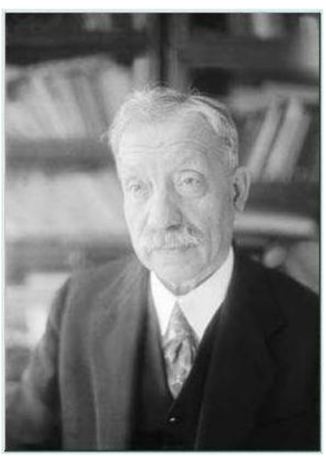
$$r^h = \nabla \cdot (a \nabla u^h) + f.$$

La technique de Galerkin consiste à annuler en moyenne le produit du résidu avec les fonctions de forme

$r^h \simeq 0$

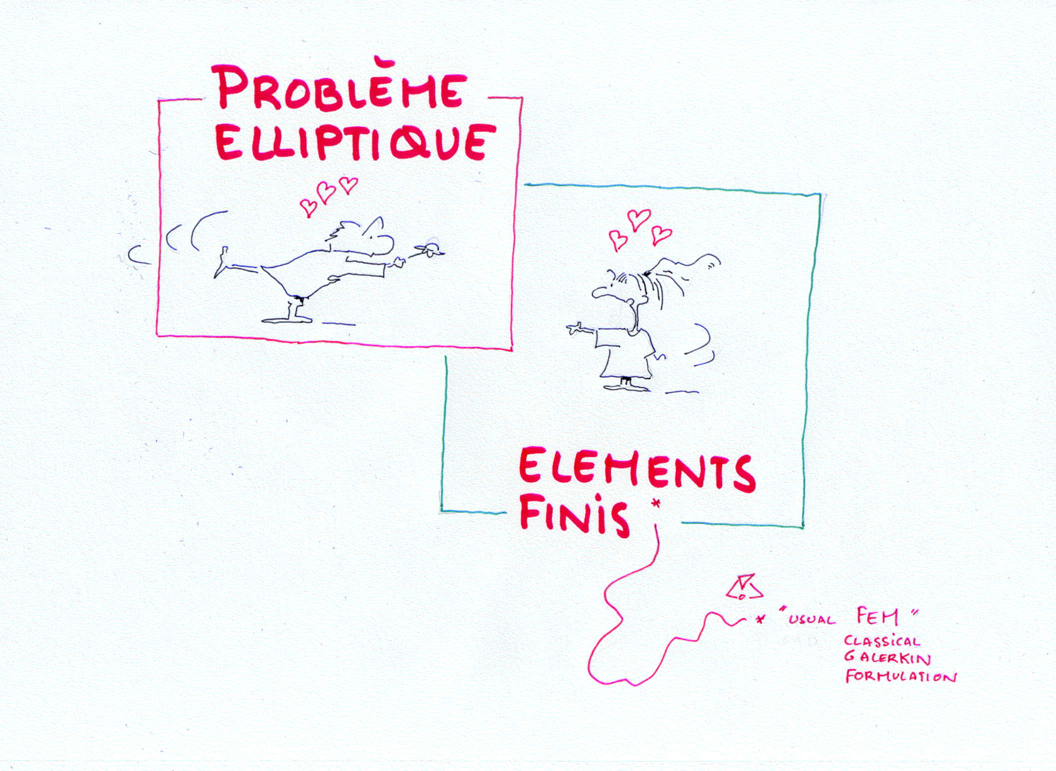
$$
\begin{aligned}
< \tau_i \, r^h > &= 0, \\
&\downarrow \\
< \tau_i \, (\nabla \cdot (a \nabla u^h)) > + < \tau_i \, f > &= 0, \\
&\downarrow \\
< (\nabla \tau_i) \cdot (a \nabla u^h) > &= < \tau_i f >, \\
&\downarrow \\
\sum_{j=1}^{n} \underbrace{< (\nabla \tau_i) \cdot (a \nabla \tau_j) >}_{A_{ij}} \, U_j &= \underbrace{< \tau_i f >}_{B_i}
\end{aligned}
$$

Et cette manière de procéder minimise l'erreur dans la norme L2 : c'est une formulation optimale en ce sens !



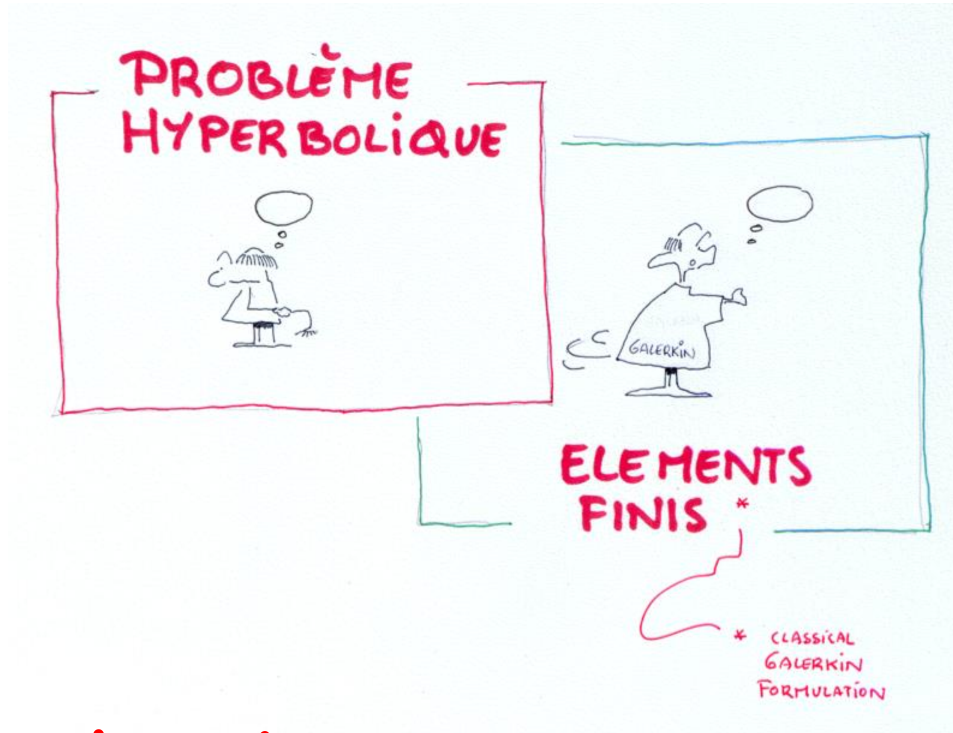
Galerkin 1871-1945

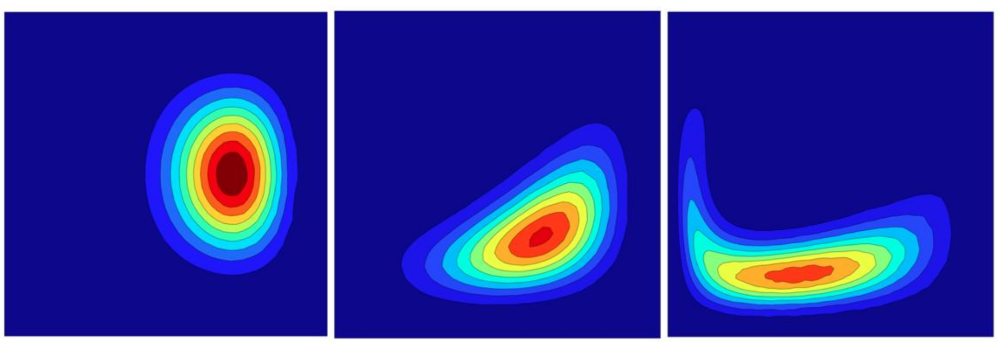
Les éléments finis sont une méthode de résidus pondérés

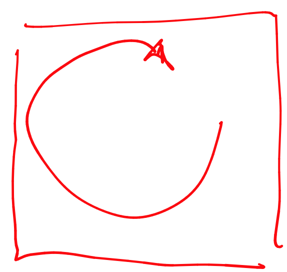


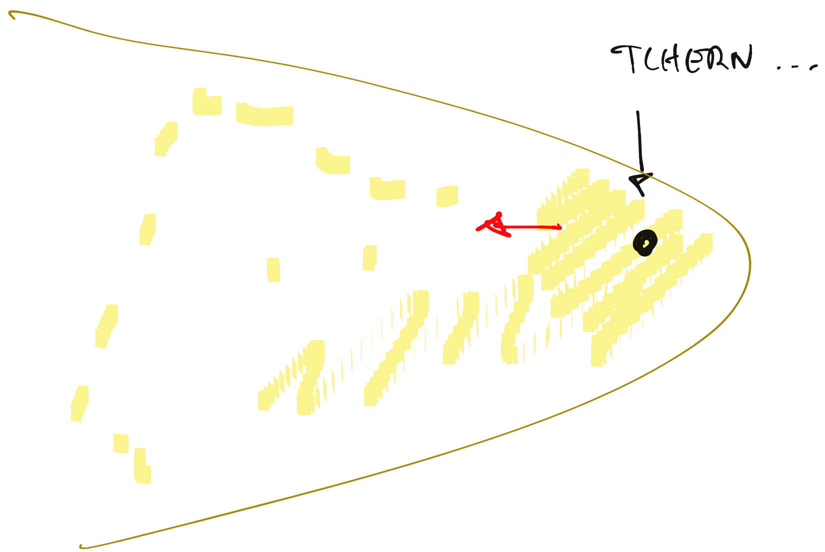
Galerkin, c'est donc optimal pour des équations elliptiques

Mais,
plus pour des
équations
d’advection diffusion !





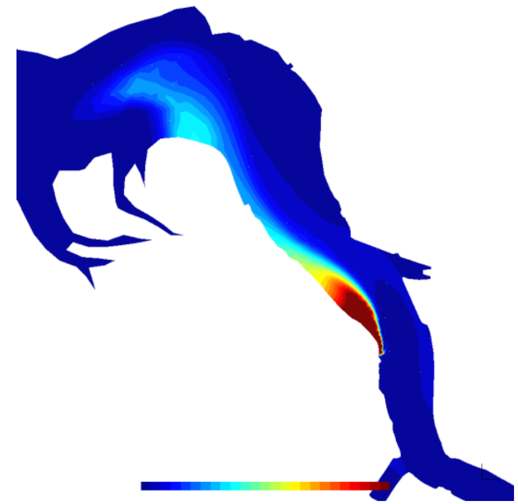


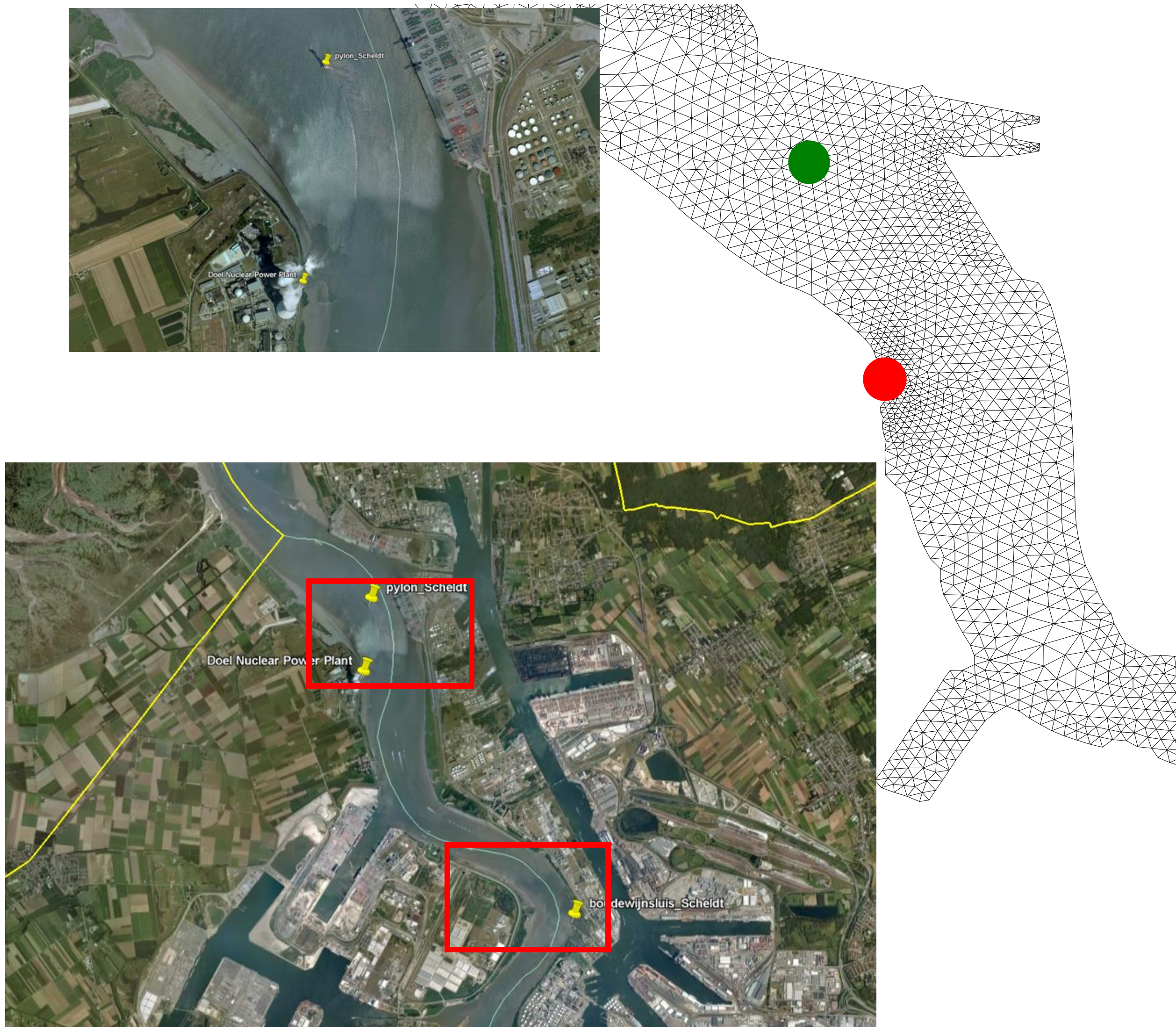

TCHERN ....

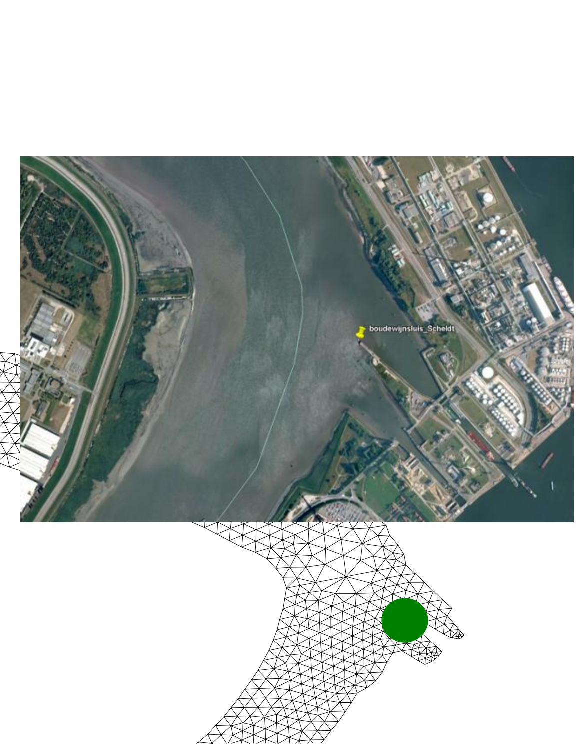

Smolensk
Minsk
Belarus
Russia
Mogilev
Krychaw
Cherykaw
Bykhov
Krasnapolle
Bryansk
Babruysk
Rahachow
Chachersk
Vetka
Novozybkov
Gomel
Klimovo
Luninets
Kalinkavičy
Mazyr
Khoiniki
Narowla
Yelsk
Vystupovichi
Chernobyl Plant
Pershotravneve
Ovruch
Lypnyky
Poliske
Chernobyl
Korosten
Chernihiv
Ukraine
Kiev
Zhytomyr
Confiscated/Closed Zone
Greater than 40 curies per square kilometer (Ci/km²) of Cesium-137
Permanent Control Zone
15 to 40 Ci/km² of Cesium-137
Periodic Control Zone
5 to 15 Ci/km² of Cesium-137
Unnamed zone
1 to 15 Ci/km² of Cesium-137

Tchernobyl…

Un petit exemple concret







Diffusion et transport d’un traceur passif

MOYENNE DE LA CONCENTRATION SUR LA PROFONDEUR

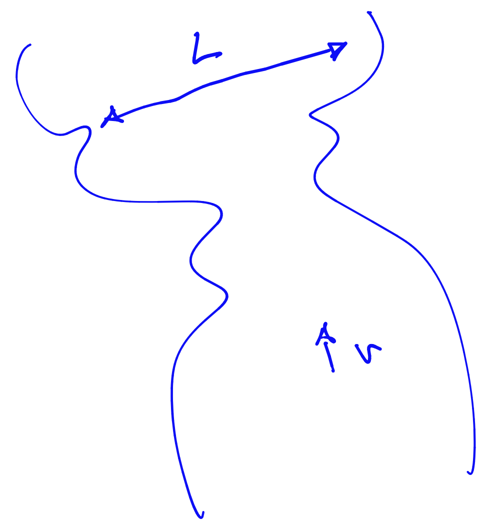
COEFFICIENT DE DIFFUSION

$$\frac{\partial c}{\partial t} + \underbrace{\underset{\sim}{v} \cdot \underset{\sim}{\nabla} c}_{\text{TERME DE TRANSPORT}} + \underbrace{\underset{\sim}{\nabla} \cdot \left(D \underset{\sim}{\nabla} c \right)}_{\text{TERME DE DIFFUSION}} + S$$

$\underset{\sim}{v}$ VITESSE HORIZONTALE

$$\frac{\partial c}{\partial t} + \boxed{u \frac{\partial c}{\partial x} + v \frac{\partial c}{\partial y}} = \boxed{k \left[\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} \right]}$$

$$\partial\left(\frac{V \Delta c}{L} \right) \qquad \partial\left(k \frac{\Delta c}{L^2} \right)$$



$$P_e = \frac{\blacksquare}{\blacksquare} = \frac{V \cancel{\Delta c}}{L} \frac{L^2}{k \cancel{\Delta c}} = \frac{VL}{k}$$

Analysons les cas particuliers !

$$\frac{\partial c}{\partial t} + v\frac{\partial c}{\partial x} = k\frac{\partial^2 c}{\partial x^2}$$

$$P_e = \frac{VL}{k} = \frac{\blacksquare}{\blacksquare}$$

	STATIONNAIRE	NON-STATIONNAIRE
P_e PETIT	$\boxed{0 = k\frac{\partial^2 c}{\partial x^2}}$ 2 ELLIPTIQUE	$\boxed{\frac{\partial c}{\partial t} = k\frac{\partial^2 c}{\partial x^2}}$ 2 PARA
	$\boxed{v\frac{\partial c}{\partial x} = k\frac{\partial^2 c}{\partial x^2}}$ 2 ELLIPTIQUE	$\boxed{\frac{\partial c}{\partial t} = v\frac{\partial c}{\partial x} + k\frac{\partial^2 c}{\partial x^2}}$ 2 PARA
P_e GRAND	$\boxed{v\frac{\partial c}{\partial x} = 0}$ 1 HYPERBOLIQUE	$\boxed{\frac{\partial c}{\partial t} = v\frac{\partial c}{\partial x}}$ 1 HYPERBOLIQUE

Le nombre de Péclet permet d'estimer l'importance du terme de transport par rapport à celui de la diffusion !

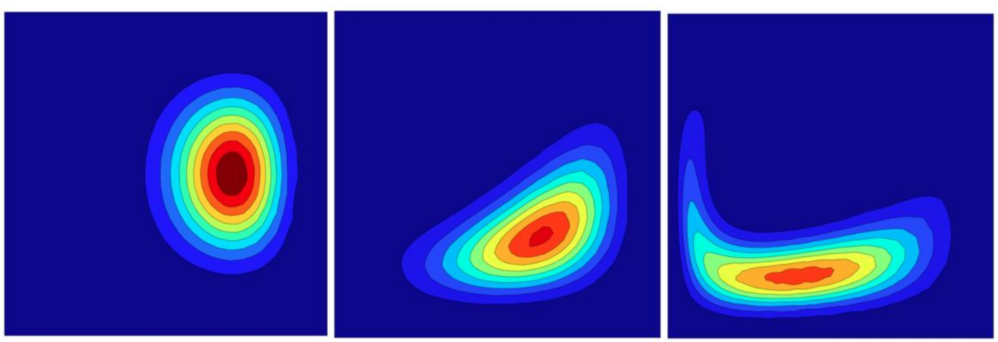
Diffusion et transport d'un traceur passif

MOYENNE DE LA CONCENTRATION SUR LA PROFONDEUR

COEFFICIENT DE DIFFUSION

$$\frac{\partial c}{\partial t} + \underbrace{\underline{v} \cdot \underline{\nabla} c}_{\text{TERME DE TRANSPORT}} = \underbrace{\underline{\nabla} \cdot (D\, \underline{\nabla} c)}_{\text{TERME DE DIFFUSION}} + S$$

$\underline{v}$ = VITESSE HORIZONTALE



C'est une équation parabolique du second ordre !

Ce n'est pas elliptique !

$$\frac{\partial c}{\partial t} + v\frac{\partial c}{\partial x} = k\frac{\partial^2 c}{\partial x^2}$$

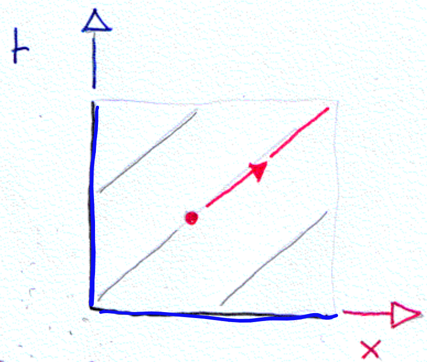
	STATIONNAIRE		INSTATIONNAIRE	
P_e PETIT	$0 = k\frac{\partial^2 c}{\partial x^2}$	ELL 2	$\frac{\partial c}{\partial t} = k\frac{\partial^2 c}{\partial x^2}$	PARA 2
	$v\frac{\partial c}{\partial x} = k\frac{\partial^2 c}{\partial x^2}$	ELL 2	$\frac{\partial c}{\partial t} + v\frac{\partial c}{\partial x} = k\frac{\partial^2 c}{\partial x^2}$	PARA 2
P_e GRAND	$v\frac{\partial c}{\partial x} = 0$	HYP 1	$\frac{\partial c}{\partial t} + v\frac{\partial c}{\partial x} = 0$	HYP 1

Le nombre de Péclet permet d'estimer l'importance du terme de transport par rapport à celui de la diffusion !

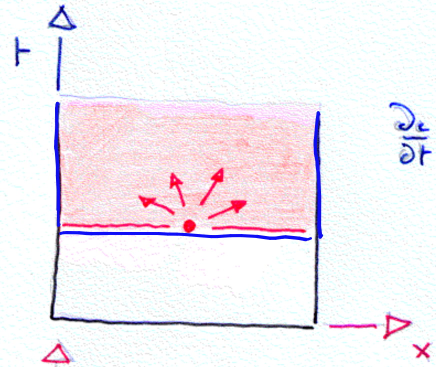
$$\boxed{P_e = \frac{vL}{k}}$$

PROBLEME BIEN POSE

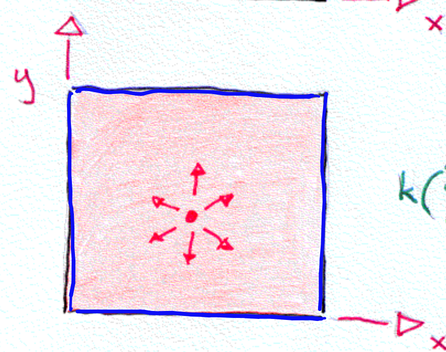
→ CONDITIONS AUX LIMITES



$$\frac{\partial c}{\partial t} + v\frac{\partial c}{\partial x} = 0$$

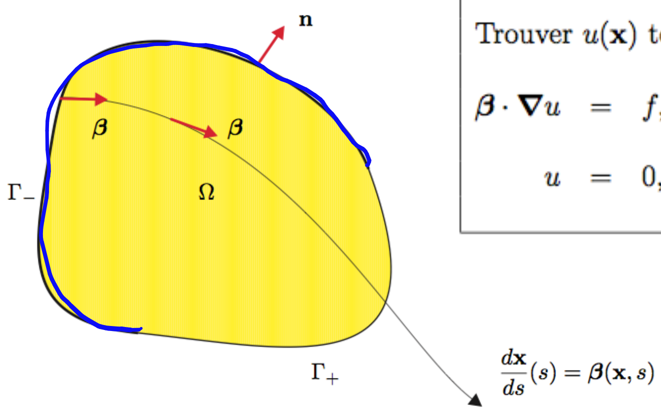


$$\frac{\partial c}{\partial t} = k\frac{\partial^2 c}{\partial x^2}$$



$$k\left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2}\right) = 0$$

Advection pure

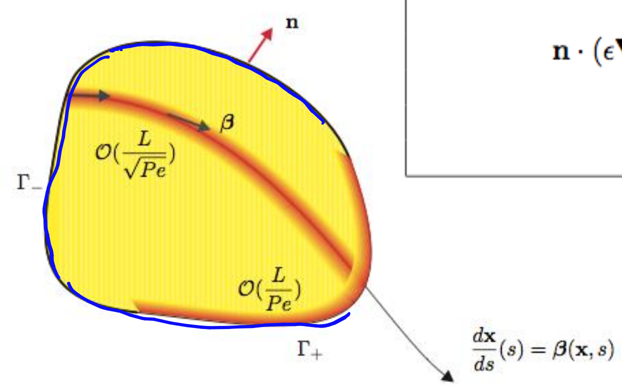


Trouver $u(\mathbf{x})$ tel que

$$
\begin{aligned}
\boldsymbol{\beta} \cdot \boldsymbol{\nabla} u &= f, \qquad \forall\, x \in \Omega, \\
u &= 0, \qquad \forall\, x \in \Gamma_-
\end{aligned}
$$

Trouver $u(\mathbf{x}) \in \mathcal{U}_s$ tel que

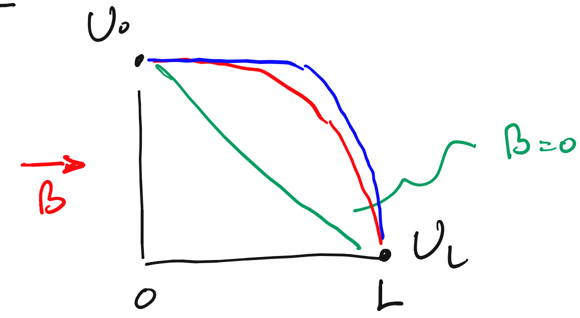
$$
\begin{aligned}
\boldsymbol{\beta} \cdot \boldsymbol{\nabla} u - \boldsymbol{\nabla} \cdot (\epsilon \boldsymbol{\nabla} u) &= f, \qquad \forall \mathbf{x} \in \Omega, \\
\mathbf{n} \cdot (\epsilon \boldsymbol{\nabla} u) &= g, \qquad \forall \mathbf{x} \in \Gamma_N, \\
u &= t, \qquad \forall \mathbf{x} \in \Gamma_D,
\end{aligned}
$$



Advection-diffusion

Equation d'advection-diffusion

$$P_e = \frac{\beta L}{\varepsilon}$$



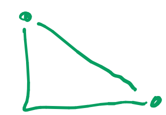
$$\beta \frac{du}{dx} - \varepsilon \frac{d^2u}{dx^2} = 0$$

$$u(x) = C \exp\left[\frac{\beta x}{\varepsilon}\right]$$

$$\varepsilon u'' = \frac{\beta^2}{\varepsilon^2} \varepsilon \exp[\dots] C$$

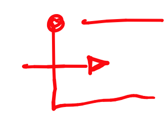
$$\beta u' = \frac{\beta}{\varepsilon} \beta \exp[\dots] C$$

$$\varepsilon \frac{d^2u}{dx^2} = 0$$

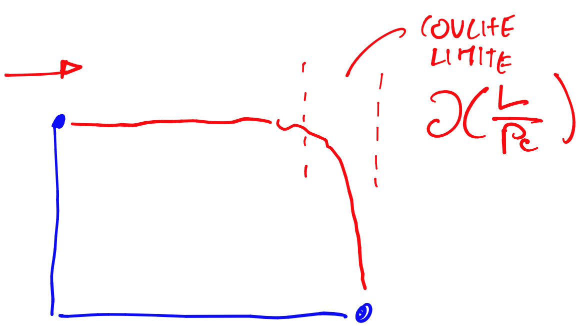


$$u(0) = U_0$$
$$u(L) = 0$$

$$\beta \frac{du}{dx} = 0$$



$$u(0) = U_0$$

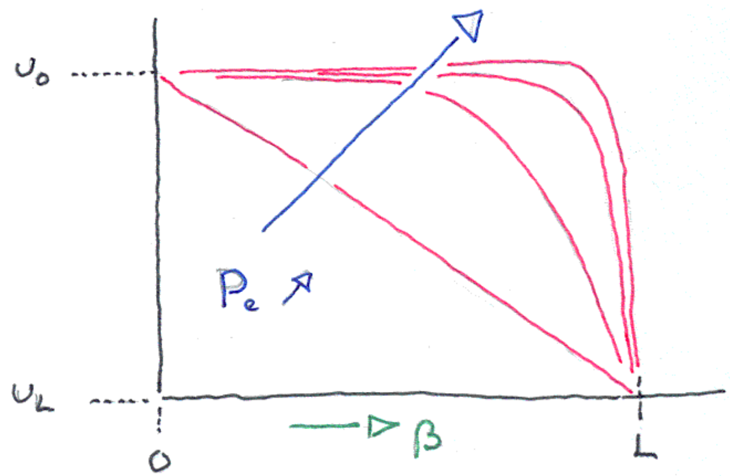


$$\boxed{\frac{u(x) - U_0}{U_L - U_0} = \frac{\exp\left[\frac{\beta x}{\varepsilon}\right] - 1}{\exp\left[\frac{\beta L}{\varepsilon}\right] - 1}}$$

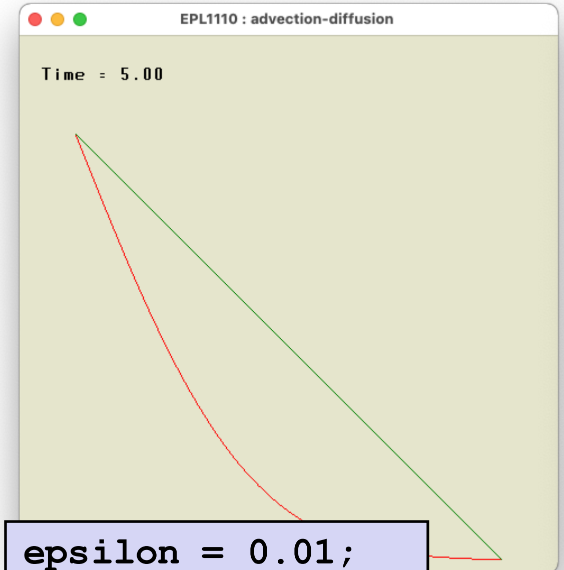
EQUATION D'ADVECTION - DIFFUSION

$$\boxed{\begin{aligned} &\beta \frac{du}{dx} - \varepsilon \frac{d^2u}{dx^2} = 0 \\ &u(0) = u_0 \\ &u(L) = u_L \end{aligned}}$$

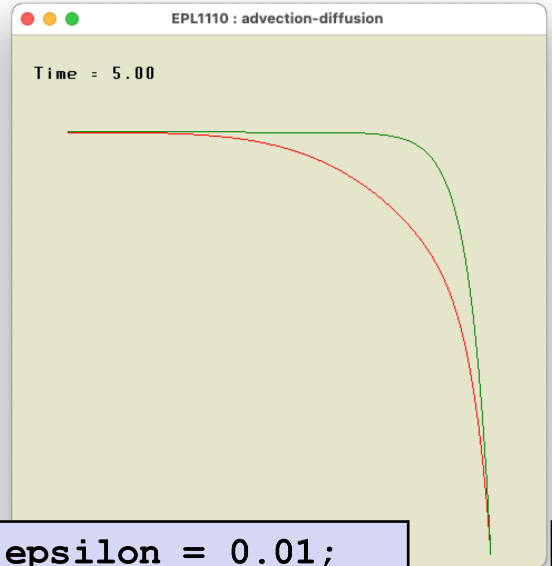
$$\boxed{P_e = \frac{\beta L}{\varepsilon}}$$



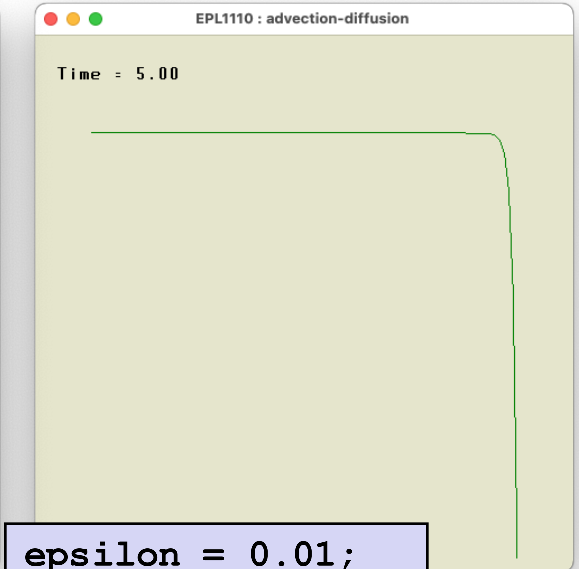
$$\frac{u - u_0}{u_L - u_0} = \frac{\exp\left(\overbrace{\beta x/\varepsilon}^{P_e\, x/L}\right) - 1}{\exp\left(\underbrace{\beta L/\varepsilon}_{P_e}\right) - 1}$$



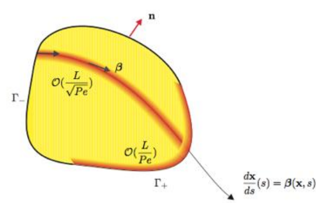
epsilon = 0.01;
beta = 0.0;



epsilon = 0.01;
beta = 0.2;



epsilon = 0.01;
beta = 1.0;



Trouver $u(\mathbf{x}) \in \mathcal{U}_s$ tel que

$$
\begin{aligned}
\boldsymbol{\beta} \cdot \boldsymbol{\nabla} u - \boldsymbol{\nabla} \cdot (\epsilon \boldsymbol{\nabla} u) &= f, &\quad \forall \mathbf{x} \in \Omega,\\
\mathbf{n} \cdot (\epsilon \boldsymbol{\nabla} u) &= g, &\quad \forall \mathbf{x} \in \Gamma_N,\\
u &= t, &\quad \forall \mathbf{x} \in \Gamma_D,
\end{aligned}
$$

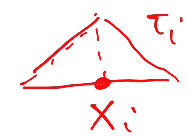
Advection-diffusion

Equation de transport !
Advection pure !

$$\frac{du}{dx} = f$$

$$u(0) = U_0$$

$$u(x) \simeq u^h(x) = \sum U_i \, \tau_i(x)$$



GALERKIN

$$\langle r^h \, \tau_i \rangle = 0$$

$$\sum_j \underbrace{\langle \tau_i \, \tau_j' \rangle}_{A_{ij}} U_j = \underbrace{\langle f \, \tau_i \rangle}_{B_i}$$

$$\underbrace{\frac{du^h}{dx} - f}_{r^h} \simeq 0$$

Appliquons la méthode de Galerkin
à notre problème de transport

$u' = f$

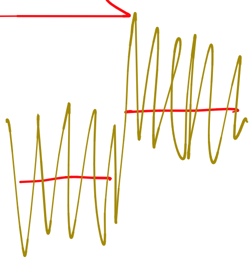
$$\frac{du}{dx} = \mu(x)$$

OK :-)

$$\frac{du}{dx} =$$

$$u =$$

KO



Bubnov-Galerkin !

Ce n'est plus un problème de minimisation

$$u \simeq u^h = \sum U_i \tau_i$$

$$\langle r^h \, \hat{\tau}_i \rangle = 0 \quad \forall i$$

$$\boxed{\hat{\tau}_i = \tau_i}$$

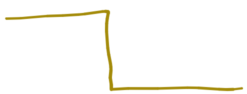
Petrov-Galerkin !
C'est plus le même problème !

$$\frac{du}{dx} = f$$

$$u(0) = U_0$$

$$\Big\langle \overbrace{\Big(\underbrace{\frac{du^h}{dx}}_{\sum U_j \tau_j'} - f\Big)}^{r^h} \tau_i' \Big\rangle = 0$$

$$\boxed{\hat{\tau}_i = \tau_i'}$$



$u' = 0$

$$\sum U_j \underbrace{\langle \tau_i' \, \tau_j' \rangle}_{A_{ij}} = \underbrace{\langle f \, \tau_i' \rangle}_{-\langle f \, \tau_i \rangle + \ll \; \gg}$$

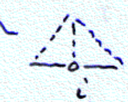
SYM
DEFINIE POSITIVE

$$\frac{d^2u}{dx^2} = \frac{df}{dx} \qquad u(0) = \qquad u(L) = \dots$$

EQUATION DE TRANSPORT

$$\boxed{\begin{array}{l} \dfrac{du}{dx} = f \\ u(0) = u_0 \end{array}}$$

$$u \simeq u^h = \sum_i U_i \underbrace{\tau_i(x)}$$



GALERKIN

? U_i TELS QUE

$$\langle \tau_i \;\; r_h \rangle = 0$$

$$\sum_j \overbrace{\langle \tau_i \;\; \tau_{j,x} \rangle}^{A_{ij}} U_j = \overbrace{\langle \tau_i \;\; f \rangle}^{B_i}$$

≠ MATRICE DEFINIE POSITIVE !

$$\boxed{\frac{du}{dx} = f}$$

$\xrightarrow{\text{GALERKIN}}$

$$\boxed{\sum_j A_{ij} U_j = B_i}$$

BUBNOV - GALERKIN

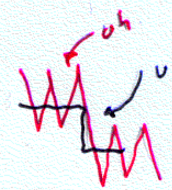
DIFFERENCES FINIES CENTREES

$$\frac{U_{j+1} - U_{j-1}}{2\Delta x} = B_j$$

$\not\Leftrightarrow$ **PROBLEME DE MINIMISATION**

$f(x) = \sin(x) \longrightarrow$ OK

$f(x) =$ $\longrightarrow$ "Boum"



PETROV-GALERKIN

? U_j TELS QUE

$$\langle \tau_{j,x} \; r^h \rangle = 0$$

INTEGRATION LE LONG DES CARACTERISTIQUES

DIFFERENCES FINIES AMONT

$$\frac{U_{j+1} - U_j}{\Delta x} = B_j$$

$$\sum_j \underbrace{\langle \tau_{i,x} \; \tau_{j,x} \rangle}_{A_{ij}} U_j = \underbrace{\langle \tau_{i,x} \; f \rangle}_{B_i}$$

MATRICE DEFINIE POSITIVE

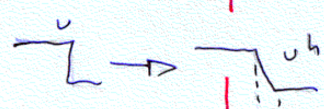
1 CONDITION AUX LIMITES !

$$\frac{du}{dx} = f$$

A ÉTÉ REMPLACÉ PAR $\frac{d^2u}{dx^2} = \frac{df}{dx}$

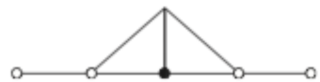
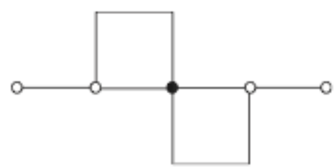
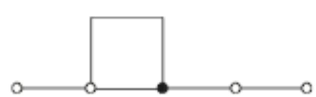
2 CONDITIONS AUX LIMITES !

HIC



En bref :-)

$$\boxed{\begin{aligned} \frac{du}{dx} &= f, \\ u(0) &= 0, \end{aligned}}$$

Galerkin $w_i = \tau_i$ 	*Différences finies centrées* Simple et donc tentant... Oscillations numériques si f n'est pas lisse ! $\frac{U_{i+1} - U_{i-1}}{2h} = \frac{F_{i+1} + 4F_i + F_{i-1}}{6},$
Petrov-Galerkin $w_i = \tau_{i,x}$ 	*Différences finies centrées d'ordre deux* Mathématiquement, tentant Condition frontière parasite ! $\frac{U_{i+1} - 2Ui + U_{i-1}}{h^2} = \frac{F_{i+1} - F_{i-1}}{2h},$
Petrov-Galerkin $w_i = \tau_{i-1}^{cst}$ 	*Différences finies amont* Quasiment optimal... Correspond à une intégration le long de la caractéristique, Pas d'oscillation numérique $\frac{U_i - U_{i-1}}{h} = \frac{F_{i+1} + F_{i-1}}{2},$

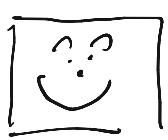
Petrov-Galerkin !
Comment sélectionner le paramètre magique ?

$$u' - \varepsilon u'' - f = 0$$

$u' \to \tau_i'$, $u'' \to \tau_i$

$$< (\tau_i + \zeta \tau_i')((u^h)' - \varepsilon (u^h)'' - f) > = 0 \quad \forall i$$

$$\sum U_j \left[\zeta <\tau_i' \tau_j'> - \varepsilon \zeta <\tau_i' \tau_j''> + \underbrace{\varepsilon <\tau_i' \tau_j'>}_{-\varepsilon <\tau_i \tau_j''> + \ll \dots \gg} + <\tau_i \tau_j'> \right] = \dots$$



$\zeta \nearrow$ AUGMENTE

MAIS

AUGMENTE AUSSI

SAUF SI $\tau_j'' = 0$

2D

$$\hat{\tau}_i = \tau_i + \zeta\, \underline{\beta} \cdot \underline{\nabla} \tau_i$$

STREAMLINE UPWINDING

PETROV-GALERKIN

$$\langle (\underbrace{\tau_i + \zeta\, \tau_{i,x}}_{\hat{\tau}_i}) (\overbrace{u^h_{,x} - \varepsilon\, u^h_{,xx} - f}^{r^h}) \rangle = 0$$

HOW TO SELECT ζ ?

$$\sum_j \left(\begin{matrix} \zeta \langle \tau_{i,x}\, \tau_{j,x} \rangle \\ \varepsilon \langle \tau_{i,x}\, \tau_{j,x} \rangle \end{matrix} \quad \begin{matrix} - \\ + \end{matrix} \quad \begin{matrix} \varepsilon\, \zeta \langle \tau_{i,x}\, \tau_{j,xx} \rangle \\ \langle \tau_i \quad \tau_{j,x} \rangle \end{matrix} \right) U_j = \dots$$

Différences finies centrées
On résout analytiquement le problème discret…

$$\beta u' = \varepsilon u''$$

$$\boxed{U_i = A r^i + B}$$

$$\beta \, \frac{U_{i+1} - U_{i-1}}{2h} = \varepsilon \, \frac{U_{i+1} - 2U_i + U_{i-1}}{h^2}$$

$$\cancel{A r^{i-1}} \, \boxed{\frac{\beta h}{2\varepsilon}} \, (r^2 - 1) = \cancel{A r^{i-1}} \, (r^2 - 2r + 1)$$

$$\sim \underbrace{\frac{Pe^h}{2}}_{\text{☺}}$$

PECLET DE MAILLE

$$0 = \underbrace{(1 - \text{☺})}_{a} \, r^2 - 2r + \underbrace{(1 + \text{☺})}_{c}$$

$$r = \frac{1 \pm \sqrt{1 - ac}}{a} = \frac{1 \pm \sqrt{\cancel{1} - \cancel{1} + (\text{☺})^2}}{(1 - \text{☺})} = \frac{1 + \text{☺}}{1 - \text{☺}}$$

$$\frac{\beta h}{2\varepsilon} = \frac{Pe^h}{2} < 1$$

DIFF. CENTRÉES

$$\beta \, \frac{U_{i+1} - U_{i-1}}{2h} = \varepsilon \, \frac{U_{i+1} - 2U_i + U_{i-1}}{h^2}$$

$$U_0 = u_0$$

$$U_N = u_L$$

EQUATION AUX RECURENCES !

$$\boxed{U_i = A r^i + B}$$

$$\cancel{A r^{i-1}} \, \underbrace{\frac{\beta h}{2\varepsilon}} \, (r^2 - 1) = \cancel{A r^{i-1}} \, (r^2 - 2r + 1)$$

$$\triangleq \frac{P_e^h}{2}$$

PECLET DE MAILLE

$$0 = \left(\frac{1 - P_e^h/2}{2} \right) r^2 - r + \left(\frac{1 + P_e^h/2}{2} \right)$$

$$r = \frac{1 \pm \sqrt{1 - (1 + P_e^h/2)(1 - P_e^h/2)}}{(1 - P_e^h/2)}$$

$$r = \frac{1 + P_e^h/2}{1 - P_e^h/2}$$

REJECT $r = 1$ OF COURSE !

$$\frac{U_i - U_0}{U_L - U_0} = \frac{\left(\frac{1 + P_e^h/2}{1 - P_e^h/2}\right)^i - 1}{\left(\frac{1 + P_e^h/2}{1 - P_e^h/2}\right)^N - 1}$$

$$\left(\frac{1 + P_e^h/2}{1 - P_e^h/2}\right)^N \simeq \exp\left(P_e^h\right)^N \simeq \exp\Big(\underbrace{\frac{\beta h N}{\varepsilon}}_{P_e}\Big)$$

[SCHAUM p 551]

$$\frac{x}{e^x - 1} \simeq 1 - x/2$$

$$\Rightarrow \quad (1 + x/2) = e^x (1 - x/2)$$

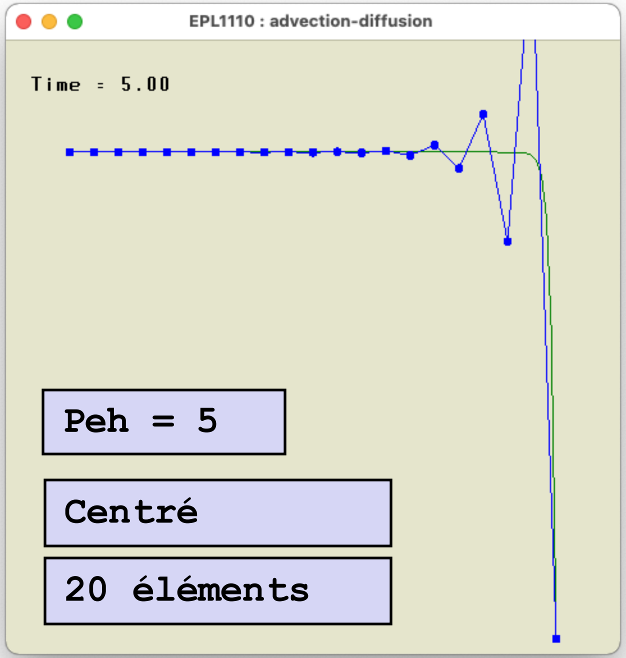
$$\boxed{\frac{P_e^h}{2} < 1}$$

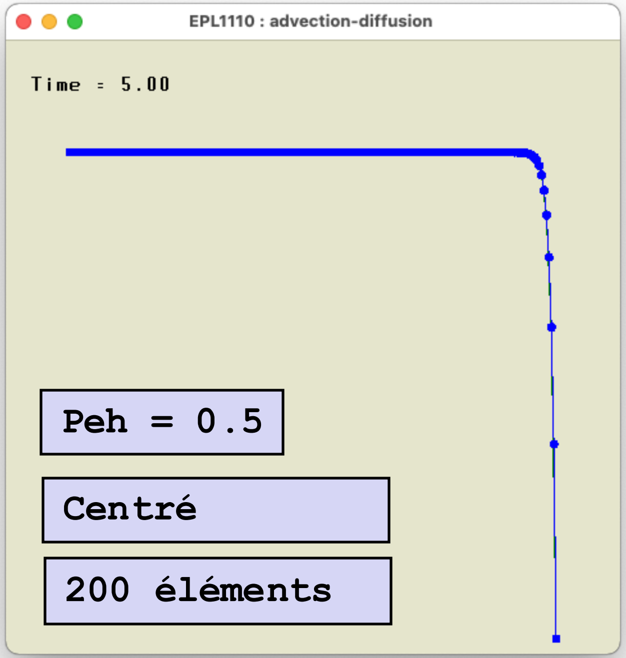
TO AVOID OSCILLATORY BEHAVIOUR

Galerkin converge si on raffine le maillage suffisament !

```
epsilon = 0.01;
beta = 1.0;
```

L’équation d’advection-diffusion est formellement une équation elliptique et donc c’était prévisible par la théorie !





Différences finies décentrées.
On résout analytiquement
le problème discret…

$$\beta \frac{U_i - U_{i-1}}{h} = \varepsilon \frac{U_{i+1} - 2U_i + U_{i-1}}{h^2}$$

$$\cancel{A r^{i-1}} \underbrace{\boxed{\frac{\beta h}{2\varepsilon}}\cancel{(r^2 - 1)}}_{\frac{\beta h}{\varepsilon}(r-1)} = \cancel{A r^{i-1}} (r^2 - 2r + 1)$$

$$\frac{\beta h}{2\varepsilon} = \underbrace{\frac{Pe^h}{2}}_{☺}$$

PECLET DE MAILLE

$$0 = r^2 - 2r(1 + ☺) - (1 - 2☺)$$

$$r = (1 + ☺) \pm \sqrt{(1 + ☺)^2 - (1 - 2☺)}$$

$$= \underbrace{1 + 2☺}_{\text{TOUJOURS} > 0\,!}$$

$$\beta u' = \left(\varepsilon + \frac{\beta h}{2}\right) u''$$

$$\beta \frac{U_i - U_{i-1}}{h} = \varepsilon \frac{U_{i+1} - 2U_i + U_{i-1}}{h^2}$$

$$\beta \left[\frac{U_{i+1}}{2h} - \frac{U_{i-1}}{2h} \right] - \left[\frac{U_{i+1}}{h^2} - \frac{2U_i}{h} + \frac{U_{i-1}}{h^2} \right] \left[\frac{\beta h}{2} \right]$$

… et on obtient la diffusion numérique !

UPWIND DIFF.

$$\boxed{\beta \frac{U_i - U_{i-1}}{h} = \varepsilon \frac{U_{i+1} - 2U_i + U_{i-1}}{h^2} \qquad U_0 = u_0 \qquad U_N = u_L}$$

$$\cancel{A r^{i-1}} \underbrace{\frac{\beta h}{\varepsilon}}_{\boxed{\triangleq Pe^h}} (r-1) = \cancel{A r^{i-1}} (r^2 - 2r + 1)$$

$$0 = \frac{r^2}{2} - \left(1 + Pe^h/2\right) r + \frac{(1 + Pe^h)}{2}$$

$$r = \left(1 + Pe^h/2\right) \pm \sqrt{\underbrace{\left(1 + Pe^h/2\right)^2 - \left(1 + Pe^h\right)}_{Pe^h/2}}$$

$$= 1 + Pe^h$$

$$\frac{U_i - U_0}{U_L - U_0} = \frac{(1 + Pe^h)^i - 1}{(1 + Pe^h)^N - 1}$$

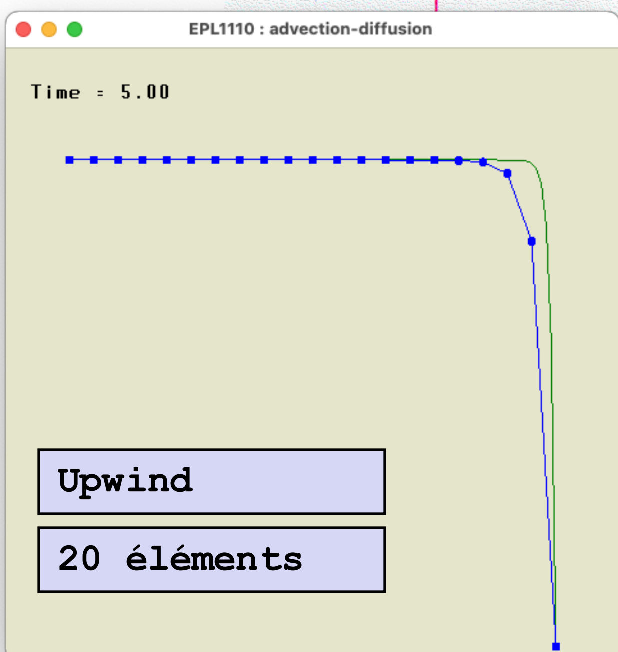
$$\boxed{(1 + Pe^h) \underset{\forall h}{>} 0 \quad \text{NO OSCILLATIONS}}$$

BUT...

NUMERICAL DIFFUSION

$$\beta \, \frac{U_i - U_{i-1}}{h} = \varepsilon \, \frac{U_{i+1} - 2U_i + U_{i-1}}{h^2}$$

$$\beta \, \frac{U_{i+1} - U_{i-1}}{2h} - \underbrace{\frac{\beta h}{2}}_{\text{NUMERICAL DIFFUSIVITY}} \frac{U_{i+1} - 2U_i + U_{i-1}}{h^2}$$



On fait le meilleur compromis !
On résout analytiquement
le problème discret…

$$\beta(1-\zeta)\frac{U_{i+1}-U_{i-1}}{2h}+\zeta\beta\frac{U_i-U_{i-1}}{h}=\varepsilon\frac{U_{i+1}-2U_i+U_{i-1}}{h^2}$$

HYBRID SCHEME

$$\boxed{(1-\zeta)\beta\,\frac{U_{i+1}-U_{i-1}}{2h} + \zeta\beta\,\frac{U_i-U_{i-1}}{2h} = \varepsilon\,\frac{U_{i+1}-2U_i+U_{i-1}}{h^2}}$$

$$0 = \left(1 - \frac{(1-\zeta)\,Pe^{h}/2}{2}\right)r^2 - \left(1+\zeta\,Pe^{h}/2\right)r + \left(1 + \frac{\overbrace{(1-\zeta)\,Pe^{h}/2 + \zeta\,Pe^{h}}^{(1+\zeta)\,Pe^{h}/2}}{2}\right)$$

$$r = \frac{\left(1+\zeta\,Pe^{h}/2\right) \pm \sqrt{\underbrace{\left(1+\zeta\,Pe^{h}/2\right)^2 - \left(1-(1-\zeta)\,Pe^{h}/2\right)\cdot\left(1+(1+\zeta)\,Pe^{h}/2\right)}_{=\,Pe^{h}/2}}}{\left(1-(1-\zeta)\,Pe^{h}/2\right)}$$

$$\boxed{\pi = \frac{1 + (1+\zeta)\,Pe\,h/2}{1 - (1-\zeta)\,Pe\,h/2}}$$

HOW TO SELECT ζ?

$$U_i = u(ih)$$

$$\underbrace{\left(\frac{1+(1+\zeta)\,Pe\,h/2}{1-(1-\zeta)\,Pe\,h/2}\right)^i}_{\pi^i} = \underbrace{\exp\left(\frac{\beta h}{\varepsilon}\, i\right)}_{\left(\exp(Pe\,h)\right)^i}$$

$$(1 + Pe\,h/2) - \zeta\, Pe\,h/2 = \exp(Pe)\left((1 - Pe\,h/2) + \zeta\, Pe\,h/2\right)$$

$$\zeta \, \frac{P_e^h}{2} \left(1 - \exp(P_e) \right) = \exp(P_e^h) \left(1 - \frac{P_e^h}{2} \right) - \left(1 + \frac{P_e^h}{2} \right)$$

$$\Downarrow$$

$$\zeta = \frac{\exp(P_e^h) \left(\frac{2}{P_e^h} - 1 \right) - \left(\frac{2}{P_e^h} + 1 \right)}{\left(1 - \exp(P_e^h) \right)}$$

$$\Downarrow$$

$$= - \underbrace{\frac{\left(1 + \exp(P_e^h) \right)}{\left(1 - \exp(P_e^h) \right)}}_{\coth(P_e^h/2)} - \frac{2}{P_e^h}$$

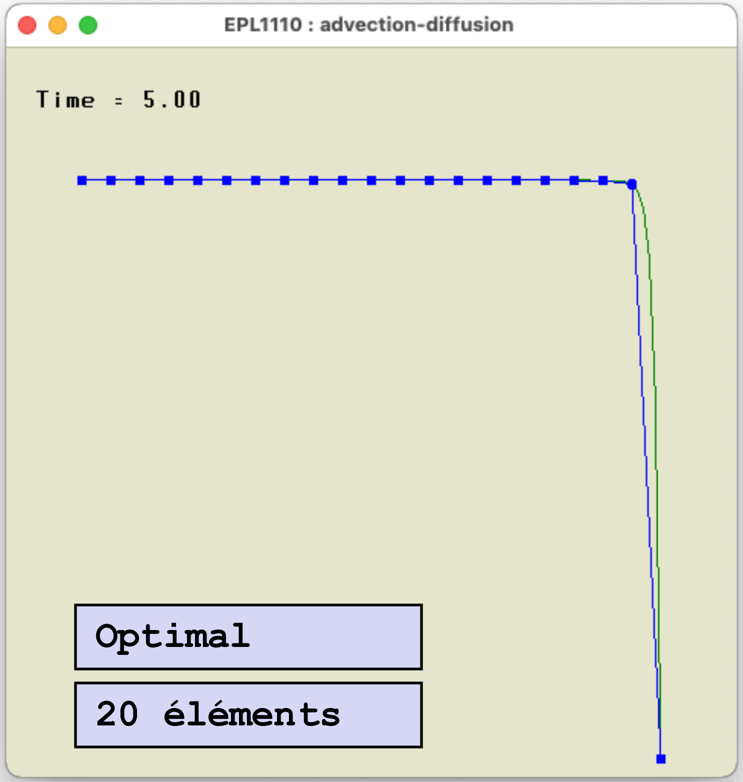
$$\boxed{\zeta = \coth\left(\frac{P_e^h}{2} \right) - \frac{2}{P_e^h}}$$

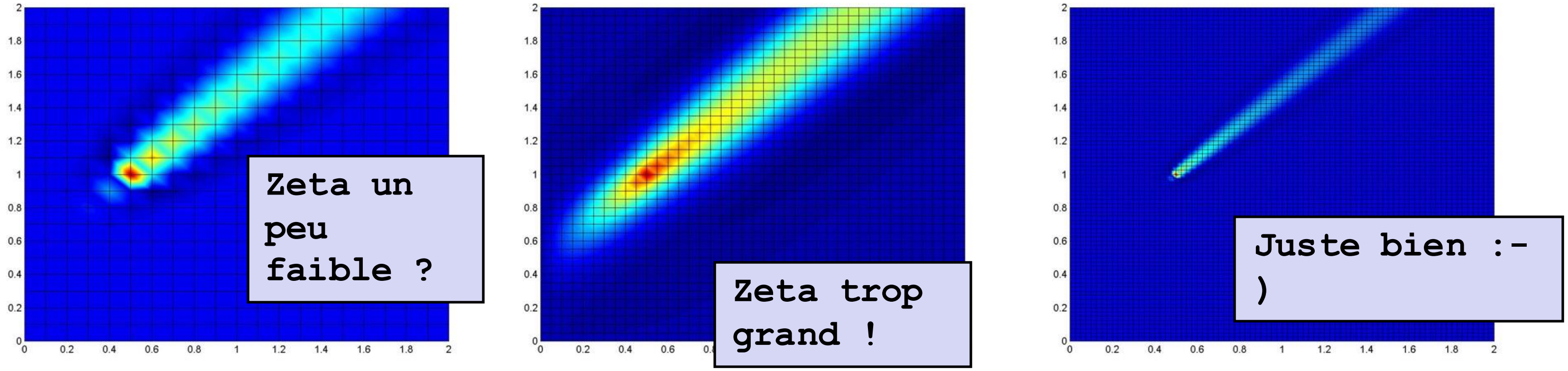
$\square$

On a trouvé
la méthode parfaite
pour des équations
unidimensionnelles !

$$
\begin{aligned}
\beta\frac{du}{dx} - \epsilon\frac{d^2u}{dx^2} &= 0,\\
u(0) &= u_0,\\
u(L) &= u_L,
\end{aligned}
$$

$$
\boxed{\zeta \;=\; \coth\!\left(\frac{Pe^h}{2}\right) - \frac{2}{Pe^h}}
$$





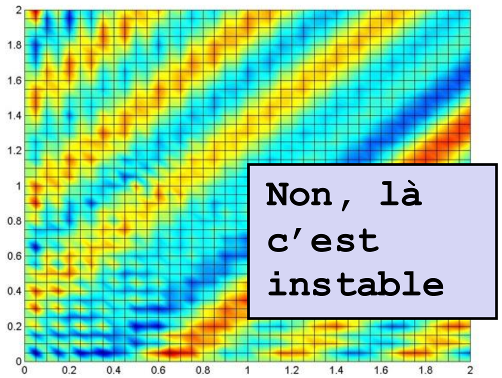
En extrapolant aux dimensions supérieures…

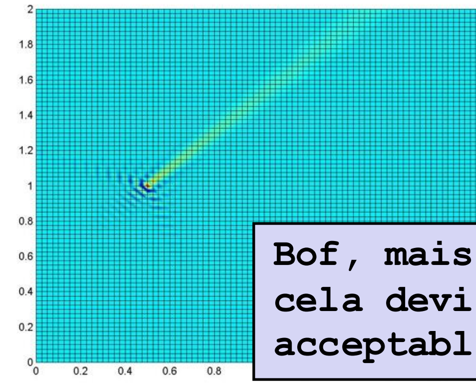
$$w_i = \tau_i + \zeta \boldsymbol{\beta} \cdot \boldsymbol{\nabla} \tau_i$$

Facteur de stabilisation
Trop grand : diffusion numérique !
Trop petit : instable !

Trouver $U_j \in \mathbb{R}^n$ tel que

$$\sum_{j=1}^{n} \underbrace{< w_i \; \boldsymbol{\beta} \cdot \boldsymbol{\nabla} \tau_j + \epsilon \boldsymbol{\nabla} w_i \cdot \boldsymbol{\nabla} \tau_j >}_{A_{ij}} \; U_j \quad = \quad \underbrace{< w_i f > + \ll w_i g \gg_N}_{B_i} , \qquad i = 1, \ldots, n,$$





Et en payant le prix, Galerkin fonctionne !

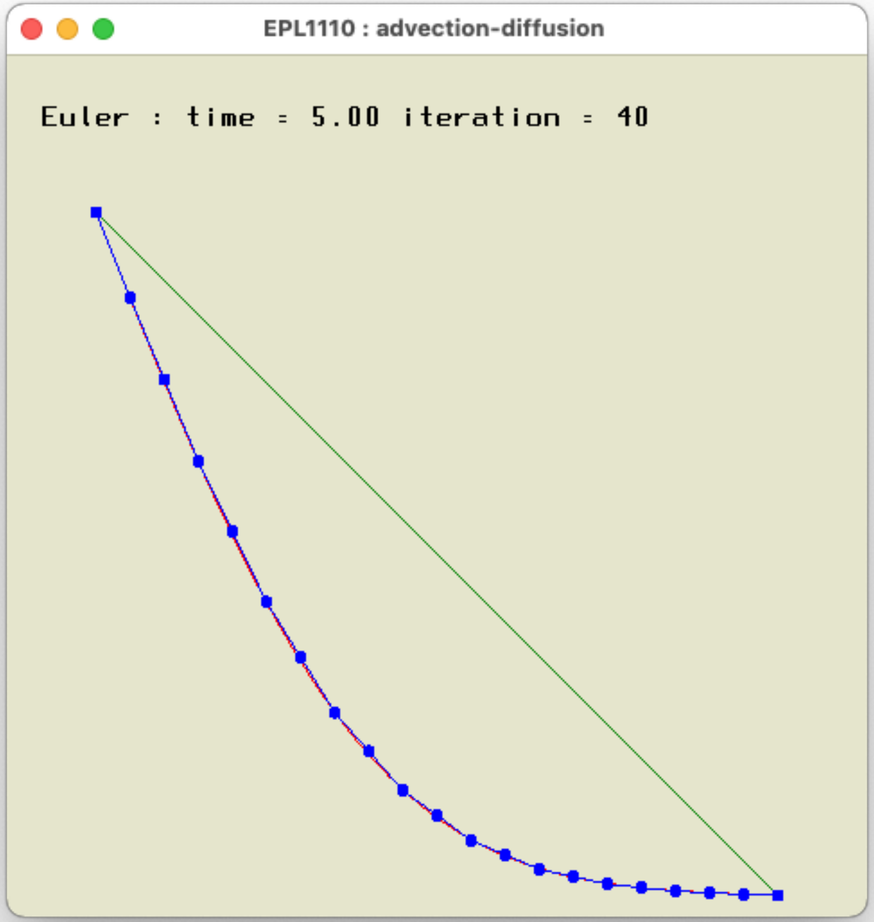
$$w_i = \tau_i + \zeta \boldsymbol{\beta} \cdot \boldsymbol{\nabla} \tau_i$$

Pas de stabilisation !
Zeta = 0 !

Trouver $U_j \in \mathbb{R}^n$ tel que

$$\sum_{j=1}^{n} \underbrace{< w_i \ \boldsymbol{\beta} \cdot \boldsymbol{\nabla} \tau_j + \epsilon \boldsymbol{\nabla} w_i \cdot \boldsymbol{\nabla} \tau_j >}_{A_{ij}} U_j \quad = \quad \underbrace{< w_i f > + \ll w_i g \gg_N}_{B_i}, \qquad i = 1, \ldots, n,$$

Et maintenant introduisons le temps…



$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \epsilon \frac{\partial^2 u}{\partial x^2} \\
u(0) &= 1 \\
u(1) &= 0
\end{aligned}
$$

```
epsilon = 0.01;
L = 1
```

Différences finies (espace)
Euler explicite (temps)

$$\left(\frac{U_i^{n+1} - U_i^n}{\Delta t}\right) = \epsilon \left(\frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{(\Delta x)^2}\right)$$

En définissant $b = \dfrac{\epsilon \Delta t}{(\Delta x)^2}$,

$$\boxed{U_i^{n+1} = U_i^n + b\Big(U_{i+1}^n + U_{i-1}^n - 2U_i^n\Big)}$$

C'est une itération pour un vecteur qui doit converger vers la solution de régime
C'est quelque chose qu'on a déjà rencontré…

On intègre un système linéaire…

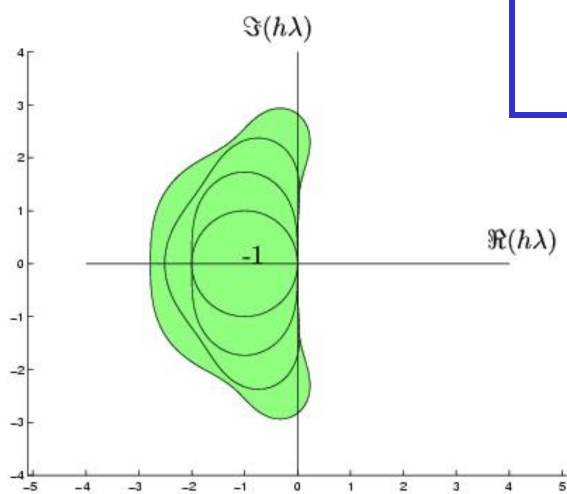
$$U_i^{n+1} = U_i^n + b\left(U_{i+1}^n + U_{i-1}^n - 2U_i^n\right)$$

En passant à une notation matricielle,

$$\begin{bmatrix} U_1^{n+1} \\ U_2^{n+1} \\ U_3^{n+1} \\ U_4^{n+1} \\ U_5^{n+1} \\ \\ U_m^{n+1} \end{bmatrix} = \begin{bmatrix} U_1^n \\ U_2^n \\ U_3^n \\ U_4^n \\ U_5^n \\ \\ U_m^n \end{bmatrix} + b \begin{bmatrix} -2 & 1 & & & & & & \\ 1 & -2 & 1 & & & & & \\ & 1 & -2 & 1 & & & & \\ & & 1 & -2 & 1 & & & \\ & & & 1 & -2 & 1 & & \\ & & & & & & & \\ & & & & & & 1 & -2 \end{bmatrix} \begin{bmatrix} U_1^n \\ U_2^n \\ U_3^n \\ U_4^n \\ U_5^n \\ \\ U_m^n \end{bmatrix}$$

En définissant adéquatement u_n et A,

$$\mathbf{u}_{n+1} = \mathbf{u}_n + b\mathbf{A}\mathbf{u}_n$$



On résout le système linéaire défini par :

$$\mathbf{u}'(t) = \frac{\epsilon}{(\Delta x)^2}\mathbf{A}\mathbf{u}(t)$$

$$\Delta x = 0.1, \Delta t = 0.005$$

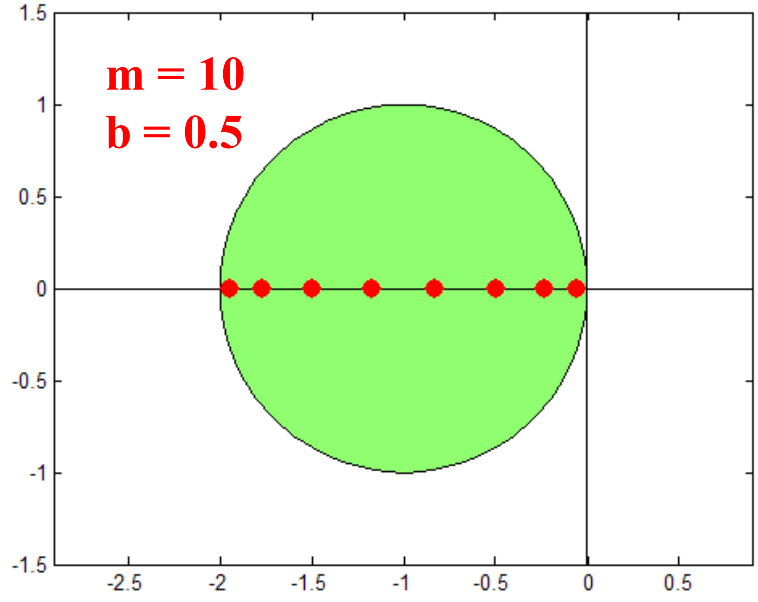
Euler explicite

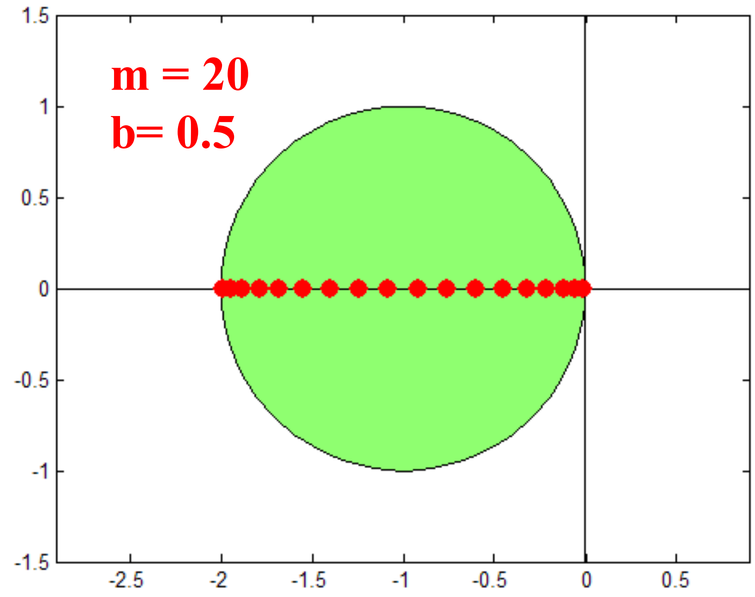
$$\mathbf{u}_{n+1} = \mathbf{u}_n + \underbrace{\frac{\epsilon \Delta t}{(\Delta x)^2}}_{b} \mathbf{A}\mathbf{u}_n$$

$$|1 + b\lambda_i| \leq 1$$

Valeurs propres de A

$$\Delta x = 0.05, \Delta t = 0.00125$$





Condition de stabilité pour la méthode d'Euler explicite….

$$\beta = \frac{\epsilon \Delta t}{(\Delta x)^2} \leq \frac{1}{2}$$

$$\boxed{\Delta t \leq \frac{(\Delta x)^2}{2\epsilon}}$$

Courant, Friedrichs et Lewy (1928)

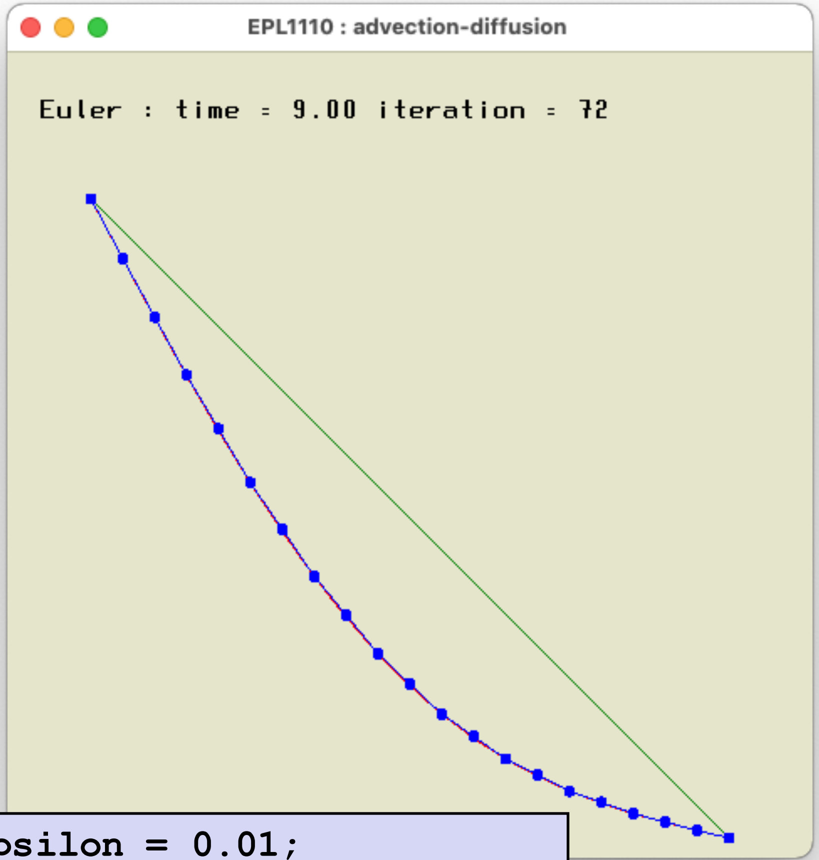

```
epsilon = 0.01;
dt = h*h/(epsilon*1.94);
```

Condition de stabilité pour la méthode d'Euler explicite….

$$\beta \quad = \quad \frac{\epsilon \Delta t}{(\Delta x)^2} \quad \leq \quad \frac{1}{2}$$

$$\boxed{\Delta t \quad \leq \quad \frac{(\Delta x)^2}{2\epsilon}}$$

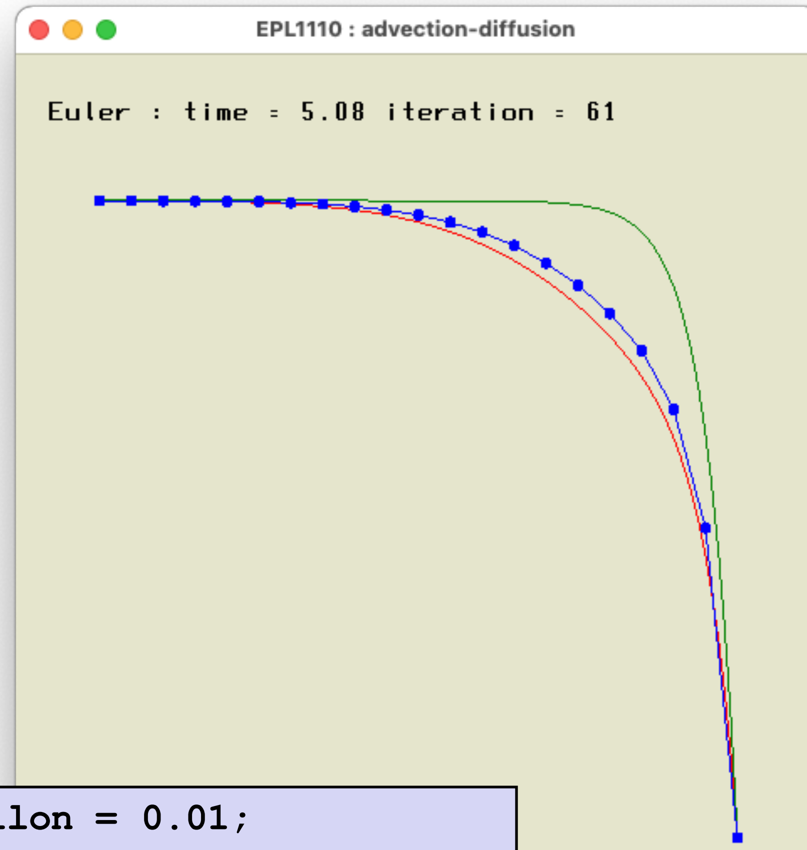
Courant, Friedrichs et Lewy (1928)




```
epsilon = 0.01;
dt = h*h/(epsilon*2.0);
```

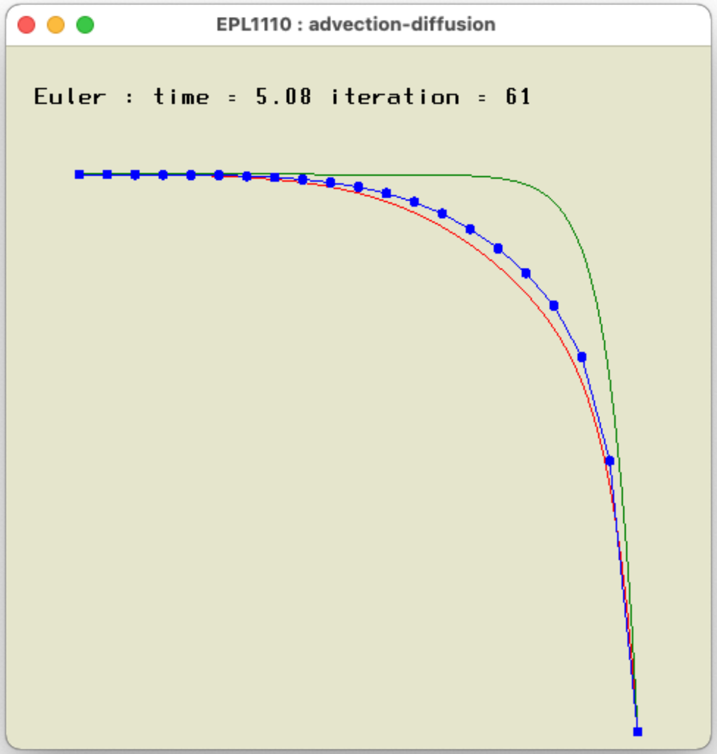
Et maintenant introduisons l'advection…

$$
\begin{aligned}
\frac{\partial u}{\partial t} + \beta \frac{\partial u}{\partial x} &= \epsilon \frac{\partial^2 u}{\partial x^2} \\
u(0) &= 1 \\
u(1) &= 0
\end{aligned}
$$




```
epsilon = 0.01;
beta = 0.2;
dt = h*h/(epsilon*3.0);
```

Et comment déduire le pas de temps ?



$$U_j^n = U^n e^{ikX_j}$$

Considérons une perturbation quelconque...

$$\begin{aligned}
U_j^{n+1} &= U_j^n + \Delta t \Big(\overbrace{\Big((\zeta - 1)\frac{\beta}{2h} + \frac{\epsilon}{h^2} \Big)}^{a} U_{j+1}^n + \Big(\overbrace{\Big(-\zeta\frac{\beta}{h} - 2\frac{\epsilon}{h^2} \Big)}^{b} U_j^n + \Big(\overbrace{\Big((\zeta + 1)\frac{\beta}{2h} + \frac{\epsilon}{h^2} \Big)}^{c} U_{j-1}^n \Big) \\
&= U_j^n \Big(1 + \Delta t \Big(a e^{ikh} + b + c e^{-ikh} \Big) \Big) \\
&= U_j^n \Big(1 + \Delta t \Big(\underbrace{(a + c)}_{-b} \cos kh + b + i \underbrace{(a - c)}_{-\beta/h} \sin kh \Big) \Big)
\end{aligned}$$

Il faut que le module du facteur d'amplification soit inférieur à l'unité :-)

$$\boxed{U = \left(1 + \Delta t \left(b - b\cos kh + b - i\frac{\beta}{h}\sin kh\right)\right)}$$

$$\left|\left(1 + \Delta t b - \Delta t b\cos(kh)\right) - i\Delta t\left(\tfrac{\beta}{h}\sin(kh)\right)\right| \quad \leq \quad 1$$

$$\downarrow$$

$$\begin{aligned}
1 + \Delta t^2 b^2 (1-\cos(kh))^2 + 2b\Delta t(1-\cos(kh)) + \Delta t^2 \tfrac{\beta^2}{h^2}\sin(kh)^2) &\leq 1\\
\Delta t^2 b^2 (1-\cos(kh))^2 + 2b\Delta t(1-\cos(kh)) + \Delta t^2 \tfrac{\beta^2}{h^2}(1-\cos(kh)^2) &\leq 0\\
\Delta t b^2 (1-\cos(kh)) + 2b + \Delta t \tfrac{\beta^2}{h^2}(1+\cos(kh)) &\leq 0
\end{aligned}$$

On déduit finalement :

$$\Delta t \leq \frac{-2b}{(1-\cos(kh))b^2 + (1+\cos(kh))\frac{\beta^2}{h^2}}$$

$$\Delta t \leq \frac{2(\zeta\beta h + 2\epsilon)h^2}{(\zeta h\beta + 2\epsilon)^2 + \beta^2 h^2 + \cos(kh)(\beta^2 h^2 - (\zeta h\beta + 2\epsilon)^2)}$$

On conclut donc : $\boxed{\Delta t \quad \leq \quad \min\left(\frac{\zeta h\beta + 2\epsilon}{\beta^2}, \frac{h^2}{\zeta h\beta + 2\epsilon}\right)}$

Notons que l'on obtient les résultats habituels $\Delta t \leq \dfrac{h}{\beta}$ pour $\epsilon = 0, \zeta = 1$ et $\Delta t \leq \dfrac{h^2}{2\epsilon}$ pour $\beta = 0$.

Pratiquement…