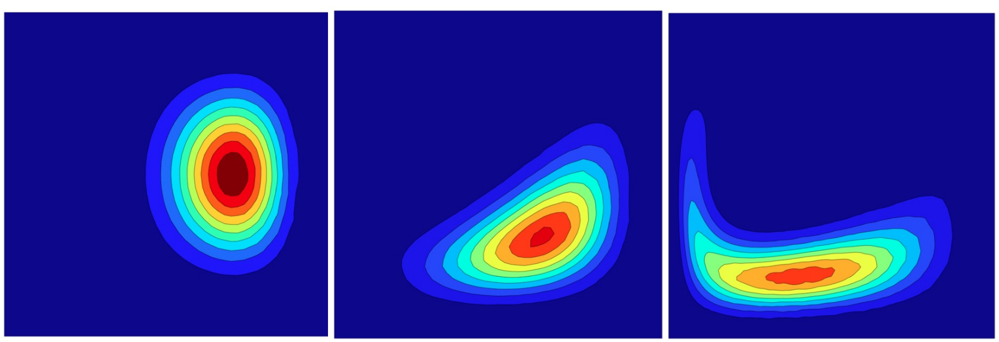
Diffusion et transport d'un traceur passif

MOYENNE DE LA CONCENTRATION SUR LA PROFONDEUR

COEFFICIENT DE DIFFUSION

$$\frac{\partial c}{\partial t} + \underbrace{\underset{\sim}{v} \cdot \underset{\sim}{\nabla} c}_{\text{TERME DE TRANSPORT}} = \underbrace{\underset{\sim}{\nabla} \cdot \left(D \, \underset{\sim}{\nabla} c \right)}_{\text{TERME DE DIFFUSION}} + S$$

$\underset{\sim}{v}$ = VITESSE HORIZONTALE



C'est une équation parabolique du second ordre !

Ce n'est pas elliptique !

$$\frac{\partial c}{\partial t} + v\frac{\partial c}{\partial x} = k\frac{\partial^2 c}{\partial x^2}$$

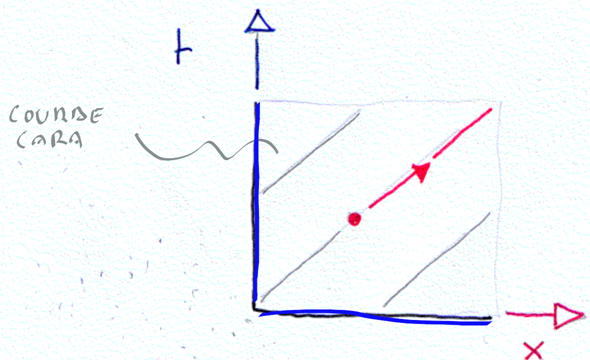
	STATIONNAIRE		INSTATIONNAIRE	
P_e PETIT	$0 = k\frac{\partial^2 c}{\partial x^2}$	ELL 2	$\frac{\partial c}{\partial t} = k\frac{\partial^2 c}{\partial x^2}$	PARA 2
	$v\frac{\partial c}{\partial x} = k\frac{\partial^2 c}{\partial x^2}$	ELL 2	$\frac{\partial c}{\partial t} + v\frac{\partial c}{\partial x} = k\frac{\partial^2 c}{\partial x^2}$	PARA 2
P_e GRAND	$v\frac{\partial c}{\partial x} = 0$	HYP 1	$\frac{\partial c}{\partial t} + v\frac{\partial c}{\partial x} = 0$	HYP 1

Le nombre de Péclet permet d'estimer l'importance du terme de transport par rapport à celui de la diffusion !

$$P_e = \frac{vL}{k}$$

PROBLEME BIEN POSE

→ CONDITIONS AUX LIMITES

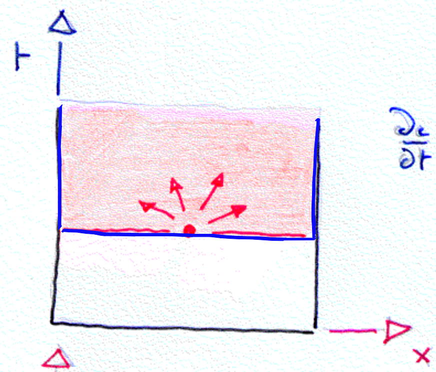


$$\frac{\partial c}{\partial t} + v\frac{\partial c}{\partial x} = 0$$

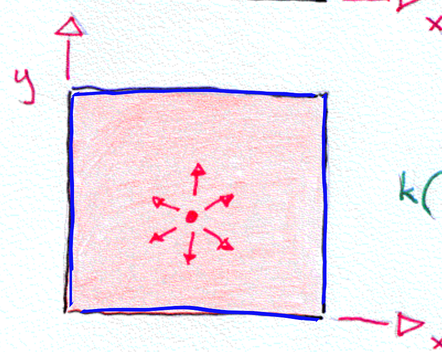
$$c(x - vt)$$

$$\frac{\partial c}{\partial t} = -v\, c'$$

$$\frac{\partial c}{\partial x} = c'$$

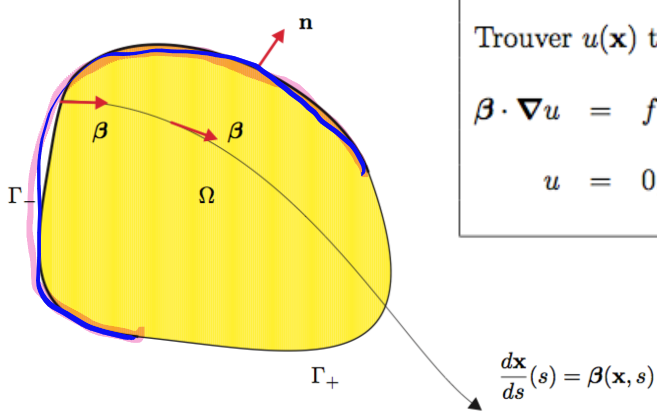


$$\frac{\partial c}{\partial t} = k\frac{\partial^2 c}{\partial x^2}$$



$$k\left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2}\right) = 0$$

Advection pure

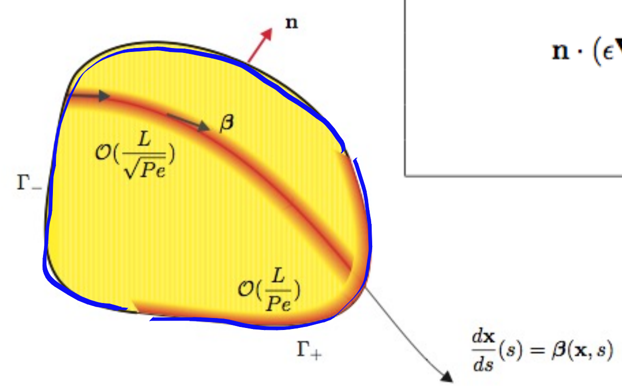


Trouver $u(\mathbf{x})$ tel que

$$\begin{aligned} \boldsymbol{\beta} \cdot \boldsymbol{\nabla} u &= f, \qquad \forall\, x \in \Omega, \\ u &= 0, \qquad \forall\, x \in \Gamma_- \end{aligned}$$

Trouver $u(\mathbf{x}) \in \mathcal{U}_s$ tel que

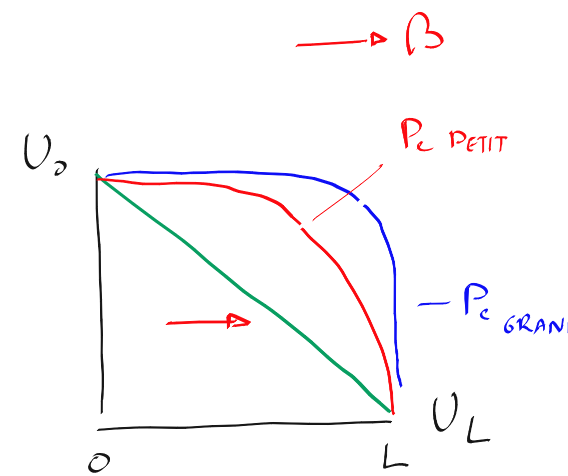
$$\begin{aligned} \boldsymbol{\beta} \cdot \boldsymbol{\nabla} u - \boldsymbol{\nabla} \cdot (\epsilon \boldsymbol{\nabla} u) &= f, \qquad \forall \mathbf{x} \in \Omega, \\ \mathbf{n} \cdot (\epsilon \boldsymbol{\nabla} u) &= g, \qquad \forall \mathbf{x} \in \Gamma_N, \\ u &= t, \qquad \forall \mathbf{x} \in \Gamma_D, \end{aligned}$$



Advection-diffusion

Equation d'advection-diffusion

$$P_e = \frac{\beta L}{\varepsilon}$$



$$\beta \frac{du}{dx} - \varepsilon \frac{d^2u}{dx^2} = 0$$

$$u(x) = C \exp\left[\frac{\beta x}{\varepsilon}\right]$$

$$O\left(\frac{L}{P_e}\right)$$

COUCHE LIMITE

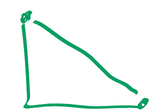
$$\varepsilon u'' = \frac{\beta^2}{\varepsilon^2}\, \varepsilon \exp[\,]\, C$$

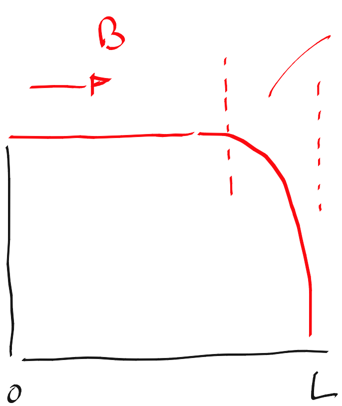
$$\beta u' = \frac{\beta}{\varepsilon}\, \beta \exp[\,]\, C$$

$$\varepsilon \frac{d^2u}{dx^2} = 0$$

$u(0) = U_0$

$u(L) = U_L$

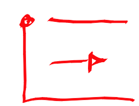




$$\boxed{\frac{u(x) - U_0}{U_L - U_0} = \frac{\exp\left[\frac{\beta x}{\varepsilon}\right] - 1}{\exp\left[\frac{\beta L}{\varepsilon}\right] - 1}}$$

$$\beta \frac{du}{dx} = 0$$

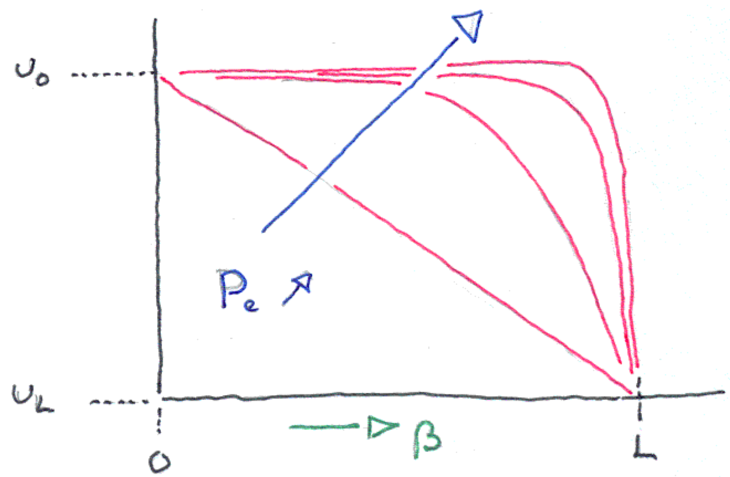
$u(0) = U_0$



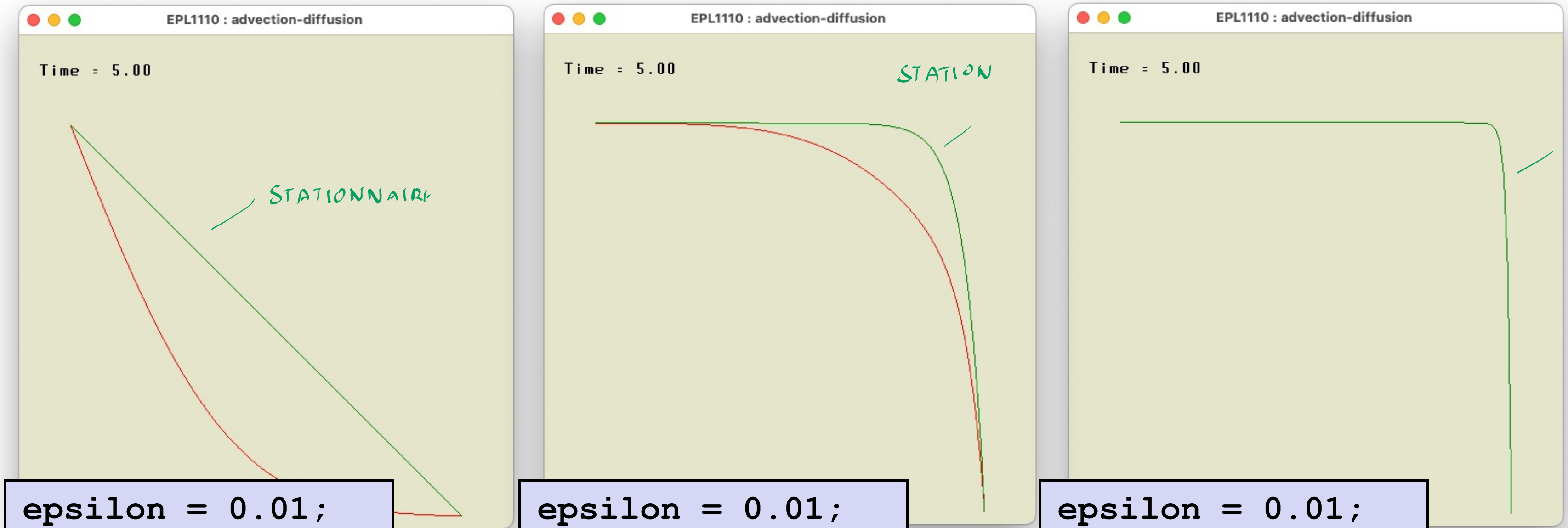
EQUATION D'ADVECTION - DIFFUSION

$$\boxed{\begin{aligned} &\beta \frac{du}{dx} - \varepsilon \frac{d^2u}{dx^2} = 0 \\ &u(0) = u_0 \\ &u(L) = u_L \end{aligned}}$$

$$\boxed{P_e = \frac{\beta L}{\varepsilon}}$$



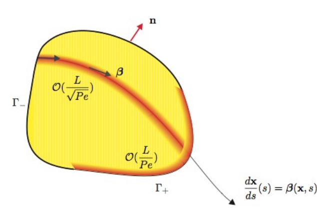
$$\frac{u - u_0}{u_L - u_0} = \frac{\exp\left(\overbrace{\beta x/\varepsilon}^{P_e\, x/L}\right) - 1}{\exp\left(\underbrace{\beta L/\varepsilon}_{P_e}\right) - 1}$$




```
epsilon = 0.01;
beta = 0.0;
```

```
epsilon = 0.01;
beta = 0.2;
```

```
epsilon = 0.01;
beta = 1.0;
```



Trouver $u(\mathbf{x}) \in \mathcal{U}_s$ tel que

$$\begin{aligned} \boldsymbol{\beta} \cdot \boldsymbol{\nabla} u - \boldsymbol{\nabla} \cdot (\epsilon \boldsymbol{\nabla} u) &= f, & \forall \mathbf{x} \in \Omega, \\ \mathbf{n} \cdot (\epsilon \boldsymbol{\nabla} u) &= g, & \forall \mathbf{x} \in \Gamma_N, \\ u &= t, & \forall \mathbf{x} \in \Gamma_D, \end{aligned}$$

Advection-diffusion

Equation de transport !
Advection pure !

$$\frac{du}{dx} = f$$

$$u(0) = U_0$$

$$u(x) \simeq u^h(x) = \sum U_i \, \tau_i(x)$$

GALERKIN

? U_i TEL QUE

$$\langle \tau_i \; r^h \rangle = 0$$

$$\sum_j \underbrace{\langle \tau_i \; \tau_{j,x} \rangle}_{A_{ij}} U_j = \overbrace{\langle \tau_i \; f \rangle}^{B_i}$$