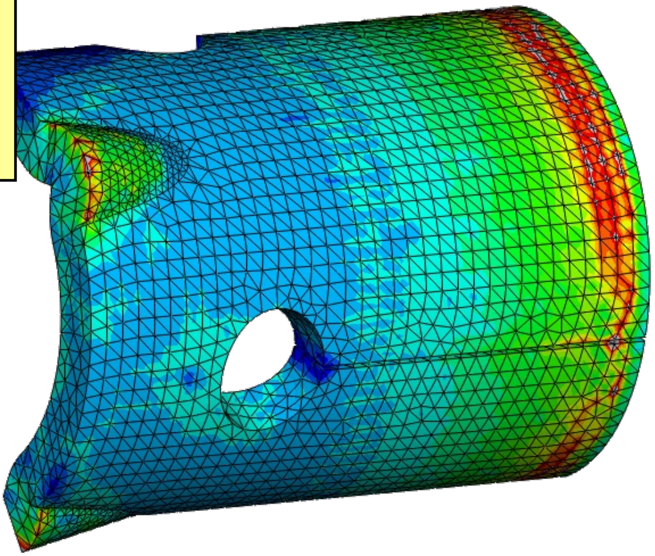
# A quoi cela sert
# les éléments finis ?

Antoine

# Plan du cours et évaluation

**Comment intégrer numériquement une fonction sur un carré ?**



**Comment résoudre l'élasticité linéaire ?**

***Evaluation continue***

***S1-S8 : 8 cours et 7 petits problèmes*** ***A***

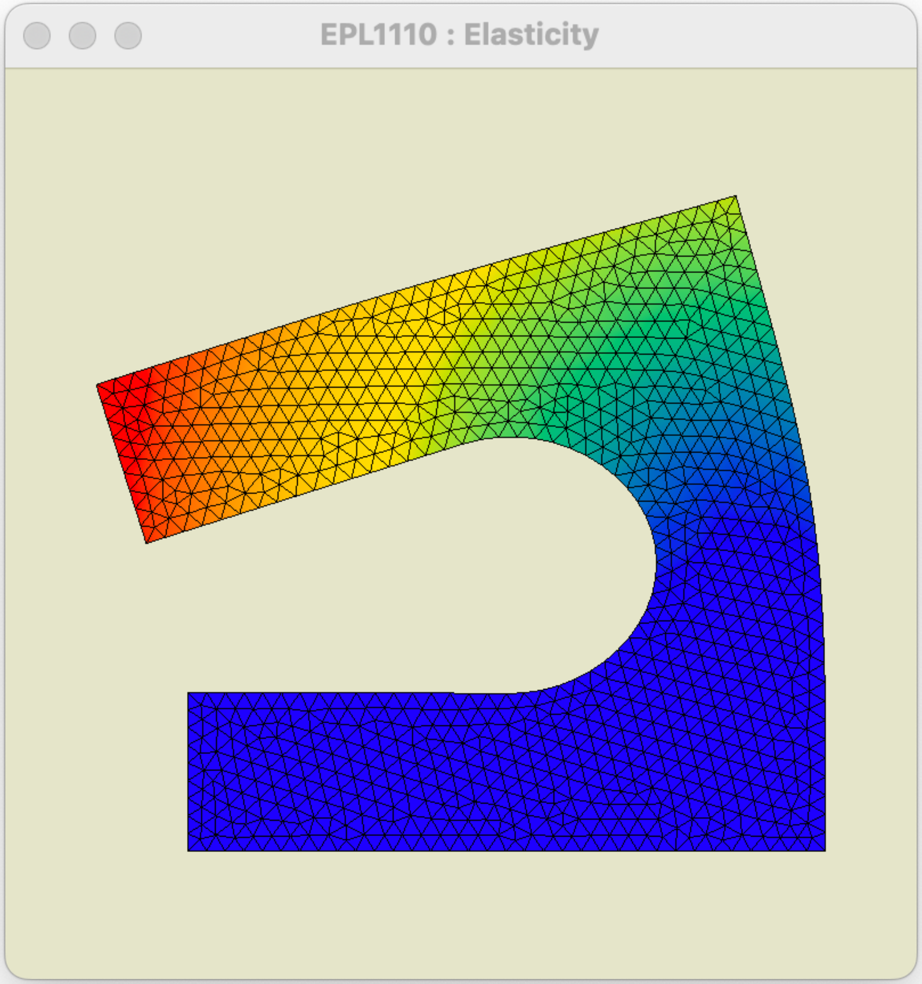
***Evaluation certificative***

***S7-S9 : mini-projet*** ***B***
***S9-S12 : 4 cours théorie éléments finis***
***En session : examen écrit*** ***C***

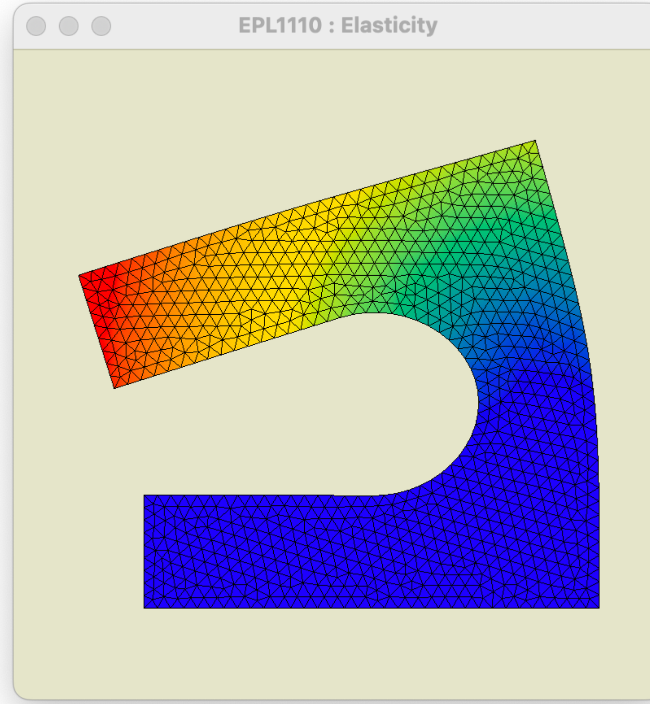
***En juin, note finale = (A+B+C)/3 si (B+C)/2 > 10***
***= (B+C)/2 sinon***
***En septembre, examen écrit***
***Pas de projet de rattrapage !***

# Evaluation

# Elasticité linéaire



# Hoping that deadlines are flexible ?




| | | |
|---|---|---|
| **Définition du problème**<br>Soumission du texte de deux pages<br>Approbation finale par l'assistants | <br>Vendredi 15 mars<br>Vendredi 22 mars | |
| **Soumission du projet**<br>Soumission du code du projet et du rapport | <br>Vendredi ~~12~~ avril | <br>19 |
| **Interview sur le projet**<br>Interview avec votre assistant de référence | <br>Jeudi 18 avril<br>Vendredi 19 avril | <br>25<br>26 |

| | |
|---|---|
| 10-08-10-10-00-00-00-00 | No problem submitted :-( |
| 10-10-10-08-00-00-00-00 | File problem.pdf :-) |
| 10-10-10-10-00-00-00-00 | No problem submitted :-( |
| 05-10-09-10-00-00-00-00 | No problem submitted :-( |
| 10-08-10-07-00-00-00-00 | No problem submitted :-( |
| 10-08-10-08-00-00-00-00 | No problem submitted :-( |
| 10-05-10-10-00-00-00-00 | No problem submitted :-( |
| 10-10-10-08-00-00-00-00 | No problem submitted :-( |
| 10-10-04-05-00-00-00-00 | No problem submitted :-( |
| 05-10-10-10-00-00-00-00 | File problem.pdf :-) |
| 10-10-00-07-00-00-00-00 | No problem submitted :-( |
| 10-10-10-10-00-00-00-00 | No problem submitted :-( |
| 10-06-10-10-00-00-00-00 | No problem submitted :-( |
| 10-10-00-10-00-00-00-00 | No problem submitted :-( |
| 10-10-10-10-00-00-00-00 | No problem submitted :-( |
| 10-08-10-10-00-00-00-00 | No problem submitted :-( |
| 09-10-10-09-00-00-00-00 | No problem submitted :-( |
| 09-10-10-09-00-00-00-00 | No problem submitted :-( |

# Votre soumission préliminaire…

## 1 Introduction

Pour le projet d'éléments finis, nous avons décidé d'analyser les forces appliquées par une personne qui est sur une échelle verticale.

## 2 Variantes

Notre problème de base concerne une personne qui est debout sur l'échellon d'une échelle classique telle que représentée à la figure 1 mais nous aimerions aussi proposer d'autres variantes:

- Notre personnage est placé sur des échellons à différents niveaux pour voir les différents cas de charge.
- Notre personnage est en train de monter sur l'échelle et ses mains appliquent également une force sur des échellons supérieurs à celui où ses pieds sont posés.
- Notre échelle a une forme différente, plus large ou avec des échellons de largeurs différentes.

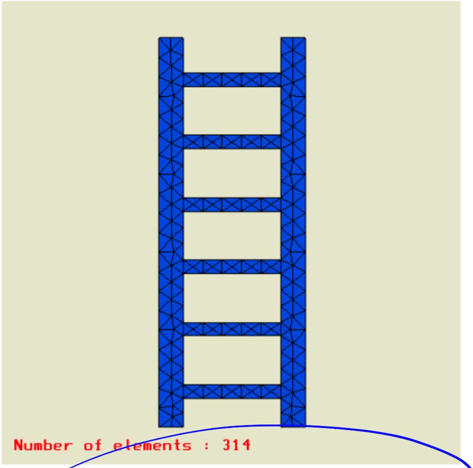



Figure 1: Résultats du filtre de Butterworth.

Votre soumission préliminaire…

$\left[\frac{N}{m^2}\right]$ $[N]$ $[N/m^2]$

1. Sur $\Omega_1$

$$\sigma.n = F_1 = p_1 * S_1$$

$[m^2]$

2. Sur $\Omega_2$

$$\sigma.n = F_2 = p_0 * S_2$$

où $S_1$ et $S_2$ représentent les surfaces à l'intérieur et à l'extérieur du réservoir, $p_1$ la pression intérieure (plus au moins 700 bars )[1] et $p_0$ la pression atmosphérique évaluée à 1 bar. Il s'agit donc de conditions frontières de Neumann.
Nous poserons également les hypothèses suivantes :

1. répartition du gaz constante (forces appliquée constantes)
2. Problème axisymétrique

Nous devrons naturellement détailler les surfaces et leur géométrie (approximation a 2 cylindres).
Il est à noter que puisque le problème contient des symétries nous pouvons réaliser une économie sur les calculs et le maillage en ne représentant qu'un quart du problème en imposant donc une condition de Dirichlet nul sur les surfaces de coupure sans perte de généralité. Nous pensions également rajouter une vanne de pression à la bonbonne afin de complexifier un peu le maillage mais il faudrait évaluer l'impact que cela aurait sur les équations et les conditions aux frontières.

## 2 Géométrie et Maillage

Notre maillage aurait la forme suivante :

$\underline{\underline{\sigma}} : \underline{n} = p\,\underline{n}$

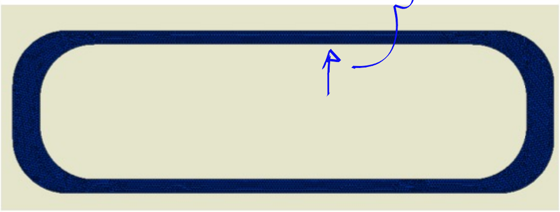


FIGURE 1 – Exemple du Maillage

Ce qui correspondrait à la géométrie de notre problème sur le schéma suivant :

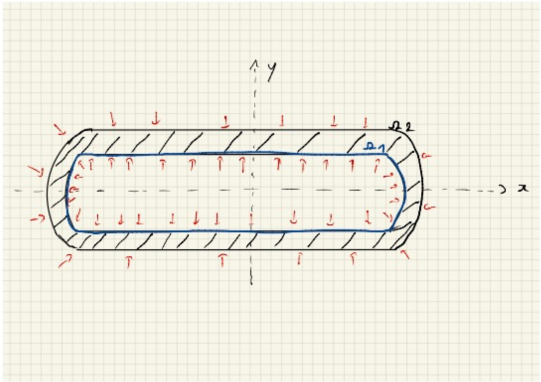



FIGURE 2 – Géométrie du problème

# 3D LINEAR ISOTROPIC ELASTICITY

$$\begin{bmatrix} \times & & \\ \circ & \times & \\ \circ & \circ & \times \end{bmatrix}$$

SHEAR STRESSES

NORMAL STRESSES

HOOKE ELASTIC BODY
SMALL DEF.

CONSTITUTIVE LAW

$$\underline{\underline{\sigma}} = \underbrace{\frac{E}{(1+\nu)}}_{2\mu} \underline{\underline{\varepsilon}} + \underbrace{\frac{E\nu}{(1+\nu)(1-2\nu)}}_{\lambda} \operatorname{tr}(\underline{\underline{\varepsilon}}) \underline{\underline{\delta}}$$

DEFORMATION TENSOR

$$\underline{\underline{\varepsilon}} \triangleq \frac{1}{2}\left(\underline{\nabla}\,\underline{u} + (\underline{\nabla}\,\underline{u})^T\right)$$

CONSERVATION LAWS

$$\underline{\nabla} \cdot \underline{\underline{\sigma}} + \underline{f} = 0$$

$$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T$$

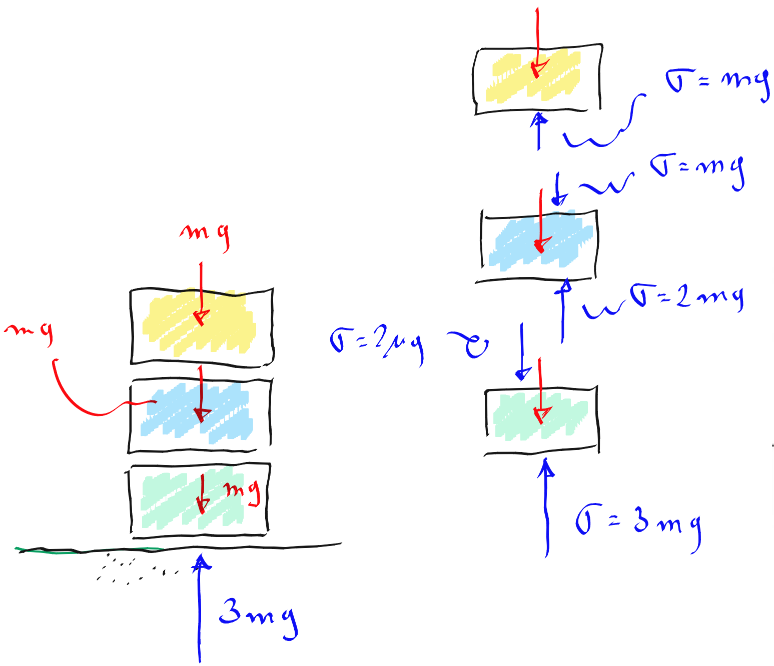
$? \, \underline{u}$

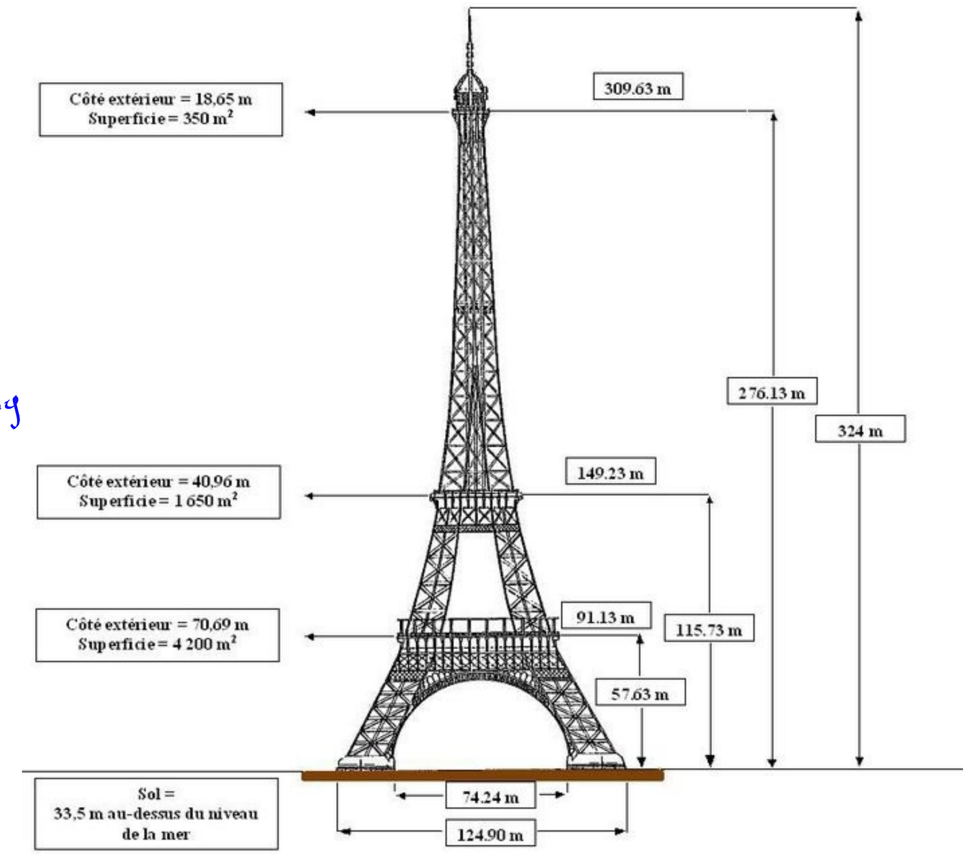
$$\underline{\nabla} \cdot \underline{\underline{\sigma}}(\underline{u}) + \underline{f} = 0 \quad \text{in } \Omega$$

$$\underline{\underline{\sigma}} \cdot \underline{n} = \underline{g} \quad \text{on } \Gamma_N$$

$$\underline{u} = 0 \quad \text{on } \Gamma_D$$

# Monsieur,
# j'ai pas eu MMC, moi ?

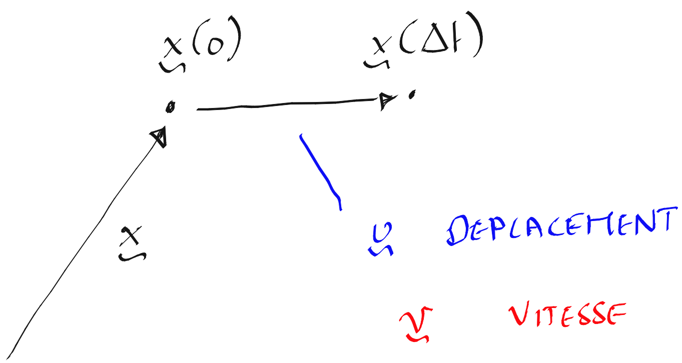


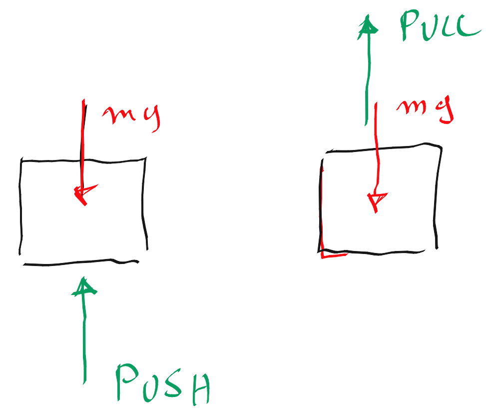




# Ouuuppss !
# Mais, MMC, MSD et Flotte : quelle horreur !

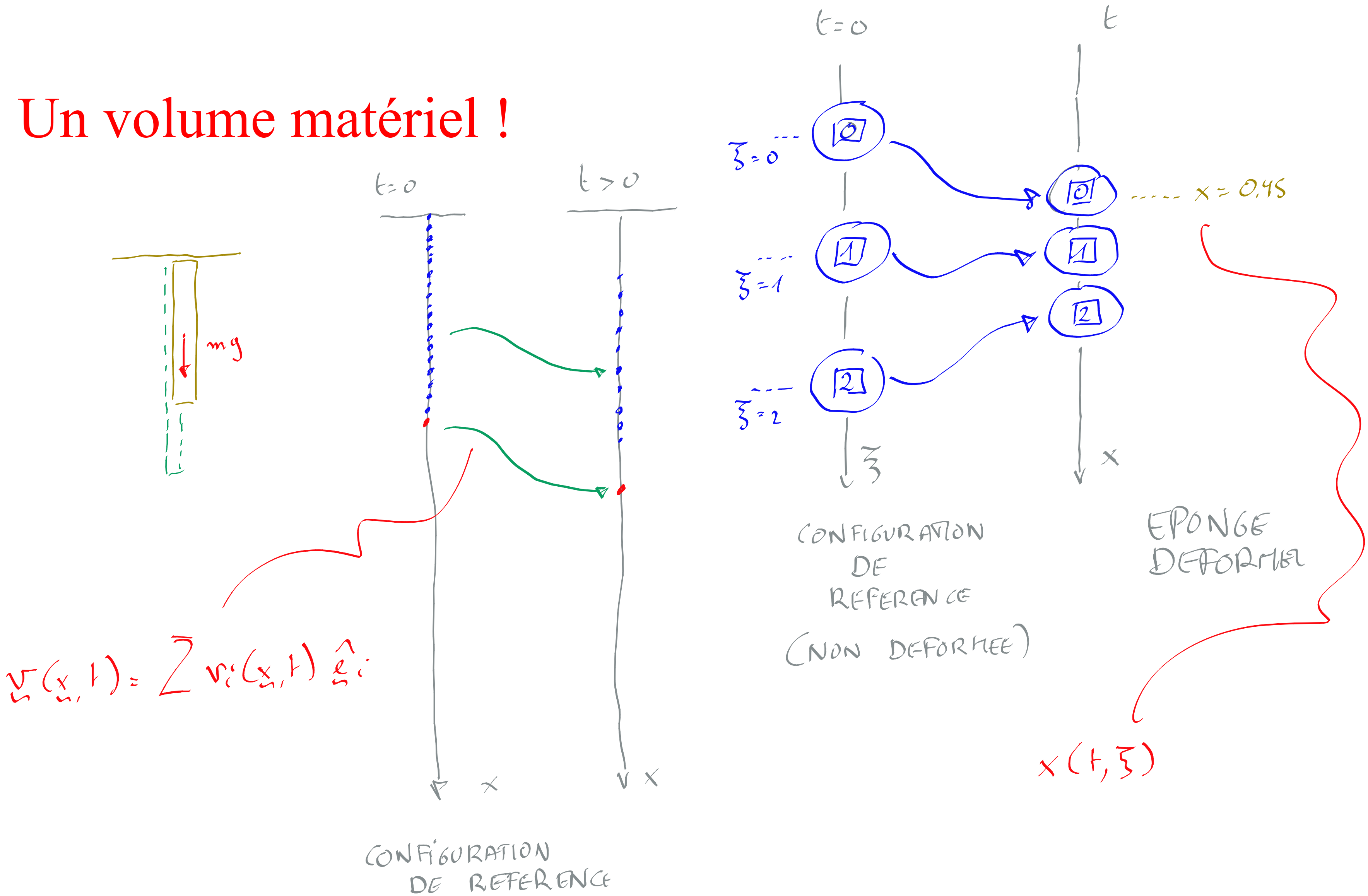
Souviens-toi des anciens !
Le grand Newton !
L’inoubliable Roland !

$$m \frac{d^2 \underset{\sim}{x}}{dt^2}(t) = \sum \underset{\sim}{F}$$

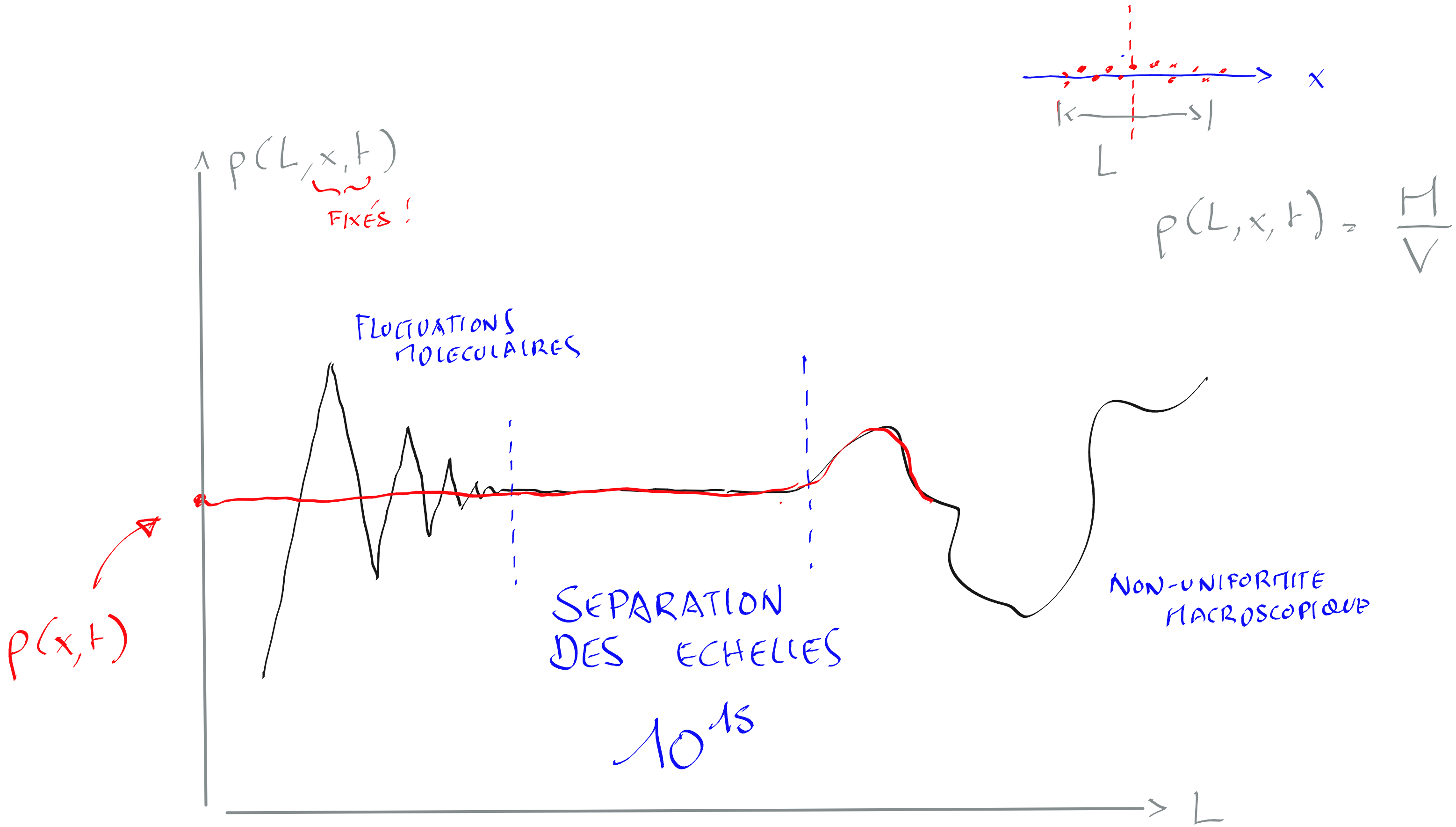


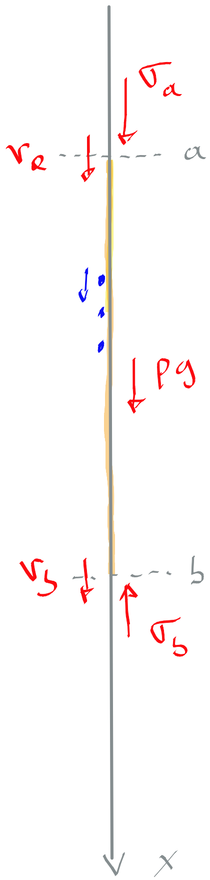
# Un volume matériel !



# La mécanique des milieux continus !
# C’est une arnaque !



# Bilan
# de quantité de mouvement




$\forall$ INTERVALLE $a,b$

ACCROISSEMENT QUANTITÉ DE MVT = CE QUI ENTRE − CE QUI SORT + FORCES

$$\frac{d}{dt}\int_a^b \rho v \, dx = \underbrace{\rho_a v_a^2 - \rho_b v_b^2}_{-\left[\rho v^2\right]_a^b} + \int_a^b \rho g + \underbrace{(-\sigma_a) - (-\sigma_b)}_{\left[\sigma\right]_a^b}$$

$$-\left[\rho v^2\right]_a^b = -\int_a^b \frac{\partial}{\partial x}(\rho v^2)$$

$$\left[\sigma\right]_a^b = \int_a^b \frac{\partial \sigma}{\partial x}$$

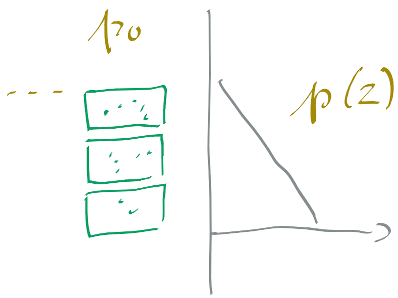
$$\underbrace{\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho v^2)}_{\rho \frac{Dv}{Dt}} = \rho g + \frac{\partial \sigma}{\partial x}$$

# Bilan
# de quantité de mouvement

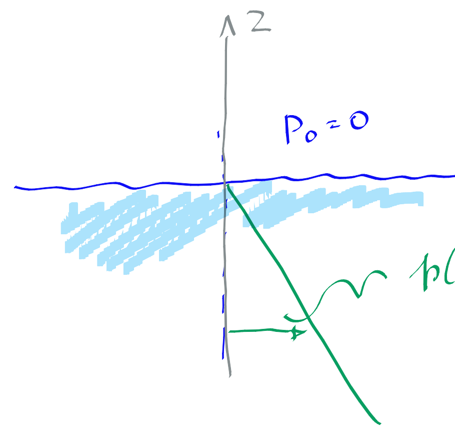
PROBLEME STATIONNAIRE
PAS DE MOUVEMENT

$$\frac{Dv}{Dt} = 0$$

$$\boxed{0 = \rho g + \frac{\partial \sigma}{\partial x}}$$




$p(z) = p_0 - \rho g z$




$p(z) = p_0 - \rho g z$

PRESSION
HYDROSTATIQUE

# Loi de Hooke !

# So easy !

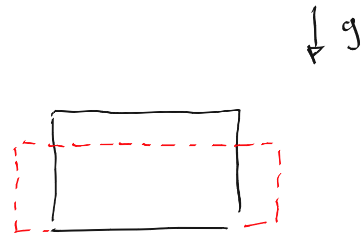
$$\frac{\partial \sigma}{\partial x} + \rho g = 0$$

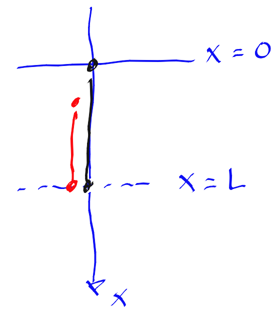
$$\sigma = E \frac{du}{dx}$$

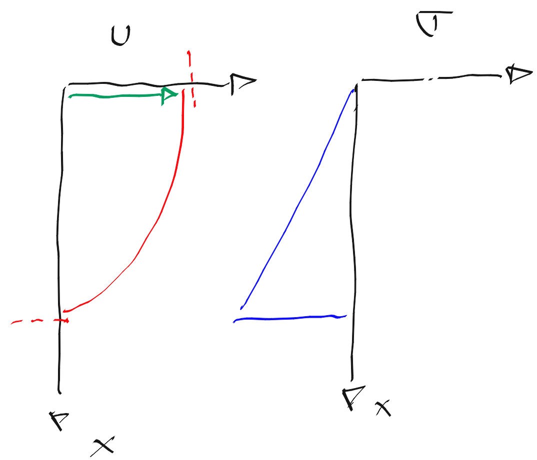
$$E \frac{d^2 u}{dx^2} = - \rho g$$

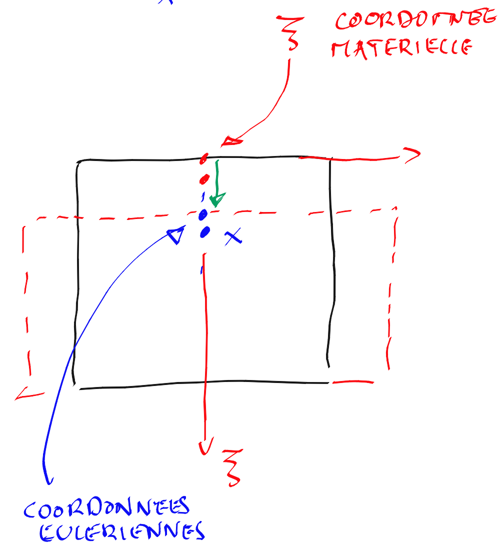
$$u(L) = 0$$

$$u'(0) = 0$$




Vitesse $\quad \underline{v} \longrightarrow \nabla \underline{v} \qquad \overbrace{\frac{1}{2}\left(\nabla \underline{v} + \nabla \underline{v}^{T}\right)}^{\underline{\underline{d}}}$

Déplacement $\quad \underline{u} \longrightarrow \nabla \underline{u} \qquad \underbrace{\frac{1}{2}\left(\nabla \underline{u} + \nabla \underline{u}^{T}\right)}_{\underline{\underline{\varepsilon}}}$

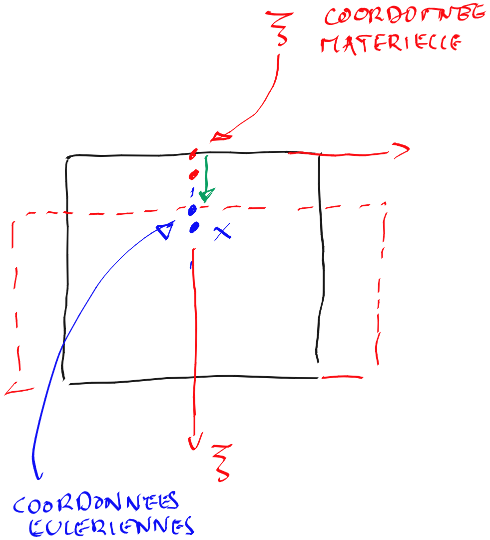
$$\underline{\underline{\sigma}} = 2\mu\, \underline{\underline{\varepsilon}} + \lambda\, \mathrm{tr}(\underline{\underline{\varepsilon}})\, \underline{\underline{\delta}}$$

$$\underline{\underline{\tau}} = \quad \underline{\underline{d}} \qquad \underline{\underline{d}}\ \cancel{0} \qquad - p\, \underline{\underline{\delta}}$$

Taux de déformation = gradient de vitesses

Déformation = gradient de déplacement

$$\sigma = E \frac{du}{dx}$$

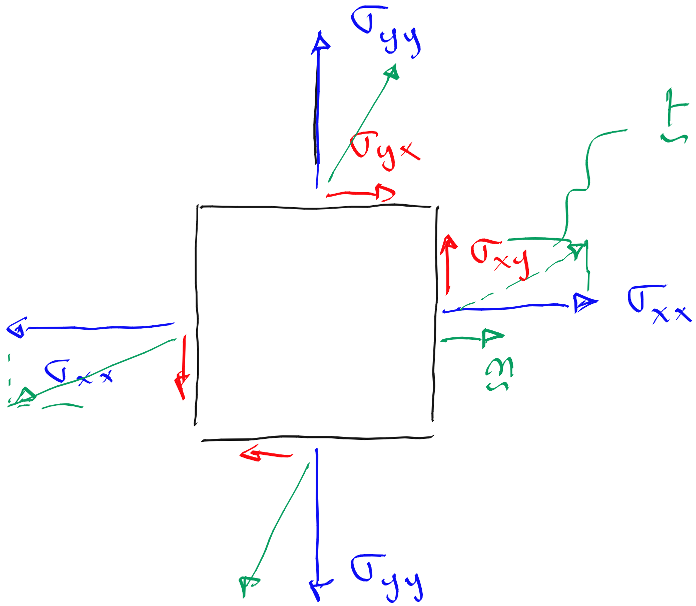



$$\frac{du}{dx} \simeq \frac{du}{d\xi}$$

ON FAIT L'HYPOTHESE DE PETITES DEFORMATIONS

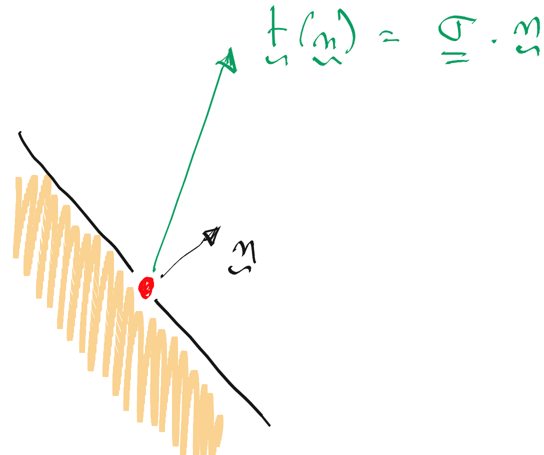
Grandes et petites déformations !

Cela peut devenir très compliqué !




$$\underline{t} = \underline{\underline{\sigma}} \cdot \underline{n}$$

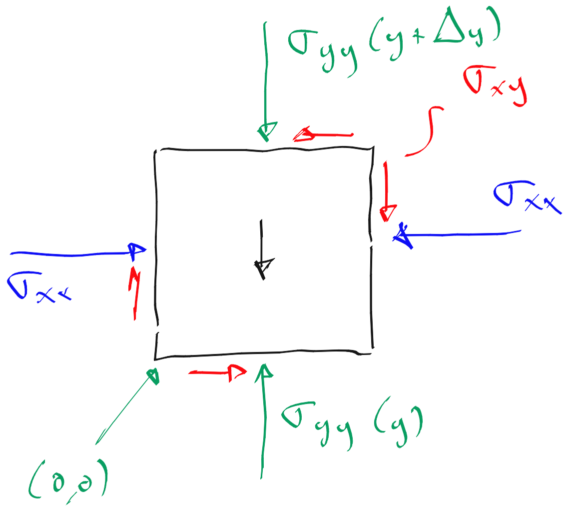
$$\underline{t}(\underline{n}) = \underline{\underline{\sigma}} \cdot \underline{n}$$




$$\underline{\underline{\sigma}} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix}$$

$$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T$$

# Tenseur de contraintes



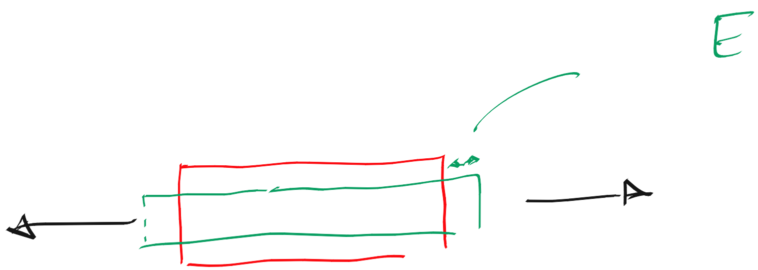

$$\rho g \, \Delta x \Delta y + \sigma_{yy}(y+\Delta y)\,\Delta x - \sigma_{yy}(y)\,\Delta x$$

$$\begin{cases} \rho g + \dfrac{\partial}{\partial y}(\sigma_{yy}) + \dfrac{\partial}{\partial x}(\sigma_{xy}) = 0 \\ \dfrac{\partial}{\partial x}(\sigma_{xx}) + \dfrac{\partial}{\partial y}(\sigma_{xy}) = 0 \end{cases}$$

# Bilan de quantité de mouvement

COEFFICIENTS DE LAMÉ

$$\underline{\underline{\sigma}} = \underbrace{2\mu}_{\frac{E}{(1+\nu)}} \, \underline{\underline{\varepsilon}} + \underbrace{\lambda}_{\frac{E\nu}{(1+\nu)(1-2\nu)}} \, \mathrm{tr}(\underline{\underline{\varepsilon}}) \, \underline{\underline{\delta}}$$




Loi de Hooke

Coefficients de Lamé

Module de Young et coefficients de Poisson

# ABSTRACT GENERIC ELLIPTIC PROBLEM

HEAT CONDUCTION
LINEAR ELASTICITY
SIMPLIFIED MODELS OF LINEAR ELASTICITY
BEAM / SHELLS
ROPE / MENBRANE

STOKES PROBLEM

NOW,
WE CONSIDER
A VECTORIAL UNKNOWN FIELD !

WEAK FORMULATION

$? \, \underset{\sim}{u} \in \mathcal{U}$ SUCH THAT

$$\underbrace{a(\underset{\sim}{u}\ \hat{\underset{\sim}{u}})}_{\text{BILINEAR CONTINUOUS SYMMETRIC COERCIVE FORM}} = \underbrace{b(\hat{\underset{\sim}{u}})}_{\text{CONTINUOUS LINEAR FORM}} \quad \forall \hat{\underset{\sim}{u}} \in \hat{\mathcal{U}}$$

ASSUMPTIONS REQUIRED TO HAVE AN ABSTRACT MINIMIZATION PROBLEM

$\Updownarrow$

$? \, \underset{\sim}{u} \in \mathcal{U}$ SUCH THAT

$$J(\underset{\sim}{u}) = \min_{\underset{\sim}{v} \in \mathcal{U}} \underbrace{\frac{1}{2} a(\underset{\sim}{v}\ \underset{\sim}{v}) - b(\underset{\sim}{v})}_{J(\underset{\sim}{v})}$$

MINIMIZATION PROBLEM

# HEAT CONDUCTION

CONSERVATION LAW

$$-\nabla \cdot q + f = 0$$

HEAT FLOW

CONSTITUTIVE LAW

FOURIER

$$q = -k \nabla u$$

TEMPERATURE

? $u$ SUCH THAT

$$\nabla \cdot \big( \overbrace{k \nabla u}^{-q(u)} \big) + f = 0 \quad \text{in } \Omega$$

$$-q \cdot n = g \quad \text{on } \Gamma_N$$

$$u = 0 \quad \text{on } \Gamma_D$$

# 3D LINEAR ISOTROPIC ELASTICITY

$$\begin{bmatrix} \times & & \\ \circ & \times & \\ \circ & \circ & \times \end{bmatrix}$$

SHEAR STRESSES

NORMAL STRESSES

HOOKE ELASTIC BODY
SMALL DEF.

CONSTITUTIVE LAW

$$\underline{\underline{\sigma}} = \underbrace{\frac{E}{(1+\nu)}}_{2\mu} \underline{\underline{\varepsilon}} + \underbrace{\frac{E\nu}{(1+\nu)(1-2\nu)}}_{\lambda} \mathrm{tr}(\underline{\underline{\varepsilon}})\, \underline{\underline{\delta}}$$

DEFORMATION TENSOR

CONSERVATION LAWS

$$\underset{\sim}{\nabla} \cdot \underline{\underline{\sigma}} + \underset{\sim}{f} = 0$$

$$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T$$

$$\underline{\underline{\varepsilon}} \triangleq \frac{1}{2}\left(\underset{\sim}{\nabla}\underset{\sim}{u} + (\underset{\sim}{\nabla}\underset{\sim}{u})^T\right)$$

? $\underset{\sim}{u}$

$$\underset{\sim}{\nabla} \cdot \underline{\underline{\sigma}}(\underset{\sim}{u}) + \underset{\sim}{f} = 0 \quad \text{in } \Omega$$

$$\underline{\underline{\sigma}} \cdot \underset{\sim}{n} = \underset{\sim}{g} \quad \text{on } \Gamma_N$$

$$\underset{\sim}{u} = 0 \quad \text{on } \Gamma_D$$

# STOKES PROBLEM

CONSERVATION LAWS

$$\underbrace{\cancel{(\underline{v}\cdot\nabla)\underline{v}}}_{} = \nabla\cdot\underline{\underline{\sigma}} + \underline{f}$$

$$\nabla\cdot\underline{v} = 0$$

CREEPING FLOW
$Re \simeq 0$

INCOMPRESSIBLE FLOW

CONSTITUTIVE LAW

NEWTONIAN FLUID

$$\underline{\underline{\sigma}} = 2\mu\,\underline{\underline{d}} - p\,\underline{\underline{I}}$$

RATE OF DEFORMATION TENSOR

$$\underline{\underline{d}} \triangleq \frac{1}{2}\left(\nabla\underline{v} + (\nabla\underline{v})^T\right)$$

? $(\underline{v}, p)$

$$\begin{cases} \nabla\cdot\left(\underline{\underline{\sigma}}(\underline{v}, p)\right) + \underline{f} = 0 \\ \nabla\cdot\underline{v} = 0 \end{cases} \quad \text{in } \Omega$$

$$\underline{\underline{\sigma}}\cdot\underline{n} = \underline{g} \quad \text{on } \Gamma_N$$

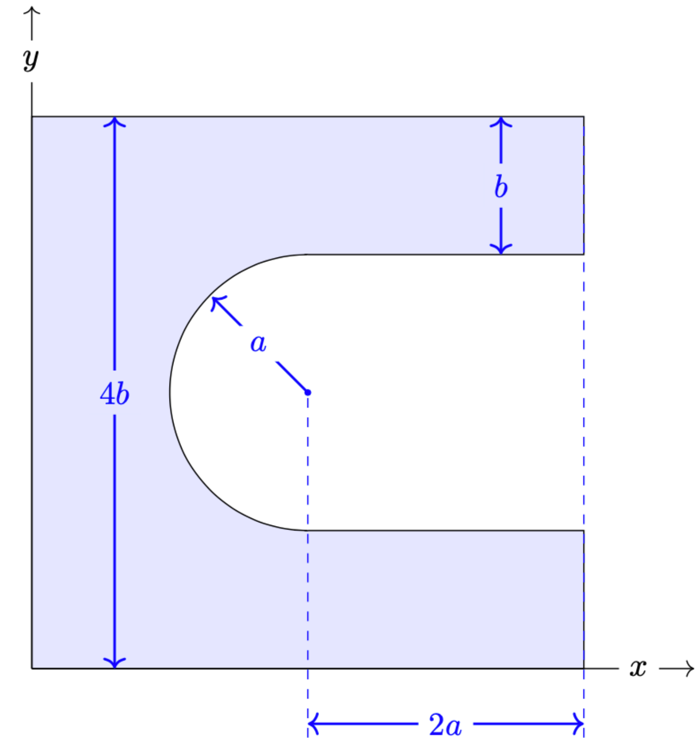
$$\underline{v} = 0 \quad \text{on } \Gamma_D$$

# Créer une géométrie !
# Construire le maillage !

**Définir la géométrie et le maillage !**

Le maillage et la géométrie sont définis comme suit :

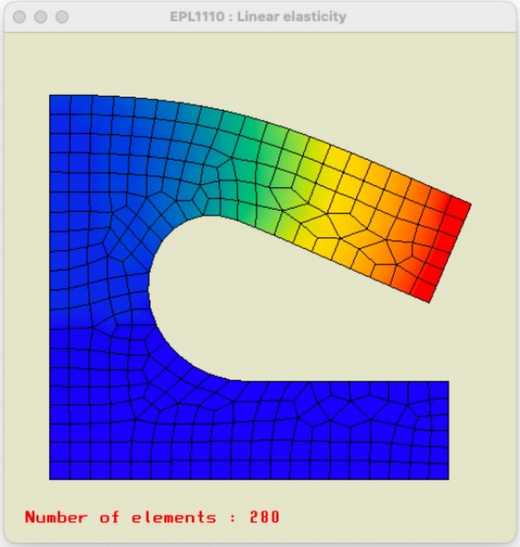
```
double Lx = 1.0;
double Ly = 1.0;

theGeometry->LxPlate     = Lx;
theGeometry->LyPlate     = Ly;
theGeometry->h           = Lx * 0.05;
theGeometry->elementType = FEM_TRIANGLE;
```

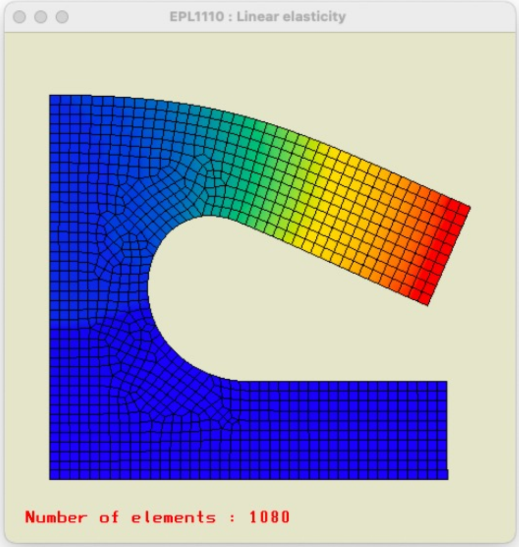


# Que va-t-on résoudre ?

Trouver $\mathbf{u}(\mathbf{x}) \in \mathcal{U}$ tel que

$$\underbrace{< \boldsymbol{\epsilon}(\widehat{\mathbf{u}}) : \mathbf{C} : \boldsymbol{\epsilon}(\mathbf{u}). >}_{a(\widehat{\mathbf{u}},\mathbf{u})} \quad = \quad \underbrace{< \widehat{\mathbf{u}} f > + \ll \widehat{\mathbf{u}} g \gg_N,}_{b(\widehat{\mathbf{u}})} \qquad \forall \widehat{\mathbf{u}} \in \widehat{\mathcal{U}},$$



# Et on minimise toujours une fonctionnelle !

Trouver $\mathbf{u}(\mathbf{x}) \in \mathcal{U}$ tel que

$$\underbrace{< \boldsymbol{\epsilon}(\widehat{\mathbf{u}}) : \mathbf{C} : \boldsymbol{\epsilon}(\mathbf{u}). >}_{a(\widehat{\mathbf{u}},\mathbf{u})} \quad = \quad \underbrace{< \widehat{\mathbf{u}} f > + \ll \widehat{\mathbf{u}} g \gg_N}_{b(\widehat{\mathbf{u}})}, \qquad \forall \widehat{\mathbf{u}} \in \widehat{\mathcal{U}},$$

$$J(\mathbf{v}) \quad = \quad \underbrace{\frac{1}{2} < \boldsymbol{\epsilon}(\mathbf{v}) : \mathbf{C} : \boldsymbol{\epsilon}(\mathbf{v}). >}_{\frac{1}{2}a(\mathbf{v},\mathbf{v})} - \underbrace{< \mathbf{v} f > + \ll \mathbf{v} g \gg_N}_{b(\mathbf{v})},$$

# En deux dimensions !

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix} = \begin{bmatrix} A\epsilon_{xx} + B\epsilon_{yy} & 2C\epsilon_{xy} \\ 2C\epsilon_{xy} & A\epsilon_{yy} + B\epsilon_{xx} \end{bmatrix}$$

| Acier |
| --- |
| $E = 2.11\ 10^{11}\ [N/m^2]$ <br> $\nu = 0.3$ <br> $\rho = 7.85\ 10^3\ [kg/m^3]$ |

| Déformations planes | Tensions planes |
| --- | --- |
| $A = \dfrac{E(1-\nu)}{(1+\nu)(1-2\nu)}$ | $A = \dfrac{E}{(1-\nu^2)}$ |
| $B = \dfrac{E\nu}{(1+\nu)(1-2\nu)}$ | $B = \dfrac{E\nu}{(1-\nu^2)}$ |

$$C = \frac{E}{2(1+\nu)}$$

# Déformations planes…

$$\begin{aligned}
\sigma_{xx} &= \frac{E}{(1+\nu)(1-2\nu)}\left((1-\nu)\epsilon_{xx}+\nu\epsilon_{yy}\right), \\
\sigma_{yy} &= \frac{E}{(1+\nu)(1-2\nu)}\left(\nu\epsilon_{xx}+(1-\nu)\epsilon_{yy}\right), \\
\sigma_{zz} &= \frac{E\nu}{(1+\nu)(1-2\nu)}\left(\epsilon_{xx}+\epsilon_{yy}\right), \\
\sigma_{xy} &= \frac{E}{(1+\nu)}\epsilon_{xy},
\end{aligned}$$

$$\begin{aligned}
\sigma_{xx} &= \frac{E}{(1-\nu^2)}\left(\epsilon_{xx}+\nu\epsilon_{yy}\right), \\
\sigma_{yy} &= \frac{E}{(1-\nu^2)}\left(\nu\epsilon_{xx}+\epsilon_{yy}\right), \\
\sigma_{xy} &= \frac{E}{(1+\nu)}\epsilon_{xy}.
\end{aligned}$$

# …tensions planes

# La fonctionnelle à minimiser !

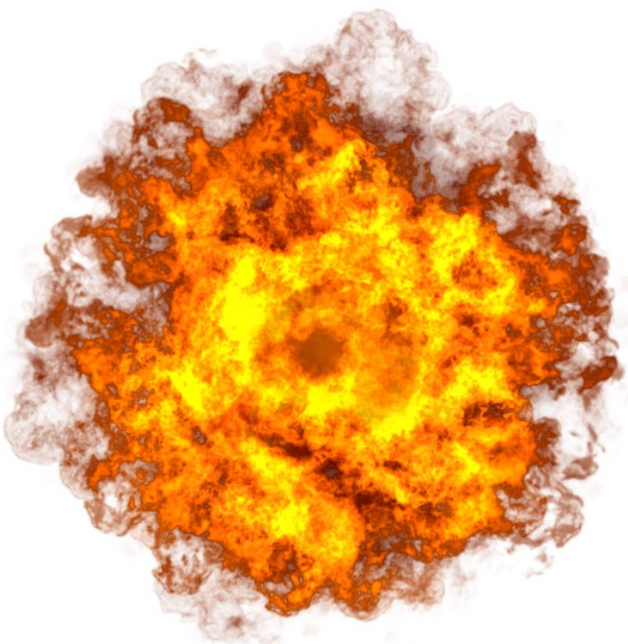
$$
\begin{aligned}
\frac{1}{2}\, a(\mathbf{u},\mathbf{u}) &= \frac{1}{2} < \boldsymbol{\epsilon}(\mathbf{u}) : \boldsymbol{\sigma}(\mathbf{u}) >, \\
&= \frac{1}{2} < \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} \\ \epsilon_{xy} & \epsilon_{yy} \end{bmatrix} : \begin{bmatrix} A\epsilon_{xx} + B\epsilon_{yy} & 2C\epsilon_{xy} \\ 2C\epsilon_{xy} & A\epsilon_{yy} + B\epsilon_{xx} \end{bmatrix} >, \\
&= \frac{1}{2} < \begin{bmatrix} \epsilon_{xx} & \epsilon_{yy} & 2\epsilon_{xy} \end{bmatrix} \cdot \begin{bmatrix} A & B & 0 \\ B & A & 0 \\ 0 & 0 & C \end{bmatrix} \cdot \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ 2\epsilon_{xy} \end{bmatrix} > .
\end{aligned}
$$

# Matrice et vecteur locaux à construire !

$$\mathbf{A}_{ij} = \left[\begin{array}{c|c} < \tau_{i,x} A \tau_{j,x} > + < \tau_{i,y} C \tau_{j,y} > & < \tau_{i,x} B \tau_{j,y} > + < \tau_{i,y} C \tau_{j,x} > \\ \hline < \tau_{i,y} B \tau_{j,x} > + < \tau_{i,x} C \tau_{j,y} > & < \tau_{i,y} A \tau_{j,y} > + < \tau_{i,x} C \tau_{j,x} > \end{array}\right],$$

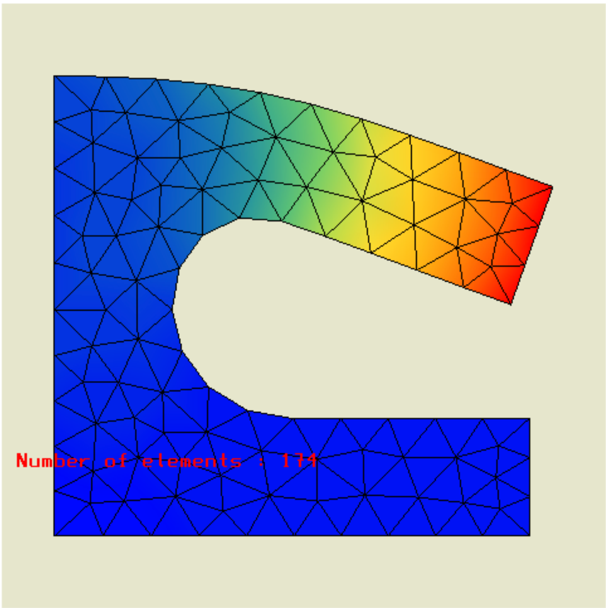
$$\mathbf{B}_{i} = \left[\begin{array}{c} < \tau_i f_x > + \ll \tau_i g_x \gg \\ < \tau_i f_y > + \ll \tau_i g_y \gg \end{array}\right].$$

# Et on soumet au serveur…



```
=====================================================================================

 Linear elasticity problem
   Young modulus   E   =  2.1100000e+11 [N/m2]
   Poisson's ratio nu  =  3.0000000e-01 [-]
   Density         rho =  7.8500000e+03 [kg/m3]
   Gravity         g   =  9.8100000e+00 [m/s2]
   Planar strains formulation
   Boundary conditions :
              Symmetry : imposing  0.00e+00 as the horizontal displacement
                Bottom : imposing  0.00e+00 as the vertical displacement
=====================================================================================

Pivot index 1  Pivot value 0.000000e+00
-------------------------------------------------------------------------------
  Error in src/fem.c at line 555 :
  Cannot eliminate with such a pivot
--------------------------------------------------------------------- Yek Yek !!



La compilation ou l'execution du test 0 a plante :-(
```

# Et après avoir fait le devoir…




**Diagnostic du serveur**

```
********** TEST 0 **********
gcc -o /tmp/work/exe src/glfem.c src/main.c src/fem.c src/mesaglfem.c homework.c -I src -lm -lrt -Wall -g -Wno-
** COMPILATION SUCCEED **

/usr/bin/time -o /work/outputStat.txt -f %R %Z %e %M sudo -ustudent ./exe
** RUN SUCCEED **
Wall-clock time : 0.24s (limit 40s)
Approximated maximum memory usage : 112.98Mb
Approximated total memory allocation : 28.8203Mb
********** RUN OUTPUT **********
Info    : Meshing 1D...
Info    : [  0%] Meshing curve 1 (Line)
Info    : [ 20%] Meshing curve 2 (Line)
Info    : [ 30%] Meshing curve 3 (Line)
```

# PROBLÈME ELLIPTIQUE

# ELEMENTS FINIS *

* "USUAL FEM"
CLASSICAL
GALERKIN
FORMULATION

# ABSTRACT GENERIC ELLIPTIC PROBLEM

NOW, WE CONSIDER A VECTORIAL UNKNOWN FIELD !

WEAK FORMULATION

$$? \underset{\sim}{u} \in \mathcal{U} \quad \text{SUCH THAT}$$

$$\underbrace{a(\underset{\sim}{u}, \underset{\sim}{\hat{u}})}_{\text{BILINEAR CONTINUOUS SYMMETRIC COERCIVE FORM}} = \underbrace{b(\underset{\sim}{\hat{u}})}_{\text{CONTINUOUS LINEAR FORM}} \quad \forall \underset{\sim}{\hat{u}} \in \hat{\mathcal{U}}$$

ASSUMPTIONS REQUIRED TO HAVE AN ABSTRACT MINIMIZATION PROBLEM

HEAT CONDUCTION
LINEAR ELASTICITY
SIMPLIFIED MODELS OF LINEAR ELASTICITY
BEAM / SHELLS
ROPE / MENBRANE
STOKES PROBLEM

$\Updownarrow$

$$? \underset{\sim}{u} \in \mathcal{U} \quad \text{SUCH THAT}$$

$$J(\underset{\sim}{u}) = \min_{\underset{\sim}{v} \in \mathcal{U}} \underbrace{\frac{1}{2} a(\underset{\sim}{v}, \underset{\sim}{v}) - b(\underset{\sim}{v})}_{J(\underset{\sim}{v})}$$

MINIMIZATION PROBLEM

# HEAT CONDUCTION

CONSERVATION LAW

$$-\nabla \cdot q + f = 0$$

HEAT FLOW

CONSTITUTIVE LAW

$$q = -k \nabla u$$

FOURIER

TEMPERATURE

?$u$ SUCH THAT

$$\nabla \cdot \big( \overbrace{k \nabla u}^{-q(u)} \big) + f = 0 \quad \text{in } \Omega$$

$$-q \cdot n = g \quad \text{on } \Gamma_N$$

$$u = 0 \quad \text{on } \Gamma_D$$

# 3D LINEAR ISOTROPIC ELASTICITY

$$\begin{bmatrix} \times & & \\ \circ & \times & \\ \circ & \circ & \times \end{bmatrix}$$

SHEAR STRESSES

NORMAL STRESSES

HOOKE ELASTIC BODY
SMALL DEF.

CONSTITUTIVE LAW

$$\underline{\underline{\sigma}} = \underbrace{\frac{E}{(1+\nu)}}_{2\mu} \underline{\underline{\varepsilon}} + \underbrace{\frac{E\nu}{(1+\nu)(1-2\nu)}}_{\lambda} \mathrm{tr}(\underline{\underline{\varepsilon}})\, \underline{\underline{\delta}}$$

DEFORMATION TENSOR

$$\underline{\underline{\varepsilon}} \triangleq \frac{1}{2}\left(\underline{\nabla}\,\underline{u} + (\underline{\nabla}\,\underline{u})^T\right)$$

CONSERVATION LAWS

$$\underline{\nabla} \cdot \underline{\underline{\sigma}} + \underline{f} = 0$$

$$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T$$

? $\underline{u}$

$$\underline{\nabla} \cdot \underline{\underline{\sigma}}(\underline{u}) + \underline{f} = 0 \quad \text{in } \Omega$$

$$\underline{\underline{\sigma}} \cdot \underline{n} = \underline{g} \quad \text{on } \Gamma_N$$

$$\underline{u} = 0 \quad \text{on } \Gamma_D$$

# STOKES PROBLEM

CONSERVATION LAWS

$$\cancel{(\underline{v}\cdot\underline{\nabla})\underline{v}} = \underline{\nabla}\cdot\underline{\underline{\sigma}} + \underline{f}$$

$$\underline{\nabla}\cdot\underline{v} = 0$$

CREEPING FLOW
$Re \simeq 0$

INCOMPRESSIBLE FLOW

CONSTITUTIVE LAW

NEWTONIAN FLUID

$$\underline{\underline{\sigma}} = 2\mu\,\underline{\underline{d}} - p\,\underline{\underline{I}}$$

RATE OF DEFORMATION TENSOR

$$\underline{\underline{d}} \triangleq \frac{1}{2}\left(\underline{\nabla}\,\underline{v} + (\underline{\nabla}\,\underline{v})^{T}\right)$$

$?\,(\underline{v}, p)$

$$\begin{cases} \underline{\nabla}\cdot\left(\underline{\underline{\sigma}}(\underline{v}, p)\right) + \underline{f} = 0 \\ \underline{\nabla}\cdot\underline{v} = 0 \end{cases} \quad \text{in } \Omega$$

$$\underline{\underline{\sigma}}\cdot\underline{n} = \underline{g} \quad \text{on } \Gamma_N$$

$$\underline{v} = 0 \quad \text{on } \Gamma_D$$

CALCULUS

$\underset{\sim}{\nabla} \cdot \underset{\sim}{v} = 0$ IN $\Omega$ $\iff$ $\exists\, \underset{\sim}{\psi}$ SUCH THAT $\underset{\sim}{v} = \underset{\sim}{\nabla} \times \underset{\sim}{\psi}$

? $\underset{\sim}{\psi}$ STREAM FUNCTION

$$\underset{\sim}{\nabla}^4 \underset{\sim}{\psi} + \underset{\sim}{\nabla} \times \underset{\sim}{f} = 0 \quad \text{in } \Omega$$

+ SUITABLE BOUNDARY CONDITIONS

# 3D LINEAR ISOTROPIC ELASTICITY

$$\begin{bmatrix} \times & & \\ \circ & \times & \\ \circ & \circ & \times \end{bmatrix}$$

SHEAR STRESSES

NORMAL STRESSES

HOOKE ELASTIC BODY

SMALL DEF.

## CONSTITUTIVE LAW

$$\underline{\underline{\sigma}} = \underbrace{\frac{E}{(1+\nu)}}_{2\mu} \underline{\underline{\varepsilon}} + \underbrace{\frac{E\nu}{(1+\nu)(1-2\nu)}}_{\lambda} \operatorname{tr}(\underline{\underline{\varepsilon}})\, \underline{\underline{\delta}}$$

DEFORMATION TENSOR

## CONSERVATION LAWS

$$\underline{\nabla} \cdot \underline{\underline{\sigma}} + \underline{f} = 0$$

$$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T$$

$$\underline{\underline{\varepsilon}} \triangleq \frac{1}{2}\left(\underline{\nabla}\,\underline{u} + (\underline{\nabla}\,\underline{u})^T\right)$$

? $\underline{u}$

$$\underline{\nabla} \cdot \underline{\underline{\sigma}}(\underline{u}) + \underline{f} = 0 \quad \text{in } \Omega$$

$$\underline{\underline{\sigma}} \cdot \underline{n} = \underline{g} \quad \text{on } \Gamma_N$$

$$\underline{u} = 0 \quad \text{on } \Gamma_D$$

$$\underline{\underline{\sigma}} = \underbrace{\frac{E}{(1+\nu)}}_{2\mu} \underline{\underline{\varepsilon}} + \underbrace{\frac{E\nu}{(1+\nu)(1-2\nu)}}_{\lambda} \mathrm{tr}(\underline{\underline{\varepsilon}})\, \underline{\underline{S}}$$

SHEAR MODULUS

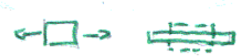
$\lambda \to \infty$

IF $\nu \to 1/2$

INCOMPRESSIBLE MATERIAL

POISSON'S COEFFICIENT $\nu \in \,]-1 \;\; 1/2[$

STRANGE MATERIALS



USUAL BEHAVIOUR

# ABSTRACT GENERIC DISCRETE FORMULATION

2D $\begin{bmatrix} u \\ v \end{bmatrix}$ 3D $\begin{bmatrix} u \\ v \\ w \end{bmatrix}$

$$\underbrace{\underset{\sim}{u}(\underset{\sim}{x})}_{\in\, \mathcal{U}} \simeq \underbrace{\underset{\sim}{u}^h(\underset{\sim}{x})}_{\in\, \mathcal{U}^h \subset\, \mathcal{U}} = \sum_{i=1}^{n} \underset{\sim}{U}_i\, \tau_i(\underset{\sim}{x})$$

$\dim(\mathcal{U}^h) = 2n$ — 2D PROBLEMS

$\ldots = 3n$ — 3D PROBLEMS

$$\mathcal{U}^h = \text{SPAN} \left\{ \underbrace{\begin{bmatrix} \tau_1 \\ 0 \end{bmatrix} \begin{bmatrix} \tau_2 \\ 0 \end{bmatrix} \cdots \begin{bmatrix} \tau_n \\ 0 \end{bmatrix}}_{n} \underbrace{\begin{bmatrix} 0 \\ \tau_1 \end{bmatrix} \cdots \begin{bmatrix} 0 \\ \tau_n \end{bmatrix}}_{n} \right\}$$

? $\underset{\sim}{u}^h$ SUCH THAT

$$\boxed{a(\underset{\sim}{u}^h, \hat{\underset{\sim}{u}}^h) = b(\hat{\underset{\sim}{u}}^h) \quad \forall \hat{\underset{\sim}{u}}^h \in \hat{\mathcal{U}}^h}$$

DISCRETE FORMULATION

TIP #1

$$\overset{\in \hat{\mathcal{U}}}{}\quad \overset{\in \mathcal{U}}{}$$

$$< \hat{\underline{u}} \cdot \left( \underline{\nabla} \cdot \underline{\underline{\sigma}}(\underline{u}) \right) > \; = \; \underbrace{< \underline{\nabla} \cdot \left( \hat{\underline{u}} \cdot \underline{\underline{\sigma}}(\underline{u}) \right) >} \; - \; \overbrace{< \underline{\nabla}\hat{\underline{u}} : \underline{\underline{\sigma}}(\underline{u}) >}^{\triangleq\, a(\hat{\underline{u}},\, \underline{u})}$$

$\hat{\underline{u}} \in \hat{\mathcal{U}}$, $\underline{u} \in \mathcal{U}$

TIP #2

$$= \ll \underline{n} \cdot \hat{\underline{u}} \cdot \underline{\underline{\sigma}}(\underline{u}) \gg$$

TIP #3

$$= \ll \hat{\underline{u}} \cdot \big( \underbrace{\underline{n} \cdot \underline{\underline{\sigma}}(\underline{u})}_{g \text{ on } \Gamma_N} \big) \gg_N$$

USUAL CALCULUS !

# IS $a(\hat{\underset{\sim}{u}}, \underset{\sim}{u})$ SYMMETRIC ?

$$\underset{=}{\sigma} = \underset{\equiv}{C} : \underset{=}{\varepsilon}$$

$$\langle \nabla \hat{\underset{\sim}{u}} : \underset{=}{\sigma}(\underset{\sim}{u}) \rangle = \langle \Big( \overset{\underset{=}{\varepsilon}(\hat{\underset{\sim}{u}})}{\overbrace{\underbrace{\frac{\nabla \hat{\underset{\sim}{u}} + (\nabla \hat{\underset{\sim}{u}})^T}{2}}_{\text{SYM}}}} + \underbrace{\frac{\nabla \hat{\underset{\sim}{u}} - (\nabla \hat{\underset{\sim}{u}})^T}{2}}_{\text{ANTI SYM.}} \Big) : \underbrace{\underset{=}{\sigma}(\underset{\sim}{u})}_{\text{SYM.}} \rangle$$

$$\nabla$$

$$= \langle \underset{=}{\varepsilon}(\hat{\underset{\sim}{u}}) : \underbrace{\underset{=}{\sigma}(\underset{\sim}{u})}_{\boxed{\underset{\equiv}{C} : \underset{=}{\varepsilon}(\underset{\sim}{u})}} \rangle$$

GENERALIZED HOOKE'S LAW

$$\nabla$$

$$= \langle \underset{=}{\varepsilon}(\hat{\underset{\sim}{u}}) : \underset{\equiv}{C} : \underset{=}{\varepsilon}(\underset{\sim}{u}) \rangle$$

# WEAK FORMULATION

? $\underline{u} \in \mathcal{U}$ SUCH THAT

$$\underbrace{\langle \underline{\underline{\varepsilon}}(\hat{\underline{v}}) : \underline{\underline{\underline{\underline{C}}}} : \underline{\underline{\varepsilon}}(\underline{u}) \rangle}_{a(\hat{\underline{v}}\,\underline{u})} = \underbrace{\langle \hat{\underline{v}} \cdot \underline{f} \rangle + \langle\!\langle \hat{\underline{v}} \cdot \underline{g} \rangle\!\rangle_N}_{b(\hat{\underline{v}})}$$

$\forall \hat{\underline{v}} \in \hat{\mathcal{U}}$

$$\Updownarrow$$

FOR SUITABLE $\underline{\underline{\underline{\underline{C}}}}$ ONLY !

? $\underline{u} \in \mathcal{U}$ SUCH THAT

$$J(\underline{u}) = \min_{\underline{v} \in \mathcal{U}} \underbrace{\frac{1}{2} a(\underline{v}\,\underline{v}) - b(\underline{v})}_{J(\underline{v})}$$

# MINIMIZATION PROBLEM

$$\underbrace{\frac{1}{2}\, a(\underline{v},\underline{v})}_{\text{ENERGY OF DEFORMATION}} = \underbrace{\left\langle \frac{1}{2}\, \lambda\, \underline{\underline{\varepsilon}}(\underline{v}) : \underline{\underline{I}} \;\; \overbrace{\underline{\underline{I}} : \underline{\underline{\varepsilon}}(\underline{v})}^{tr(\underline{\underline{\varepsilon}}(\underline{v}))} + \mu\, \underline{\underline{\varepsilon}}(\underline{v}) : \underline{\underline{\varepsilon}}(\underline{v}) \right\rangle}_{\text{MUST BE A QUADRATIC POSITIVE FORM IN ORDER TO OBTAIN A MINIMIZATION PROBLEM}}$$

$$\boxed{\begin{aligned} \mu &> 0 \\ \frac{3}{2}\lambda + \mu &> 0 \end{aligned}} \qquad \begin{aligned} &(\varepsilon_{12} \neq 0) \\ &(\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} \neq 0) \end{aligned}$$

# ADMISSIBLE VALUES FOR LAMÉ COEFFICIENTS

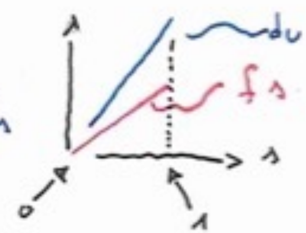
# "DEFORMATION ENERGY"

WHAT IS IT ?

ENERGY OF DEFORMATION
= WORK PERFORMED TO DEFORM THE STRUCTURE AND IS STORED INSIDE.
SUCH AN ENERGY IS RECOVERED WHEN EXTERNAL FORCES ARE REMOVED

$f$ FORCE, $u$ DISPLACEMENT

$$\text{ENERGY} = \underbrace{\text{FORCE} \cdot \text{DISPLACEMENT}}_{\text{WORK PERFORMED}}$$

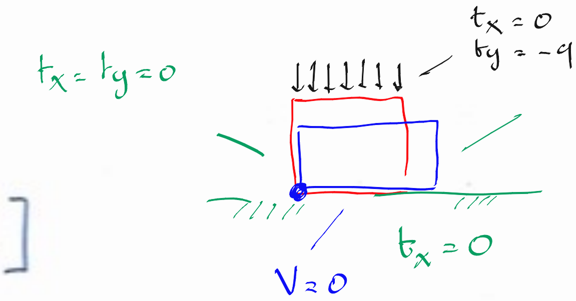

$$\text{WORK PERFORMED} = \int_0^u \lambda f \, du = \int_0^1 f u \, \lambda \, d\lambda = f u \underbrace{\int_0^1 \lambda \, d\lambda}_{1/2} = \frac{1}{2} f u \;!$$

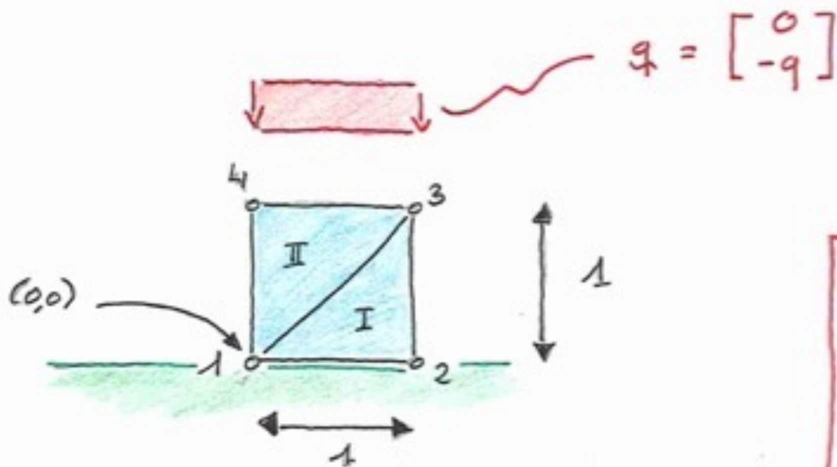
VERY VERY SLOW LOADING
→ QUASI-STATIC APPROACH
- IRREVERSIBILITY
- DISSIPATION

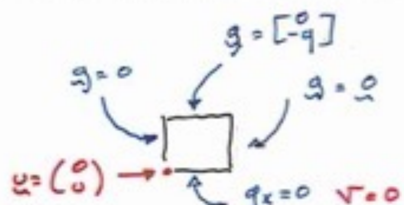
} ARE NEGLECTED !

# NUMERICAL EXAMPLE

$\underset{\sim}{u} = \begin{bmatrix} u \\ v \end{bmatrix}$




5 UNKNOWN DISPLACEMENTS

$$\begin{bmatrix} U_i \\ V_i \end{bmatrix}$$

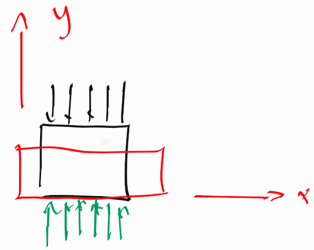
$$? \; \underset{\sim}{u}^h = \sum_{i=1}^{m} \underset{\sim}{U}_i \, \tau_i$$

$$\langle \underset{\approx}{\varepsilon}(\hat{\underset{\sim}{u}}^h) : \underset{\approx}{\sigma}(\underset{\sim}{u}^h) \rangle = \langle \underset{\sim}{f} \cdot \hat{\underset{\sim}{u}}^h \rangle + \langle\!\langle \underset{\sim}{g} \cdot \hat{\underset{\sim}{u}}^h \rangle\!\rangle_N \qquad \forall \hat{\underset{\sim}{u}}^h$$

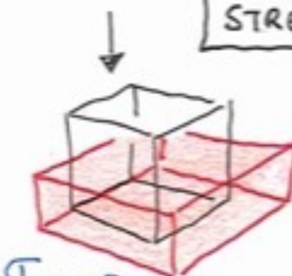
$$\begin{bmatrix} \tau_i \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ \tau_i \end{bmatrix}$$

2m TEST FUNCTIONS

# ANALYTICAL SOLUTION



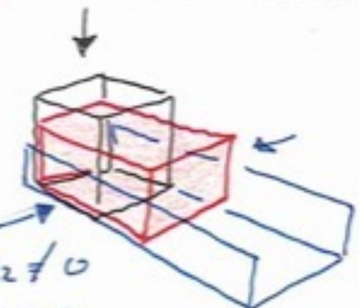

PLANAR STRESSES



$\sigma_{zz} = 0$

$\varepsilon_{zz} \neq 0$

PLANAR DEFORMATIONS



$\sigma_{zz} \neq 0$

$\varepsilon_{zz} = 0$

PLANAR STRESSES

$$\underset{\sim}{u} = \frac{q}{E}\begin{bmatrix} \nu x \\ -y \end{bmatrix}$$

$$\underset{\approx}{\varepsilon} = \frac{q}{E}\begin{bmatrix} \nu & 0 \\ 0 & -1 \end{bmatrix}$$

$$\underbrace{\underbrace{\nabla}\cdot\underbrace{\underset{\approx}{\sigma}}}_{=0\,!} = 0$$

$$\sigma_{xx} = \frac{E}{(1-\nu^2)}\Big(\underbrace{\varepsilon_{xx}}_{\frac{q\nu}{E}} + \underbrace{\nu\,\varepsilon_{yy}}_{-\frac{q\nu}{E}}\Big) = 0$$

$$\sigma_{yy} = \frac{E}{(1-\nu^2)}\Big(\underbrace{\nu\,\varepsilon_{xx} - \varepsilon_{yy}}_{\frac{q}{E}(\nu^2-1)}\Big) = -q \;!$$

$$\begin{bmatrix} \tau_i \\ 0 \end{bmatrix} \text{ OR } \begin{bmatrix} 0 \\ \tau_i \end{bmatrix}$$

$$\boxed{\left\langle \underline{\underline{\varepsilon}}(\hat{\underline{u}}^h) : \underline{\underline{\sigma}}(\underline{u}^h) \right\rangle \quad \forall \hat{\underline{u}}^h}$$

$$\sum_j U_j \, \underline{\underline{\sigma}}\begin{pmatrix} \tau_j \\ 0 \end{pmatrix} + V_j \, \underline{\underline{\sigma}}\begin{pmatrix} 0 \\ \tau_j \end{pmatrix}$$

$$\sum_j \begin{bmatrix} U_j \\ V_j \end{bmatrix} \tau_j = \sum U_j \begin{bmatrix} \tau_j \\ 0 \end{bmatrix} + V_j \begin{bmatrix} 0 \\ \tau_j \end{bmatrix}$$

$$\boxed{\sum_j \underline{\underline{A}}_{ij} \cdot \underline{U}_j}$$

$$\sum_j \begin{bmatrix} \left\langle \underline{\underline{\varepsilon}}\begin{pmatrix} \tau_i \\ 0 \end{pmatrix} : \underline{\underline{\sigma}}\begin{pmatrix} \tau_j \\ 0 \end{pmatrix} \right\rangle & \left\langle \underline{\underline{\varepsilon}}\begin{pmatrix} \tau_i \\ 0 \end{pmatrix} : \underline{\underline{\sigma}}\begin{pmatrix} 0 \\ \tau_j \end{pmatrix} \right\rangle \\ \left\langle \underline{\underline{\varepsilon}}\begin{pmatrix} 0 \\ \tau_i \end{pmatrix} : \underline{\underline{\sigma}}\begin{pmatrix} \tau_j \\ 0 \end{pmatrix} \right\rangle & \left\langle \underline{\underline{\varepsilon}}\begin{pmatrix} 0 \\ \tau_i \end{pmatrix} : \underline{\underline{\sigma}}\begin{pmatrix} 0 \\ \tau_j \end{pmatrix} \right\rangle \end{bmatrix} \cdot \begin{bmatrix} U_j \\ V_j \end{bmatrix} \quad \forall i$$

**DISCRETE OPERATOR**

$$\underline{\underline{\varepsilon}}\begin{pmatrix}\tau_i\\ 0\end{pmatrix} = \begin{bmatrix}\tau_{i,x} & \tau_{i,y}/2\\ \tau_{i,y}/2 & 0\end{bmatrix}$$

$$\underline{\underline{\varepsilon}}\begin{pmatrix}0\\ \tau_i\end{pmatrix} = \begin{bmatrix}0 & \tau_{i,x}/2\\ \tau_{i,x}/2 & \tau_{i,y}\end{bmatrix}$$

$$\underline{\underline{\sigma}}(\underline{\underline{\varepsilon}}) = \begin{bmatrix}A\varepsilon_{xx} + B\varepsilon_{yy} & 2C\varepsilon_{xy}\\ 2C\varepsilon_{xy} & A\varepsilon_{yy} + B\varepsilon_{xx}\end{bmatrix}$$

# HOW TO CALCULATE IT ?

| | PLANAR DEFORMATIONS | PLANAR STRESSES |
|---|---|---|
| A | $\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$ | $\frac{E}{(1-\nu^2)}$ |
| B | $\frac{E\nu}{(1+\nu)(1-2\nu)}$ | $\frac{E\nu}{(1-\nu^2)}$ |
| C | $\frac{E}{2(1+\nu)}$ | $\frac{E}{2(1+\nu)}$ |

$$\underline{\underline{\varepsilon}} : \underline{\underline{\sigma}} = \varepsilon_{xx}\left(A\,\varepsilon_{xx} + B\,\varepsilon_{yy}\right) + 4C\,\varepsilon_{xy}\,\varepsilon_{xy} + \varepsilon_{yy}\left(A\,\varepsilon_{yy} + B\,\varepsilon_{xx}\right)$$

$$\underline{\underline{A}}_{ij} = \begin{bmatrix} \langle \tau_{i,x}\, A\, \tau_{j,x} \rangle + \langle \tau_{i,y}\, C\, \tau_{j,y} \rangle & \langle \tau_{i,x}\, B\, \tau_{j,y} \rangle + \langle \tau_{i,y}\, C\, \tau_{j,x} \rangle \\ \underbrace{\langle \tau_{i,y}\, B\, \tau_{j,x} \rangle + \langle \tau_{i,x}\, C\, \tau_{j,y} \rangle}_{\underline{\underline{\varepsilon}}\binom{0}{\tau_i} : \underline{\underline{\sigma}}\left(\underline{\underline{\varepsilon}}\binom{\tau_j}{0}\right)} & \langle \tau_{i,y}\, A\, \tau_{j,y} \rangle + \langle \tau_{i,x}\, C\, \tau_{j,x} \rangle \end{bmatrix}$$

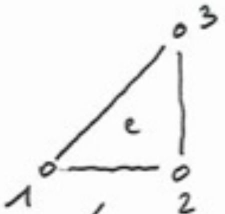
LOCAL ELASTICITY MATRIX

# COMPUTING A LOCAL ELASTICITY MATRIX

$$\begin{bmatrix} A\langle \tau_{i,x}\,\tau_{j,x}\rangle + C\langle \tau_{i,y}\,\tau_{j,y}\rangle & B\langle \tau_{i,x}\,\tau_{j,y}\rangle + C\langle \tau_{i,y}\,\tau_{j,x}\rangle \\ B\langle \tau_{i,y}\,\tau_{j,x}\rangle + C\langle \tau_{i,x}\,\tau_{j,y}\rangle & A\langle \tau_{i,y}\,\tau_{j,y}\rangle + C\langle \tau_{i,x}\,\tau_{j,x}\rangle \end{bmatrix}$$

$\underline{\underline{A}}^e_{11}$: $A_{xx11} = A$, $A_{yy11} = C$, $A_{xy11} = 0$, $A_{yx11} = 0$

$$\underline{\underline{A}}^e_y = \begin{bmatrix} \boxed{A} & \boxed{0} & -A & B & 0 & -B \\ \boxed{0} & \boxed{C} & C & -C & -C & 0 \\ & & A+C & -B-C & -C & B \\ & & -B-C & A+C & C & -A \\ & & & & C & 0 \\ & & & & 0 & A \end{bmatrix} \frac{1}{2}$$



| | $\tau_{i,x}$ | $\tau_{i,y}$ |
|---|---|---|
| 1 | -1 | 0 |
| 2 | 1 | -1 |
| 3 | 0 | 1 |

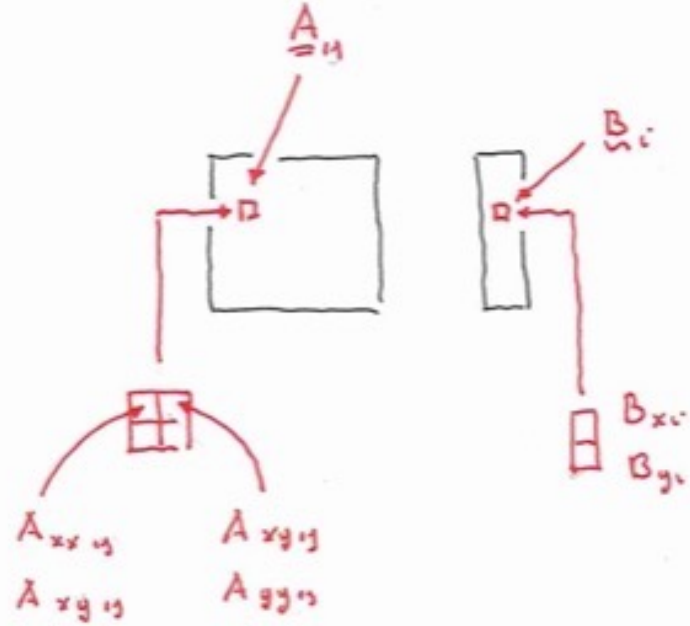
$$\underset{\sim}{B}_i = \langle\!\langle g \cdot \hat{\underset{\sim}{u}}^h \rangle\!\rangle_N$$

$$\begin{bmatrix} g_x \\ g_y \end{bmatrix}$$

$$\begin{bmatrix} \tau_i \\ 0 \end{bmatrix} \text{ OR } \begin{bmatrix} 0 \\ \tau_i \end{bmatrix}$$

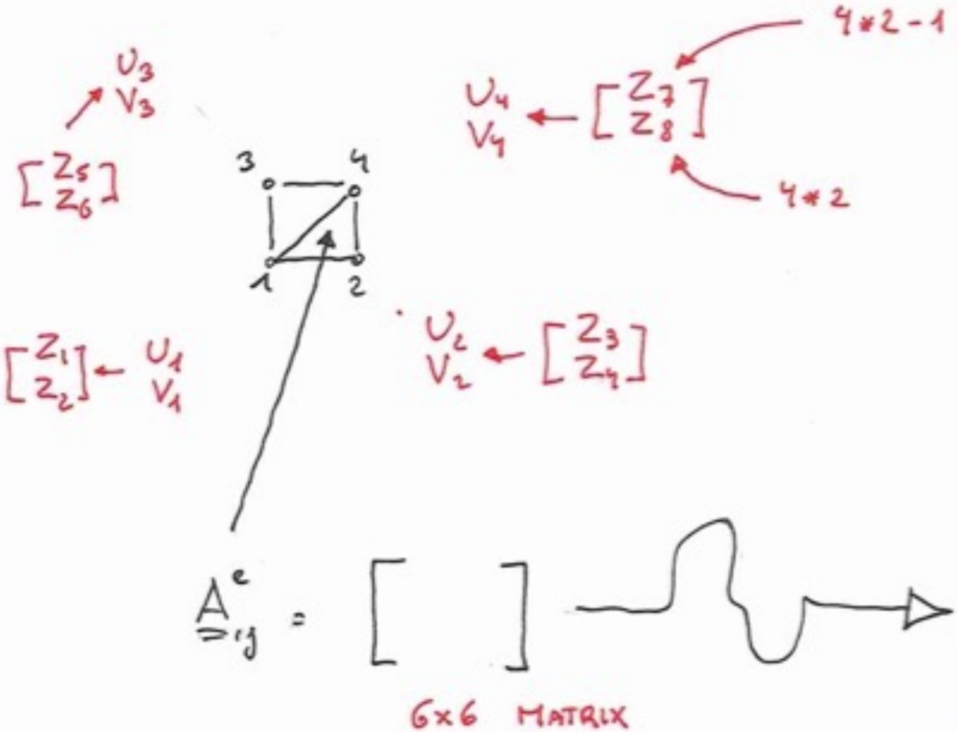
$$\nabla$$

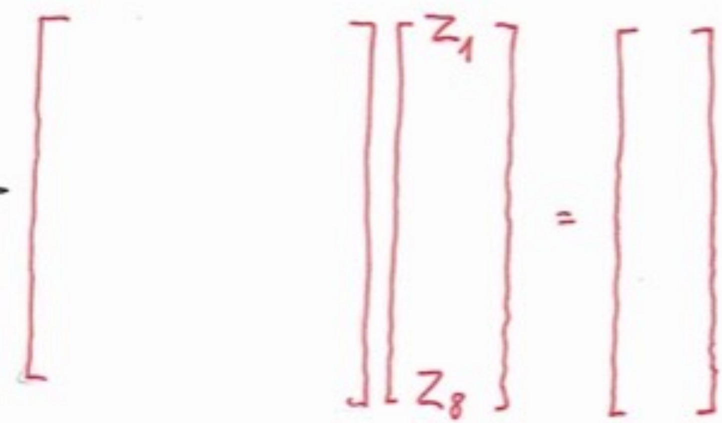
$$\boxed{\begin{bmatrix} \langle\!\langle g_x \, \tau_i \rangle\!\rangle \\ \langle\!\langle g_y \, \tau_i \rangle\!\rangle \end{bmatrix}}$$




AND HYPER VECTOR !

# ASSEMBLING PROCEDURE




$$\text{GLOBAL HORIZONTAL INDEX} = \underbrace{2 * \text{GLOBAL SCALAR INDEX}}_{\text{GLOBAL}} - 1$$