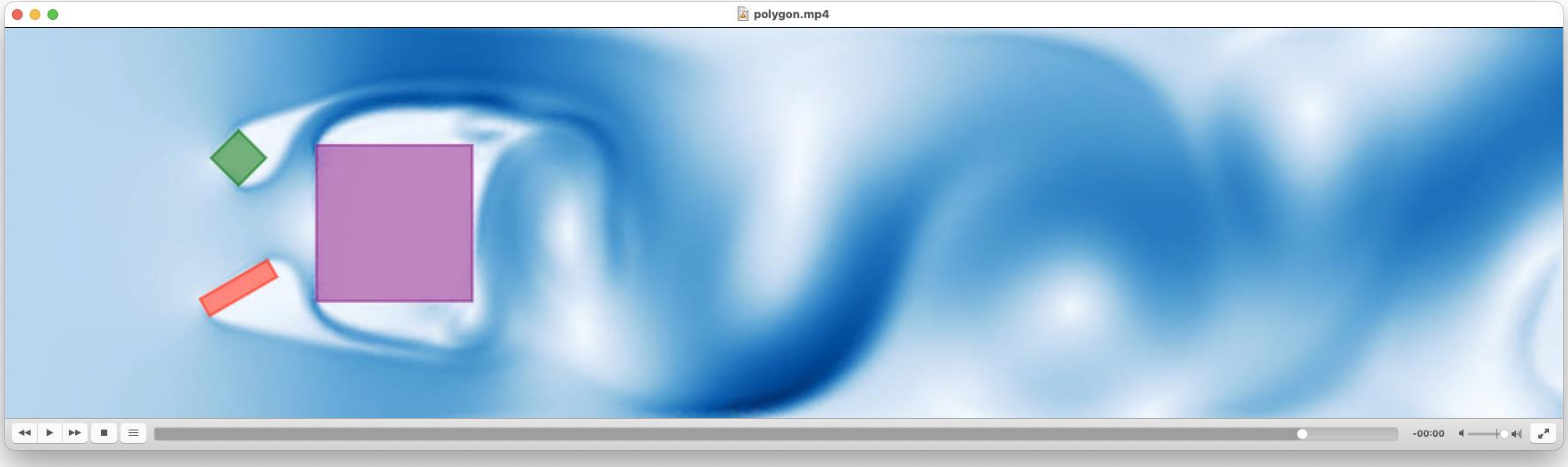
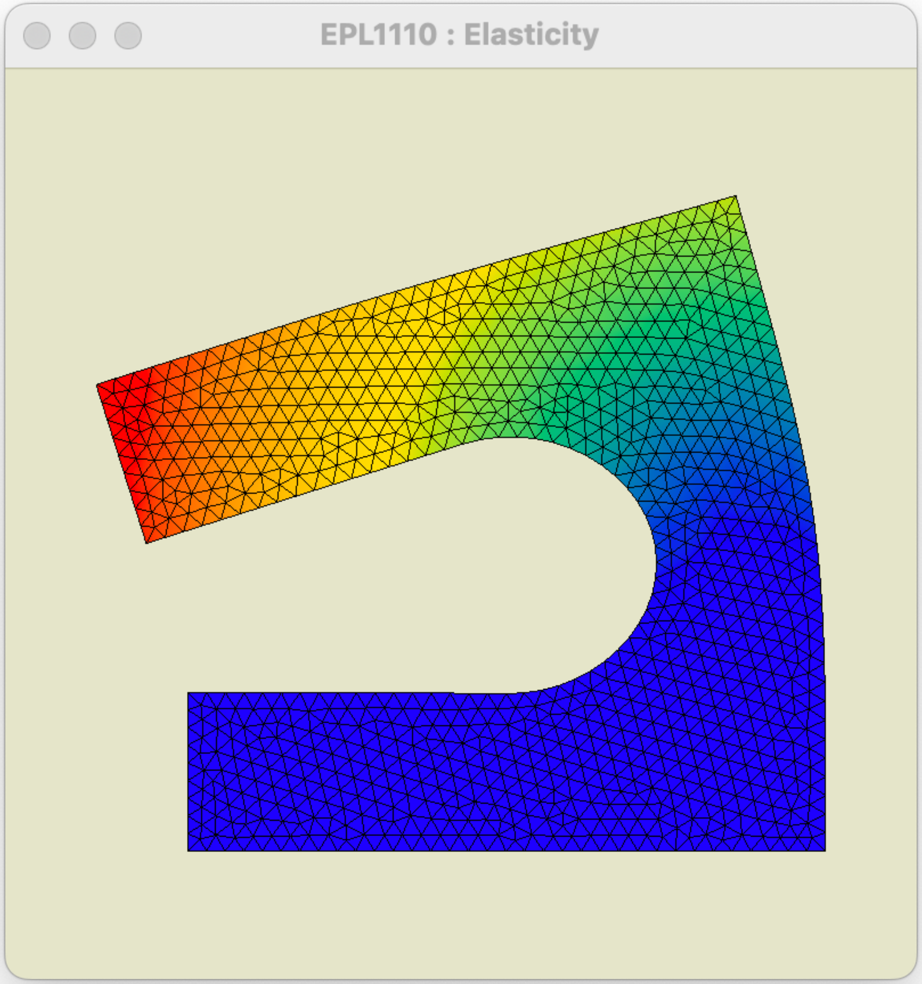
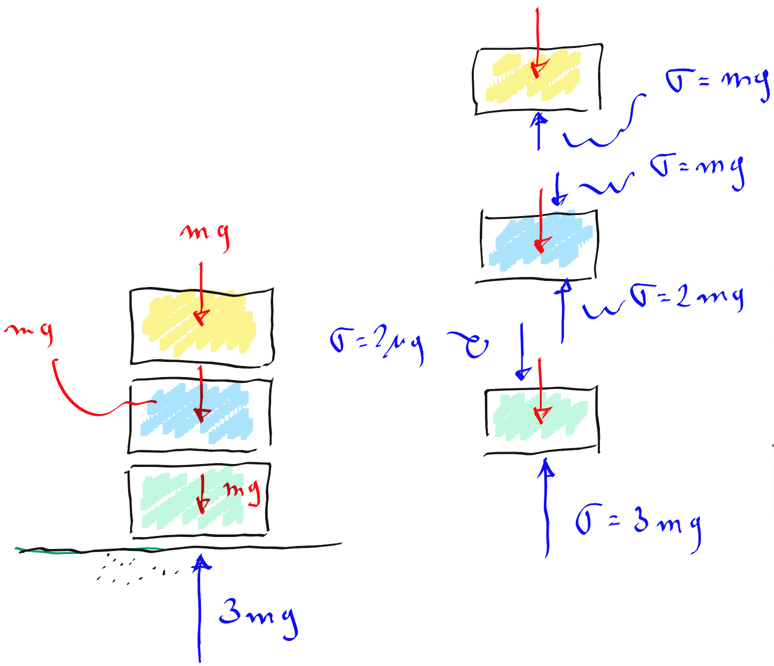
A quoi cela sert
les éléments finis ?

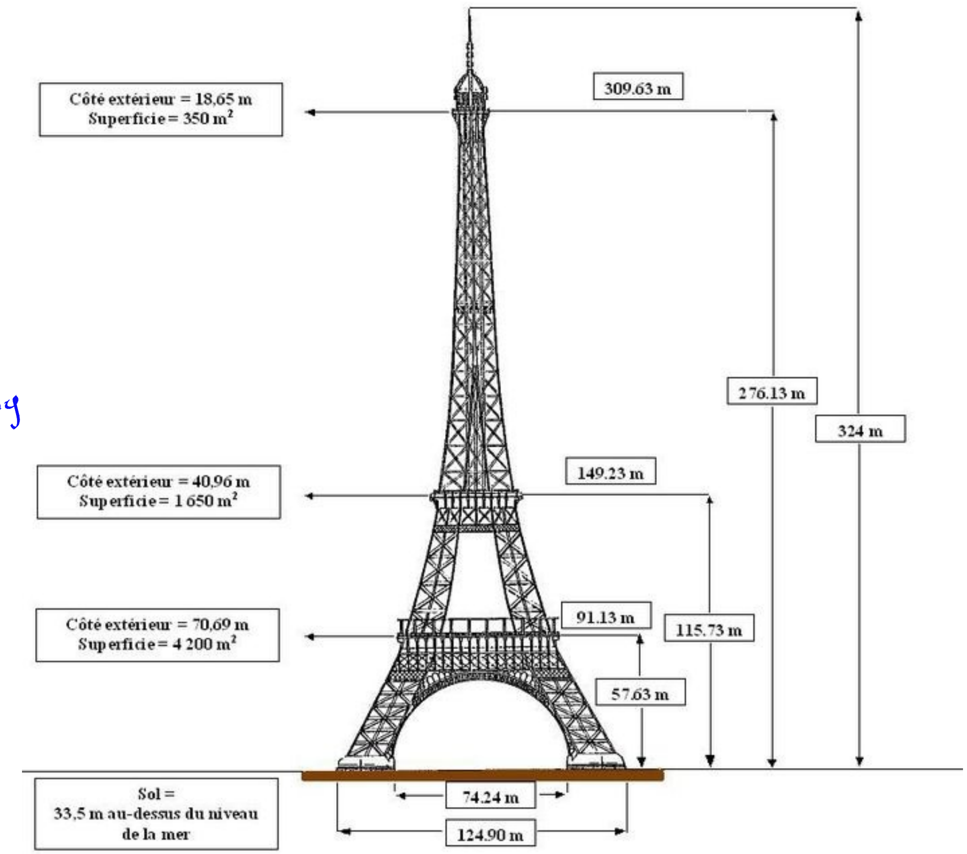


Elasticité linéaire



Monsieur,
j'ai pas eu MMC, moi ?

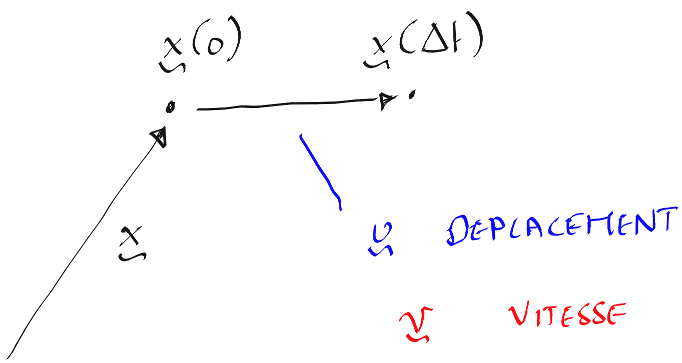


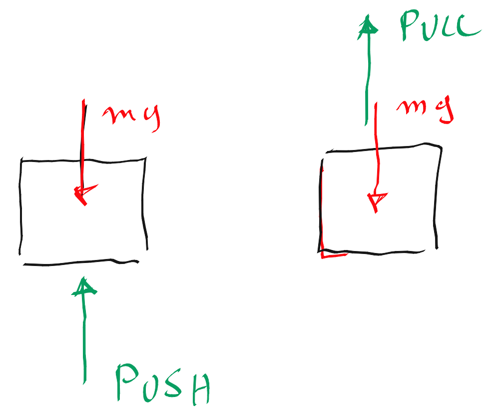


Ouuuppss !
Mais, MMC, MSD et Flotte : quelle horreur !

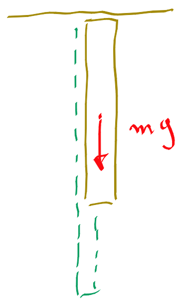
Souviens-toi des anciens !
Le grand Newton !
L'inoubliable Roland !

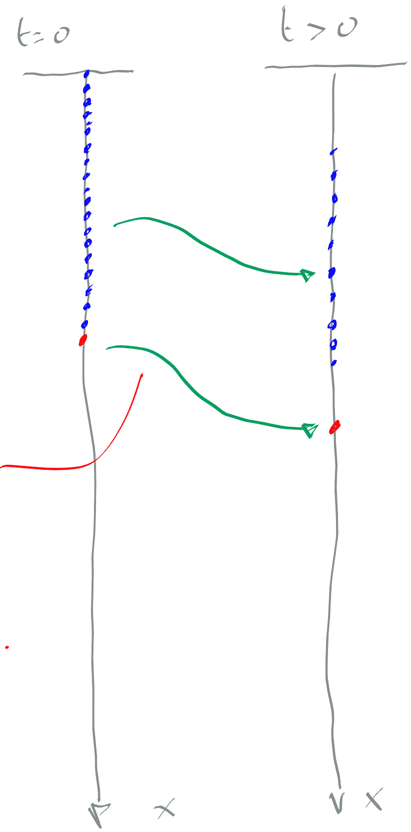
$$\boxed{m \frac{d^2 \underset{\sim}{x}}{dt^2}(t) = \sum \underset{\sim}{F}}$$





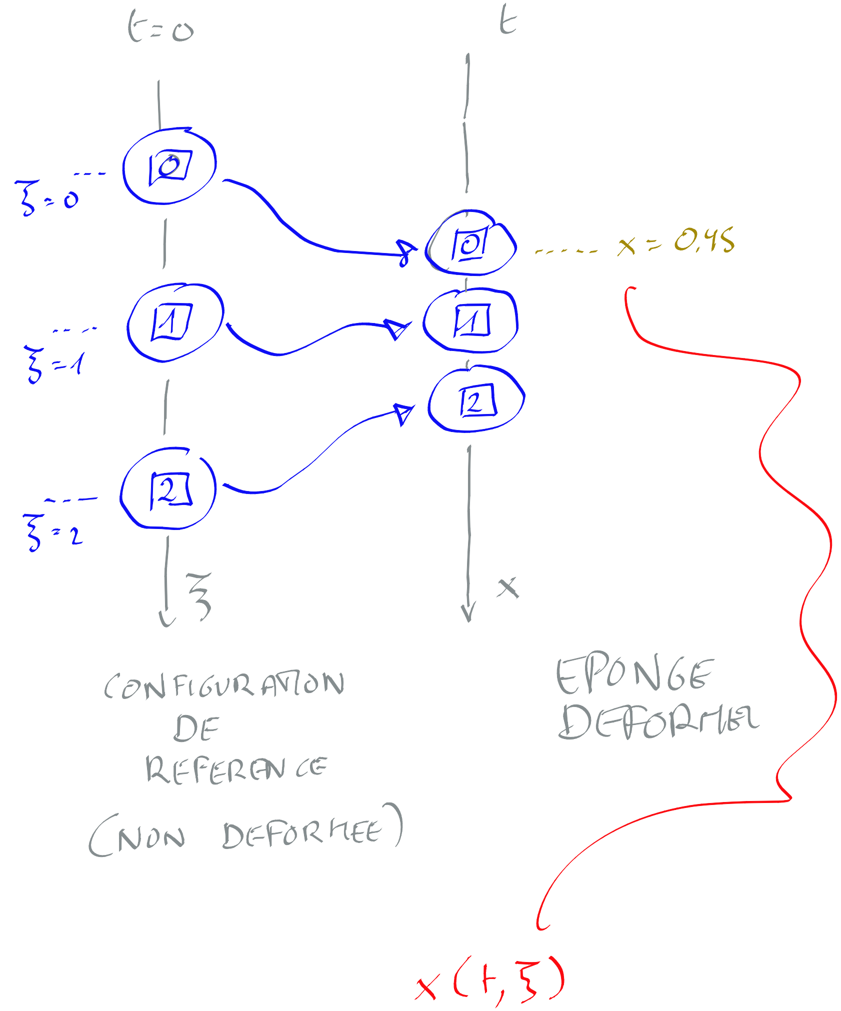
Un volume matériel !



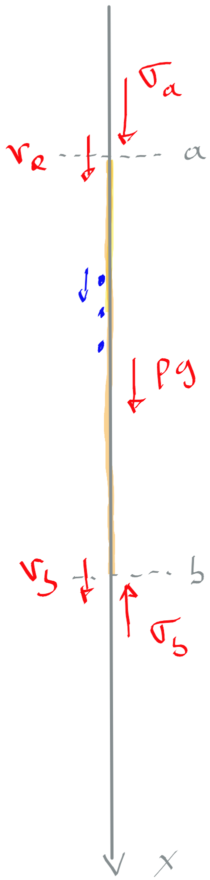


CONFIGURATION DE REFERENCE

$$\underline{v}(\underline{x},t) = \sum v_i(\underline{x},t)\,\hat{\underline{e}}_i$$



Bilan
de quantité de mouvement



$\forall$ intervalle a, b ACCROISSEMENT QUANTITÉ DE MVT = CE QUI ENTRE − CE QUI SORT + FORCES

$$\frac{d}{dt}\int_a^b \rho v \, dx = \underbrace{\rho_a v_a^2 - \rho_b v_b^2}_{-\left[\rho v^2\right]_a^b} + \int_a^b \rho g + \underbrace{(-\sigma_a) - (-\sigma_b)}_{\left[\sigma\right]_a^b}$$

$$= \int_a^b \frac{\partial}{\partial x}(\rho v^2) \qquad \int_a^b \frac{\partial \sigma}{\partial x}$$

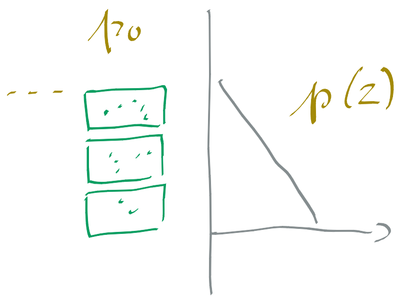
$$\underbrace{\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho v^2)}_{\rho \frac{Dv}{Dt}} = \rho g + \frac{\partial \sigma}{\partial x}$$

Bilan
de quantité de mouvement

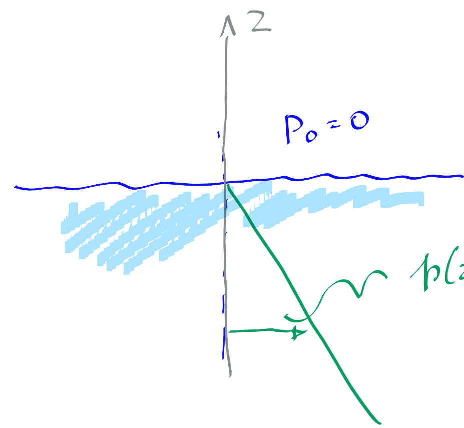
PROBLEME STATIONNAIRE
PAS DE MOUVEMENT

$$\frac{Dv}{Dt} = 0$$

$$\boxed{0 = \rho g + \frac{\partial \sigma}{\partial x}}$$



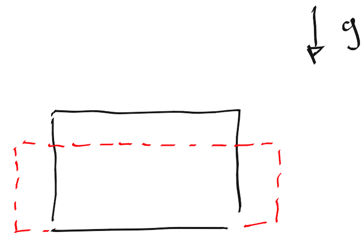
$$p(z) = p_0 - \rho g z$$

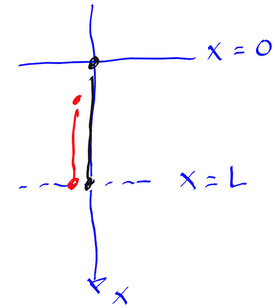


$$p(z) = p_0 - \rho g z$$

PRESSION
HYDROSTATIQUE

Loi de Hooke !
So easy !



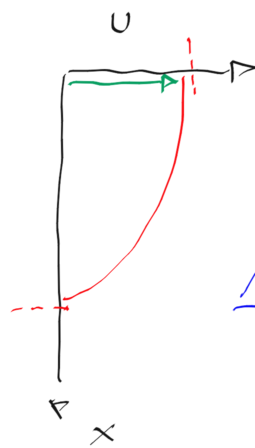


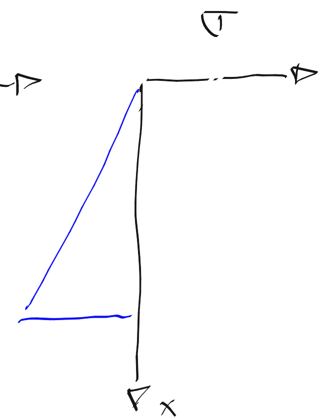
$$\frac{\partial \sigma}{\partial x} + \rho g = 0$$

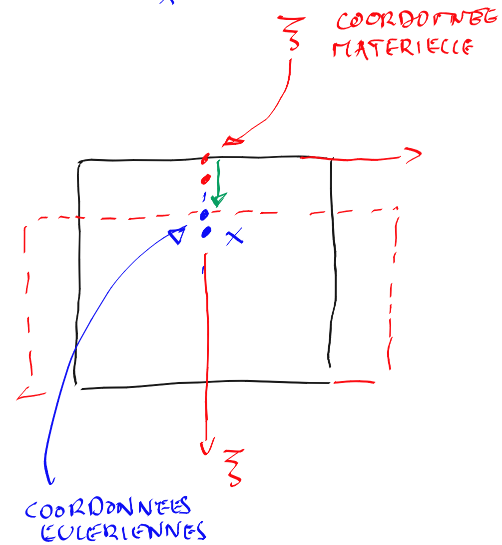
$$\sigma = E \frac{du}{dx}$$

$$E \frac{d^2 u}{dx^2} = - \rho g$$

$$u(L) = 0$$
$$u'(0) = 0$$







Vitesse $\quad \underline{v} \longrightarrow \nabla \underline{v} \qquad \overbrace{\frac{1}{2}\left(\nabla \underline{v} + \nabla \underline{v}^{T}\right)}^{\underline{\underline{d}}}$

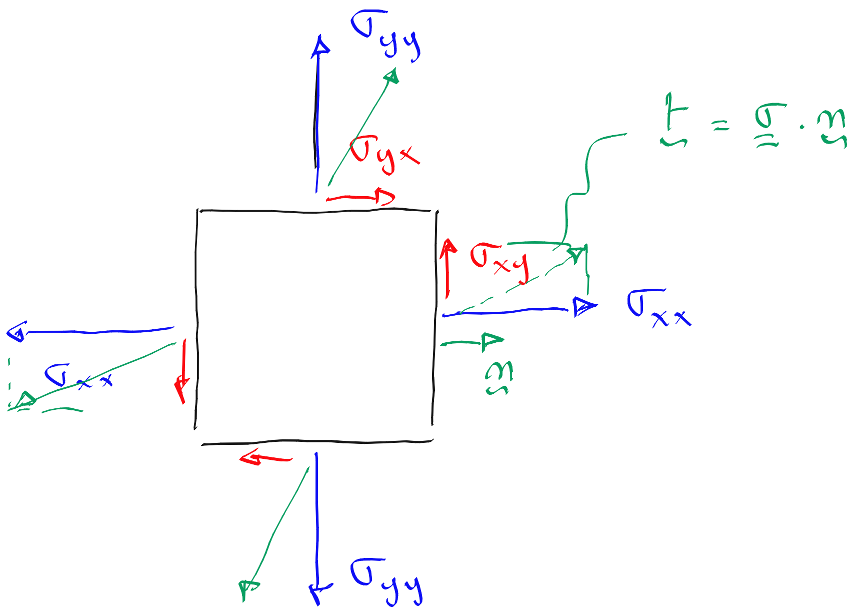
Déplacement $\quad \underline{u} \longrightarrow \nabla \underline{u} \qquad \underbrace{\frac{1}{2}\left(\nabla \underline{u} + \nabla \underline{u}^{T}\right)}_{\underline{\underline{\varepsilon}}}$

$$\underline{\underline{\sigma}} = 2\mu\, \underline{\underline{\varepsilon}} + \lambda\, \mathrm{tr}(\underline{\underline{\varepsilon}})\, \underline{\underline{\delta}}$$

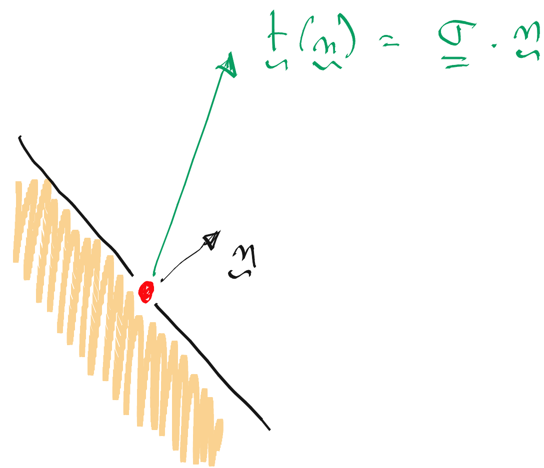
$$\underline{\underline{\tau}} = \underline{\underline{d}} \qquad \underline{\underline{d}} \qquad - p\, \underline{\underline{\delta}}$$

Taux de déformation = gradient de vitesses

Déformation = gradient de déplacement



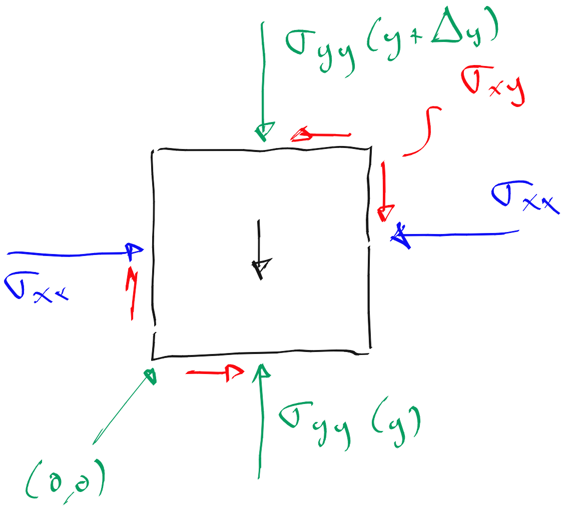
$$\underset{\sim}{t}(\underset{\sim}{n}) = \underline{\underline{\sigma}} \cdot \underset{\sim}{n}$$



$$\underline{\underline{\sigma}} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix}$$

$$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T$$

Tenseur de contraintes



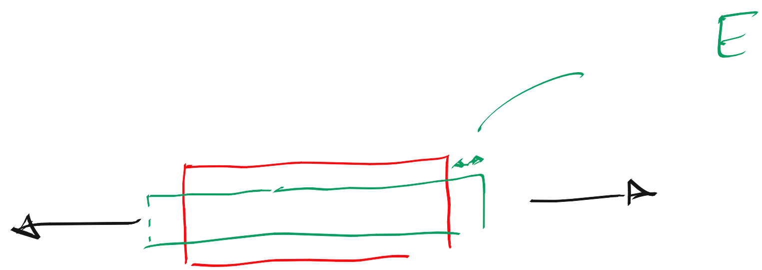
$$\rho g \,\Delta x \Delta y + \sigma_{yy}(y+\Delta y)\,\Delta x - \sigma_{yy}(y)\,\Delta x$$

$$\begin{cases} \rho g + \dfrac{\partial}{\partial y}(\sigma_{yy}) + \dfrac{\partial}{\partial x}(\sigma_{xy}) = 0 \\ \dfrac{\partial}{\partial x}(\sigma_{xx}) + \dfrac{\partial}{\partial y}(\sigma_{xy}) = 0 \end{cases}$$

Bilan
de quantité de mouvement

COEFFICIENTS DE LAMÉ

$$\underline{\underline{\sigma}} = \underbrace{2\mu}_{\frac{E}{(1+\nu)}}\,\underline{\underline{\varepsilon}} + \underbrace{\lambda}_{\frac{E\nu}{(1+\nu)(1-2\nu)}}\,\mathrm{tr}(\underline{\underline{\varepsilon}})\,\underline{\underline{\delta}}$$



Loi de Hooke

Coefficients de Lamé

Module de Young et coefficients de Poisson

$$\underline{\underline{\sigma}} = \underbrace{\frac{E}{(1+\nu)}}_{2\mu} \underline{\underline{\varepsilon}} + \underbrace{\frac{E\nu}{(1+\nu)(1-2\nu)}}_{\lambda} \operatorname{tr}(\underline{\underline{\varepsilon}})\ \underline{\underline{S}}$$

SHEAR MODULUS

$\lambda \to \infty$

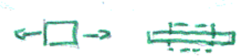
IF $\nu \to 1/2$

INCOMPRESSIBLE MATERIAL

POISSON'S COEFFICIENT

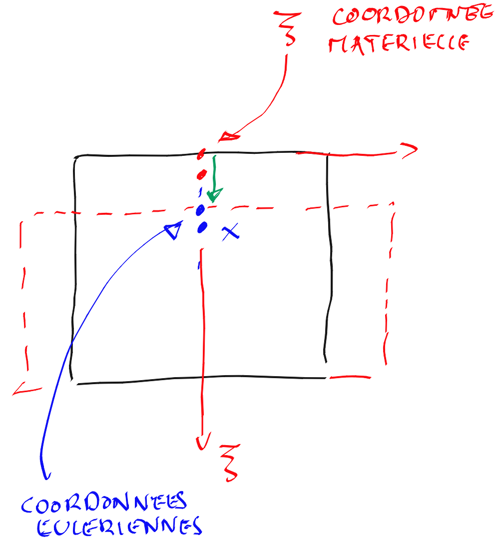
$\nu \in \left]-1 \quad 1/2\right[$

STRANGE MATERIALS



USUAL BEHAVIOUR

$$\sigma = E \frac{du}{dx}$$



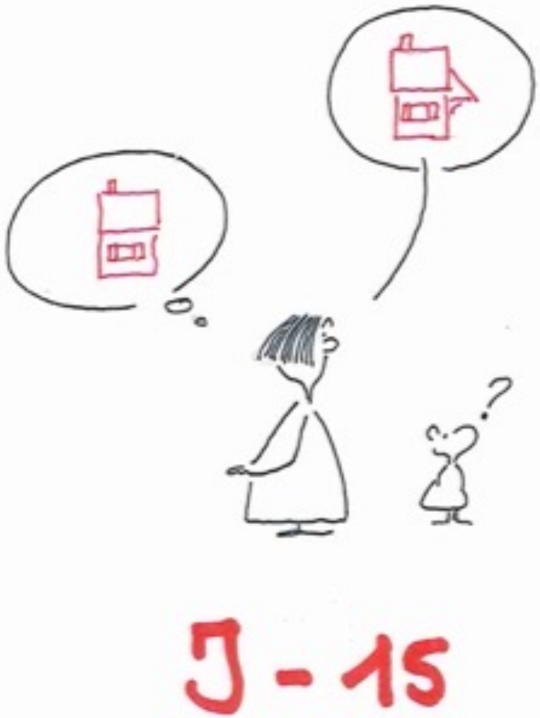
$$\frac{du}{dx} \simeq \frac{du}{d\xi}$$

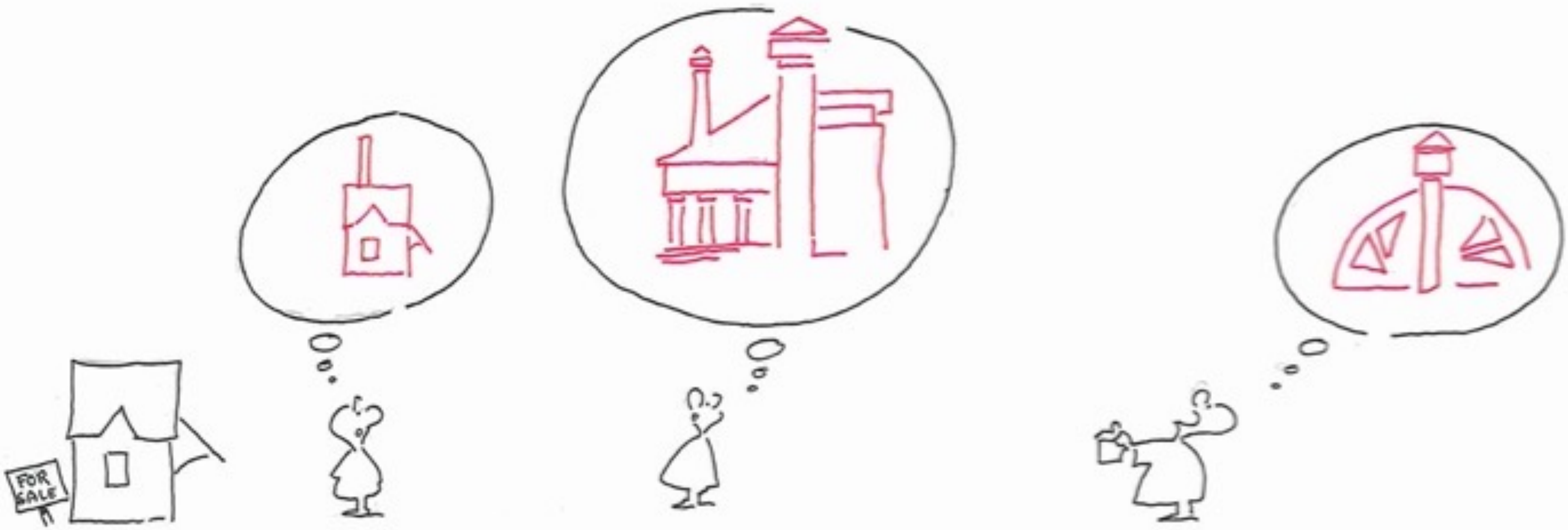
ON FAIT L'HYPOTHESE DE PETITES DEFORMATIONS

Grandes et petites déformations !

Cela peut devenir très compliqué !

Le projet

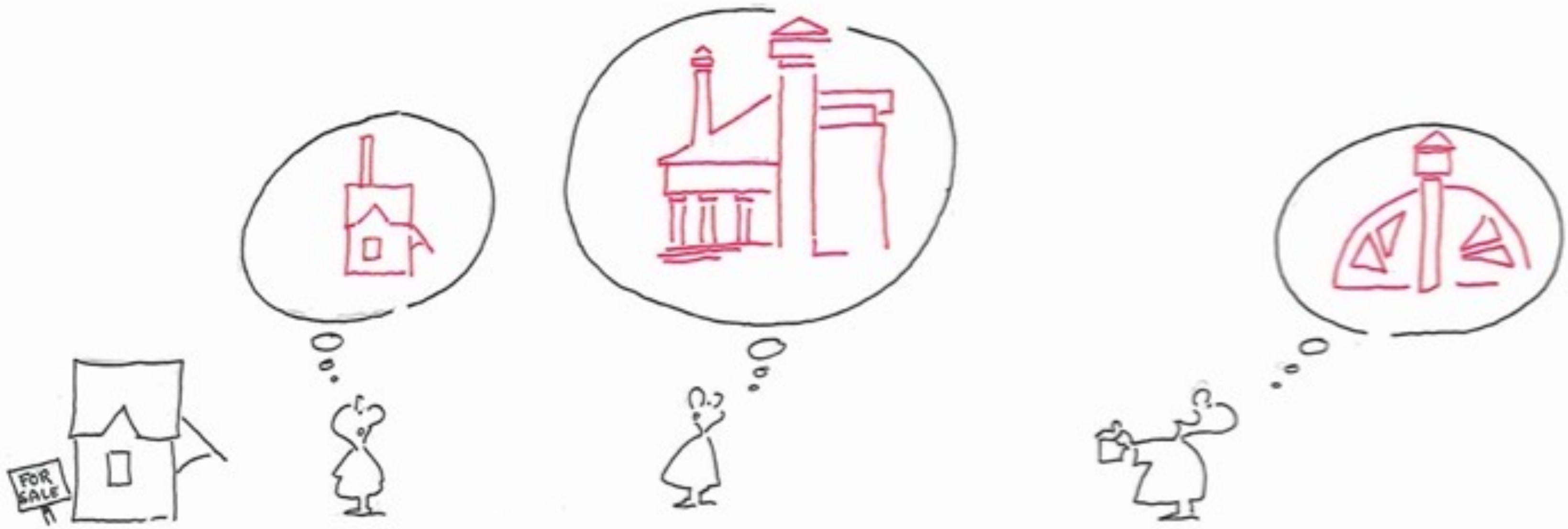




J-14

J-12

J-10

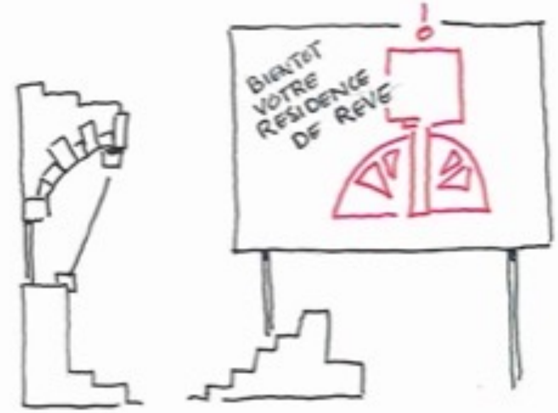


J-14

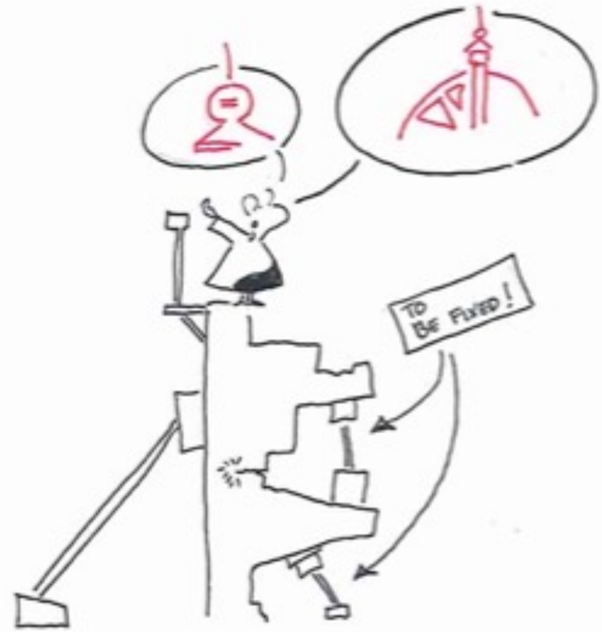
J-12

J-10

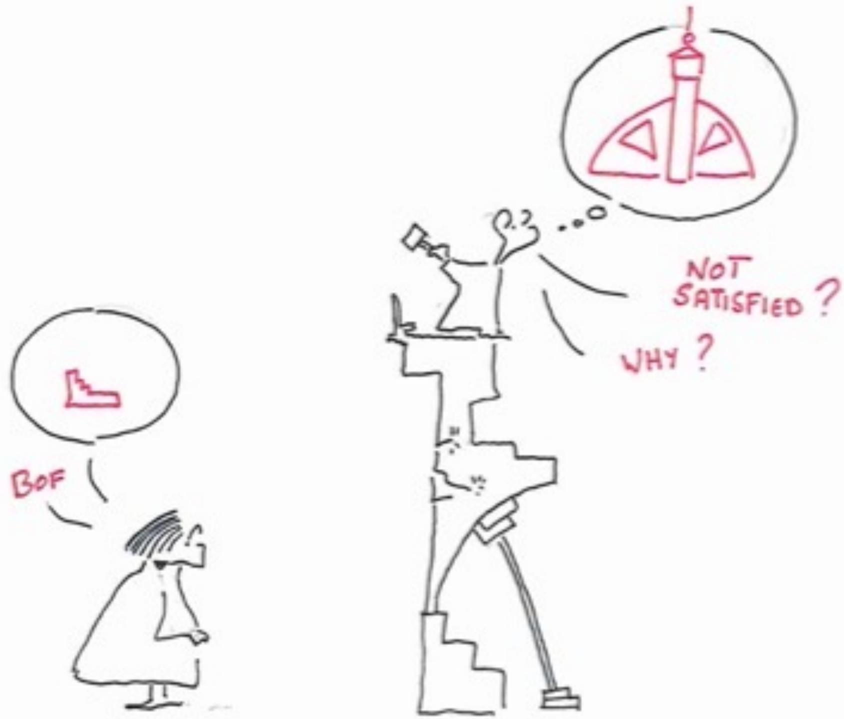
J-5



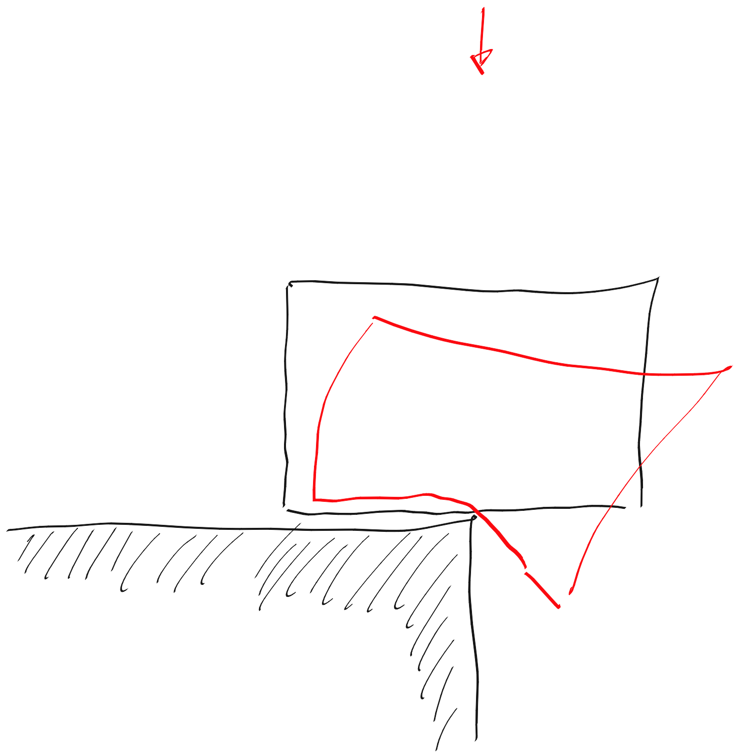
J-4

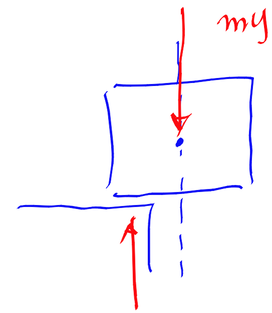


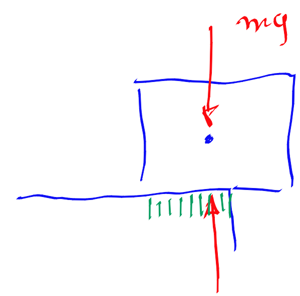
J-1



J







Et plein d’autres idées sont possibles !
Regardez autour de vous !

Votre soumission préliminaire…

2 Géométrie et Maillage

Notre maillage aurait la forme suivante :

$\underline{\underline{\sigma}} : \underline{n} = p \, \underline{n}$

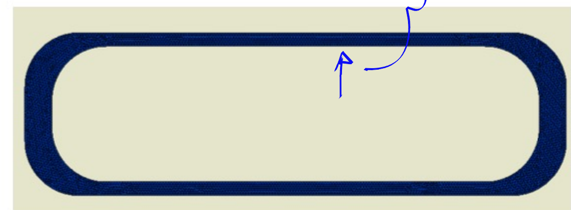


FIGURE 1 – Exemple du Maillage

Ce qui correspondrait à la géométrie de notre problème sur le schéma suivant :

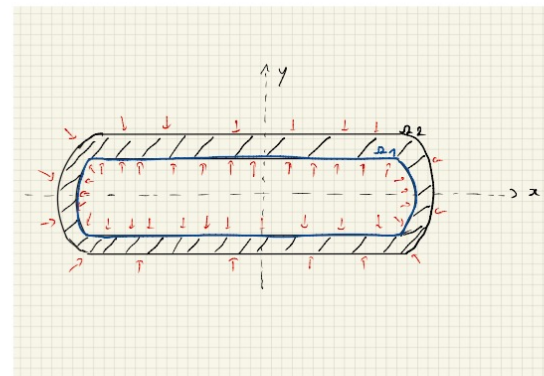


FIGURE 2 – Géométrie du problème

$[N/m^2]$ $[N]$ $[N/m^2]$

1. Sur Ω_1

$$\sigma.n = F_1 = p_1 * S_1$$

$[m^2]$

2. Sur Ω_2

$$\sigma.n = F_2 = p_0 * S_2$$

où S_1 et S_2 représentent les surfaces à l'intérieur et à l'extérieur du réservoir, p_1 la pression intérieure (plus au moins 700 bars)[1] et p_0 la pression atmosphérique évaluée à 1 bar. Il s'agit donc de conditions frontières de Neumann.
Nous poserons également les hypothèses suivantes :

1. répartition du gaz constante (forces appliquée constantes)
2. Problème axisymétrique

Nous devrons naturellement détailler les surfaces et leur géométrie (approximation a 2 cylindres).
Il est à noter que puisque le problème contient des symétries nous pouvons réaliser une économie sur les calculs et le maillage en ne représentant qu'un quart du problème en imposant donc une condition de Dirichlet nul sur les surfaces de coupure sans perte de généralité. Nous pensions également rajouter une vanne de pression à la bonbonne afin de complexifier un peu le maillage mais il faudrait évaluer l'impact que cela aurait sur les équations et les conditions aux frontières.

ABSTRACT GENERIC ELLIPTIC PROBLEM

HEAT CONDUCTION
LINEAR ELASTICITY
SIMPLIFIED MODELS OF LINEAR ELASTICITY
BEAM / SHELLS
ROPE / MENBRANE
STOKES PROBLEM

NOW,
WE CONSIDER
A VECTORIAL UNKNOWN FIELD !

WEAK FORMULATION

$? \, \underset{\sim}{u} \in \mathcal{U}$ SUCH THAT

$$a(\underset{\sim}{u}, \hat{\underset{\sim}{u}}) = b(\hat{\underset{\sim}{u}}) \quad \forall \hat{\underset{\sim}{u}} \in \hat{\mathcal{U}}$$

BILINEAR CONTINUOUS SYMMETRIC COERCIVE FORM

CONTINUOUS LINEAR FORM

ASSUMPTIONS REQUIRED TO HAVE AN ABSTRACT MINIMIZATION PROBLEM

$\Updownarrow$

$? \, \underset{\sim}{u} \in \mathcal{U}$ SUCH THAT

$$J(\underset{\sim}{u}) = \min_{\underset{\sim}{v} \in \mathcal{U}} \underbrace{\frac{1}{2} a(\underset{\sim}{v}, \underset{\sim}{v}) - b(\underset{\sim}{v})}_{J(\underset{\sim}{v})}$$

MINIMIZATION PROBLEM

HEAT CONDUCTION

CONSERVATION LAW

$$-\underset{\sim}{\nabla} \cdot q + f = 0$$

HEAT FLOW

CONSTITUTIVE LAW

FOURIER

$$q = -k \, \underset{\sim}{\nabla} u$$

TEMPERATURE

? u SUCH THAT

$$\underset{\sim}{\nabla} \cdot \big(\underbrace{k \, \underset{\sim}{\nabla} u}_{-q(u)} \big) + f = 0 \quad \text{in } \Omega$$

$$-q \cdot \underset{\sim}{n} = g \quad \text{on } \Gamma_N$$

$$u = 0 \quad \text{on } \Gamma_D$$

3D LINEAR ISOTROPIC ELASTICITY

$$\begin{bmatrix} \times & & \\ \circ & \times & \\ \circ & \circ & \times \end{bmatrix}$$

SHEAR STRESSES

NORMAL STRESSES

HOOKE ELASTIC BODY
SMALL DEF.

CONSTITUTIVE LAW

$$\underline{\underline{\sigma}} = \underbrace{\frac{E}{(1+\nu)}}_{2\mu} \underline{\underline{\varepsilon}} + \underbrace{\frac{E\nu}{(1+\nu)(1-2\nu)}}_{\lambda} \mathrm{tr}(\underline{\underline{\varepsilon}})\, \underline{\underline{\delta}}$$

DEFORMATION TENSOR

$$\underline{\underline{\varepsilon}} \triangleq \frac{1}{2}\left(\underline{\nabla}\,\underline{u} + (\underline{\nabla}\,\underline{u})^T\right)$$

CONSERVATION LAWS

$$\underline{\nabla} \cdot \underline{\underline{\sigma}} + \underline{f} = 0$$

$$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T$$

? $\underline{u}$

$$\underline{\nabla} \cdot \underline{\underline{\sigma}}(\underline{u}) + \underline{f} = 0 \quad \text{in } \Omega$$

$$\underline{\underline{\sigma}} \cdot \underline{n} = \underline{g} \quad \text{on } \Gamma_N$$

$$\underline{u} = 0 \quad \text{on } \Gamma_D$$

STOKES PROBLEM

CONSERVATION LAWS

$$\cancel{(\underset{\sim}{v}\cdot\underset{\sim}{\nabla})\underset{\sim}{v}} = \underset{\sim}{\nabla}\cdot\underline{\underline{\sigma}} + \underset{\sim}{f}$$

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{v} = 0$$

CREEPING FLOW

$Re \simeq 0$

INCOMPRESSIBLE FLOW

CONSTITUTIVE LAW

NEWTONIAN FLUID

$$\underline{\underline{\sigma}} = 2\mu\,\underline{\underline{d}} - p\,\underline{\underline{\delta}}$$

RATE OF DEFORMATION TENSOR

$$\underline{\underline{d}} \triangleq \frac{1}{2}\left(\underset{\sim}{\nabla}\underset{\sim}{v} + (\underset{\sim}{\nabla}\underset{\sim}{v})^T\right)$$

$?\,(\underset{\sim}{v}, p)$

$$\begin{cases} \underset{\sim}{\nabla}\cdot\left(\underline{\underline{\sigma}}(\underset{\sim}{v}, p)\right) + \underset{\sim}{f} = 0 \\ \underset{\sim}{\nabla}\cdot\underset{\sim}{v} = 0 \end{cases} \quad \text{in } \Omega$$

$$\underline{\underline{\sigma}}\cdot\underset{\sim}{n} = g \quad \text{on } \Gamma_N$$

$$\underset{\sim}{v} = 0 \quad \text{on } \Gamma_D$$

CALCULUS

$\underset{\sim}{\nabla} \cdot \underset{\sim}{v} = 0$ IN Ω $\iff$ $\exists\, \underset{\sim}{\psi}$ SUCH THAT $\underset{\sim}{v} = \underset{\sim}{\nabla} \times \underset{\sim}{\psi}$

? $\underset{\sim}{\psi}$ STREAM FUNCTION

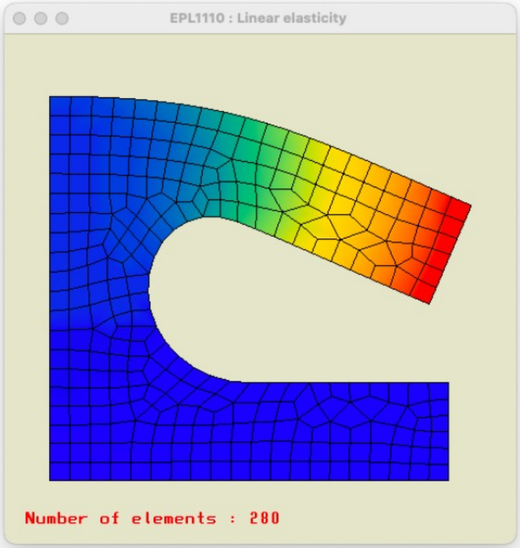
$$\underset{\sim}{\nabla}^4 \underset{\sim}{\psi} + \underset{\sim}{\nabla} \times \underset{\sim}{f} = 0 \quad \text{in } \Omega$$

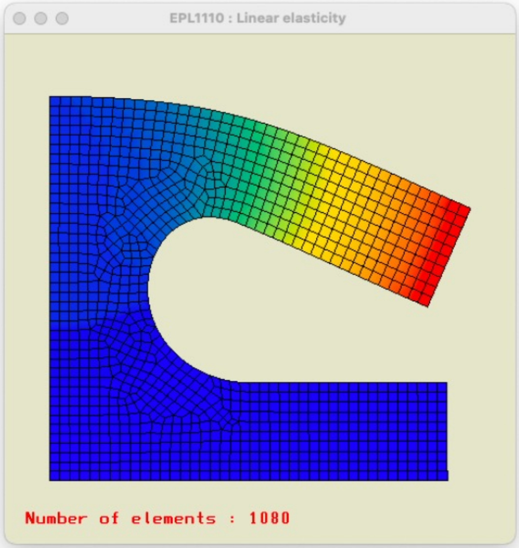
+ SUITABLE BOUNDARY CONDITIONS

Que va-t-on résoudre ?

Trouver $\mathbf{u}(\mathbf{x}) \in \mathcal{U}$ tel que

$$\underbrace{< \boldsymbol{\epsilon}(\hat{\mathbf{u}}) : \mathbf{C} : \boldsymbol{\epsilon}(\mathbf{u}). >}_{a(\hat{\mathbf{u}},\, \mathbf{u})} \quad = \quad \underbrace{< \hat{\mathbf{u}} f > + \ll \hat{\mathbf{u}} g \gg_N,}_{b(\hat{\mathbf{u}})} \qquad \forall \hat{\mathbf{u}} \in \widehat{\mathcal{U}},$$



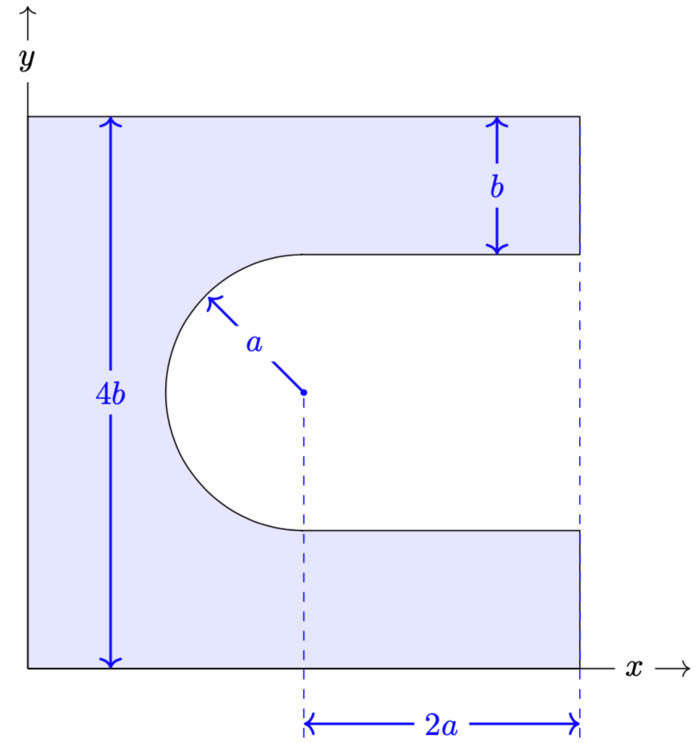


Créer une géométrie !
Construire le maillage !

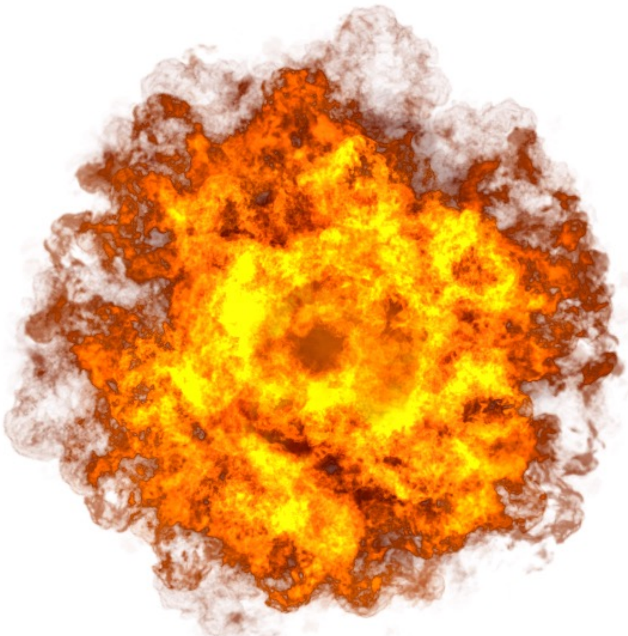
Définir la géométrie et le maillage !

Le maillage et la géométrie sont définis comme suit :

```
double Lx = 1.0;
double Ly = 1.0;

theGeometry->LxPlate     = Lx;
theGeometry->LyPlate     = Ly;
theGeometry->h           = Lx * 0.05;
theGeometry->elementType = FEM_TRIANGLE;
```

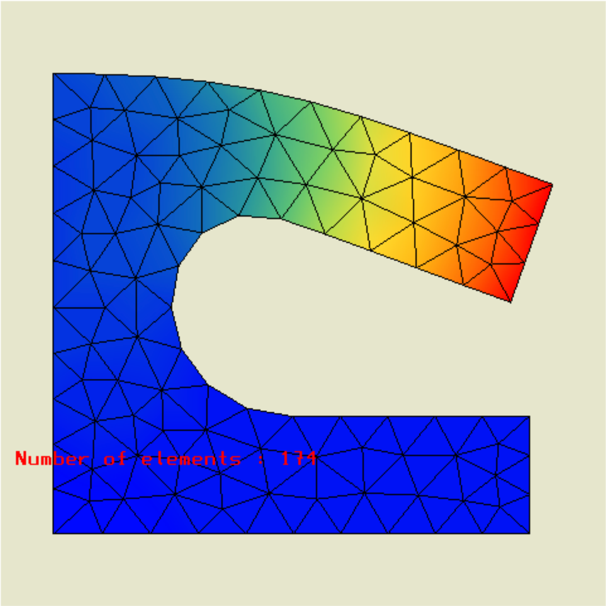


Et on soumet au serveur…



```
====================================================================================

 Linear elasticity problem
   Young modulus   E   =  2.1100000e+11 [N/m2]
   Poisson's ratio nu  =  3.0000000e-01 [-]
   Density         rho =  7.8500000e+03 [kg/m3]
   Gravity         g   =  9.8100000e+00 [m/s2]
   Planar strains formulation
   Boundary conditions :
              Symmetry : imposing  0.00e+00 as the horizontal displacement
                Bottom : imposing  0.00e+00 as the vertical displacement
====================================================================================

Pivot index 1  Pivot value 0.000000e+00
--------------------------------------------------------------------------------
  Error in src/fem.c at line 555 :
  Cannot eliminate with such a pivot
--------------------------------------------------------------------- Yek Yek !!



La compilation ou l'execution du test 0 a plante :-(
```

Et après avoir fait le devoir…



Diagnostic du serveur

```
********** TEST 0 **********
gcc -o /tmp/work/exe src/glfem.c src/main.c src/fem.c src/mesaglfem.c homework.c -I src -lm -lrt -Wall -g -Wno-
** COMPILATION SUCCEED **

/usr/bin/time -o /work/outputStat.txt -f %R %Z %e %M sudo -ustudent ./exe
** RUN SUCCEED **
Wall-clock time : 0.24s (limit 40s)
Approximated maximum memory usage : 112.98Mb
Approximated total memory allocation : 28.8203Mb
********** RUN OUTPUT **********
Info    : Meshing 1D...
Info    : [  0%] Meshing curve 1 (Line)
Info    : [ 20%] Meshing curve 2 (Line)
Info    : [ 30%] Meshing curve 3 (Line)
```

Et on minimise toujours une fonctionnelle !

Trouver $\mathbf{u}(\mathbf{x}) \in \mathcal{U}$ tel que

$$\underbrace{< \boldsymbol{\epsilon}(\widehat{\mathbf{u}}) : \mathbf{C} : \boldsymbol{\epsilon}(\mathbf{u}). >}_{a(\widehat{\mathbf{u}},\mathbf{u})} \quad = \quad \underbrace{< \widehat{\mathbf{u}} f > + \ll \widehat{\mathbf{u}} g \gg_N ,}_{b(\widehat{\mathbf{u}})} \qquad \forall \widehat{\mathbf{u}} \in \widehat{\mathcal{U}},$$

$$J(\mathbf{v}) \quad = \quad \underbrace{\frac{1}{2} < \boldsymbol{\epsilon}(\mathbf{v}) : \mathbf{C} : \boldsymbol{\epsilon}(\mathbf{v}). >}_{\frac{1}{2}a(\mathbf{v},\mathbf{v})} \; - \; \underbrace{< \mathbf{v} f > + \ll \mathbf{v} g \gg_N ,}_{b(\mathbf{v})}$$

En deux dimensions !

$$\left[\begin{array}{cc} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{array}\right] = \left[\begin{array}{cc} A\epsilon_{xx} + B\epsilon_{yy} & 2C\epsilon_{xy} \\ 2C\epsilon_{xy} & A\epsilon_{yy} + B\epsilon_{xx} \end{array}\right]$$

Acier
$E = 2.11\ 10^{11}\ [N/m^2]$ $\nu = 0.3$ $\rho = 7.85\ 10^3\ [kg/m^3]$

Déformations planes	Tensions planes
$A = \dfrac{E(1-\nu)}{(1+\nu)(1-2\nu)}$	$A = \dfrac{E}{(1-\nu^2)}$
$B = \dfrac{E\nu}{(1+\nu)(1-2\nu)}$	$B = \dfrac{E\nu}{(1-\nu^2)}$

$$C = \frac{E}{2(1+\nu)}$$

Déformations planes…

$$
\begin{aligned}
\sigma_{xx} &= \frac{E}{(1+\nu)(1-2\nu)}\left((1-\nu)\epsilon_{xx} + \nu\epsilon_{yy}\right),\\
\sigma_{yy} &= \frac{E}{(1+\nu)(1-2\nu)}\left(\nu\epsilon_{xx} + (1-\nu)\epsilon_{yy}\right),\\
\sigma_{zz} &= \frac{E\nu}{(1+\nu)(1-2\nu)}\left(\epsilon_{xx} + \epsilon_{yy}\right),\\
\sigma_{xy} &= \frac{E}{(1+\nu)}\epsilon_{xy},
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{xx} &= \frac{E}{(1-\nu^2)}\left(\epsilon_{xx} + \nu\epsilon_{yy}\right),\\
\sigma_{yy} &= \frac{E}{(1-\nu^2)}\left(\nu\epsilon_{xx} + \epsilon_{yy}\right),\\
\sigma_{xy} &= \frac{E}{(1+\nu)}\epsilon_{xy}.
\end{aligned}
$$

…tensions planes

La fonctionnelle à minimiser !

$$
\begin{aligned}
\tfrac{1}{2}\, a(\mathbf{u},\mathbf{u}) &= \tfrac{1}{2} < \boldsymbol{\epsilon}(\mathbf{u}) : \boldsymbol{\sigma}(\mathbf{u}) >, \\
&= \tfrac{1}{2} < \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} \\ \epsilon_{xy} & \epsilon_{yy} \end{bmatrix} : \begin{bmatrix} A\epsilon_{xx} + B\epsilon_{yy} & 2C\epsilon_{xy} \\ 2C\epsilon_{xy} & A\epsilon_{yy} + B\epsilon_{xx} \end{bmatrix} >, \\
&= \tfrac{1}{2} < \begin{bmatrix} \epsilon_{xx} & \epsilon_{yy} & 2\epsilon_{xy} \end{bmatrix} \cdot \begin{bmatrix} A & B & 0 \\ B & A & 0 \\ 0 & 0 & C \end{bmatrix} \cdot \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ 2\epsilon_{xy} \end{bmatrix} > .
\end{aligned}
$$

Matrice et vecteur locaux à construire !

$$\mathbf{A}_{ij} = \left[\begin{array}{c|c} <\tau_{i,x}A\tau_{j,x}> + <\tau_{i,y}C\tau_{j,y}> & <\tau_{i,x}B\tau_{j,y}> + <\tau_{i,y}C\tau_{j,x}> \\ \hline <\tau_{i,y}B\tau_{j,x}> + <\tau_{i,x}C\tau_{j,y}> & <\tau_{i,y}A\tau_{j,y}> + <\tau_{i,x}C\tau_{j,x}> \end{array}\right],$$

$$\mathbf{B}_{i} = \left[\begin{array}{c} <\tau_i f_x> + \ll \tau_i g_x \gg \\ <\tau_i f_y> + \ll \tau_i g_y \gg \end{array}\right].$$

ABSTRACT GENERIC DISCRETE FORMULATION

$\begin{bmatrix} u \\ v \end{bmatrix}$ 2D $\begin{bmatrix} u \\ v \\ w \end{bmatrix}$ 3D

$$\underbrace{\underset{\sim}{u}(\underset{\sim}{x})}_{\in\, \mathcal{U}} \simeq \underbrace{\underset{\sim}{u}^h(\underset{\sim}{x})}_{\in\, \mathcal{U}^h \subset\, \mathcal{U}} = \sum_{i=1}^{n} \underset{\sim}{U}_i\, \tau_i(\underset{\sim}{x})$$

$\dim(\mathcal{U}^h) = 2n$ — 2D PROBLEMS

$\ldots = 3n$ — 3D PROBLEMS

$$\mathcal{U}^h = \text{SPAN} \left\{ \underbrace{\begin{bmatrix} \tau_1 \\ 0 \end{bmatrix} \begin{bmatrix} \tau_2 \\ 0 \end{bmatrix} \cdots \begin{bmatrix} \tau_n \\ 0 \end{bmatrix}}_{n} \underbrace{\begin{bmatrix} 0 \\ \tau_1 \end{bmatrix} \cdots \begin{bmatrix} 0 \\ \tau_n \end{bmatrix}}_{n} \right\}$$

$?\ \underset{\sim}{u}^h$ SUCH THAT

$$a(\underset{\sim}{u}^h, \hat{\underset{\sim}{u}}^h) = b(\hat{\underset{\sim}{u}}^h)$$

$$\forall\, \hat{\underset{\sim}{u}}^h \in \hat{\mathcal{U}}^h$$

DISCRETE FORMULATION

TIP #1

$$\langle \underline{\hat{u}} \cdot (\underline{\nabla} \cdot \underline{\underline{\sigma}}(\underline{u})) \rangle = \langle \underline{\nabla} \cdot (\underline{\hat{u}} \cdot \underline{\underline{\sigma}}(\underline{u})) \rangle - \overbrace{\langle \underline{\nabla}\underline{\hat{u}} : \underline{\underline{\sigma}}(\underline{u}) \rangle}^{\triangleq\, a(\underline{\hat{u}}, \underline{u})}$$

$\underline{\hat{u}} \in \hat{\mathcal{U}}$ $\qquad$ $\underline{u} \in \mathcal{U}$

TIP #2

$$= \langle\!\langle \underline{n} \cdot \underline{\hat{u}} \cdot \underline{\underline{\sigma}}(\underline{u}) \rangle\!\rangle$$

TIP #3

$$= \langle\!\langle \underline{\hat{u}} \cdot \underbrace{(\underline{n} \cdot \underline{\underline{\sigma}}(\underline{u}))}_{g \text{ on } \Gamma_N} \rangle\!\rangle_N$$

USUAL CALCULUS !

IS $a(\hat{\underset{\sim}{u}}, \underset{\sim}{u})$ SYMMETRIC ?

$$\langle \underset{\sim}{\nabla} \hat{\underset{\sim}{u}} : \underset{=}{\sigma}(\underset{\sim}{u}) \rangle = \langle \left(\underbrace{\overbrace{\frac{\underset{\sim}{\nabla}\hat{\underset{\sim}{u}} + (\underset{\sim}{\nabla}\hat{\underset{\sim}{u}})^T}{2}}^{\underset{=}{\varepsilon}(\hat{\underset{\sim}{u}})}}_{\text{SYM}} + \underbrace{\frac{\underset{\sim}{\nabla}\hat{\underset{\sim}{u}} - (\underset{\sim}{\nabla}\hat{\underset{\sim}{u}})^T}{2}}_{\text{ANTI SYM.}} \right) : \underbrace{\underset{=}{\sigma}(\underset{\sim}{u})}_{\text{SYM.}} \rangle$$

$$= \langle \underset{=}{\varepsilon}(\hat{\underset{\sim}{u}}) : \underbrace{\underset{=}{\sigma}(\underset{\sim}{u})}_{\boxed{\underset{\equiv}{C} : \underset{=}{\varepsilon}(\underset{\sim}{u}) \quad \text{GENERALIZED HOOKE'S LAW}}} \rangle$$

$$= \langle \underset{=}{\varepsilon}(\hat{\underset{\sim}{u}}) : \underset{\equiv}{C} : \underset{=}{\varepsilon}(\underset{\sim}{u}) \rangle$$

WEAK FORMULATION

? $\underline{u} \in \mathcal{U}$ SUCH THAT

$$\underbrace{\langle \underline{\underline{\varepsilon}}(\hat{\underline{u}}) : \underline{\underline{\underline{\underline{C}}}} : \underline{\underline{\varepsilon}}(\underline{u}) \rangle}_{a(\hat{\underline{u}}\;\underline{u})} = \underbrace{\langle \hat{\underline{u}} \cdot \underline{f} \rangle + \langle\!\langle \hat{\underline{u}} \cdot \underline{g} \rangle\!\rangle_N}_{b(\hat{\underline{u}})}$$

$\forall \hat{\underline{u}} \in \hat{\mathcal{U}}$

$$\Updownarrow$$

FOR SUITABLE $\underline{\underline{\underline{\underline{C}}}}$ ONLY !

? $\underline{u} \in \mathcal{U}$ SUCH THAT

$$J(\underline{u}) = \min_{\underline{v} \in \mathcal{U}} \underbrace{\frac{1}{2} a(\underline{v}\;\underline{v}) - b(\underline{v})}_{J(\underline{v})}$$

MINIMIZATION PROBLEM

$$\frac{1}{2}\underbrace{a(\underset{\sim}{v},\underset{\sim}{v})}_{\text{ENERGY OF DEFORMATION}} = \underbrace{\langle \frac{1}{2}\lambda\, \underline{\underline{\varepsilon}}(\underset{\sim}{v}) : \underline{\underline{I}} \; \overbrace{\underline{\underline{I}} : \underline{\underline{\varepsilon}}(\underset{\sim}{v})}^{tr(\underline{\underline{\varepsilon}}(\underset{\sim}{v}))} + \mu\, \underline{\underline{\varepsilon}}(\underset{\sim}{v}) : \underline{\underline{\varepsilon}}(\underset{\sim}{v}) \rangle}_{\text{MUST BE A QUADRATIC POSITIVE FORM IN ORDER TO OBTAIN A MINIMIZATION PROBLEM}}$$

$$\downarrow$$

$$\boxed{\begin{array}{r} \mu > 0 \\ \frac{3}{2}\lambda + \mu > 0 \end{array}} \quad \begin{array}{l} (\varepsilon_{12} \neq 0) \\ (\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} \neq 0) \end{array}$$

ADMISSIBLE VALUES FOR LAMÉ COEFFICIENTS

"DEFORMATION ENERGY"

WHAT IS IT ?

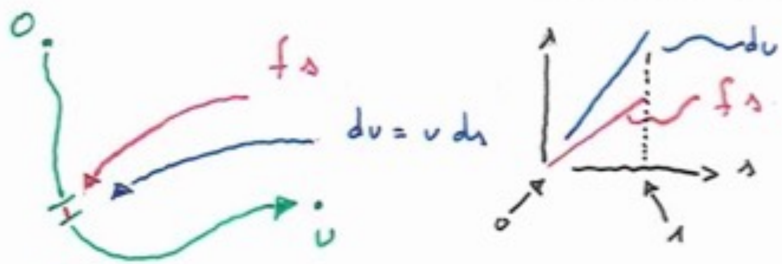
ENERGY OF DEFORMATION = WORK PERFORMED TO DEFORM THE STRUCTURE AND IS STORED INSIDE. SUCH AN ENERGY IS RECOVERED WHEN EXTERNAL FORCES ARE REMOVED

f — FORCE

v — DISPLACEMENT

$$\text{ENERGY} = \underbrace{\text{FORCE} \cdot \text{DISPLACEMENT}}_{\text{WORK PERFORMED}}$$

$$\text{WORK PERFORMED} = \int_0^v s f \, du = \int_0^1 f v \, s \, ds = f v \underbrace{\int_0^1 s \, ds}_{1/2} = \frac{1}{2} f v \quad !$$



VERY VERY SLOW LOADING

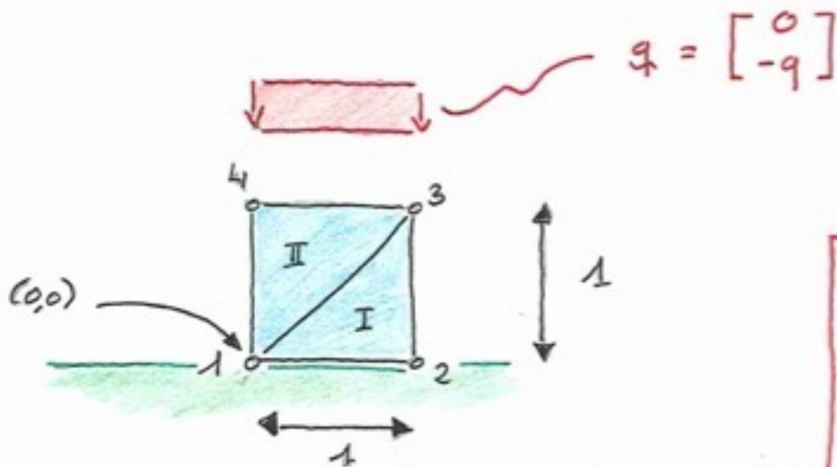
→ QUASI-STATIC APPROACH

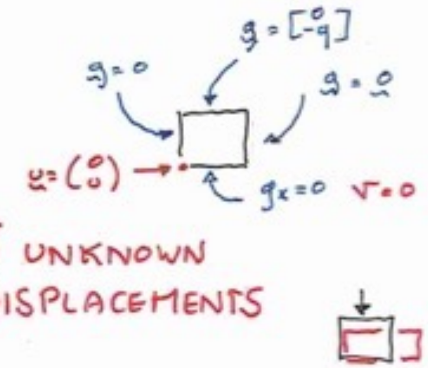
- IRREVERSIBILITY
- DISSIPATION

} ARE NEGLECTED !

NUMERICAL EXAMPLE

$\underset{\sim}{u} = \begin{bmatrix} u \\ v \end{bmatrix}$





5 UNKNOWN DISPLACEMENTS

$$\underset{\sim}{U}_i = \begin{bmatrix} U_i \\ V_i \end{bmatrix}$$

$$? \; \underset{\sim}{u}^h = \sum_{i=1}^{m} \underset{\sim}{U}_i \, \tau_i$$

$$\langle \underline{\underline{\varepsilon}}(\hat{\underset{\sim}{u}}^h) : \underline{\underline{\sigma}}(\underset{\sim}{u}^h) \rangle = \langle f \cdot \hat{\underset{\sim}{u}}^h \rangle + \langle\!\langle g \cdot \hat{\underset{\sim}{u}}^h \rangle\!\rangle_N \quad \forall \hat{\underset{\sim}{u}}^h$$

$$\begin{bmatrix} \tau_i \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ \tau_i \end{bmatrix}$$

$2m$ TEST FUNCTIONS

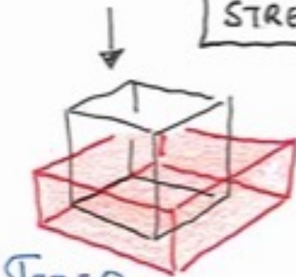
ANALYTICAL SOLUTION

PLANAR STRESSES

$$\underset{\sim}{u} = \frac{q}{E}\begin{bmatrix} \nu x \\ -y \end{bmatrix}$$

$$\underset{\approx}{\varepsilon} = \frac{q}{E}\begin{bmatrix} \nu & 0 \\ 0 & -1 \end{bmatrix}$$

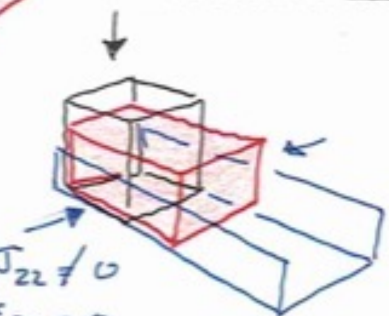
PLANAR STRESSES



$\sigma_{zz} = 0$

$\varepsilon_{zz} \neq 0$

PLANAR DEFORMATIONS



$\sigma_{zz} \neq 0$

$\varepsilon_{zz} = 0$

$$\underbrace{\underset{\sim}{\nabla}\cdot\underset{\approx}{\sigma}}_{=0\,!} = 0$$

$$\sigma_{xx} = \frac{E}{(1-\nu^2)}\left(\underbrace{\varepsilon_{xx}}_{\frac{q\nu}{E}} + \underbrace{\nu\,\varepsilon_{yy}}_{-\frac{q\nu}{E}}\right) = 0$$

$$\sigma_{yy} = \frac{E}{(1-\nu^2)}\left(\underbrace{\nu\,\varepsilon_{xx} - \varepsilon_{yy}}_{\frac{q}{E}(\nu^2-1)}\right) = -q\ !$$

$$\begin{bmatrix} \tau_i \\ 0 \end{bmatrix} \text{ OR } \begin{bmatrix} 0 \\ \tau_i \end{bmatrix}$$

$$\boxed{\left\langle \underline{\underline{\varepsilon}}(\underline{\hat{u}}^h) : \underline{\underline{\sigma}}(\underline{u}^h) \right\rangle \quad \forall \underline{\hat{u}}^h}$$

$$\sum_j U_j \, \underline{\underline{\sigma}}\begin{pmatrix} \tau_j \\ 0 \end{pmatrix} + V_j \, \underline{\underline{\sigma}}\begin{pmatrix} 0 \\ \tau_j \end{pmatrix}$$

$$\boxed{\sum_j \underline{\underline{A}}_{ij} \cdot \underline{U}_j}$$

$$\sum_j \begin{bmatrix} U_j \\ V_j \end{bmatrix} \tau_j = \sum U_j \begin{bmatrix} \tau_j \\ 0 \end{bmatrix} + V_j \begin{bmatrix} 0 \\ \tau_j \end{bmatrix}$$

$$\sum_j \begin{bmatrix} \left\langle \underline{\underline{\varepsilon}}\begin{pmatrix} \tau_i \\ 0 \end{pmatrix} : \underline{\underline{\sigma}}\begin{pmatrix} \tau_j \\ 0 \end{pmatrix} \right\rangle & \left\langle \underline{\underline{\varepsilon}}\begin{pmatrix} \tau_i \\ 0 \end{pmatrix} : \underline{\underline{\sigma}}\begin{pmatrix} 0 \\ \tau_j \end{pmatrix} \right\rangle \\ \left\langle \underline{\underline{\varepsilon}}\begin{pmatrix} 0 \\ \tau_i \end{pmatrix} : \underline{\underline{\sigma}}\begin{pmatrix} \tau_j \\ 0 \end{pmatrix} \right\rangle & \left\langle \underline{\underline{\varepsilon}}\begin{pmatrix} 0 \\ \tau_i \end{pmatrix} : \underline{\underline{\sigma}}\begin{pmatrix} 0 \\ \tau_j \end{pmatrix} \right\rangle \end{bmatrix} \cdot \begin{bmatrix} U_j \\ V_j \end{bmatrix} \quad \forall i$$

DISCRETE OPERATOR

$$\underline{\underline{\varepsilon}}\begin{pmatrix}\tau_i\\ 0\end{pmatrix} = \begin{bmatrix}\tau_{i,x} & \tau_{i,y}/2\\ \tau_{i,y}/2 & 0\end{bmatrix}$$

$$\underline{\underline{\varepsilon}}\begin{pmatrix}0\\ \tau_i\end{pmatrix} = \begin{bmatrix}0 & \tau_{i,x}/2\\ \tau_{i,x}/2 & \tau_{i,y}\end{bmatrix}$$

$$\underline{\underline{\sigma}}(\underline{\underline{\varepsilon}}) = \begin{bmatrix}A\,\varepsilon_{xx} + B\,\varepsilon_{yy} & 2C\,\varepsilon_{xy}\\ 2C\,\varepsilon_{xy} & A\,\varepsilon_{yy} + B\,\varepsilon_{xx}\end{bmatrix}$$

HOW TO CALCULATE IT ?

	PLANAR DEFORMATIONS	PLANAR STRESSES
A	$\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$	$\frac{E}{(1-\nu^2)}$
B	$\frac{E\nu}{(1+\nu)(1-2\nu)}$	$\frac{E\nu}{(1-\nu^2)}$
C	$\frac{E}{2(1+\nu)}$	$\frac{E}{2(1+\nu)}$

$$\underline{\underline{\varepsilon}} : \underline{\underline{\sigma}} = \varepsilon_{xx}\left(A\,\varepsilon_{xx} + B\,\varepsilon_{yy}\right) + 4C\,\varepsilon_{xy}\,\varepsilon_{xy} + \varepsilon_{yy}\left(A\,\varepsilon_{yy} + B\,\varepsilon_{xx}\right)$$

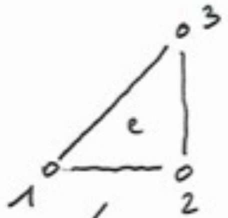
$$\underline{\underline{A}}_{y} = \begin{bmatrix} \langle \tau_{i,x}\, A\, \tau_{j,x} \rangle + \langle \tau_{i,y}\, C\, \tau_{j,y} \rangle & \langle \tau_{i,x}\, B\, \tau_{j,y} \rangle + \langle \tau_{i,y}\, C\, \tau_{j,x} \rangle \\ \underbrace{\langle \tau_{i,y}\, B\, \tau_{j,x} \rangle + \langle \tau_{i,x}\, C\, \tau_{j,y} \rangle}_{\underline{\underline{\varepsilon}}\binom{0}{\tau_i} : \underline{\underline{\sigma}}\left(\underline{\underline{\varepsilon}}\binom{\tau_j}{0}\right)} & \langle \tau_{i,y}\, A\, \tau_{j,y} \rangle + \langle \tau_{i,x}\, C\, \tau_{j,x} \rangle \end{bmatrix}$$

LOCAL ELASTICITY MATRIX

COMPUTING A LOCAL ELASTICITY MATRIX

$$\begin{bmatrix} A\langle \tau_{i,x}\,\tau_{j,x}\rangle + C\langle\tau_{i,y}\,\tau_{j,y}\rangle & B\langle\tau_{i,x}\,\tau_{j,y}\rangle + C\langle\tau_{i,y}\,\tau_{j,x}\rangle \\ B\langle\tau_{i,y}\,\tau_{j,x}\rangle + C\langle\tau_{i,x}\,\tau_{j,y}\rangle & A\langle\tau_{i,y}\,\tau_{j,y}\rangle + C\langle\tau_{i,x}\,\tau_{j,x}\rangle \end{bmatrix}$$

$\underline{\underline{A}}^e_{11}$: $A_{xx11} = A$, $A_{yy11} = C$, $A_{xy11} = 0$, $A_{yx11} = 0$



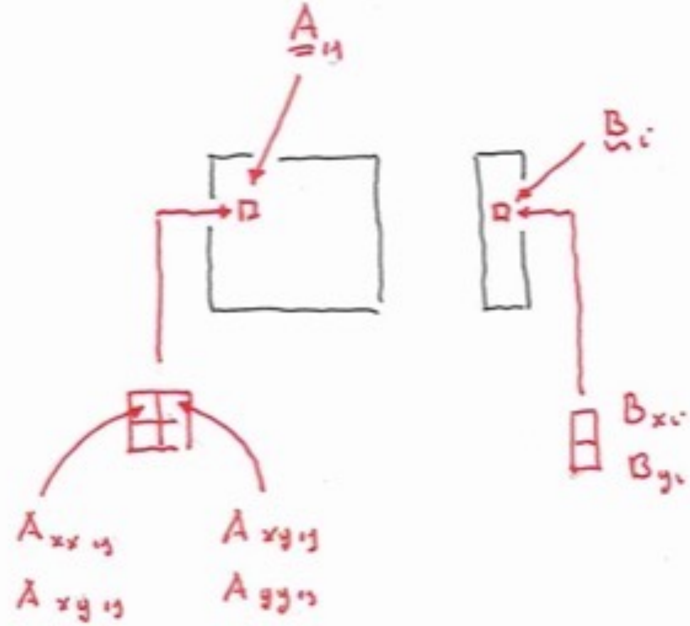
	$\tau_{i,x}$	$\tau_{i,y}$
1	-1	0
2	1	-1
3	0	1

$$\underline{\underline{A}}^e_y = \begin{bmatrix} \boxed{A} & \boxed{0} & -A & B & 0 & -B \\ \boxed{0} & \boxed{C} & C & -C & -C & 0 \\ & & A+C & -B-C & -C & B \\ & & -B-C & A+C & C & -A \\ & & & & C & 0 \\ & & & & 0 & A \end{bmatrix} \frac{1}{2}$$

$$\underset{\sim}{B}_i = \langle\langle g \cdot \hat{\underset{\sim}{u}}^h \rangle\rangle_N$$

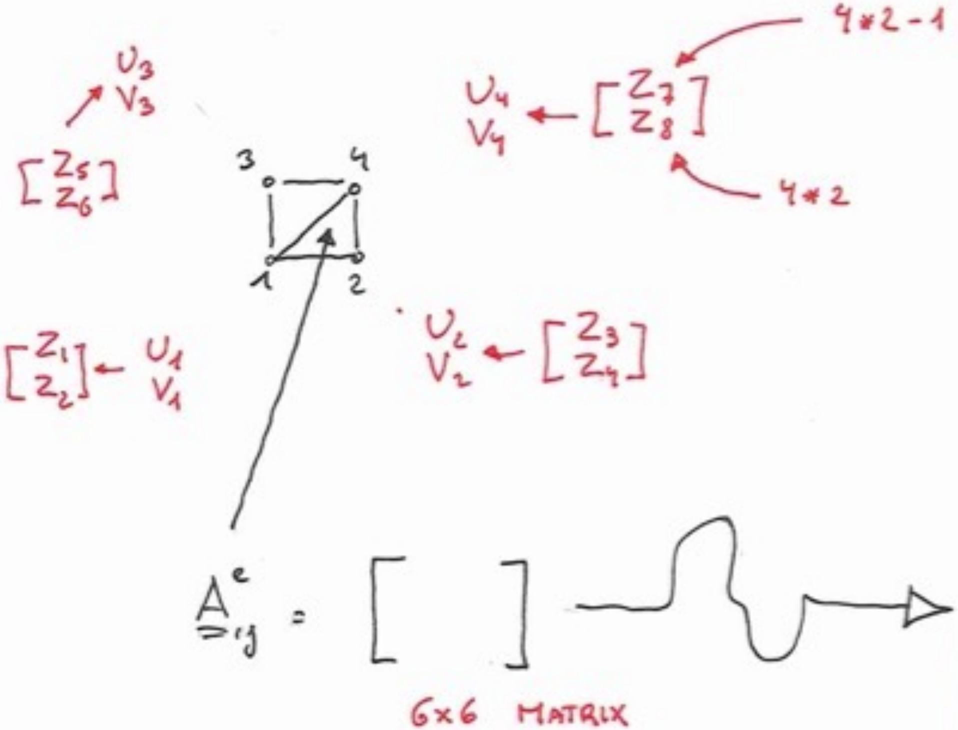
$$\begin{bmatrix} g_x \\ g_y \end{bmatrix} \qquad \begin{bmatrix} \tau_i \\ 0 \end{bmatrix} \text{ OR } \begin{bmatrix} 0 \\ \tau_i \end{bmatrix}$$

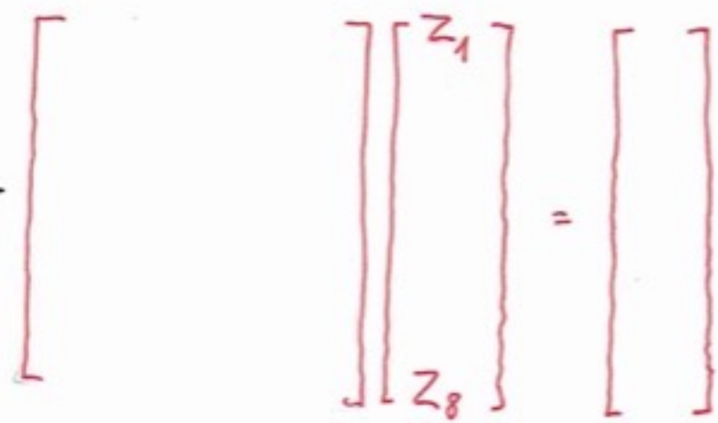
$$\boxed{\begin{bmatrix} \langle\langle g_x \, \tau_i \rangle\rangle \\ \langle\langle g_y \, \tau_i \rangle\rangle \end{bmatrix}}$$



AND
HYPER
VECTOR !

ASSEMBLING PROCEDURE





$$\text{GLOBAL HORIZONTAL INDEX} = \underbrace{2 * \text{GLOBAL SCALAR INDEX}}_{\text{GLOBAL}} - 1$$