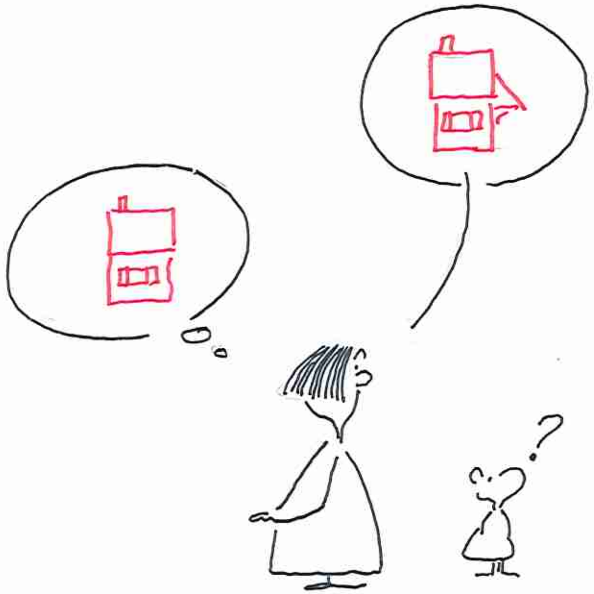
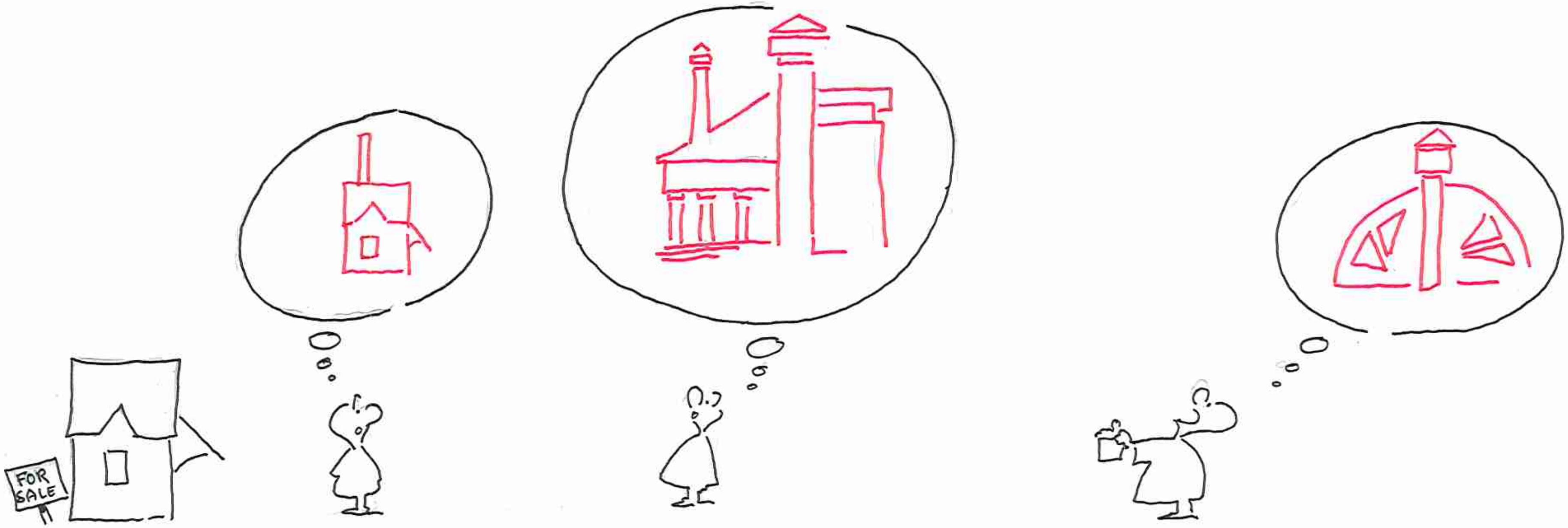


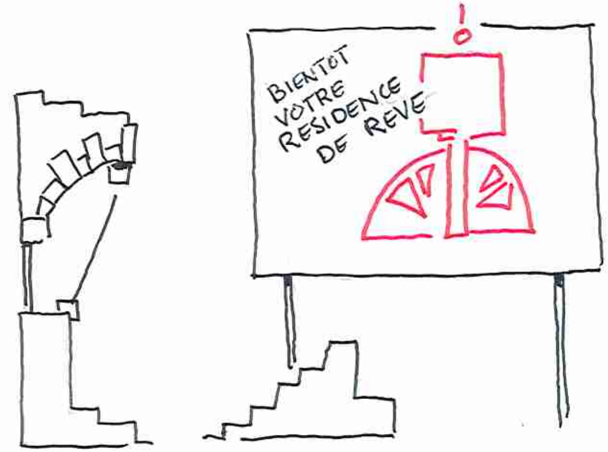
J - 15



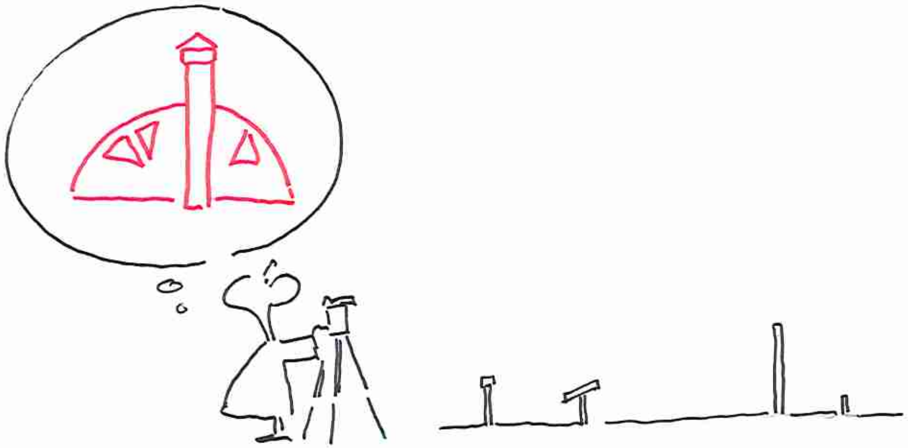
J-14

J-12

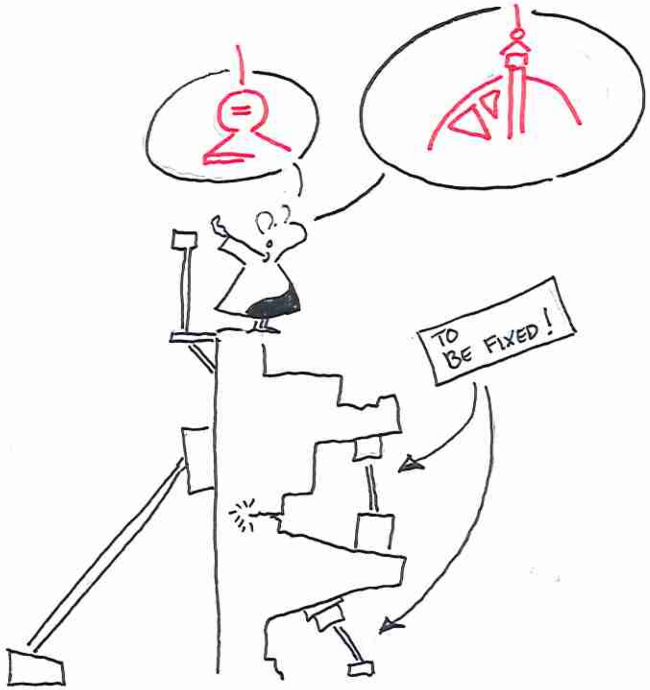
J-10



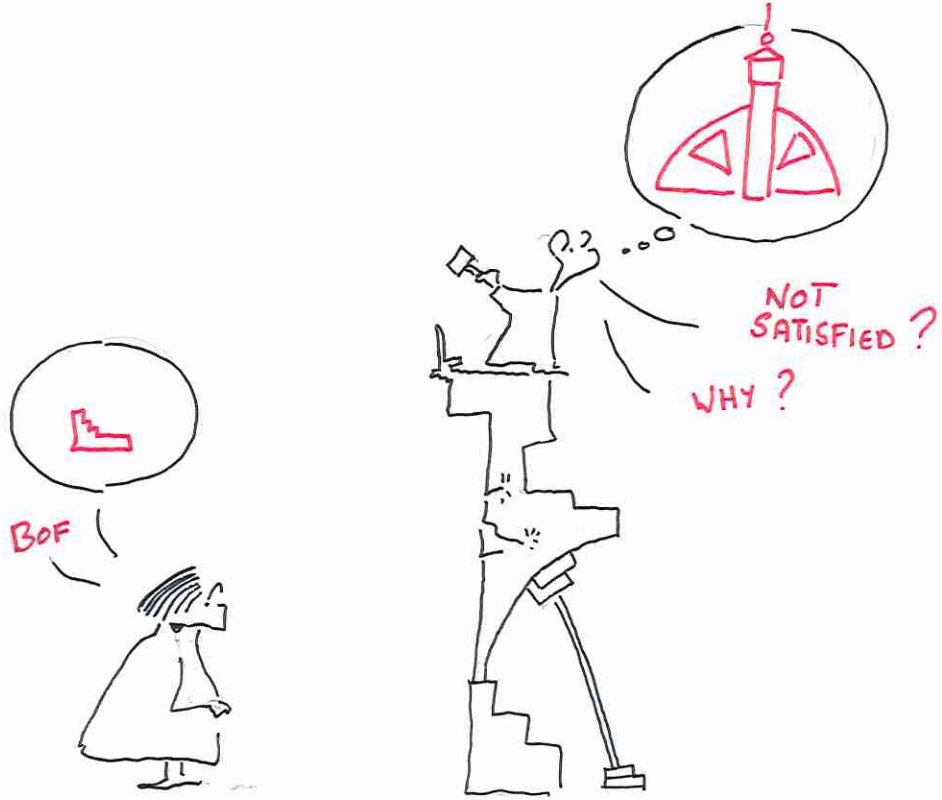
J-4



J-5

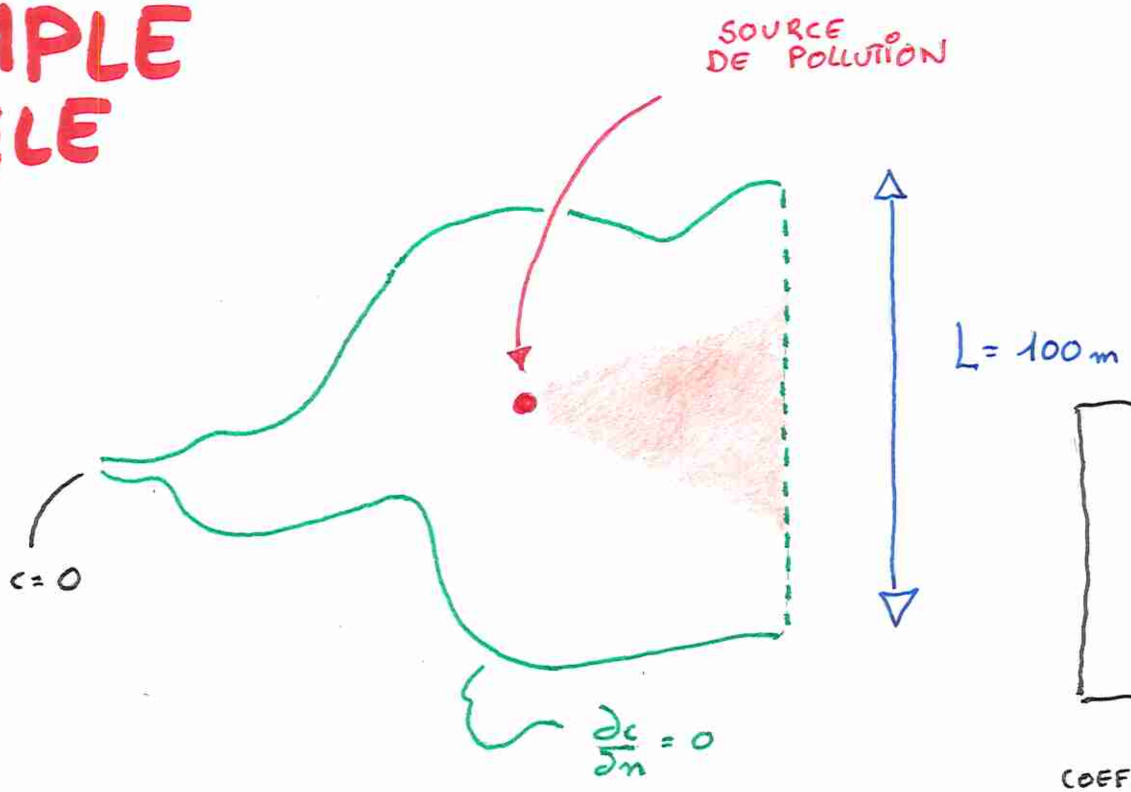


J-1



J

EXEMPLE
MODELE



$$P_e = \frac{v L}{D}$$

MOYENNE DE LA CONCENTRATION SUR LA PROFONDEUR

COEFFICIENT DE DIFFUSION

$$\frac{\partial c}{\partial t} + \underbrace{\underset{\sim}{v} \cdot \underset{\sim}{\nabla} c}_{\text{TERME DE TRANSPORT}} = \underbrace{\underset{\sim}{\nabla} \cdot \left(D \, \underset{\sim}{\nabla} c \right)}_{\text{TERME DE DIFFUSION}} + S$$

$\underset{\sim}{v}$ = VITESSE HORIZONTALE

$$\frac{\partial c}{\partial t} + v\frac{\partial c}{\partial x} = k\frac{\partial^2 c}{\partial x^2}$$

STATIONNAIRE

Pe PETIT

$$0 = k\frac{\partial^2 c}{\partial x^2}$$

ELL 2

$$v\frac{\partial c}{\partial x} = k\frac{\partial^2 c}{\partial x^2}$$

2 ELL

Pe GRAND

$$v\frac{\partial c}{\partial x} = 0$$

HYP 1

INSTATIONNAIRE

PARA 2

$$\frac{\partial c}{\partial t} = k\frac{\partial^2 c}{\partial x^2}$$

$$\frac{\partial c}{\partial t} + v\frac{\partial c}{\partial x} = k\frac{\partial^2 c}{\partial x^2}$$

PARA 2

$$\frac{\partial c}{\partial t} + v\frac{\partial c}{\partial x} = 0$$

HYP 1

$$Pe = \frac{vL}{k}$$

EDP
ORDRE 1
QUASI-LINEAIRE

$$\frac{\partial u}{\partial t} + b\,\frac{\partial u}{\partial x} = d$$

$u(x,t)$

$fct(u,x,t)$

COURBE CARACTERISTIQUE $\frac{dx}{dt} = b$

$$\frac{du}{dt} = \underbrace{\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\overbrace{\frac{dx}{dt}}^{b}}_{d}$$

UNE EDP SUR UNE COURBE CAR. SE REDUIT À UNE EDO

EDP
ORDRE n
QUASI-LINEAIRE

$$\overbrace{\underset{n \times n}{\underline{\underline{A}}} \cdot \frac{\partial \underline{u}}{\partial t} + \underline{\underline{B}} \cdot \frac{\partial \underline{u}}{\partial x} = \underline{d}}^{fct(\underline{u},\, x,\, t)}$$

n
COURBES CARACTERISTIQUES $\dfrac{dx}{dt} = \lambda_i$

$$\underset{\neq 0}{\underline{z}_i} \cdot \left(\underline{\underline{B}} - \lambda_i \underline{\underline{A}} \right) = 0$$

$$\rightarrow \det\left(\underline{\underline{B}} - \lambda_i \underline{\underline{A}} \right) = 0$$

$$\boxed{\underline{z}_i \cdot \underline{\underline{A}} \cdot \frac{d\underline{u}}{dt} = \underline{z}_i \cdot \underline{d}}$$

$$A\frac{\partial^2 u}{\partial t^2} + B\frac{\partial^2 u}{\partial x\,\partial t} + C\frac{\partial^2 u}{\partial x^2} = D$$

$$\underbrace{\begin{bmatrix} 0 & A \\ 1 & 0 \end{bmatrix}}_{\underline{\underline{A}}} \cdot \frac{\partial \underset{\sim}{u}}{\partial t} + \underbrace{\begin{bmatrix} C & B \\ 0 & -1 \end{bmatrix}}_{\underline{\underline{B}}} \cdot \frac{\partial \underset{\sim}{u}}{\partial x} = \underbrace{\begin{bmatrix} D \\ 0 \end{bmatrix}}_{\underline{\underline{d}}}$$

$$\underset{\sim}{u} = \begin{bmatrix} \partial u/\partial x \\ \partial u/\partial t \end{bmatrix}$$

$$\boxed{\underbrace{\det \begin{bmatrix} C & B - \lambda A \\ -\lambda & -1 \end{bmatrix}}_{-C + \lambda B - \lambda^2 A} = 0}$$

$$B^2 - 4AC \quad \begin{matrix} > \\ = \\ < \end{matrix} \quad 0$$

HYPERBOLIQUE

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad u = f(x-ct) + g(x+ct)$$

PARABOLIQUE

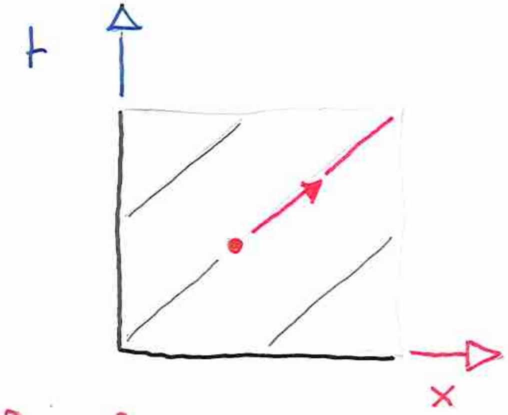
$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

ELLIPTIQUE

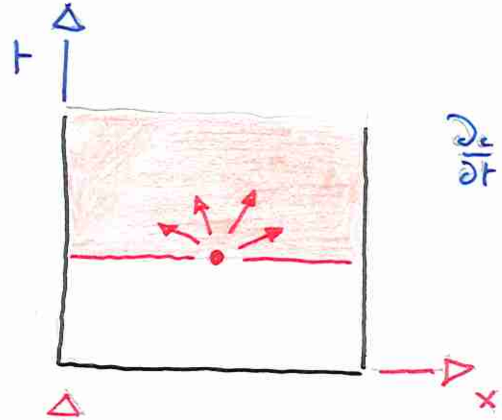
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

PROBLEME BIEN POSE

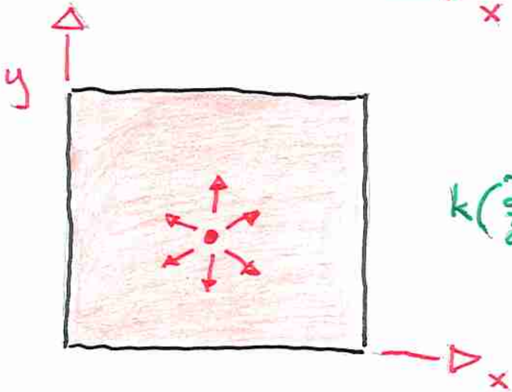
→ CONDITIONS AUX LIMITES



$$\frac{\partial c}{\partial t} + v\frac{\partial c}{\partial x} = 0$$

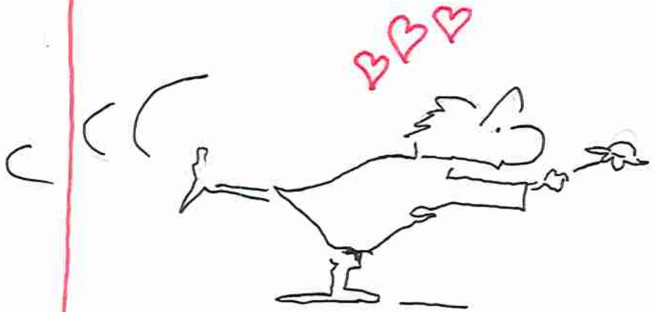


$$\frac{\partial c}{\partial t} = k\frac{\partial^2 c}{\partial x^2}$$



$$k\left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2}\right) = 0$$

PROBLÈME ELLIPTIQUE



ELEMENTS FINIS *

* "USUAL FEM"
CLASSICAL GALERKIN FORMULATION

HEAT CONDUCTION

CONSTITUTIVE LAW

$$\underset{\sim}{q} = -k \underset{\sim}{\nabla} u$$

FOURIER

TEMPERATURE

CONSERVATION LAW

$$-\underset{\sim}{\nabla} \cdot \underset{\sim}{q} + f = 0$$

HEAT FLOW

? u SUCH THAT

$$\underset{\sim}{\nabla} \cdot \big(\underbrace{k \underset{\sim}{\nabla} u}_{-q(u)} \big) + f = 0 \quad \text{IN } \Omega$$

$$-\underset{\sim}{q} \cdot \underset{\sim}{n} = g \quad \text{ON } \Gamma_N$$

$$u = 0 \quad \text{ON } \Gamma_D$$

TYPICAL ELLIPTIC BOUNDARY VALUE PROBLEM

? u SUCH THAT

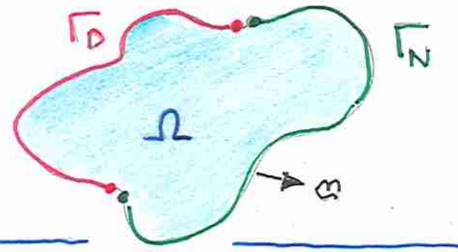
$$\underset{\sim}{\nabla} \cdot (a \underset{\sim}{\nabla} u) + f = 0 \quad \text{in } \Omega$$

$$\underset{\sim}{n} \cdot (a \underset{\sim}{\nabla} u) = g \quad \text{ON } \Gamma_N$$

NEUMANN

$$u = t \quad \text{ON } \Gamma_D$$

DIRICHLET



1D EXAMPLE

? u SUCH THAT

$$\frac{d^2u}{dx^2} + f = 0 \quad \forall x \in]0,1[$$

$$u(0) = u(1) = 0$$

Some new notations ....

$$\langle \cdot \rangle \triangleq \int_{\Omega} \cdot \, d\Omega$$

$$\langle\!\langle \cdot \rangle\!\rangle \triangleq \int_{\partial\Omega} \cdot \, ds$$

$$\langle\!\langle \cdot \rangle\!\rangle_N \triangleq \int_{\Gamma_N} \cdot \, ds$$

$$\langle\!\langle \cdot \rangle\!\rangle_D \triangleq \int_{\Gamma_D} \cdot \, ds$$

Not very rigorous !

$$\mathcal{U} \triangleq \left\{ u \ldots \text{ such that } u = t \text{ on } \Gamma_D \right\}$$

$$\hat{\mathcal{U}} \triangleq \left\{ u \text{ such that } u = 0 \text{ on } \Gamma_D \right\}$$

Functions spaces

.... to do calculus !

Tip #1

$(fg)' = fg' + f'g$

INTEGRAL BY PARTS

$$\langle \hat{u}\ \underset{\sim}{\nabla}\cdot(a\underset{\sim}{\nabla}u)\rangle = \underbrace{\langle \underset{\sim}{\nabla}\cdot(\hat{u}\,a\underset{\sim}{\nabla}u)\rangle} - \langle(\underset{\sim}{\nabla}\hat{u})\cdot(a\underset{\sim}{\nabla}u)\rangle$$

$\in \hat{\mathcal{U}}$

$\in \mathcal{U}$

Tip #2

DIVERGENCE'S THEOREM

$\langle \underset{\sim}{\nabla}\cdot\underset{\sim}{a}\rangle = \langle\!\langle \underset{\sim}{n}\cdot\underset{\sim}{a}\rangle\!\rangle$

$$= \langle\!\langle \underset{\sim}{n}\cdot(a\underset{\sim}{\nabla}u)\,\hat{u}\rangle\!\rangle$$

Tip #3

BY DEFINITION OF $\hat{\mathcal{U}}$

$$= \langle\!\langle \underbrace{\underset{\sim}{n}\cdot(a\underset{\sim}{\nabla}u)}_{g \text{ on } \Gamma_N\ !}\,\hat{u}\rangle\!\rangle_N$$

STRONG FORMULATION

? u SUCH THAT

$$\underline{\nabla}\cdot(a\underline{\nabla}u)+f=0 \quad \text{IN } \Omega$$

$$\underline{n}\cdot(a\underline{\nabla}u)=g \quad \text{ON } \Gamma_N$$

$$u=t \quad \text{ON } \Gamma_D$$

$\Downarrow \quad \not\Uparrow \quad ?$

WEAK FORMULATION

? $u\in\mathcal{U}$ SUCH THAT

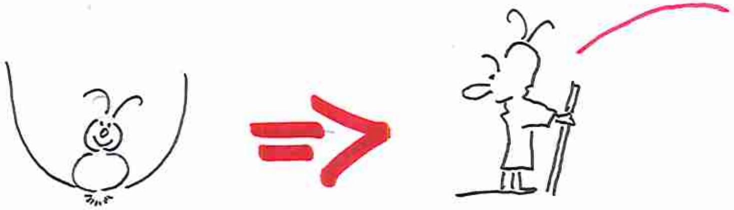
$$\underbrace{\langle \underline{\nabla}\hat{u}\cdot a\underline{\nabla}u\rangle}_{a(\hat{u},u)} = \underbrace{\langle f\hat{u}\rangle + \langle\langle g\hat{u}\rangle\rangle_N}_{b(\hat{u})} \quad \forall \hat{u}\in\hat{\mathcal{U}}$$

$\Updownarrow$

MINIMIZATION PROBLEM

? u SUCH THAT

$$J(u)=\min_{v\in\mathcal{U}}\underbrace{\left(\frac{1}{2}a(v,v)-b(v)\right)}_{J(v)}$$



WEAK
OF VARIATIONAL
FORMULATION

$u \in \mathcal{U}$
AND MINIMIZES
J

$\in \mathbb{R}$

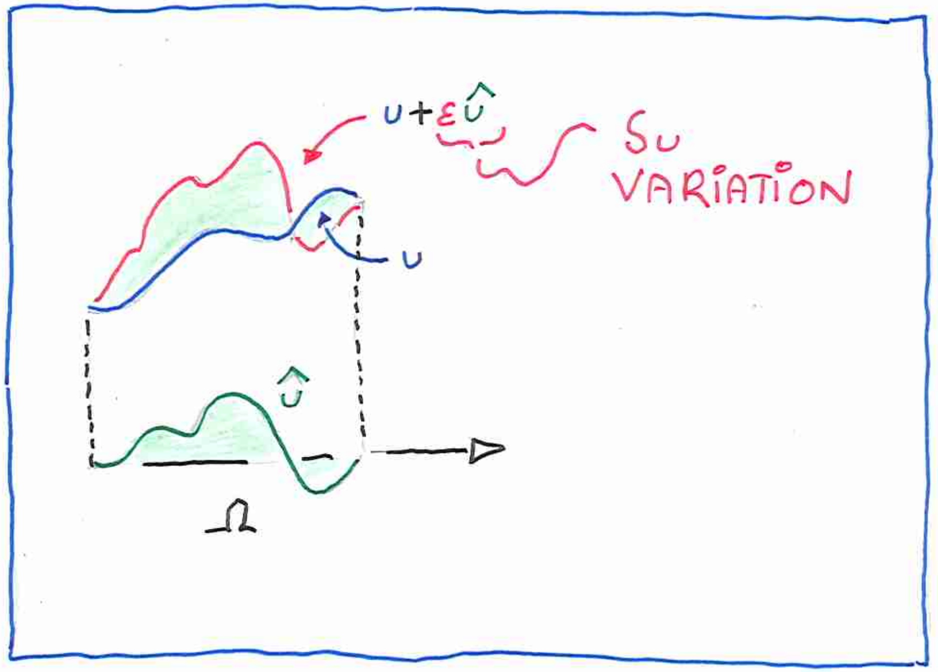
VARIATION
$\in \hat{\mathcal{U}}$

$$\frac{dJ(u+\varepsilon\hat{u})}{d\varepsilon}\Big|_{\varepsilon=0} = 0 \qquad \forall\, \hat{u} \in \hat{\mathcal{U}}$$

FUNCTIONAL

$$\delta J(u) = 0$$

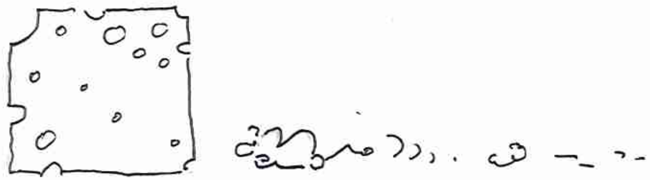
VARIATIONAL CALCULUS

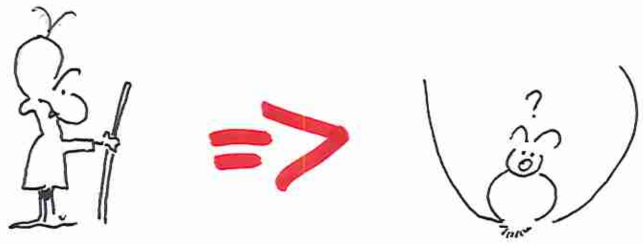


$$
J(u+\varepsilon\hat{u}) = \underbrace{\frac{1}{2}\langle \underline{\nabla} u \cdot a\, \underline{\nabla} u \rangle - \langle f u \rangle - \langle\!\langle g u \rangle\!\rangle_N}_{J(u)} + \varepsilon \langle \underline{\nabla} u \cdot a\, \underline{\nabla} \hat{u} \rangle + \frac{\varepsilon^2}{2} \langle \underline{\nabla} \hat{u} \cdot a\, \underline{\nabla} \hat{u} \rangle - \varepsilon \langle f \hat{u} \rangle - \varepsilon \langle\!\langle g \hat{u} \rangle\!\rangle_N
$$

$$
\underbrace{\left.\frac{d \overbrace{J(u+\varepsilon\hat{u})}^{g(\varepsilon)}}{d\varepsilon}\right|_{\varepsilon=0}}_{=0} = \langle \underline{\nabla} u \cdot a\, \underline{\nabla} \hat{u} \rangle - \langle f \hat{u} \rangle - \langle\!\langle g \hat{u} \rangle\!\rangle_N
$$

$$
\forall \hat{u} \in \hat{\mathcal{U}}
$$





$$J(u+\hat{u}) = \underbrace{\frac{1}{2}\langle \nabla u \cdot a \nabla u \rangle - \langle f u \rangle - \langle\!\langle g u \rangle\!\rangle_N}_{J(u)} + \underbrace{\langle \nabla u \cdot a \nabla \hat{u} \rangle - \langle f \hat{u} \rangle - \langle\!\langle g \hat{u} \rangle\!\rangle_N}_{=0} + \underbrace{\frac{1}{2}\langle \nabla \hat{u} \cdot a \nabla \hat{u} \rangle}_{\geqslant 0}$$

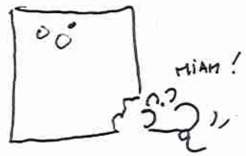
SOLUTION OF THE WEAK PROBLEM

$\in \hat{U}$

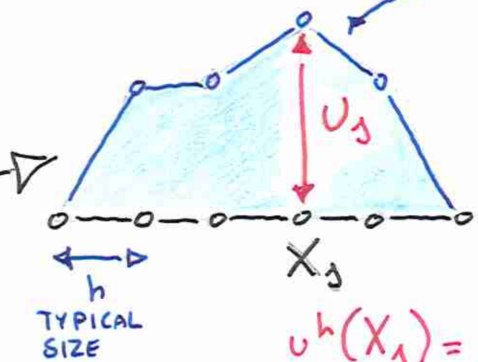
U SOLUTION OF THE WEAK PROBLEM

$\boxed{\text{IF } a > 0 \text{ !}}$

$$\geqslant J(u)$$



THE FINITE ELEMENT METHOD



$$u^h(X_j) = \underbrace{\sum_i U_i \,\tau_i(\delta_{ij})}_{U_j} \;!$$

$$\boxed{u \simeq u^h = \sum_{j=1}^{n} \underbrace{U_j}_{} \; \underbrace{\tau_j(x)}_{}}$$

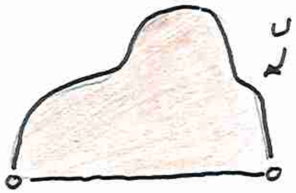
$\in \mathcal{U}$

UNKNOWN NODAL VALUES

KNOWN SHAPE FUNCTIONS

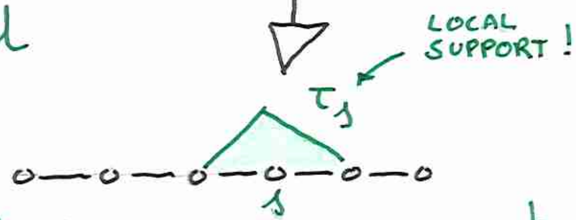
$\in \mathcal{U}$

$\tau_j(X_i) = \delta_{ij}$



$\in \mathcal{U}^h \subset \mathcal{U}$

$\dim(\mathcal{U}^h) = n$



$\{\tau_j\}$ IS A BASIS FOR $\mathcal{U}^h$

$u \longrightarrow \breve{u}^h \quad \breve{u}^h$

INTERPOLATION
APPROXIMATION

BUT u !
IT IS
NOT KNOWN !

$\breve{u}^h \quad \breve{u}^h$ .....
THOSE FUNCTIONS
ARE ONLY
VIRTUAL CONCEPTS ...

INTERPOLATION

$$U_i = u(X_i)$$

$$\longrightarrow \breve{u}^h(x)$$

LEAST SQUARES
APPROXIMATION

$$\langle \tau_j \tau_i \rangle U_i = \langle \tau_j u \rangle$$

$$\longrightarrow \breve{u}^h(x)$$

FINITE ELEMENT = VARIATIONAL METHOD

$$? \, u^h \in \mathcal{U}^h$$
$$a(u^h, \hat{u}^h) = b(\hat{u}^h) \quad \forall \hat{u}^h \in \hat{\mathcal{U}}^h$$

GALERKIN METHOD

$$\Updownarrow$$

$$? \, u^h \in \mathcal{U}^h$$
$$J(u^h) = \min_{v^h \in \mathcal{U}^h} \underbrace{\frac{1}{2} a(v^h, v^h) - b(v^h)}_{J(v^h)}$$

RITZ METHOD

DISCRETE FORM.

DISCRETE FORMULATION = LINEAR SYSTEM !

$$J(u^h) = \frac{1}{2} \langle \underbrace{\sum_i U_i \nabla \tau_i}_{\nabla u^h} \cdot \sum_j U_j \nabla \tau_j \rangle - \langle f \sum_i U_i \tau_i \rangle$$

$$= \frac{1}{2} \sum_i \sum_j U_i U_j \underbrace{\langle \nabla \tau_i \cdot \nabla \tau_j \rangle}_{A_{ij}} - \sum_i U_i \underbrace{\langle f \tau_i \rangle}_{B_i}$$

$$\underbrace{0 = \frac{\partial J(u^h)}{\partial U_i}}$$

$$\boxed{\sum_{j=1}^{n} A_{ij} U_j = B_i}$$

ASSUME
$a = 1$
$g = 0$

FINITE ELEMENT = WEIGHTED RESIDUALS METHOD

BOUNDARY APPROACH
PDE OK
DIRICHLET KO

INTERIOR APPROACH
DIRICHLET OK
PDE KO

!

MIXED APPROACH
DIRICHLET KO
PDE KO

? U_i IN ORDER TO MINIMIZE "IN A GIVEN SENSE" * THE RESIDUAL FUNCTION

$$\underbrace{\underset{\sim}{\nabla} \cdot (a \underset{\sim}{\nabla} u^h) + f}_{\triangleq\, r^h}$$

IN GENERAL $\neq 0$!

* COLLOCATION $r^h(X_i) = 0$

GALERKIN $\langle \tau_i \, r^h \rangle = 0$

$$\overbrace{\underbrace{\langle \tau_i (\nabla \cdot a \nabla u^h) \rangle}_{\textstyle -\langle \nabla \tau_i \cdot a \underbrace{\nabla u^h}_{\sum_{j=1}^{n} U_j \nabla \tau_j} \rangle + \langle \nabla \cdot (\tau_i a \nabla u^h) \rangle} + \langle \tau_i f \rangle}^{\langle r^h \tau_i \rangle} = 0$$

$$+ \langle\!\langle \underbrace{n \cdot a \nabla u^h}_{g} \tau_i \rangle\!\rangle_N$$

BECAUSE U_i ON Γ_D ARE GIVEN A PRIORI !

$$- \sum_{j=1}^{n} U_j \underbrace{\langle \nabla \tau_j \cdot \nabla \tau_i \rangle}_{A_{ij}} + \langle \tau_i f \rangle + \underbrace{\langle\!\langle g \tau_i \rangle\!\rangle_N}_{B_i} = 0$$

SPARSE MATRIX BECAUSE τ_i HAVE LOCAL SUPPORT !

d^2u/dx^2 LINEAR ELEMENTS

$$\begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix} \frac{1}{h^2}$$

1D

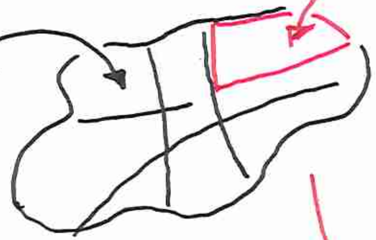
THE 2 BASIC IDEAS OF THE F.E.M.

$$\overline{\Omega} = \bigcup_{i=1}^{N} \{\overline{\Omega}_e\}$$

$$\Omega_e \cap \Omega_f = \phi \quad \text{if } e \neq f$$

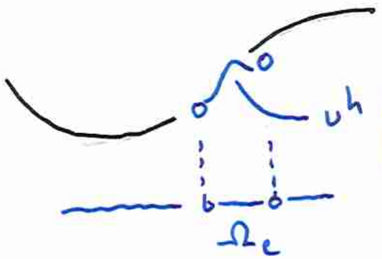
ELEMENT Ω_e

Ω



THE PROBLEM GEOMETRY IS DIVIDED INTO SMALL ELEMENTS

1 PIECEWISE APPROXIMATION OF THE UNKNOWN FUNCTION



CONSTANT
LINEAR
QUADRATIC } LOW ORDER

HIGH ORDER

LOCAL POLYNOMIAL APPROXIMATION

2 SUMMABILITY OF INTEGRALS

$$\int_{\Omega} = \sum_{e} \underbrace{\int_{\Omega_e}}$$

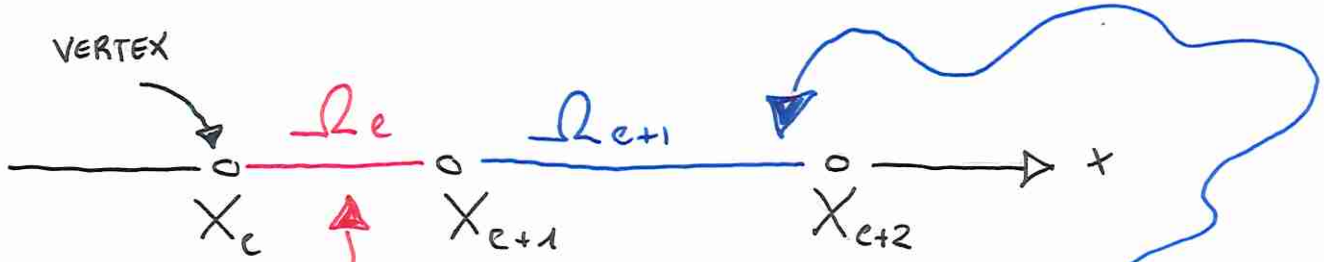
2bis PARENT ELEMENT

$\int_{\hat{\Omega}}$

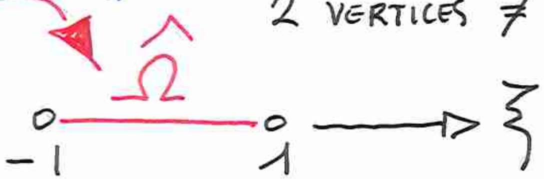
PARENT ELEMENT $\hat{\Omega}$

"THE FATHER"

HE RECEIVED A BEAUTIFUL HAT BECAUSE HE IS THE MASTER COPY !



2 VERTICES $\neq$ NODES !

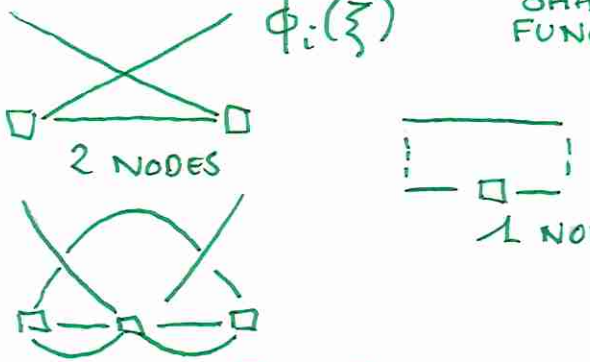


$$x(\xi) = \xi \frac{(X_{e+1} - X_e)}{2} + \frac{(X_{e+1} + X_e)}{2}$$

$$\xi(x) = \frac{2x - (X_{e+1} + X_e)}{(X_{e+1} - X_e)}$$

ISOMORPHISM

LOCAL SHAPE FUNCTIONS

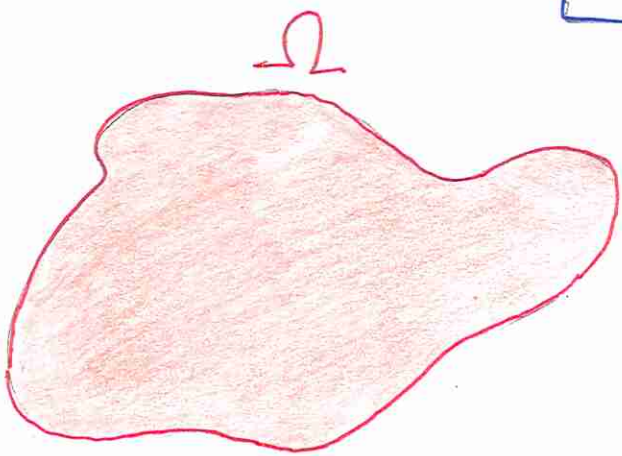


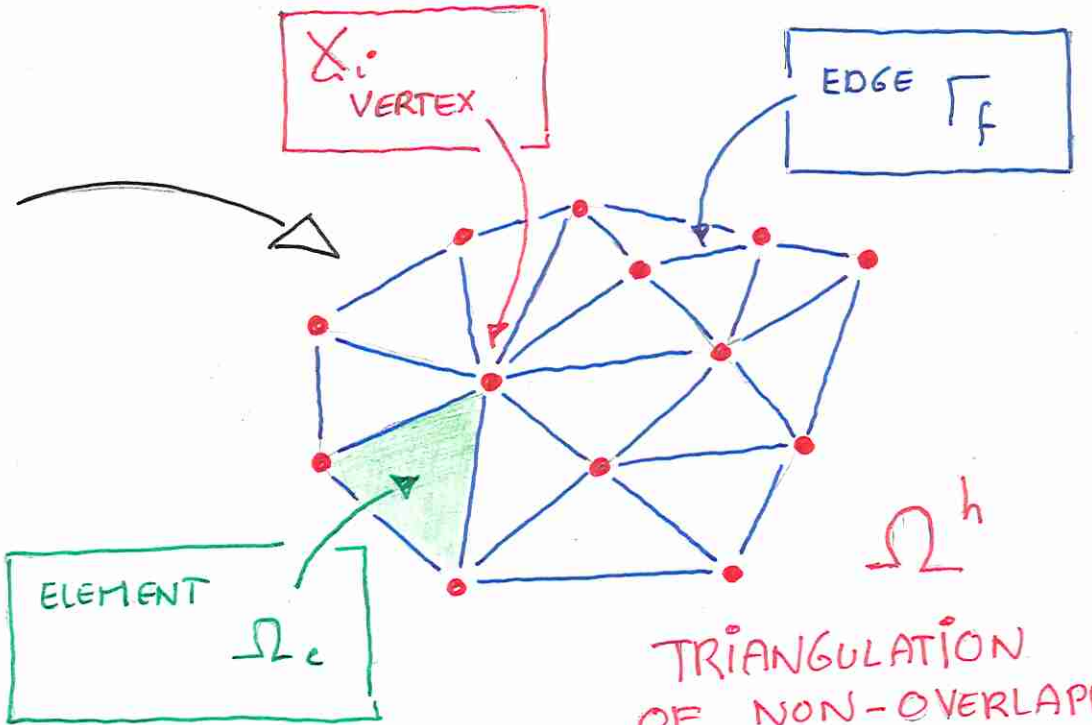
POISSON EQUATION

? u SUCH THAT

$$\nabla^2 u + f = 0 \quad \text{IN } \Omega$$

$$u = 0 \quad \text{ON } \partial\Omega$$





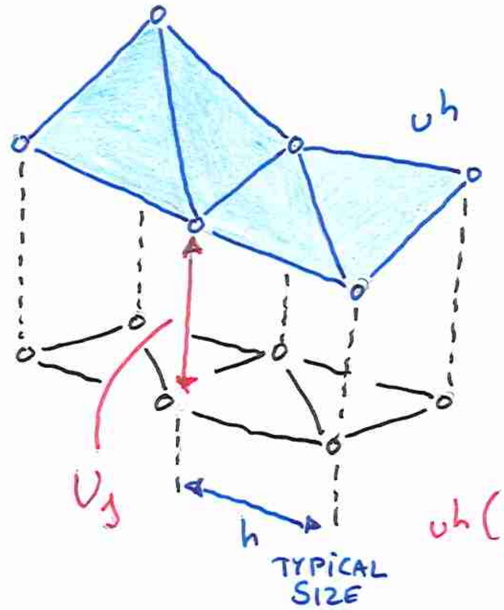
TRIANGULATION OF NON-OVERLAPPING TRIANGLES

N_0 VERTICES
N_1 EDGES
N_2 TRIANGLES

MESH PARAMETER

$$h = \max_e (\text{longest side of } \Omega_e)$$

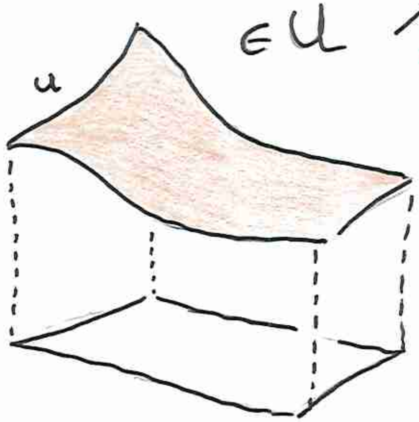
2D FINITE ELEMENT APPROXIMATION



$u^h(\underset{\sim}{X}_j) = U_j$

$$u \simeq u^h = \sum_{i=1}^{n} U_i \; \tau_i(\underset{\sim}{x})$$

$\tau_j(\underset{\sim}{X}_i) = \delta_{ij}$



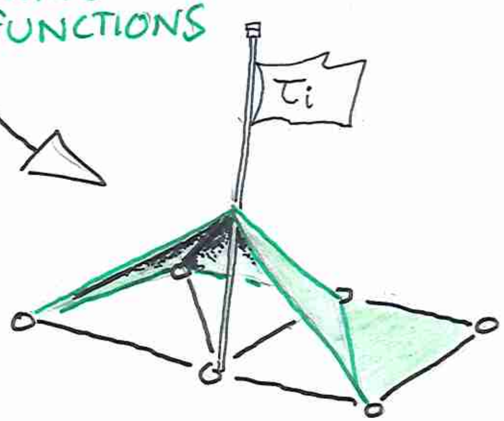
$\in \mathcal{U}$

$\in \mathcal{U}^h \subset \mathcal{U}$

$\dim(\mathcal{U}^h) = n$

UNKNOWN NODAL VALUES

KNOWN SHAPE FUNCTIONS



GLOBAL .... AND LOCAL SHAPE FUNCTIONS

$$\tau_{72}(\underset{\sim}{x}) = \phi_3^{e}(\underset{\sim}{x})$$

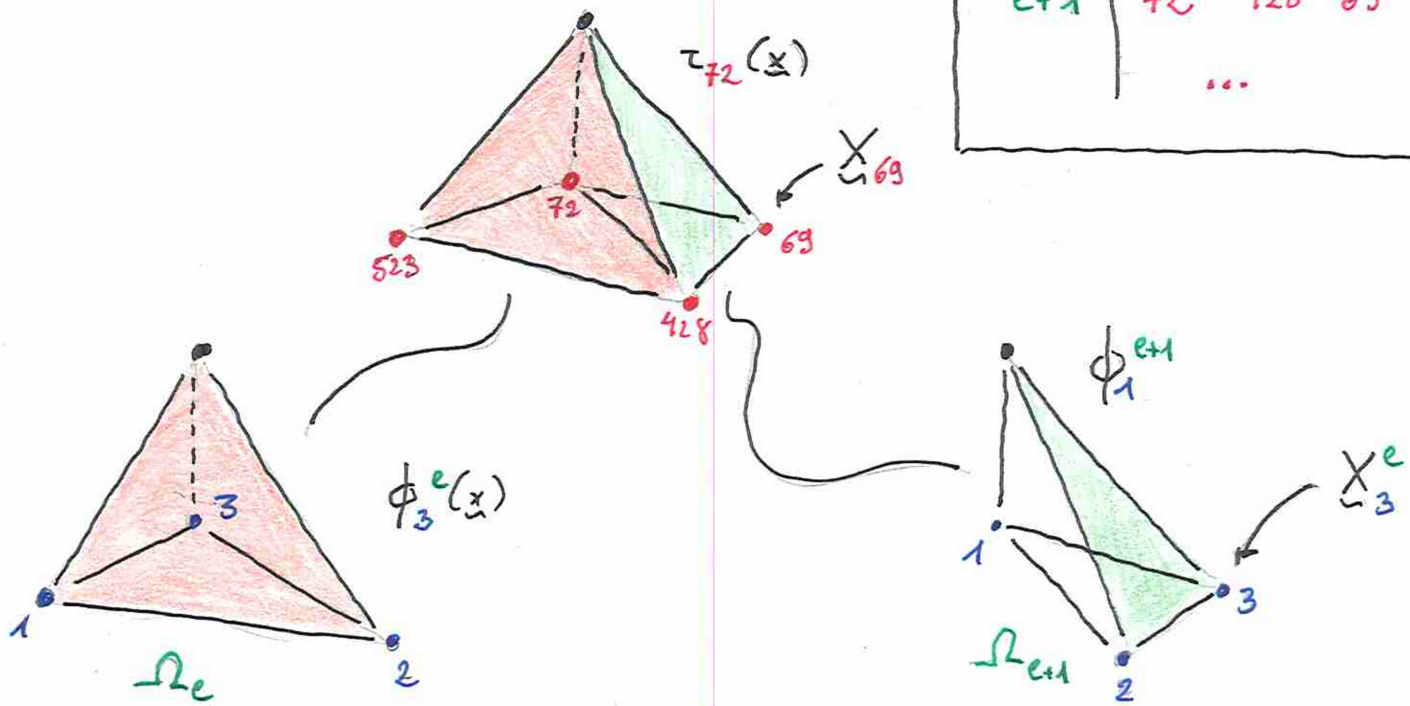
ELEMENT INDEX

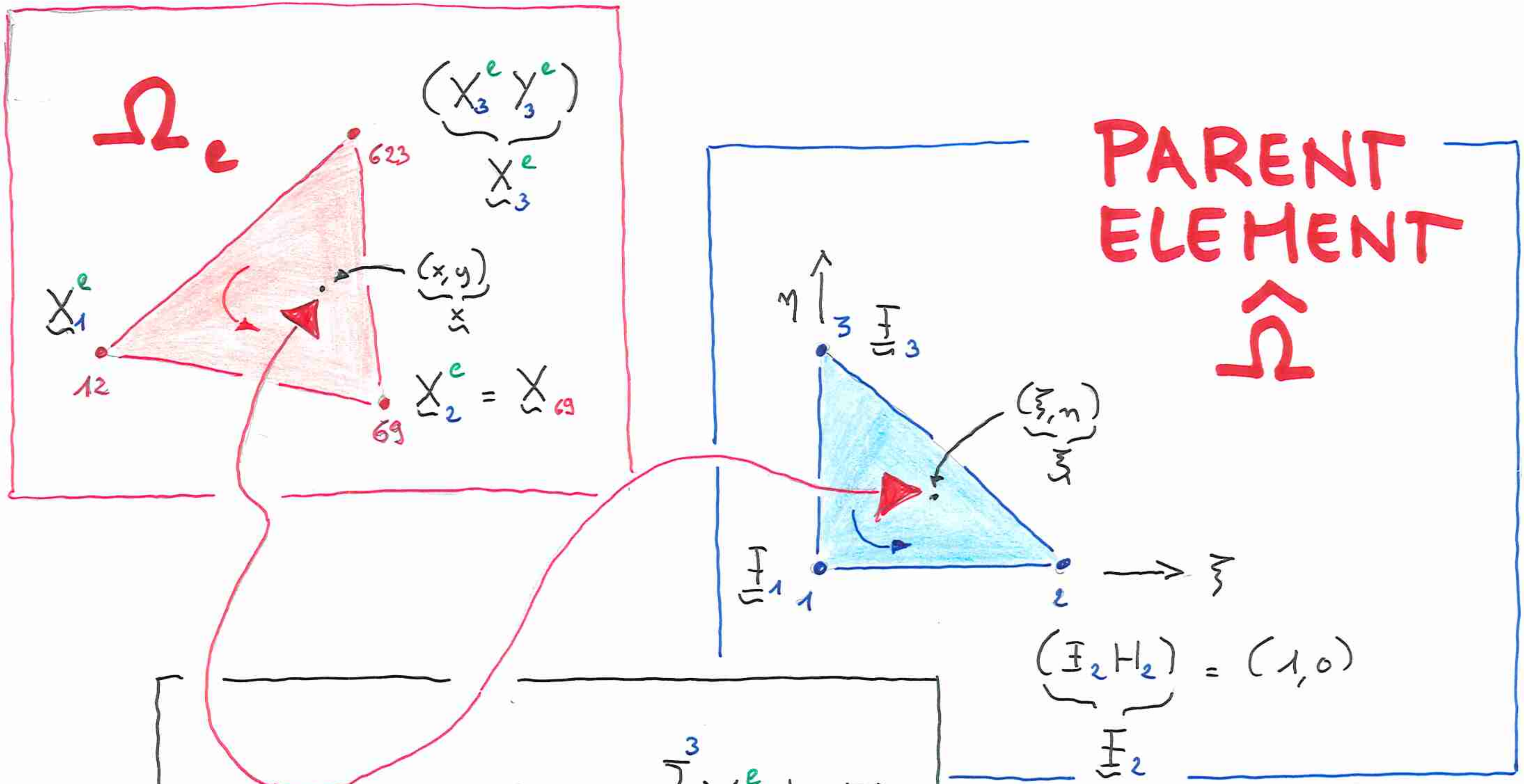
GLOBAL LABEL

ONLY ON Ω_e

LOCAL LABEL

	1	2	3
e	523	428	72
$e+1$	72	428	69
	...		





$$\underset{\sim}{x}(\underset{\sim}{\xi}) = \sum_{i=1}^{3} \underset{\sim}{X}_i^e \, \phi_i(\underset{\sim}{\xi})$$

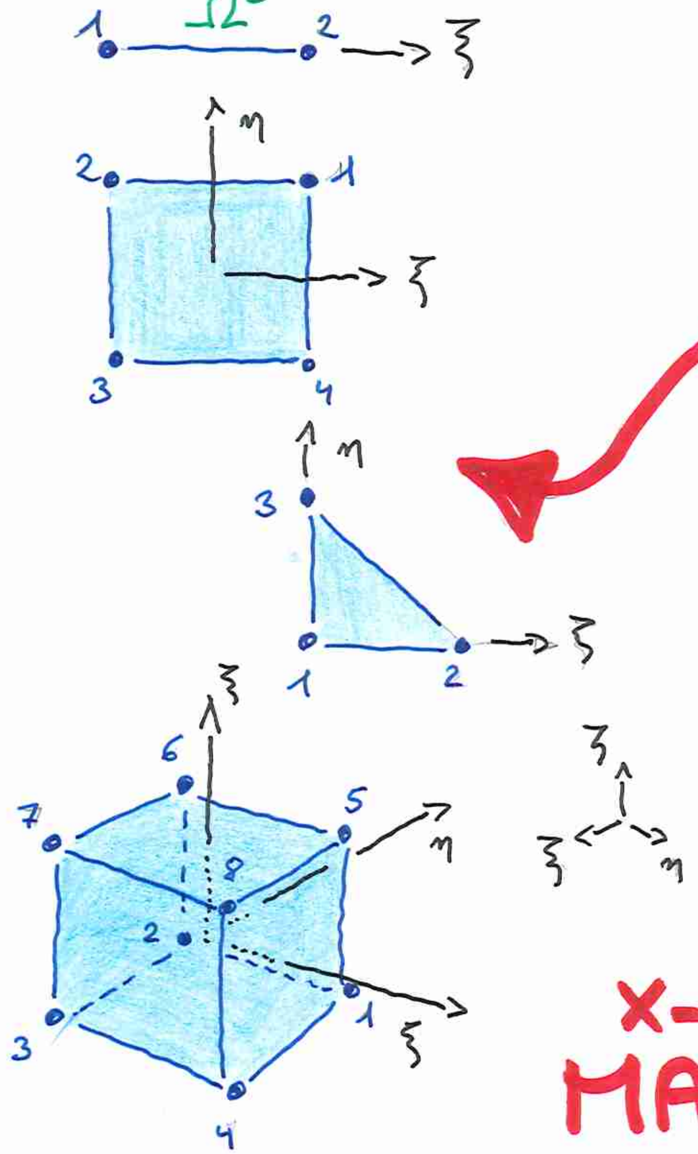
ISOMORPHISM

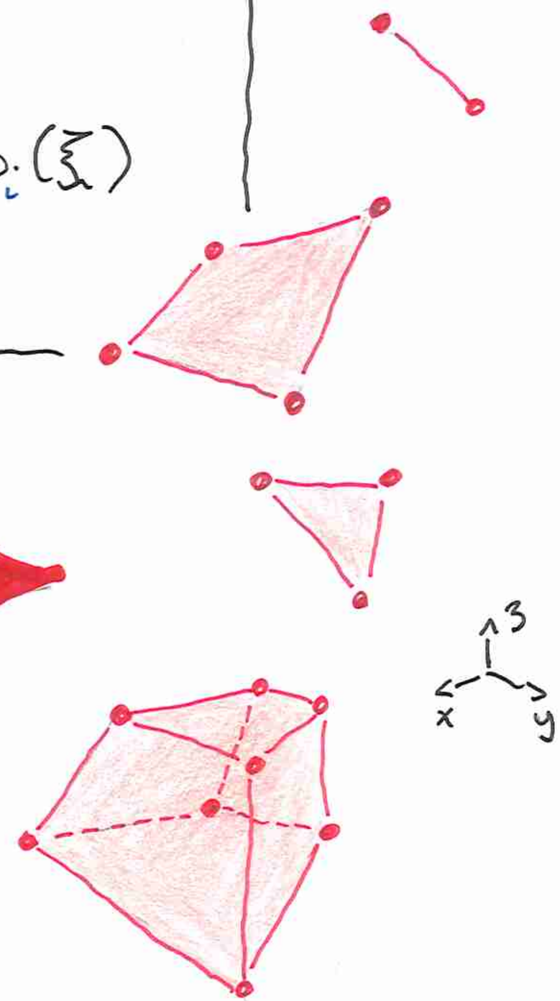
1D

$$x(\xi) = \frac{X_{e+1} - X_e}{2} \xi + \frac{(X_{e+1} + X_e)}{2}$$

$$= \underbrace{(1+\xi)/2}_{\phi_2(\xi)} X_2^e + \underbrace{(1-\xi)/2}_{\phi_1(\xi)} X_1^e$$

$$\underline{x}(\underline{\xi}) = \sum_{i=1}^{n} \underline{X}_i^{e}\, \phi_i(\underline{\xi})$$





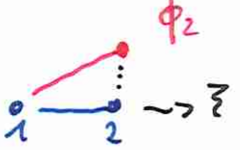
X-LINEAR MAPPING

$$\phi_i(\underline{\Xi}_j) = \delta_{ij}$$

ϕ_i REDUCES TO A LINEAR FUNCTION ON ALL EDGES

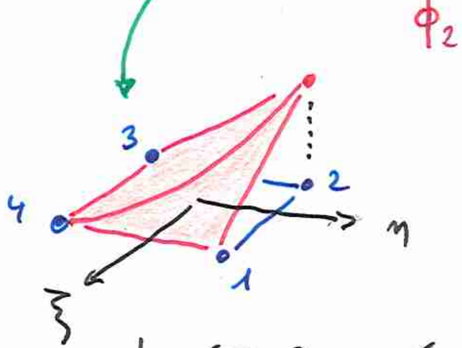
"LINEAR" SHAPE FUNCTIONS

P_1-C_0



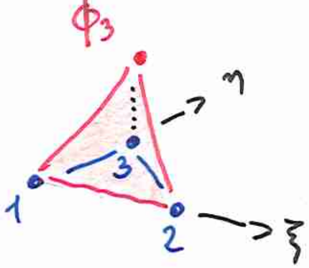
$$\phi_1(\xi) = (1-\xi)/2$$
$$\phi_2(\xi) = (1+\xi)/2$$

BILINEAR SHAPE FUNCTIONS



$$\phi_1(\xi,\eta) = (1+\xi)(1+\eta)/4$$
$$\phi_2(\xi,\eta) = (1-\xi)(1+\eta)/4$$
$$\phi_3(\xi,\eta) = (1-\xi)(1-\eta)/4$$
$$\phi_4(\xi,\eta) = (1+\xi)(1-\eta)/4$$

Q_1-C_0



$$\phi_1(\xi,\eta) = 1-\xi-\eta$$
$$\phi_2(\xi,\eta) = \xi$$
$$\phi_3(\xi,\eta) = \eta$$

TURNER TRIANGLES

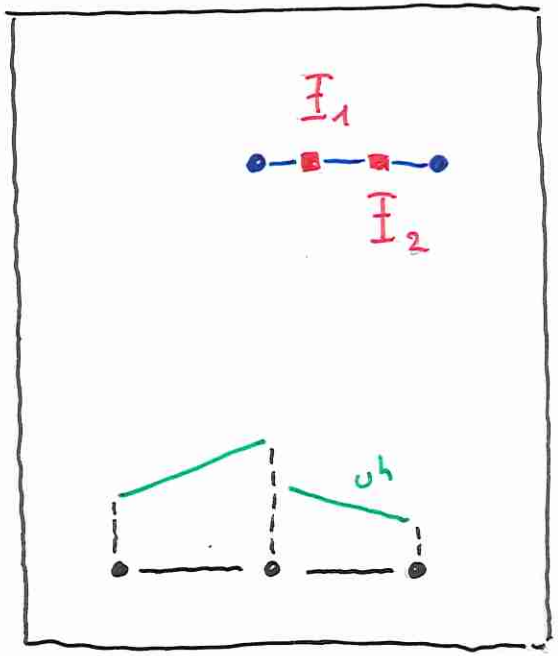
P_1-C_0

$\rightarrow u^h$ CONTINUOUS PIECEWISE "LINEAR"

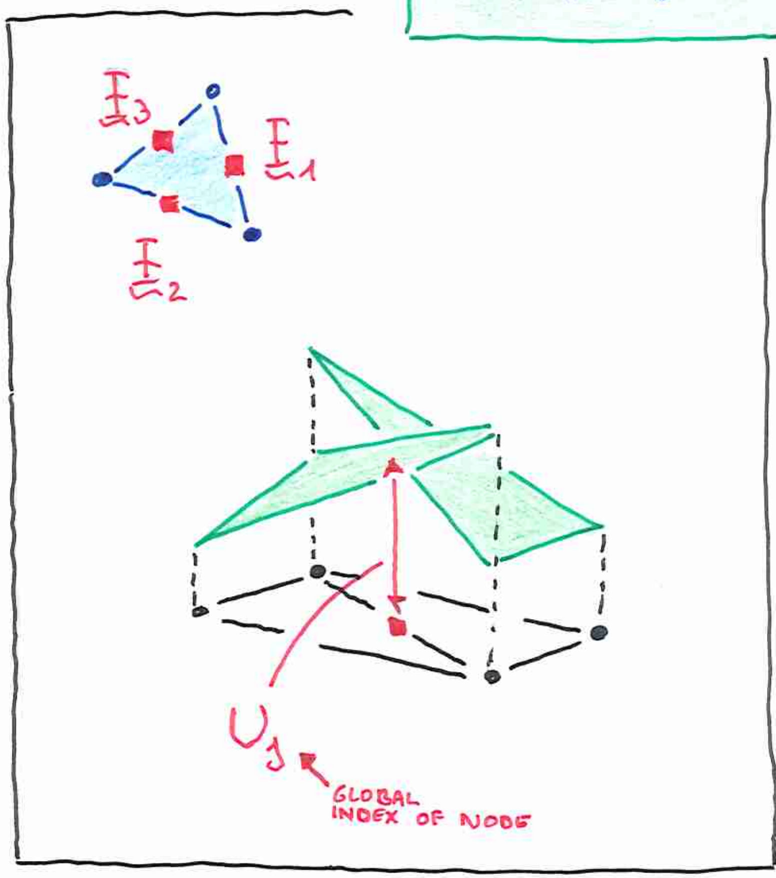
DISCONTINUOUS LINEAR SHAPE FUNCTIONS

$$\tau_i(\underline{\xi}_1) = S_{ij}$$

NODES $\neq$ VERTICES

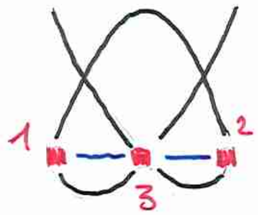


$P_1 - C_{-1}$



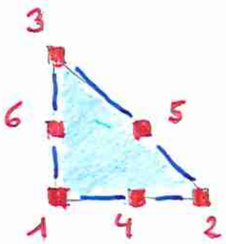
$P_1 - C_{-1}$

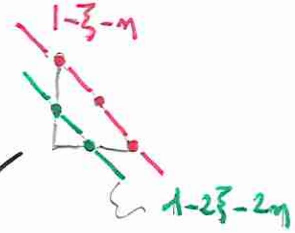
CONTINUOUS QUADRATIC SHAPE FUNCTIONS



$$\phi_1(\xi) = -\xi(1-\xi)/2$$
$$\phi_2(\xi) = \xi(1+\xi)/2$$
$$\phi_3(\xi) = (1-\xi)(1+\xi)$$

$P_2 - C_0$



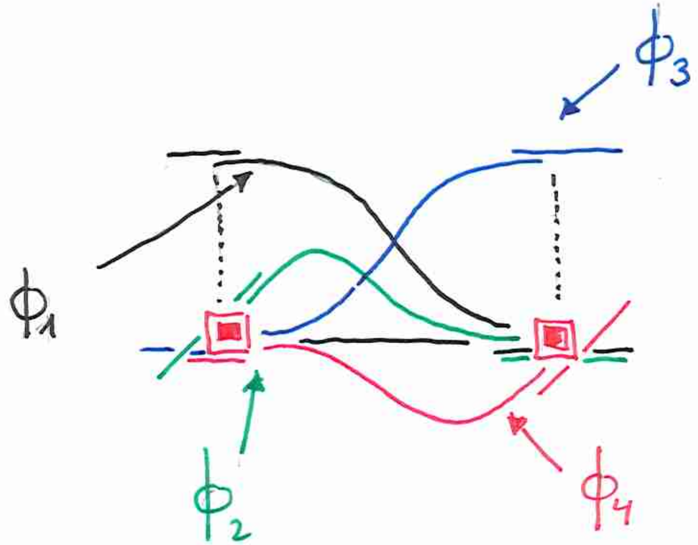


$$\phi_1(\xi,\eta) = \overbrace{(1-\xi-\eta)(1-2\xi-2\eta)}^{1-3(\xi+\eta)+2(\xi+\eta)^2}$$
$$\phi_2(\xi,\eta) = \xi(2\xi-1)$$
$$\phi_3(\xi,\eta) = \eta(2\eta-1)$$
$$\phi_4(\xi,\eta) = 4\xi(1-\xi-\eta)$$
$$\phi_5(\xi,\eta) = 4\xi\eta$$
$$\phi_6(\xi,\eta) = 4\eta(1-\xi-\eta)$$

$P_1 - C_0$

C^1 CUBIC SHAPE FUNCTIONS

$P_3 - C_1$



HERMITIAN ELEMENT

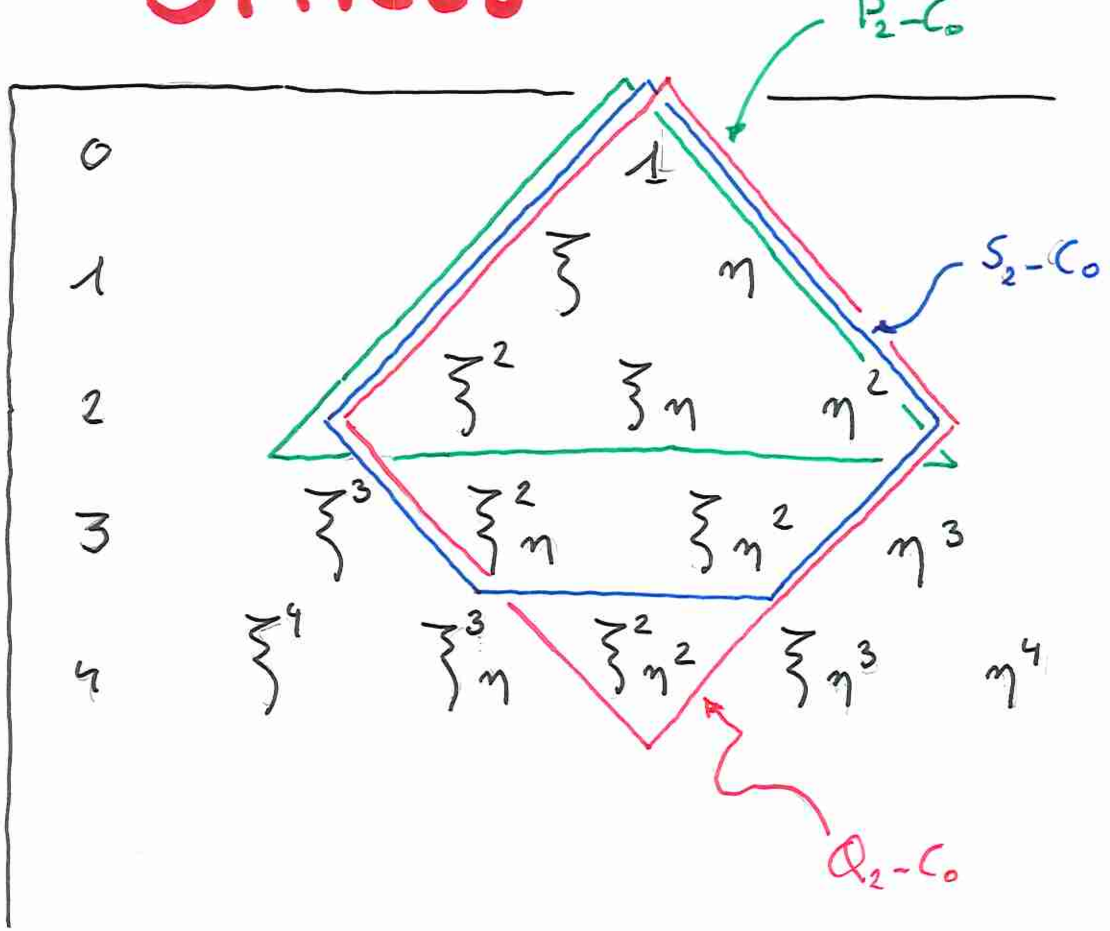
$$\phi_1 = (1-\xi)^2 (2+\xi)/4$$

$$\phi_2 = (1-\xi)^2 (1+\xi)/4$$

$$\phi_3 = (1+\xi)^2 (2-\xi)/4$$

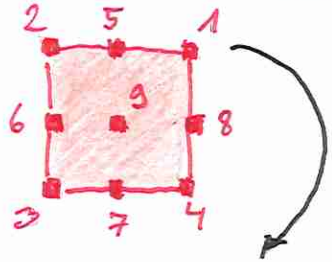
$$\phi_4 = -(1+\xi)^2 (1-\xi)/4$$

COMPLETE POLYNOMIAL DISCRETE SPACES



$Q_2 - C_0$

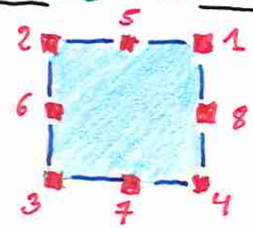
LAGRANGIAN BIQUADRATIC ELEMENT



$$\phi_1(\xi,\eta) = \frac{\xi(1+\xi)}{2}\,\eta\,\frac{(1+\eta)}{2}$$

$S_2 - C_0$

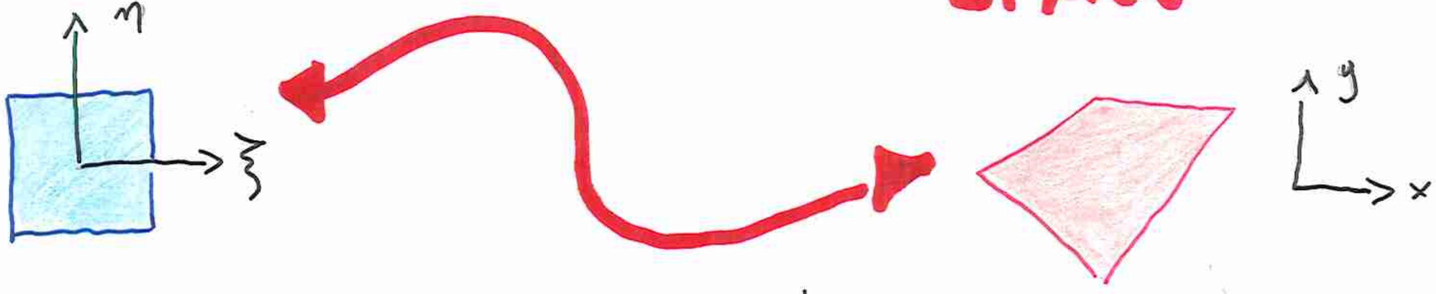
SERENDIP 2° ORDER ELEMENT



$$\phi_1(\xi,\eta) = \frac{(1+\eta)(1+\xi)(\eta+\xi-1)}{4}$$

$S_2 - C_0$ NOT $x-y$ COMPLETE
$Q_2 - C_0$ $x-y$ COMPLETE

$\xi-\eta$ COMPLETE DISCRETE SPACES $\neq$ $x-y$ COMPLETE DISCRETE SPACE



$$a_1 + a_2\xi + a_3\eta + a_4\xi\eta + a_5\xi^2 + a_6\eta^2$$

$$b_1 + b_2\, x(\xi,\eta) + b_3\, y(\xi,\eta) + b_4\, x(\xi,\eta)\, y(\xi,\eta) + b_5\, x^2(\xi,\eta) + b_6\, y^2(\xi,\eta)$$

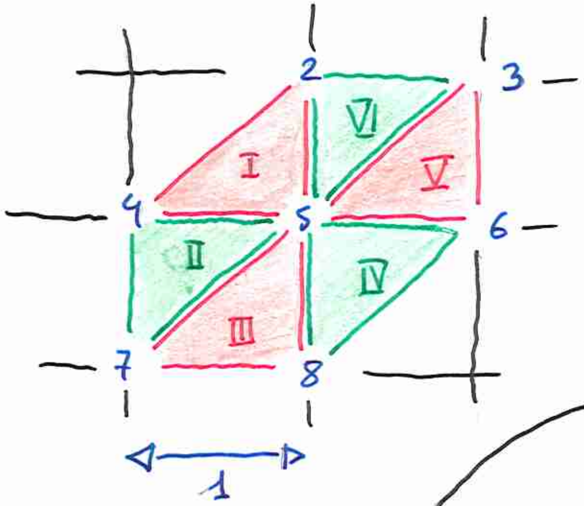
$$\boxed{\begin{aligned} c_1 &+ c_2\xi \\ &+ c_3\eta \\ &+ c_4\xi\eta \\ &+ c_5\xi^2 \\ &+ c_6\eta^2 \\ &+ c_7\xi\eta^2 \\ &+ c_8\eta\xi^2 \\ &+ c_9\xi^2\eta^2 \end{aligned}}$$

$$\underline{x}(\xi,\eta) = \sum_{i=1}^{4} \underline{X}_i^e \underbrace{\frac{(1\pm\xi)(1\pm\eta)}{4}}_{\frac{1\pm\xi\pm\eta\pm\xi\eta}{4}}$$

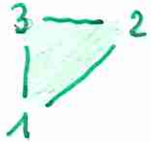
STIFFNESS MATRIX CALCULATION

$$< \frac{\partial \phi_i^e}{\partial x}\frac{\partial \phi_j^e}{\partial x} + \frac{\partial \phi_i^e}{\partial y}\frac{\partial \phi_j^e}{\partial y} >_{\Omega_e}$$

$$A_{ij} = \oplus \underbrace{A_{ij}^e}$$



$$\begin{array}{c|cc} & \phi_{i,x} & \phi_{i,y} \\ \hline 1 & -1 & 0 \\ 2 & 1 & -1 \\ 3 & 0 & 1 \end{array} \longrightarrow \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$



$$\begin{array}{c|cc} & \phi_{i,x} & \phi_{i,y} \\ \hline 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & -1 & 1 \end{array} \longrightarrow \underbrace{\frac{1}{2}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix}}_{A_{ij}^e}$$

CHECK !

$\sum \phi_i = 1$

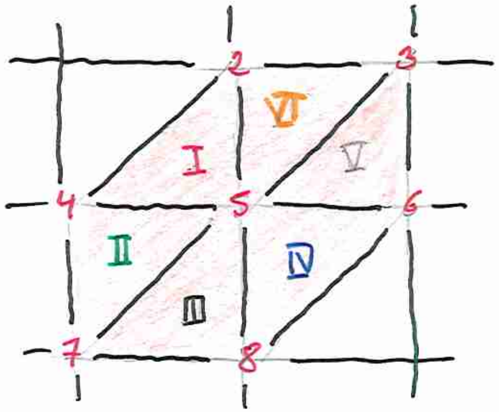
$\sum \phi_{i,x} = 0$

$\sum \phi_{i,y} = 0$

$\phi_1 = (1-x)$ + cst

$\phi_{1,x} = -1$

$\phi_{1,y} = 0$



ASSEMBLING LOCAL MATRICES

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \tfrac{1}{2}$$

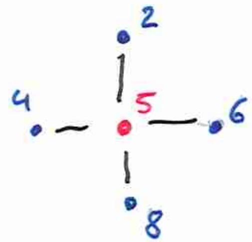
$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -2 & -1 & 2 \end{bmatrix} \tfrac{1}{2}$$

LOCATION TABLE

I	4	5	2
II	7	5	4
III	7	8	5
IV	8	6	5
V	5	6	3
VI	5	3	2

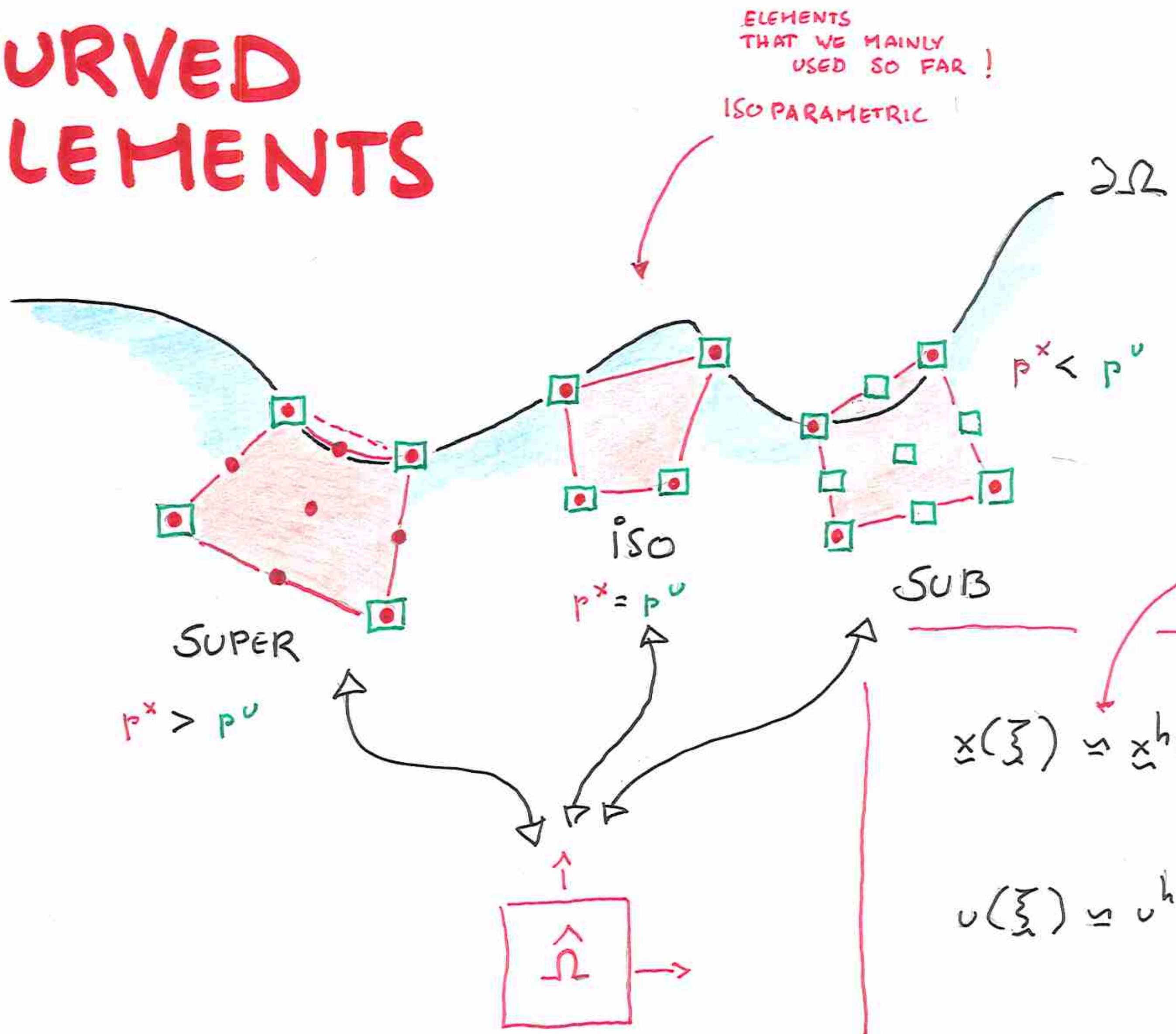
	2	3	4	5	6	7	8
2	1 2	-1		-1 -1			
3	-1	1 1			-1		
4			1 2	-1 -1		-1	
5	-1 -1		-1 -1	2 1 1 2 1 1	-1 -1		-1 -1
6		-1		-1 -1	1 2		
7			-1			1 1	-1
8				-1 -1		-1	2 1

$$\frac{1}{2}\left(8U_5 - 2U_2 - 2U_4 - 2U_8 - 2U_6\right)$$



REGULAR LATTICE OF TURNER TRIANGLES = CENTRAL FINITE DIFFERENCES

CURVED ELEMENTS



IN FACT, ONLY ON $\partial\Omega$!

$$\underset{\sim}{x}(\underset{\sim}{\xi}) \simeq \underset{\sim}{x}^h(\underset{\sim}{\xi}) = \sum_{j=1}^{p^x} \underset{\sim}{X}_i^e \, \phi_i^x(\underset{\sim}{\xi})$$

$$u(\underset{\sim}{\xi}) \simeq u^h(\underset{\sim}{\xi}) = \sum_{i=1}^{p^u} U_i^e \, \phi_i^u(\underset{\sim}{\xi})$$

SUPER - ISO - SUB PARAMETRIC ?

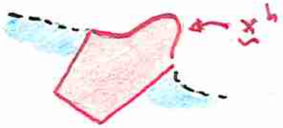
2 CONFLICTING GOALS

TYPICALLY ISO OR SUB-PARAMETRIC ELEMENTS ARE USED

COMPROMISE REQUIRED

IMPROVING THE APPROXIMATION OF THE BOUNDARY

$\phi_i^x \nearrow$



USING A MAPPING $\hat{\Omega} \leftrightarrow \Omega_e$ AS SIMPLE AS POSSIBLE

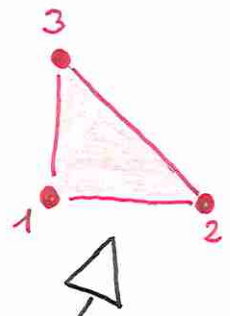
ϕ_i^x = X - LINEAR

TO PRESERVE X-Y COMPLETENESS OF THE DISCRETE SPACE

TO MAINTAIN ONE-TO-ONE MAPPING

DO NOT INTRODUCE INTERNAL CURVED EDGES !

WHEN IS $\underset{\sim}{x}(\underset{\sim}{\xi})$ ONE-TO-ONE ?

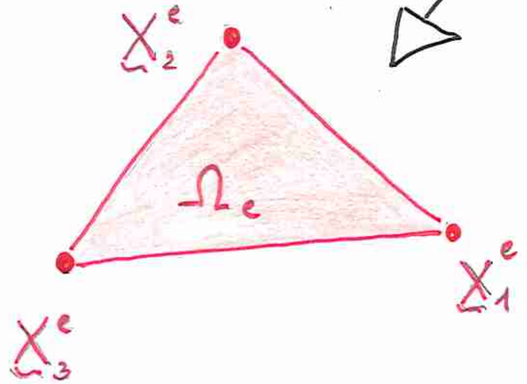


ANSWER

$$\underbrace{\det\left(\frac{\partial \underset{\sim}{x}}{\partial \underset{\sim}{\xi}}(\underset{\sim}{\xi})\right)}_{\triangleq\, J_e(\underset{\sim}{\xi})} \neq 0$$

$$\forall\, \underset{\sim}{\xi} \in \hat{\Omega}$$

JACOBIAN OF ELEMENT Ω_e



$$\underset{\sim}{x}(\underset{\sim}{\xi}) = \sum_{i=1}^{3} \underset{\sim}{X}_i^e\, \phi_i(\underset{\sim}{\xi})$$

$$\frac{\partial \underset{\sim}{x}}{\partial \underset{\sim}{\xi}} = \sum_{i=1}^{3} \underset{\sim}{X}_i^e\, \frac{\partial \phi_i(\underset{\sim}{\xi})}{\partial \underset{\sim}{\xi}}$$

$$\phi_1(\xi,\eta) = 1-\xi-\eta$$
$$\phi_2(\xi,\eta) = \xi$$
$$\phi_3(\xi,\eta) = \eta$$

	$\phi_{i;\xi}$	$\phi_{i;\eta}$
1	-1	-1
2	1	0
3	0	1

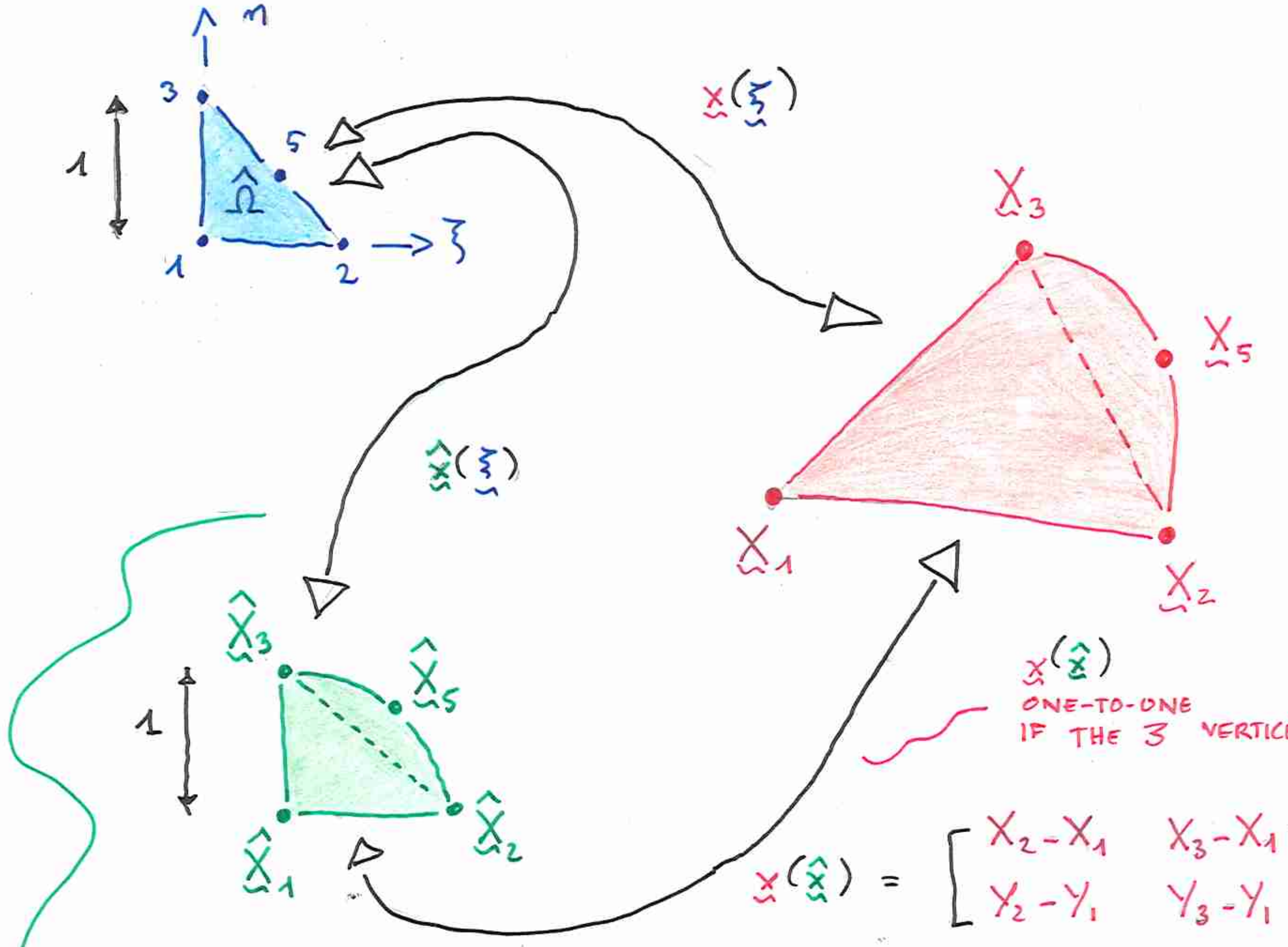
$$\frac{\partial \underset{\sim}{x}}{\partial \underset{\sim}{\xi}} = \sum_{i=1}^{3} \underset{\sim}{X}_i^e \, \frac{\partial \phi_i}{\partial \underset{\sim}{\xi}}$$

$$\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} X_2^e - X_1^e & X_3^e - X_1^e \\ Y_2^e - Y_1^e & Y_3^e - Y_1^e \end{bmatrix}$$

J_e CONSTANT BECAUSE LINEAR MAPPING

$J_e \neq 0$ IF $\underset{\sim}{X}_i^e$ NOT COLLINEAR

$$J_e = \det\left(\frac{\partial \underset{\sim}{x}}{\partial \underset{\sim}{\xi}}\right) = (X_2^e - X_1^e)(Y_3^e - Y_1^e) - (Y_2^e - Y_1^e)(X_3^e - X_1^e)$$



$\boldsymbol{x}(\hat{\boldsymbol{x}})$
ONE-TO-ONE
IF THE 3 VERTICES ARE NOT COLLINEAR

OK

$$\boldsymbol{x}(\hat{\boldsymbol{x}}) = \begin{bmatrix} X_2-X_1 & X_3-X_1 \\ Y_2-Y_1 & Y_3-Y_1 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} + \begin{bmatrix} X_1 \\ Y_1 \end{bmatrix}$$

$$\hat{\boldsymbol{x}}(\boldsymbol{\xi}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} + \begin{bmatrix} 4\hat{X}_5 - 2 \\ 4\hat{Y}_5 - 2 \end{bmatrix} \begin{bmatrix} \xi\eta \\ \xi\eta \end{bmatrix}$$

$\hat{\boldsymbol{x}}(\boldsymbol{\xi})$
ONE-TO-ONE ?

CFR $\hat{\boldsymbol{x}}(1/2, 1/2) = 1/2 - (4\hat{\boldsymbol{X}}_5 - 2)\,1/4 = \hat{\boldsymbol{X}}_5$

$$\frac{\partial \hat{\underline{x}}}{\partial \underline{\xi}} = \begin{bmatrix} 1 + (4\hat{X}_s - 2)\eta & (4\hat{X}_s - 2)\xi \\ (4\hat{Y}_s - 2)\eta & 1 + (4\hat{Y}_s - 2)\xi \end{bmatrix}$$

$$\begin{aligned} \hat{J}_e &= \big(1 + (4\hat{X}_s - 2)\eta\big)\big(1 + (4\hat{Y}_s - 2)\xi\big) - (4\hat{X}_s - 2)(4\hat{Y}_s - 2)\xi\eta \\ &= \underbrace{1 + (4\hat{X}_s - 2)\eta + (4\hat{Y}_s - 2)\xi}_{\text{LINEAR}} \end{aligned}$$

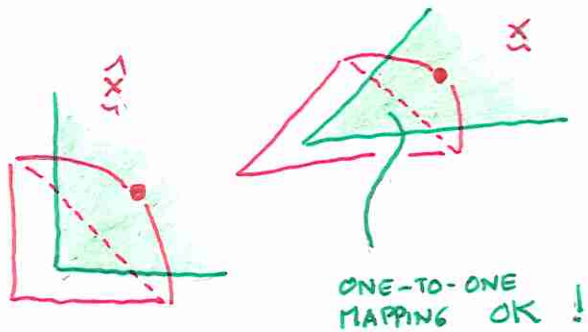
> 0 IF $J(0,0) > 0$

$J(1,0) > 0$

$J(0,1) > 0$

$$4\hat{X}_s - 2 > -1$$
$$4\hat{Y}_s - 2 > -1$$

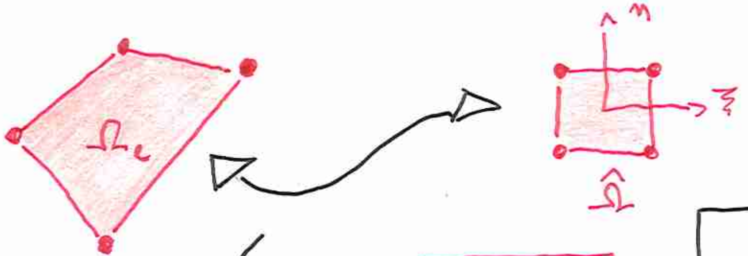
$\longrightarrow$



$$\hat{X}_s > 1/4$$
$$\hat{Y}_s > 1/4$$

COMPUTATION OF THE ELEMENT STIFFNESS MATRIX

$$A^e_{ij} = \int_{\Omega_e} \underset{\sim}{\nabla} \phi^e_i \cdot \underset{\sim}{\nabla} \phi^e_j \, d\Omega$$



$\int_{\Omega} \rightarrow \int_{\hat{\Omega}}$

$\frac{\partial}{\partial \underset{\sim}{x}} \rightarrow \frac{\partial}{\partial \underset{\sim}{\xi}}$

$\int_{\hat{\Omega}} f(\xi) \rightarrow \sum_k w_k f(\xi_k)$

3 DIFFICULTIES TO HANDLE !

MORE DIFFICULT IN THE GENERAL 2D-3D CASE

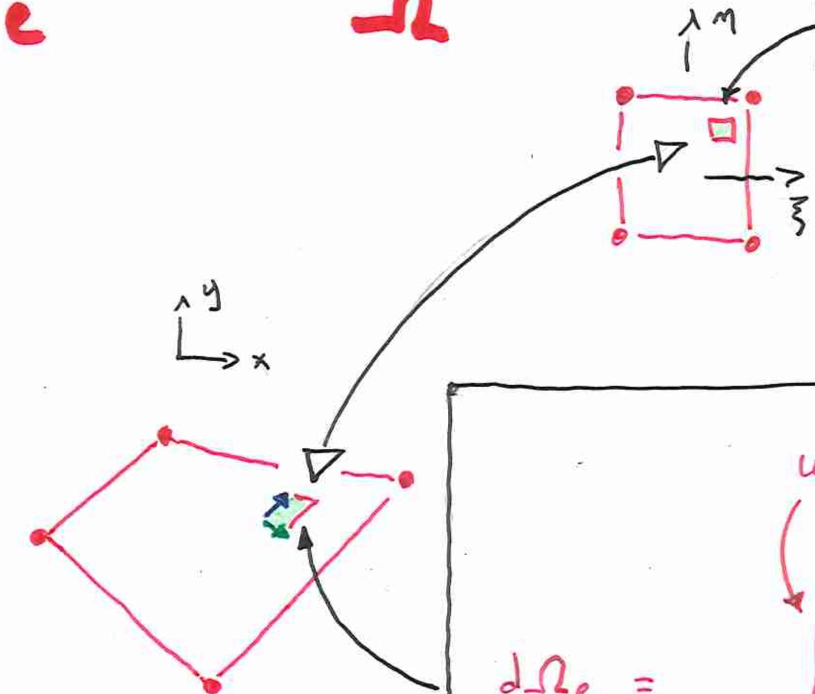
EASY IN 1D

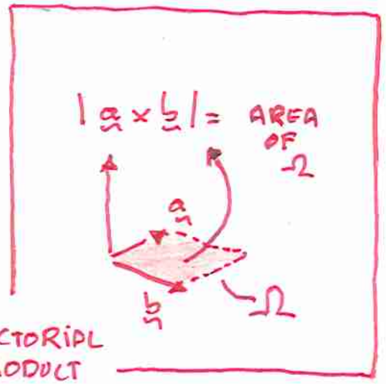
EASY FOR 2D LINEAR MAPPING

$$\underset{\sim}{x}(\underset{\sim}{\xi}) = \sum_{i=1}^{4} \underset{\sim}{X}^e_i \, \phi_i(\underset{\sim}{\xi})$$

$$\int_{\Omega_e} \longrightarrow \int_{\hat{\Omega}}$$

$$d\hat{\Omega} = d\xi \, d\eta$$



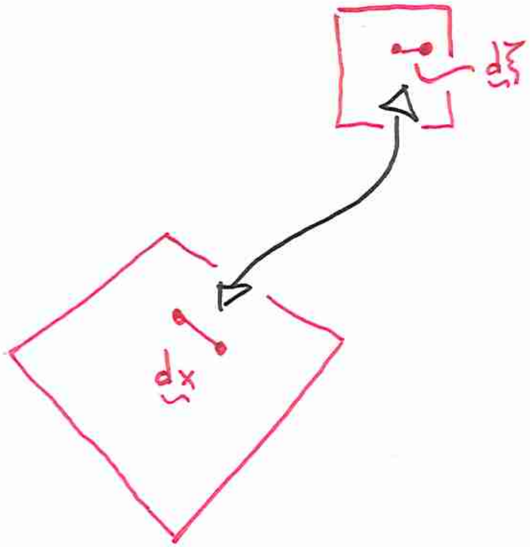


LENGTH

VECTORIAL PRODUCT

$$d\Omega_e = \left| \left(\frac{\partial \underset{\sim}{x}}{\partial \xi} d\xi \times \frac{\partial \underset{\sim}{x}}{\partial \eta} d\eta \right) \right|$$

$$= \underbrace{\left| \frac{\partial \underset{\sim}{x}}{\partial \xi} \times \frac{\partial \underset{\sim}{x}}{\partial \eta} \right|}_{\det\left(\frac{\partial \underset{\sim}{x}}{\partial \underset{\sim}{\xi}}\right)} \underbrace{d\xi \, d\eta}_{d\hat{\Omega}}$$



$$\underline{x}(\underline{\xi}) = \sum_{i=1}^{4} \underline{X}_i^e \, \phi_i(\underline{\xi})$$

$$\underbrace{d\underline{x}(\underline{\xi})}_{\begin{bmatrix} dx \\ dy \end{bmatrix}} = \boxed{\underbrace{\sum_{i=1}^{4} \underline{X}_i^e \, \frac{\partial \phi_i}{\partial \underline{\xi}}(\underline{\xi})}_{\frac{\partial \underline{x}}{\partial \underline{\xi}} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix}}} \cdot \underbrace{d\underline{\xi}}_{\begin{bmatrix} d\xi \\ d\eta \end{bmatrix}}$$

How to calculate J_e ?

$$J^e = \det\left(\frac{\partial \underline{x}}{\partial \underline{\xi}}\right)$$

$$\frac{\partial}{\partial \underline{x}} \to \frac{\partial}{\partial \underline{\xi}}$$

IT IS NOT NECESSARY TO CALCULATE $\underline{\xi}(\underline{x})$!

$\underline{\nabla} \phi_i$

$$\frac{\partial \phi_i}{\partial \underline{x}} = \frac{\partial \phi_i}{\partial \underline{\xi}} \cdot \frac{\partial \underline{\xi}}{\partial \underline{x}}$$

$$\frac{\partial \phi_i}{\partial \underline{\xi}} = \frac{\partial \phi_i}{\partial \underline{x}} \cdot \frac{\partial \underline{x}}{\partial \underline{\xi}}$$

$$= \sum_{i=1}^{4} \underline{x}_i^e \frac{\partial \phi_i}{\partial \underline{\xi}}$$

$\triangleright$  $=$ ( $)^{-1}$

$$\begin{bmatrix} \partial\xi/\partial x & \partial\xi/\partial y \\ \partial\eta/\partial x & \partial\eta/\partial y \end{bmatrix} = \frac{1}{J_e} \begin{bmatrix} \partial y/\partial\eta & -\partial x/\partial\eta \\ -\partial y/\partial\xi & \partial x/\partial\xi \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial \phi_i}{\partial x} & \frac{\partial \phi_i}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial \phi_i}{\partial \xi} & \frac{\partial \phi_i}{\partial \eta} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{bmatrix}$$

$$\frac{\partial \phi_i}{\partial x} = \underbrace{\frac{1}{J_e}\left(\overbrace{\frac{\partial y}{\partial \eta}}^{\sum y_i \frac{\partial \phi_i}{\partial \eta}} \frac{\partial \phi_i}{\partial \xi} - \overbrace{\frac{\partial y}{\partial \xi}}^{\sum y_i \frac{\partial \phi_i}{\partial \xi}} \frac{\partial \phi_i}{\partial \eta} \right)}_{\text{FCT}(\xi,\eta)}$$

$$\boxed{A^e_{ij} = \int_{\hat{\Omega}} \underbrace{\underbrace{\nabla \phi_i \cdot \nabla \phi_j}_{\text{FCT}(\xi,\eta)} \; J_e(\xi,\eta)}_{\text{FCT}(\xi,\eta)} \, d\hat{\Omega}}$$

YOU HAVE JUST TO INTEGRATE NOW !

$$\int_{\hat{\Omega}} f(\underline{\xi}) \longrightarrow \sum_k w_k \, f(\underline{\xi}_k)$$

NUMERICAL INTEGRATION REQUIRED !

ANALYTICAL WAY IS IMPOSSIBLE !

WEIGHTS

$\sum w_k = 1$!

IF $\int_{\hat{\Omega}} = 1$

INTEGRATION POINTS

TO BE SELECTED IN A CLEVER WAY

$$\boxed{\underbrace{\underline{\nabla}\phi_i \cdot \underline{\nabla}\phi_j \; J_e(\underline{\xi})}_{f(\underline{\xi})}}$$

GAUSS LEGENDRE QUADRATURE !

NUMERICAL INTEGRATION OF $p(\xi)$

$$\boxed{\sum_{i=0}^{2p+1} a_i \xi^i}$$

2p+2 COEFFICIENTS

$\boxed{p+1 \quad \text{WEIGHT}}$ $\boxed{p+1 \quad \text{POINTS}}$

$$\int_{-1}^{1} \widehat{p}(\xi)\, d\xi = \sum_{k=0}^{p} w_k \; p(\xi_k)$$

$$\sum_{i=0}^{2p+1} \int_{-1}^{1} a_i \xi^i \, d\xi = \sum_{k=0}^{p} w_k \sum_{i=0}^{2p+1} a_i \xi_k^i$$

$$\underbrace{\sum_{j=0}^{p} \int_{-1}^{1} a_{2j} \xi^{2j}}_{\sum_{j=0}^{p} \frac{2}{(2j+1)} a_{2j}} + \underbrace{\sum_{j=0}^{p} \int_{-1}^{1} a_{2j+1} \xi^{2j+1}}_{=0} = \sum_{k=0}^{p} w_k \sum_{j=0}^{p} a_{2j} \xi_k^{2j} + \sum_{k=0}^{p} w_k \sum_{j=0}^{p} a_{2j+1} \xi_k^{2j+1}$$

$$\boxed{\begin{aligned} \sum_{k=0}^{p} w_k \xi_k^{2j} &= \frac{2j}{(2j+1)} \\ \sum_{k=0}^{p} w_k \xi_k^{2j+1} &= 0 \end{aligned}}$$

2p+2 EQUATIONS

$$\sum_{k=0}^{p} w_k \xi_k^{2j} = \frac{2}{2j+1} \qquad j = 0 \dots p$$

$$\sum_{k=0}^{p} w_k \xi_k^{2j+1} = 0 \qquad j = 0 \dots p$$

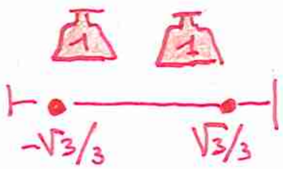
$p = 0$

$$w_0 = 2$$
$$\xi_0 w_0 = 0$$

$2p+1 = 1$!

$p = 1$

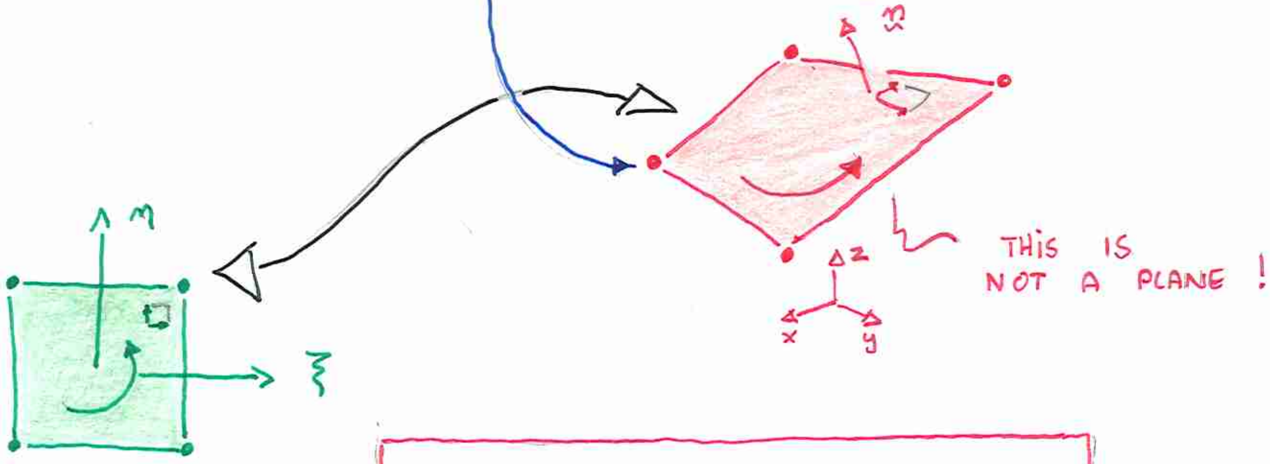
$$w_0 + w_1 = 2$$
$$\xi_0^2 w_0 + \xi_1^2 w_1 = 2/3$$
$$\xi_0 w_0 + \xi_1 w_1 = 0$$
$$\xi_0^3 w_0 + \xi_1^3 w_1 = 0$$



$2p+1 = 3$!

$$\underset{\sim}{X}_i = \begin{bmatrix} X_i \\ Y_i \\ Z_i \end{bmatrix}$$

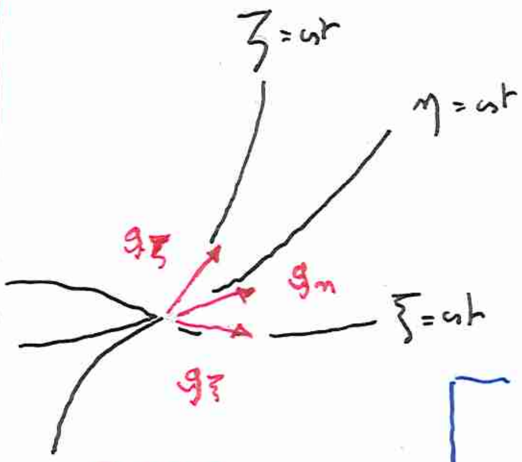
$$\begin{bmatrix} \frac{\partial y}{\partial \xi}\frac{\partial z}{\partial \eta} - \frac{\partial y}{\partial \eta}\frac{\partial z}{\partial \xi} \\ \frac{\partial z}{\partial \xi}\frac{\partial x}{\partial \eta} - \frac{\partial z}{\partial \eta}\frac{\partial x}{\partial \xi} \\ \frac{\partial x}{\partial \xi}\frac{\partial z}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi} \end{bmatrix}$$



3D FACE

$$\boxed{\frac{\partial \underset{\sim}{x}}{\partial \underset{\sim}{\xi}} = \sum_{i=1}^{4} \underset{\sim}{X}_i \frac{\partial \phi_i}{\partial \underset{\sim}{\xi}}}$$

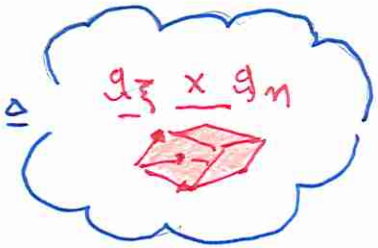
$$\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} \end{bmatrix} \qquad \begin{bmatrix} X_i \\ Y_i \\ Z_i \end{bmatrix} \qquad \begin{bmatrix} \frac{\partial \phi_i}{\partial \xi} & \frac{\partial \phi_i}{\partial \eta} \end{bmatrix}$$

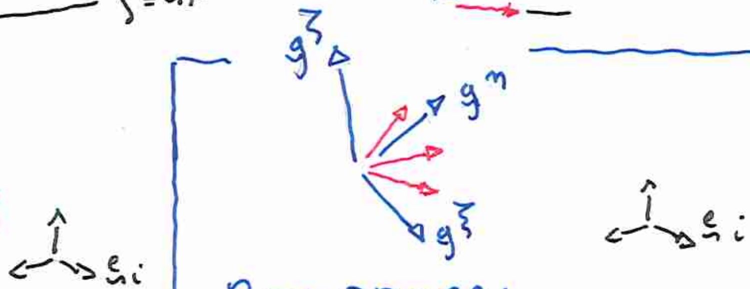


BASE VECTORS

g_i

$\triangleq$





RECIPROCAL BASE VECTORS g^i

$$g^i \cdot g_j = \delta^i_j$$

$$g^\xi = \underbrace{\frac{\partial \xi}{\partial \underline{x}}}$$

$$\underbrace{\frac{\partial \xi}{\partial x}}_{g^\xi_x} e_x + \underbrace{\frac{\partial \xi}{\partial y}}_{g^\xi_y} e_y + \underbrace{\frac{\partial \xi}{\partial z}}_{g^\xi_z} e_z$$

$$d\underline{x}(\underline{\xi}) = \frac{\partial \underline{x}}{\partial \underline{\xi}} \cdot d\underline{\xi}$$

$$\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \xi} & & \\ \frac{\partial y}{\partial \xi} & & \end{bmatrix}$$

$$g_\xi = \frac{\partial \underline{x}}{\partial \xi}$$

$$\underbrace{\det\left(\frac{\partial \underline{x}}{\partial \underline{\xi}}\right) = J}$$

METRICS

$$g_{ij} = g_i \cdot g_j$$

$$g^{ij} = g^i \cdot g^j$$

$$g_{ij}^{-1} = g^{ij}$$

$$\det(g_{ij}) = \det(g^{ij}) = g$$

$$\sqrt{g} = J$$

$$g^i = g^{ij} \cdot g_j$$

CONTRAVARIANT COMPONENTS

COVARIANT COMPONENTS

$$\underset{\sim}{v} = g_i v^i = g^i v_i$$

$$v^i = g^{ij} v_j$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$u \underset{\sim}{e}_x + v \underset{\sim}{e}_y + w \underset{\sim}{e}_z$$

CARTESIAN COMPONENTS

$$\frac{\partial \xi}{\partial \underset{\sim}{x}}$$

$$g^\xi \cdot \frac{\partial}{\partial \xi}$$

$$\underset{\sim}{\nabla} = g^\xi \frac{\partial}{\partial \xi} + g^\eta \frac{\partial}{\partial \eta} + g^\zeta \frac{\partial}{\partial \zeta}$$

3D FACE

$$s_{11} = \left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2 + \left(\frac{\partial z}{\partial \xi}\right)^2$$

$$s_{22} = \left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2 + \left(\frac{\partial z}{\partial \eta}\right)^2$$

$$s_{12} = \left(\frac{\partial x}{\partial \xi}\frac{\partial x}{\partial \eta}\right) + \left(\frac{\partial y}{\partial \xi}\frac{\partial y}{\partial \eta}\right) + \left(\frac{\partial z}{\partial \xi}\frac{\partial z}{\partial \eta}\right)$$

$$J = \sqrt{\underbrace{s_{11}\, s_{22} - s_{12}^2}_{g}}$$

$$\underbrace{\begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}}_{\text{METRIC} \quad g_{ij}} \triangleq s$$

g^{ij}

$$S \triangleq s^{-1} \qquad \text{METRIC}$$

$$= \begin{bmatrix} s_{22} & -s_{12} \\ -s_{12} & s_{11} \end{bmatrix} / J^2$$

$\frac{\partial \xi}{\partial x}$

$g^i = g^{ij} \cdot g_j$

$(g_\xi)_x$

$g^{\xi\xi}$

s_{22}/J^2

$-s_{12}/J^2$

$$g^{\xi}_{x} = \left(\frac{\partial x}{\partial \xi} S_{11} + \frac{\partial x}{\partial \eta} S_{12}\right) = \frac{\partial y}{\partial \eta} / J$$

?

$$\boxed{\frac{\partial \psi_i}{\partial x} = \frac{\partial \psi_i}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \psi_i}{\partial \eta}\frac{\partial \eta}{\partial x}}$$

$$\frac{\partial \xi}{\partial x} = \left(\cancel{\frac{\partial x}{\partial \xi}\left(\frac{\partial x}{\partial \eta}\right)^2} + \frac{\partial x}{\partial \xi}\left(\frac{\partial y}{\partial \eta}\right)^2 - \cancel{\frac{\partial x}{\partial \eta}\left(\frac{\partial x}{\partial \xi}\frac{\partial x}{\partial \eta}\right)} - \frac{\partial x}{\partial \eta}\left(\frac{\partial y}{\partial \xi}\frac{\partial y}{\partial \eta}\right)\right) \Big/ J^2$$

$$= \left(\frac{\partial y}{\partial \eta} \underbrace{\left(\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi} \right)}_{J} \right) \Big/ J^2$$

$$= \frac{\partial y}{\partial \eta} \Big/ J$$

$\square$ 

ABSTRACT GENERIC ELLIPTIC PROBLEM

HEAT CONDUCTION
LINEAR ELASTICITY
SIMPLIFIED MODELS OF LINEAR ELASTICITY
BEAM / SHELLS
ROPE / MENBRANE
STOKES PROBLEM

NOW, WE CONSIDER A VECTORIAL UNKNOWN FIELD !

WEAK FORMULATION

$$? \, \underset{\sim}{u} \in \mathcal{U} \quad \text{SUCH THAT}$$

$$\underbrace{a(\underset{\sim}{u}, \hat{\underset{\sim}{u}})}_{\text{BILINEAR CONTINUOUS SYMMETRIC COERCIVE FORM}} = \underbrace{b(\hat{\underset{\sim}{u}})}_{\text{CONTINUOUS LINEAR FORM}} \quad \forall \hat{\underset{\sim}{u}} \in \hat{\mathcal{U}}$$

ASSUMPTIONS REQUIRED TO HAVE AN ABSTRACT MINIMIZATION PROBLEM

$\Updownarrow$

$$? \, \underset{\sim}{u} \in \mathcal{U} \quad \text{SUCH THAT}$$

$$J(\underset{\sim}{u}) = \min_{\underset{\sim}{v} \in \mathcal{U}} \underbrace{\frac{1}{2} a(\underset{\sim}{v}, \underset{\sim}{v}) - b(\underset{\sim}{v})}_{J(\underset{\sim}{v})}$$

MINIMIZATION PROBLEM

ABSTRACT GENERIC DISCRETE FORMULATION

$\begin{bmatrix} u \\ v \end{bmatrix}$ 2D

$\begin{bmatrix} u \\ v \\ w \end{bmatrix}$ 3D

$$\underbrace{\underset{\sim}{u}(\underset{\sim}{x})}_{\in\, \mathcal{U}} \simeq \underbrace{\underset{\sim}{u}^h(\underset{\sim}{x})}_{\in\, \mathcal{U}^h \subset\, \mathcal{U}} = \sum_{i=1}^{n} \underset{\sim}{U}_i \, \tau_i(\underset{\sim}{x})$$

$\dim(\mathcal{U}^h) = 2n$ — 2D PROBLEMS

$\ldots = 3n$ — 3D PROBLEMS

$$\mathcal{U}^h = \text{SPAN} \left\{ \underbrace{\begin{bmatrix} \tau_1 \\ 0 \end{bmatrix} \begin{bmatrix} \tau_2 \\ 0 \end{bmatrix} \cdots \begin{bmatrix} \tau_n \\ 0 \end{bmatrix}}_{n} \underbrace{\begin{bmatrix} 0 \\ \tau_1 \end{bmatrix} \cdots\cdots \begin{bmatrix} 0 \\ \tau_n \end{bmatrix}}_{n} \right\}$$

? $\underset{\sim}{u}^h$ SUCH THAT

$$a(\underset{\sim}{u}^h, \hat{\underset{\sim}{u}}^h) = b(\hat{\underset{\sim}{u}}^h)$$

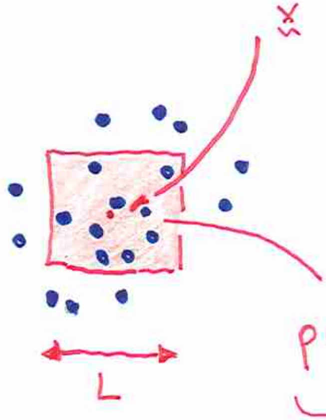
$$\forall\, \hat{\underset{\sim}{u}}^h \in \hat{\mathcal{U}}^h$$

DISCRETE FORMULATION

CONTINUUM MECHANICS

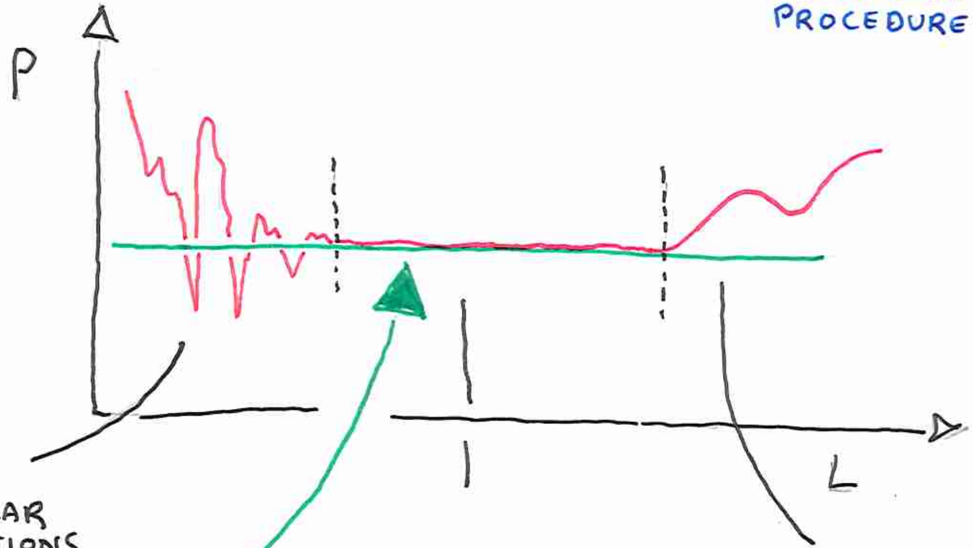
LINEAR ELASTICITY

THE REAL WORLD IS NOT CONTINUOUS !



MASS OF MOLECULES OF THE BOX

$$\underbrace{\rho(L, \underset{\sim}{x}, t)}_{\text{DENSITY}} = \underbrace{\frac{m(L, \underset{\sim}{x}, t)}{L^2}}_{\text{AVERAGING PROCEDURE}}$$



MOLECULAR FLUCTUATIONS

ESSENTIALLY CONSTANT

MACROSCOPIC NON-UNIFORMITY

$$\rho(L, \underset{\sim}{x}, t) \simeq \rho(\underset{\sim}{x}, t)$$

CONTINUUM MECHANICS ASSUMPTION

BEHAVIOUR OF MANY PHYSICAL SYSTEMS IS "ESSENTIALLY" THE SAME AS IF THEY ARE PERFECTLY CONTINUOUS

SIMPLIFIED MODELS

"EXACT" CONTINUUM MODEL

3D

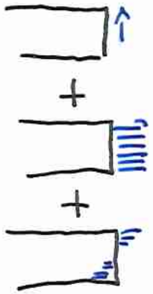
CONTINUOUS DISPLACEMENT

$u \in \mathcal{U}$

$J(u) = \min_{v \in \mathcal{U}} J(v)$

DISCRETE 3D APPROXIMATIONS

$u \simeq u^h \in \mathcal{U}^h \subset \mathcal{U}$



A HIERARCHICAL VIEW

1D / 2D

ROPE / MENBRANE MODELS

$u \simeq u_2 \in \mathcal{U}_1 \not\subset \mathcal{U}_2 \subset \mathcal{U}$

DISCRETE APPROXIMATION

$u_1 \simeq u_1^h$

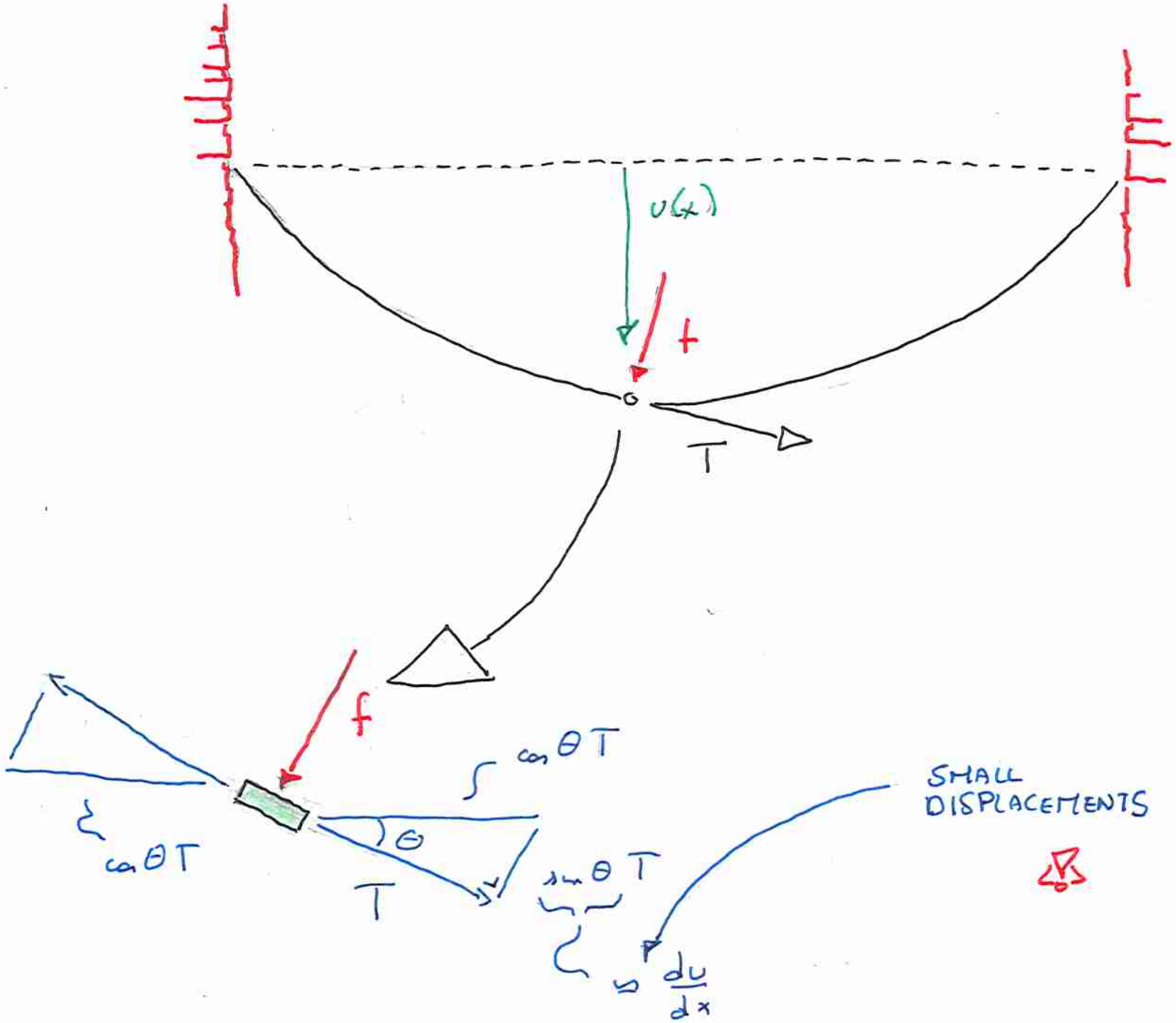
1D / 2D

BEAM / SHELLS MODELS

$u \simeq u_2 \in \mathcal{U}_2 \subset \mathcal{U}$

$u_2 \simeq u_2^h$

ROPE MODEL



VERTICAL FORCE BALANCE

$$\frac{d}{dx}\left(T \frac{du}{dx} \right) = - f$$

$$T \frac{d^2 u}{dx^2} + f = 0$$

THEORY OF BEAMS

FORCE & MOMENTUM BALANCES

$$\frac{dQ}{dx} = -q$$

$$\frac{dM}{dx} = Q$$

CONSTITUTIVE LAW

- BENDING BEAM
- SMALL DEF
- ST VENANT PRINCIPLE
- NAVIER-BERNOULLI ASS.

$$M = EI \underbrace{\frac{1}{R}}_{\text{CURVATURE OF BEAM}}$$

$$1/R = -\frac{d^2u}{dx^2}$$

? u

$$\frac{d^4u}{dx^4} - \frac{q}{EI} = 0$$

+ 4 BOUNDARY CONDITIONS

$$\langle EI \frac{d^4 u}{dx^4} \hat{u} \rangle = \langle q \hat{u} \rangle$$

$$\downarrow$$

$$- \langle EI \frac{d^3 u}{dx^3} \frac{d\hat{u}}{dx} \rangle = \langle q \hat{u} \rangle - \Big[\overbrace{EI \frac{d^3 u}{dx^3}}^{Q} \hat{u} \Big]$$

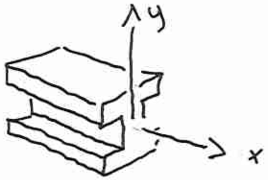
$$\downarrow$$

$$\underbrace{\langle EI \frac{d^2 u}{dx^2} \frac{d^2\hat{u}}{dx^2} \rangle}_{a(u,\hat{u})} = \underbrace{\langle q \hat{u} \rangle - \Big[Q \hat{u} \Big] + \Big[\underbrace{EI \frac{d^2 u}{dx^2}}_{M} \frac{d\hat{u}}{dx} \Big]}_{b(\hat{u})}$$

$$\boxed{\begin{array}{llll} \text{IMPOSING} & u & \text{OR} & Q \\ & \frac{du}{dx} & \text{OR} & M \\ \text{AT BOTH ENDS IS EASY !} & & & \end{array}}$$

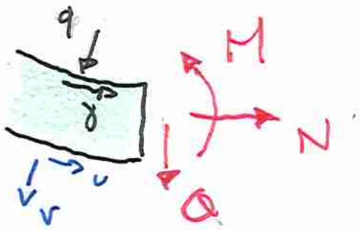
IT IS A MINIMIZATION PROBLEM...

$$J(u) = \underbrace{\frac{1}{2} \left\langle \underbrace{EI \frac{d^2u}{dx^2}}_{\substack{M(u) \\ \text{MOMENTUM}}} \underbrace{\frac{d^2u}{dx^2}}_{\substack{\chi(u) \\ \text{CURVATURE}}} \right\rangle}_{\text{DEFORMATION ENERGY}} \underbrace{- \langle q u \rangle + [Q u] - \left[M \frac{du}{dx}\right]}_{\text{WORK OF EXTERNAL FORCES}}$$



$$\frac{1}{2} \int_0^L \int_S \underbrace{\sigma}_{E y \frac{d^2u}{dx^2}} \underbrace{\varepsilon}_{y \frac{d^2u}{dx^2}} \, dS \, dx$$

TIMOSHENKO BEAM THEORY



CONSERVATION BALANCES

$$\frac{dN}{dx} = -\gamma$$

$$\frac{dQ}{dx} = -q$$

$$\frac{dM}{dx} = Q$$

CONSTITUTIVE LAW

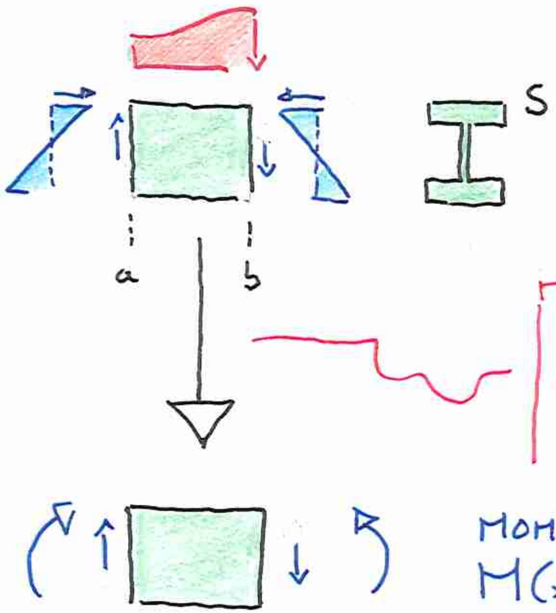
$$M = -EI\,\frac{d^2 v}{dx^2}$$

$$N = ES\,\frac{du}{dx}$$

$$EI\,\frac{d^4 v}{dx^2} = q$$

$$ES\,\frac{d^2 u}{dx^2} = -\gamma$$

BENDING OF A BEAM



MOMENTUM
$M(x)$

$Q(x)$

SHEAR STRESS

THE PATTERN OF INTERNAL STRESS PRODUCES A TWISTING FORCE, A MOMENT ON THE MATERIAL

HORIZONTAL FORCE OBSERVED

$$\overbrace{\int_S (\sigma)\, dS}^{=0} = 0$$

MOMENTUM OBSERVED

$$\underbrace{\int_S (\sigma)\,\eta\, dS}_{M} \overset{\text{HOOKE}}{=} \int_S \frac{E\eta^2}{R}\, dS$$

$$= \frac{E}{R} \underbrace{\int_S \eta^2\, dS}_{I = \frac{bh^3}{12}}$$

$$= EI \underbrace{\frac{1}{R}}_{-\frac{d^2u}{dx^2}}$$

FORCE BALANCE

$$\int_a^b q\,dx - Q_a + Q_b = 0$$

$$\int_a^b q + \frac{dQ}{dx}\,dx = 0$$

$$\boxed{\frac{dQ}{dx} = -q}$$

$$\boxed{\frac{dM}{dx} = Q}$$

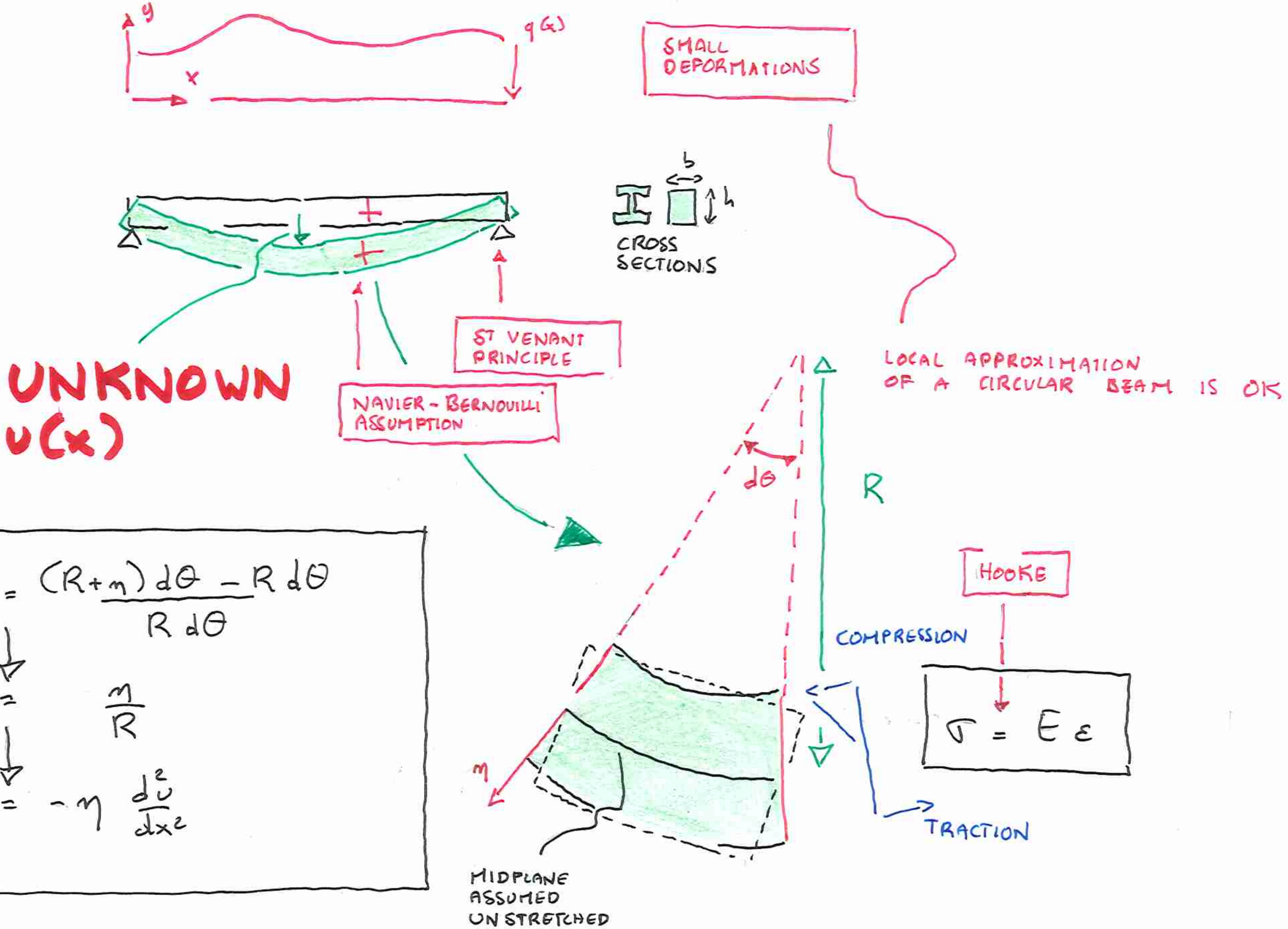
MOMENTUM BALANCE

$$M_b - M_a - \int_a^b (x-a)\,q\,dx - (b-a)\,Q_b = 0$$

$$\int_a^b \left(\frac{dM}{dx} + (x-a)\frac{dQ}{dx}\right)dx - (b-a)\,Q_b = 0$$

$$\int_a^b \left(\frac{dM}{dx} - Q\right)dx - \cancel{(b-a)\,Q_b} = 0$$

$$+ \cancel{\Big[(x-a)\,Q\Big]_a^b}$$



$$\varepsilon = \frac{(R+\eta)\,d\theta - R\,d\theta}{R\,d\theta}$$

$$= \frac{\eta}{R}$$

$$= -\eta \frac{d^2 u}{dx^2}$$

$$\boxed{\sigma = E\varepsilon}$$

3D LINEAR ISOTROPIC ELASTICITY

$$\begin{bmatrix} \times & & \\ \circ & \times & \\ \circ & \circ & \times \end{bmatrix}$$

SHEAR STRESSES

NORMAL STRESSES

HOOKE ELASTIC BODY

SMALL DEF.

CONSTITUTIVE LAW

$$\underline{\underline{\sigma}} = \underbrace{\frac{E}{(1+\nu)}}_{2\mu} \underline{\underline{\varepsilon}} + \underbrace{\frac{E\nu}{(1+\nu)(1-2\nu)}}_{\lambda} \mathrm{tr}(\underline{\underline{\varepsilon}}) \, \underline{\underline{\delta}}$$

DEFORMATION TENSOR

CONSERVATION LAWS

$$\underline{\nabla} \cdot \underline{\underline{\sigma}} + \underline{f} = 0$$

$$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T$$

$$\underline{\underline{\varepsilon}} \triangleq \frac{1}{2}\left(\underline{\nabla}\,\underline{u} + (\underline{\nabla}\,\underline{u})^T\right)$$

? $\underline{u}$

$$\underline{\nabla} \cdot \underline{\underline{\sigma}}(\underline{u}) + \underline{f} = 0 \quad \text{in } \Omega$$

$$\underline{\underline{\sigma}} \cdot \underline{n} = \underline{g} \quad \text{on } \Gamma_N$$

$$\underline{u} = 0 \quad \text{on } \Gamma_D$$

STOKES PROBLEM

CONSERVATION LAWS

$$\underbrace{\cancel{(\underset{\sim}{v} \cdot \underset{\sim}{\nabla})\underset{\sim}{v}}} = \underset{\sim}{\nabla} \cdot \underline{\underline{\sigma}} + \underset{\sim}{f}$$

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{v} = 0$$

CREEPING FLOW
$Re \simeq 0$

INCOMPRESSIBLE FLOW

CONSTITUTIVE LAW

NEWTONIAN FLUID

$$\underline{\underline{\sigma}} = 2\mu \underline{\underline{d}} - p \underline{\underline{\delta}}$$

RATE OF DEFORMATION TENSOR

$$\underline{\underline{d}} \triangleq \frac{1}{2}\left(\underset{\sim}{\nabla}\underset{\sim}{v} + (\underset{\sim}{\nabla}\underset{\sim}{v})^T\right)$$

$?\ (\underset{\sim}{v}, p)$

$$\begin{cases} \underset{\sim}{\nabla} \cdot \left(\underline{\underline{\sigma}}(\underset{\sim}{v}, p)\right) + \underset{\sim}{f} = 0 \\ \underset{\sim}{\nabla} \cdot \underset{\sim}{v} = 0 \end{cases} \quad \text{in } \Omega$$

$$\underline{\underline{\sigma}} \cdot \underset{\sim}{n} = \underset{\sim}{g} \quad \text{on } \Gamma_N$$

$$\underset{\sim}{v} = 0 \quad \text{on } \Gamma_D$$

CALCULUS

$$\nabla \cdot \underset{\sim}{v} = 0 \quad \text{in } \Omega \iff \exists\, \underset{\sim}{\psi} \text{ such that } \underset{\sim}{v} = \nabla \times \underset{\sim}{\psi}$$

STREAM FUNCTION

? $\underset{\sim}{\psi}$

$$\nabla^4 \underset{\sim}{\psi} + \nabla \times \underset{\sim}{f} = 0 \quad \text{in } \Omega$$

+ SUITABLE BOUNDARY CONDITIONS

$$\overset{\text{TIP \#1}}{}$$

$$\langle \underset{\in \hat{\mathcal{U}}}{\hat{\underset{\sim}{u}}} \cdot (\underset{\sim}{\nabla} \cdot \underline{\underline{\sigma}}(\underset{\in \mathcal{U}}{\underset{\sim}{u}})) \rangle = \underbrace{\langle \underset{\sim}{\nabla} \cdot (\hat{\underset{\sim}{u}} \cdot \underline{\underline{\sigma}}(\underset{\sim}{u})) \rangle} - \overbrace{\langle \underset{\sim}{\nabla} \hat{\underset{\sim}{u}} : \underline{\underline{\sigma}}(\underset{\sim}{u}) \rangle}^{\triangleq a(\hat{\underset{\sim}{u}}, \underset{\sim}{u})}$$

TIP #2

$$= \langle\!\langle \underset{\sim}{n} \cdot \hat{\underset{\sim}{u}} \cdot \underline{\underline{\sigma}}(\underset{\sim}{u}) \rangle\!\rangle$$

TIP #3

$$= \langle\!\langle \hat{\underset{\sim}{u}} \cdot \underbrace{(\underset{\sim}{n} \cdot \underline{\underline{\sigma}}(\underset{\sim}{u}))}_{g \text{ on } \Gamma_N} \rangle\!\rangle_N$$

USUAL CALCULUS !

IS $a(\hat{\underset{\sim}{u}}\ \underset{\sim}{u})$ SYMMETRIC ?

$$\langle \underset{\sim}{\nabla}\hat{\underset{\sim}{u}} : \underline{\underline{\sigma}}(\underset{\sim}{u}) \rangle = \langle \Bigg(\overbrace{\underbrace{\frac{\underset{\sim}{\nabla}\hat{\underset{\sim}{u}} + (\underset{\sim}{\nabla}\hat{\underset{\sim}{u}})^T}{2}}_{\text{SYM}}}^{\underline{\underline{\varepsilon}}(\hat{\underset{\sim}{u}})} + \underbrace{\frac{\underset{\sim}{\nabla}\hat{\underset{\sim}{u}} - (\underset{\sim}{\nabla}\hat{\underset{\sim}{u}})^T}{2}}_{\text{ANTI SYM.}} \Bigg) : \underbrace{\underline{\underline{\sigma}}(\underset{\sim}{u})}_{\text{SYM.}} \rangle$$

$$= \langle \underline{\underline{\varepsilon}}(\hat{\underset{\sim}{u}}) : \underbrace{\underline{\underline{\sigma}}(\underset{\sim}{u})}_{\boxed{\underline{\underline{\underline{\underline{C}}}} : \underline{\underline{\varepsilon}}(\underset{\sim}{u})}} \rangle$$

GENERALIZED HOOKE'S LAW

$$= \langle \underline{\underline{\varepsilon}}(\hat{\underset{\sim}{u}}) : \underline{\underline{\underline{\underline{C}}}} : \underline{\underline{\varepsilon}}(\underset{\sim}{u}) \rangle$$

WEAK FORMULATION

? $\underset{\sim}{u} \in \mathcal{U}$ SUCH THAT

$$\underbrace{< \underset{\approx}{\varepsilon}(\hat{\underset{\sim}{v}}) : \underset{\approx}{C} : \underset{\approx}{\varepsilon}(\underset{\sim}{u}) >}_{a(\hat{\underset{\sim}{v}}\ \underset{\sim}{u})} = < \hat{\underset{\sim}{v}} \cdot \underset{\sim}{f} > \underbrace{+ \ll \hat{\underset{\sim}{v}} \cdot \underset{\sim}{g} \gg_N}_{b(\hat{\underset{\sim}{v}})}$$

$\forall \hat{v} \in \hat{\mathcal{U}}$

$\Updownarrow$

FOR SUITABLE $\underset{\approx}{C}$ ONLY !

? $\underset{\sim}{u} \in \mathcal{U}$ SUCH THAT

$$J(\underset{\sim}{u}) = \min_{v \in \mathcal{U}} \underbrace{\frac{1}{2} a(\underset{\sim}{v}\ \underset{\sim}{v}) - b(\underset{\sim}{v})}_{J(\underset{\sim}{v})}$$

MINIMIZATION PROBLEM

$$\underbrace{\frac{1}{2}\, a(\underset{\sim}{v}, \underset{\sim}{v})}_{\text{ENERGY OF DEFORMATION}} = \underbrace{\left\langle \frac{1}{2}\, \lambda\, \underline{\underline{\varepsilon}}(\underset{\sim}{v}) : \underline{\underline{I}} \;\; \overbrace{\underline{\underline{I}} : \underline{\underline{\varepsilon}}(\underset{\sim}{v})}^{tr(\underline{\underline{\varepsilon}}(\underset{\sim}{v}))} + \mu\, \underline{\underline{\varepsilon}}(\underset{\sim}{v}) : \underline{\underline{\varepsilon}}(\underset{\sim}{v}) \right\rangle}_{\text{MUST BE A QUADRATIC POSITIVE FORM IN ORDER TO OBTAIN A MINIMIZATION PROBLEM}}$$

$$\boxed{\begin{array}{rcl} \mu & > & 0 \\ \frac{3}{2}\lambda + \mu & > & 0 \end{array}} \qquad \begin{array}{l} (\varepsilon_{12} \neq 0) \\ (\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} \neq 0) \end{array}$$

ADMISSIBLE VALUES FOR LAMÉ COEFFICIENTS

"DEFORMATION ENERGY"

WHAT IS IT ?

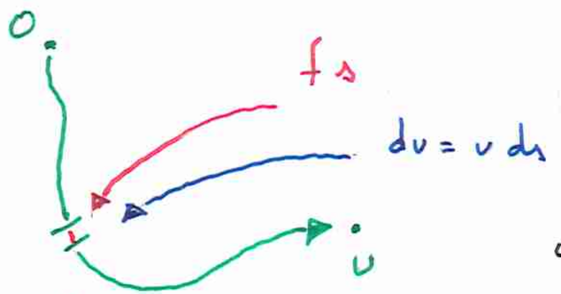
ENERGY OF DEFORMATION

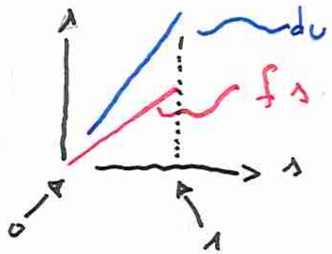
= WORK PERFORMED TO DEFORM THE STRUCTURE AND IS STORED INSIDE.

SUCH AN ENERGY IS RECOVERED WHEN EXTERNAL FORCES ARE REMOVED

f FORCE $\downarrow$ u DISPLACEMENT

ENERGY = $\underbrace{\text{FORCE} \cdot \text{DISPLACEMENT}}_{\text{WORK PERFORMED}}$





WORK PERFORMED $= \int_0^u \lambda f \, du$

$= \int_0^1 f u \lambda \, d\lambda$

$= f u \underbrace{\int_0^1 \lambda \, d\lambda}_{1/2}$

$= \frac{1}{2} f u$!

VERY VERY SLOW LOADING

→ QUASI-STATIC APPROACH

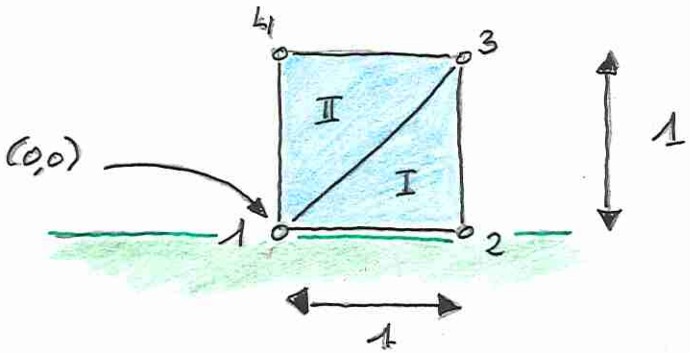
- IRREVERSIBILITY
- DISSIPATION

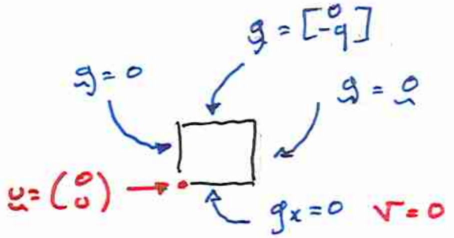
} ARE NEGLECTED !

NUMERICAL EXAMPLE

$$\underline{u} = \begin{bmatrix} u \\ v \end{bmatrix}$$

$$g = \begin{bmatrix} 0 \\ -q \end{bmatrix}$$





5 UNKNOWN DISPLACEMENTS

$$\begin{bmatrix} U_i \\ V_i \end{bmatrix}$$

$$? \; \underline{u}^h = \sum_{i=1}^{n} \underline{U}_i \, \tau_i$$

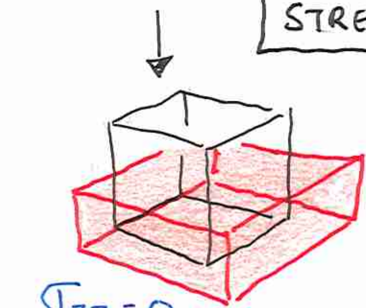
$$\langle \underline{\underline{\varepsilon}}(\hat{\underline{u}}^h) : \underline{\underline{\sigma}}(\underline{u}^h) \rangle = \langle f \cdot \hat{\underline{u}}^h \rangle + \langle\!\langle g \cdot \hat{\underline{u}}^h \rangle\!\rangle_N \qquad \forall \hat{\underline{u}}^h$$

$$\begin{bmatrix} \tau_i \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ \tau_i \end{bmatrix}$$

2n TEST FUNCTIONS

ANALYTICAL SOLUTION

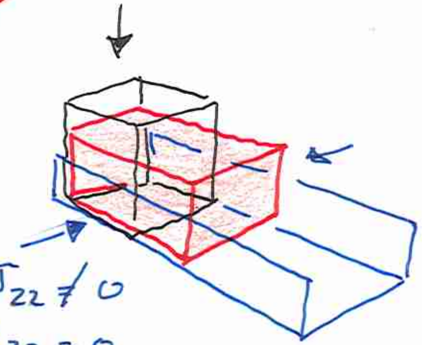
PLANAR STRESSES



$\sigma_{zz} = 0$

$\varepsilon_{zz} \neq 0$

PLANAR DEFORMATIONS



$\sigma_{zz} \neq 0$

$\varepsilon_{zz} = 0$

PLANAR STRESSES

$$\underset{\sim}{u} = \frac{q}{E}\begin{bmatrix} \nu x \\ -y \end{bmatrix}$$

$$\underset{\approx}{\varepsilon} = \frac{q}{E}\begin{bmatrix} \nu & 0 \\ 0 & -1 \end{bmatrix}$$

$$\sigma_{xx} = \frac{E}{(1-\nu^2)}\Big(\underbrace{\varepsilon_{xx}}_{\frac{q\nu}{E}} + \underbrace{\nu\,\varepsilon_{yy}}_{-\frac{q\nu}{E}}\Big) = 0$$

$$\sigma_{yy} = \frac{E}{(1-\nu^2)}\underbrace{\Big(\nu\,\varepsilon_{xx} - \varepsilon_{yy}\Big)}_{\frac{q}{E}(\nu^2-1)} = -q \;!$$

$$\underbrace{\underbrace{\underset{\sim}{\nabla}}\cdot\underbrace{\underset{\approx}{\sigma}}}_{=0\,!} = 0$$

$\begin{bmatrix}\tau_i\\0\end{bmatrix}$ OR $\begin{bmatrix}0\\\tau_i\end{bmatrix}$

$$\langle \underline{\underline{\varepsilon}}(\underline{\hat{u}}^h) : \underline{\underline{\sigma}}(\underline{u}^h) \rangle \qquad \forall \underline{\hat{u}}^h$$

$$\sum_j U_j\, \underline{\underline{\sigma}}\begin{pmatrix}\tau_j\\0\end{pmatrix} + V_j\, \underline{\underline{\sigma}}\begin{pmatrix}0\\\tau_j\end{pmatrix}$$

$$\sum_j \underline{\underline{A}}_{ij} \cdot \underline{U}_j$$

$$\sum_j \begin{bmatrix}U_j\\V_j\end{bmatrix}\tau_j = \sum U_j \begin{bmatrix}\tau_j\\0\end{bmatrix} + V_j \begin{bmatrix}0\\\tau_j\end{bmatrix}$$

$$\sum_j \begin{bmatrix} \langle \underline{\underline{\varepsilon}}\begin{pmatrix}\tau_i\\0\end{pmatrix} : \underline{\underline{\sigma}}\begin{pmatrix}\tau_j\\0\end{pmatrix}\rangle & \langle \underline{\underline{\varepsilon}}\begin{pmatrix}\tau_i\\0\end{pmatrix} : \underline{\underline{\sigma}}\begin{pmatrix}0\\\tau_j\end{pmatrix}\rangle \\ \langle \underline{\underline{\varepsilon}}\begin{pmatrix}0\\\tau_i\end{pmatrix} : \underline{\underline{\sigma}}\begin{pmatrix}\tau_j\\0\end{pmatrix}\rangle & \langle \underline{\underline{\varepsilon}}\begin{pmatrix}0\\\tau_i\end{pmatrix} : \underline{\underline{\sigma}}\begin{pmatrix}0\\\tau_j\end{pmatrix}\rangle \end{bmatrix} \cdot \begin{bmatrix}U_j\\V_j\end{bmatrix}$$

$\forall i$

DISCRETE OPERATOR

$$\underline{\underline{\varepsilon}}\begin{pmatrix}\tau_i\\ 0\end{pmatrix} = \begin{bmatrix}\tau_{i,x} & \tau_{i,y}/2\\ \tau_{i,y}/2 & 0\end{bmatrix}$$

$$\underline{\underline{\varepsilon}}\begin{pmatrix}0\\ \tau_i\end{pmatrix} = \begin{bmatrix}0 & \tau_{i,x}/2\\ \tau_{i,x}/2 & \tau_{i,y}\end{bmatrix}$$

$$\underline{\underline{\sigma}}(\underline{\underline{\varepsilon}}) = \begin{bmatrix}A\,\varepsilon_{xx} + B\,\varepsilon_{yy} & 2C\,\varepsilon_{xy}\\ 2C\,\varepsilon_{xy} & A\,\varepsilon_{yy} + B\,\varepsilon_{xx}\end{bmatrix}$$

HOW TO CALCULATE IT ?

	PLANAR DEFORMATIONS	PLANAR STRESSES
A	$\dfrac{E(1-\nu)}{(1+\nu)(1-2\nu)}$	$\dfrac{E}{(1-\nu^2)}$
B	$\dfrac{E\nu}{(1+\nu)(1-2\nu)}$	$\dfrac{E\nu}{(1-\nu^2)}$
C	$\dfrac{E}{2(1+\nu)}$	$\dfrac{E}{2(1+\nu)}$

$$\underline{\underline{\varepsilon}} : \underline{\underline{\sigma}} = \varepsilon_{xx}\left(A\,\varepsilon_{xx} + B\,\varepsilon_{yy}\right) + 4C\,\varepsilon_{xy}\,\varepsilon_{xy} + \varepsilon_{yy}\left(A\,\varepsilon_{yy} + B\,\varepsilon_{xx}\right)$$

$$\underline{\underline{A}}_{ij} = \begin{bmatrix} \langle \tau_{i,x}\, A\, \tau_{j,x}\rangle + \langle \tau_{i,y}\, C\, \tau_{j,y}\rangle & \langle \tau_{i,x}\, B\, \tau_{j,y}\rangle + \langle \tau_{i,y}\, C\, \tau_{j,x}\rangle \\ \underbrace{\langle \tau_{i,y}\, B\, \tau_{j,x}\rangle + \langle \tau_{i,x}\, C\, \tau_{j,y}\rangle}_{\underline{\underline{\varepsilon}}\left(\begin{smallmatrix}0\\ \tau_i\end{smallmatrix}\right) : \underline{\underline{\sigma}}\left(\underline{\underline{\varepsilon}}\left(\begin{smallmatrix}\tau_j\\ 0\end{smallmatrix}\right)\right)} & \langle \tau_{i,y}\, A\, \tau_{j,y}\rangle + \langle \tau_{i,x}\, C\, \tau_{j,x}\rangle \end{bmatrix}$$

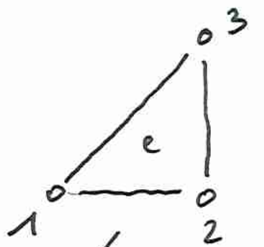
LOCAL ELASTICITY MATRIX

COMPUTING A LOCAL ELASTICITY MATRIX

$$\begin{bmatrix} A\langle \tau_{i,x}\,\tau_{j,x}\rangle + C\langle\tau_{i,y}\,\tau_{j,y}\rangle & B\langle\tau_{i,x}\,\tau_{j,y}\rangle + C\langle\tau_{i,y}\,\tau_{j,x}\rangle \\ B\langle\tau_{i,y}\,\tau_{j,x}\rangle + C\langle\tau_{i,x}\,\tau_{j,y}\rangle & A\langle\tau_{i,y}\,\tau_{j,y}\rangle + C\langle\tau_{i,x}\,\tau_{j,x}\rangle\end{bmatrix}$$

$\underline{\underline{A}}^e_{11}$

$A_{xx11} = A$
$A_{yy11} = C$
$A_{xy11} = 0$
$A_{yx11} = 0$



	$\tau_{i,x}$	$\tau_{i,y}$
1	-1	0
2	1	-1
3	0	1

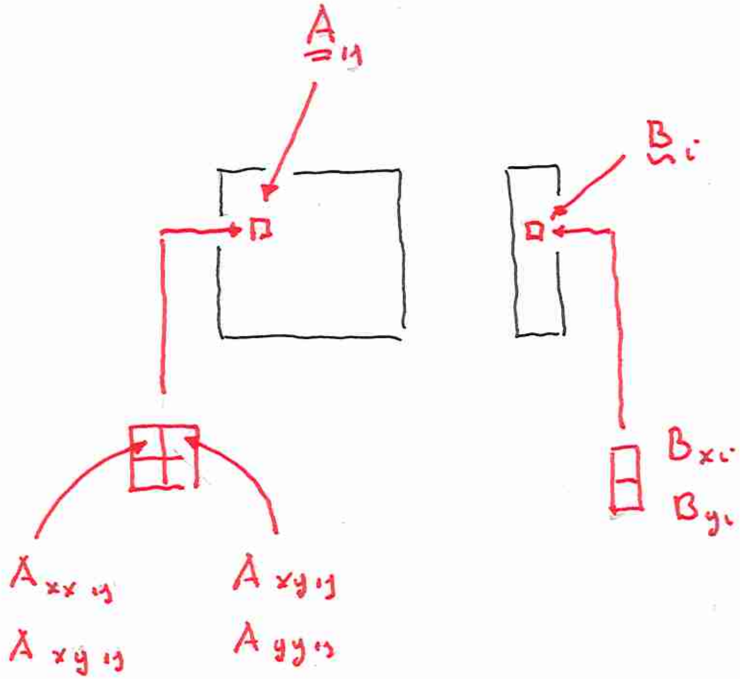
$$\underline{\underline{A}}^e_{ij} = \begin{bmatrix} A & 0 & -A & B & 0 & -B \\ 0 & C & C & -C & -C & 0 \\ & & A+C & -B-C & -C & B \\ & & -B-C & A+C & C & -A \\ & & & & C & 0 \\ & & & & 0 & A \end{bmatrix} \frac{1}{2}$$

$$\underset{\sim}{B}_i = \langle\!\langle g \cdot \hat{\underset{\sim}{u}}^h \rangle\!\rangle_N$$

$$\begin{bmatrix} \tau_i \\ 0 \end{bmatrix} \text{ OR } \begin{bmatrix} 0 \\ \tau_i \end{bmatrix}$$

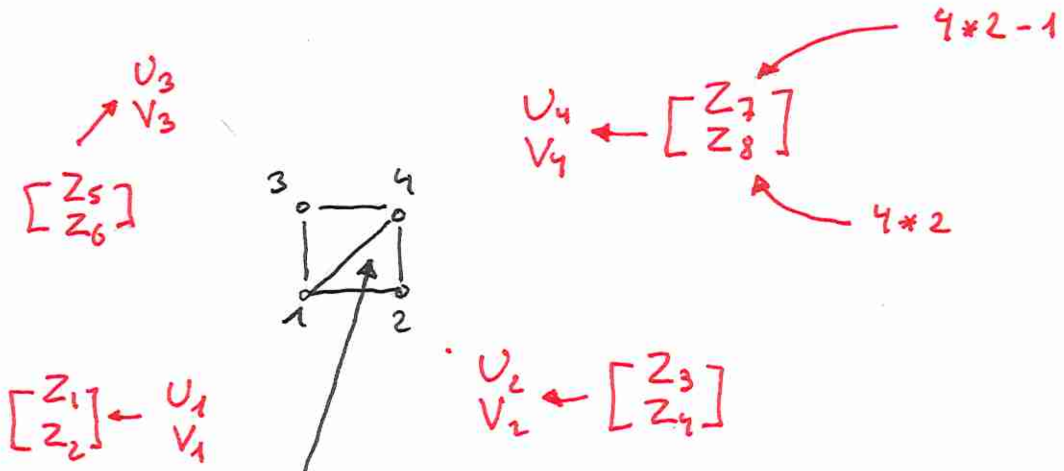
$$\begin{bmatrix} g_x \\ g_y \end{bmatrix}$$

$$\boxed{\begin{bmatrix} \langle\!\langle g_x \; \tau_i \rangle\!\rangle \\ \langle\!\langle g_y \; \tau_i \rangle\!\rangle \end{bmatrix}}$$



AND HYPER VECTOR !

ASSEMBLING PROCEDURE



$$\underline{\underline{A}}^e_{ij} = \begin{bmatrix} & \\ & \end{bmatrix} \longrightarrow \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} Z_1 \\ \vdots \\ Z_8 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

6x6 MATRIX

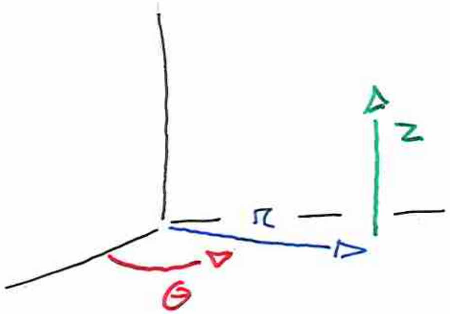
$$\text{GLOBAL HORIZONTAL INDEX} = \underbrace{2 * \text{GLOBAL SCALAR INDEX}}_{\text{GLOBAL}} - 1$$

AXISYMMETRIC PROBLEMS

AXISYMMETRY !

$u_\theta = 0$

$\frac{\partial}{\partial \theta} = 0$



$$\underset{\sim}{u} = \begin{bmatrix} u_r \\ u_\theta \\ u_z \end{bmatrix} = \begin{bmatrix} u(r,z) \\ 0 \\ v(r,z) \end{bmatrix}$$

$$\underset{=}{\varepsilon} = \begin{bmatrix} u_{r,r} & 0 & (u_{r,z} + u_{z,r})/2 \\ & u_r/r & 0 \\ & & u_{z,z} \end{bmatrix}$$

A, B, C
OF PLANAR DEFORMATIONS !

$$\underset{=}{\sigma} = \begin{bmatrix} A\varepsilon_{rr} + B(\varepsilon_{\theta\theta} + \varepsilon_{zz}) & 0 & 2C\varepsilon_{rz} \\ & A\varepsilon_{\theta\theta} + B(\varepsilon_{rr} + \varepsilon_{zz}) & 0 \\ & & A\varepsilon_{zz} + B(\varepsilon_{rr} + \varepsilon_{\theta\theta}) \end{bmatrix}$$

$$\underline{\underline{\varepsilon}}\left(\begin{bmatrix} \tau_i & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} \tau_{i,r} & 0 & \tau_{i,z}/2 \\ & \tau_i/r & 0 \\ & & 0 \end{bmatrix}$$

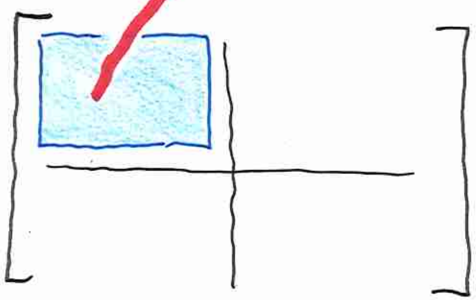
$$\underline{\underline{\varepsilon}}\left(\begin{bmatrix} 0 & 0 & \tau_i \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & \tau_{i,r}/2 \\ & 0 & 0 \\ & & \tau_{i,z} \end{bmatrix}$$

$$2\pi \underbrace{\left[\begin{array}{c|c} \left\langle r\, \underline{\underline{\varepsilon}}\begin{pmatrix}\tau_i\\0\\0\end{pmatrix} : \underline{\underline{\sigma}}\begin{pmatrix}\tau_j\\0\\0\end{pmatrix}\right\rangle & \left\langle r\, \underline{\underline{\varepsilon}}\begin{pmatrix}\tau_i\\0\\0\end{pmatrix} : \underline{\underline{\sigma}}\begin{pmatrix}0\\0\\\tau_j\end{pmatrix}\right\rangle \\ \hline \left\langle r\, \underline{\underline{\varepsilon}}\begin{pmatrix}0\\0\\\tau_i\end{pmatrix} : \underline{\underline{\sigma}}\begin{pmatrix}\tau_j\\0\\0\end{pmatrix}\right\rangle & \left\langle r\, \underline{\underline{\varepsilon}}\begin{pmatrix}0\\0\\\tau_i\end{pmatrix} : \underline{\underline{\sigma}}\begin{pmatrix}0\\0\\\tau_j\end{pmatrix}\right\rangle \end{array}\right]}_{\underline{\underline{A}}_{ij}}$$

$$\underline{\underline{\varepsilon}}\begin{pmatrix}\tau_i\\0\\0\end{pmatrix} = \begin{bmatrix}\tau_{i,r} & 0 & \tau_{i,z}/2\\ & \tau_i/r & 0\\ & & 0\end{bmatrix}$$

$$\underline{\underline{\sigma}}\begin{pmatrix}\tau_j\\0\\0\end{pmatrix} = \begin{bmatrix} A\tau_{j,r}+B\tau_j/r & 0 & C\tau_{j,z}\\ & A\tau_j/r + B\tau_{j,r} & 0 \\ & & B\tau_j/r + B\tau_{j,r}\end{bmatrix}$$

$$\boxed{\underline{\underline{\varepsilon}}\begin{pmatrix}\tau_i\\0\\0\end{pmatrix} : \underline{\underline{\sigma}}\begin{pmatrix}\tau_j\\0\\0\end{pmatrix}} = \left\langle \tau_{i,r}\left(A\tau_{j,r} + B\tau_j/r\right) r \right\rangle + \left\langle \tau_{i,z}\, C\, \tau_{j,z}\, r \right\rangle + \left\langle \frac{\tau_i}{r}\left(A\frac{\tau_j}{r} + B\tau_{j,r}\right) r \right\rangle$$



$\underline{\underline{A}}_{ij}$

IT IS ALMOST LIKE A 2D PROBLEM !

$$
\begin{bmatrix}
\begin{array}{l}
A \langle \tau_{i,r} \, \tau_{j,r} \, r \rangle \\
\quad + C \langle \tau_{i,z} \, \tau_{j,z} \, r \rangle \\
\quad + B \langle \tau_{i,r} \, \tau_j \rangle \\
\quad + \langle \underbrace{\tau_i}_{\varepsilon_{\theta\theta}/r} \underbrace{\left(B \, \tau_{j,r} + A \frac{\tau_j}{r} \right)}_{\sigma_{\theta\theta}} \rangle
\end{array}
&
\begin{array}{l}
B \langle \tau_{i,r} \, \tau_{j,z} \, r \rangle \\
\quad + C \langle \tau_{i,z} \, \tau_{j,r} \, r \rangle \\
\quad + B \langle \tau_i \, \tau_{j,z} \rangle
\end{array}
\\
\begin{array}{l}
B \langle \tau_{i,z} \, \tau_{j,r} \, r \rangle \\
\quad + C \langle \tau_{i,r} \, \tau_{j,z} \, r \rangle \\
\quad + B \langle \tau_{i,z} \, \tau_j \rangle
\end{array}
&
\begin{array}{l}
A \langle \tau_{i,z} \, \tau_{j,z} \, r \rangle \\
\quad + C \langle \tau_{i,r} \, \tau_{j,r} \, r \rangle
\end{array}
\end{bmatrix}
$$

$$J(\underset{\sim}{u}^h) = \text{MINIMUM OF THE POTENTIAL ENERGY}$$

DEFORMATION ENERGY

− WORK OF EXTERNAL FORCES

$\underset{\sim}{u}^h$ MINIMIZES THE ENERGY

DISPLACEMENTS FORMULATION

BUT $\underset{=}{\sigma}(\underset{\sim}{u}^h)$

DISCONTINUOUS ! → SEVERAL VALUES AT EACH VERTEX !

TYPICALLY CONTINUOUS LINEAR OR QUADRATIC

MUST BE VIEWED AS A LEAST-SQUARE FIT OF THE EXACT SOLUTION $\underset{=}{\sigma}(\underset{\sim}{u})$

→ ALLOWS US TO USE THE BEST STRATEGY FOR THE INTERPRETATION OF $\underset{=}{\sigma}(\underset{\sim}{u}^h)$

$\underline{\underline{\sigma}}(\underset{\sim}{u}^h)$ = SOLUTION OF A LEAST-SQUARES FIT PROBLEM

$$a(\hat{\underset{\sim}{u}}, \underset{\sim}{u}) = b(\hat{\underset{\sim}{u}}) \quad \forall \hat{u} \in \hat{\mathcal{U}}$$

$$a(\hat{\underset{\sim}{u}}^h, \underset{\sim}{u}^h) = b(\hat{\underset{\sim}{u}}^h) \quad \forall \underset{\sim}{u}^h \in \hat{\mathcal{U}}^h$$

? $\underset{\sim}{u}^h$

$$a(\hat{\underset{\sim}{u}}^h, \underset{\sim}{u}^h - \underset{\sim}{u}) = 0 \quad \forall \hat{\underset{\sim}{u}}^h \in \hat{\mathcal{U}}^h$$

CFR $\hat{\mathcal{U}}^h \subset \hat{\mathcal{U}}$!

CFR LINEARITY !

$\Updownarrow$

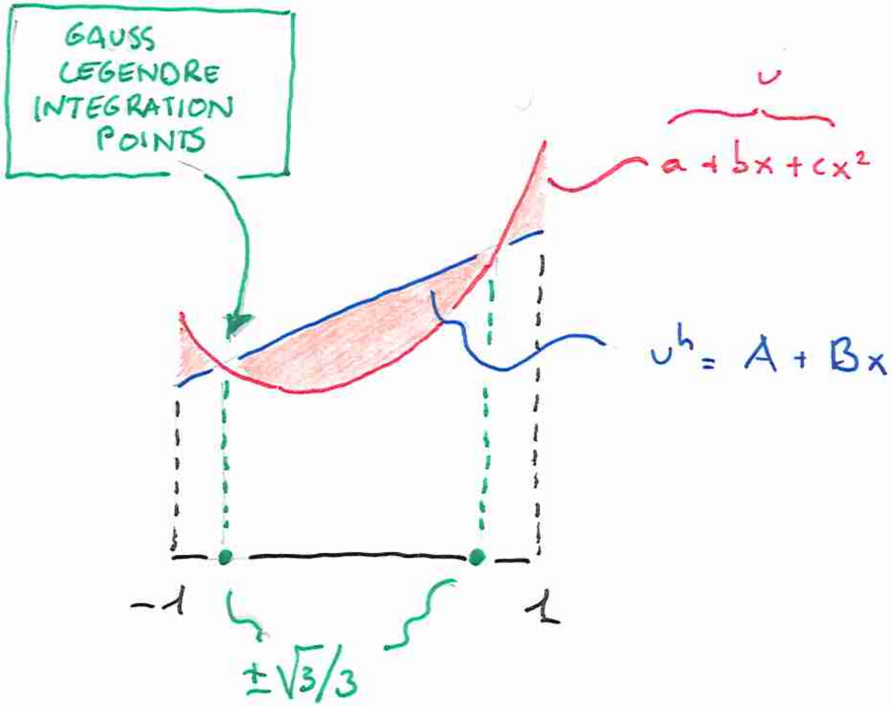
$$\hat{b}(\hat{\underset{\sim}{u}}^h) = a(\hat{\underset{\sim}{u}}^h, \underset{\sim}{u})$$

$$\hat{a}(\hat{\underset{\sim}{u}}^h, \underset{\sim}{u}^h) = a(\hat{\underset{\sim}{u}}^h, \underset{\sim}{u}^h)$$

? $\underset{\sim}{u}^h$

$$G(\underset{\sim}{u}^h) = \min_{\underset{\sim}{v}^h \in \mathcal{U}^h} \underbrace{\frac{1}{2} \left\langle \left(\underset{\approx}{\varepsilon}(\underset{\sim}{v}^h) - \underset{\approx}{\varepsilon}(\underset{\sim}{u})\right) : \underset{\approx}{\underset{\approx}{C}} : \left(\underset{\approx}{\varepsilon}(\underset{\sim}{v}^h) - \underset{\approx}{\varepsilon}(\underset{\sim}{v})\right) \right\rangle}_{G(\underset{\sim}{v}^h)}$$

WHAT ARE THE "BEST VALUES" OF A LEAST-SQUARES POLYNOMIAL FIT ?



$$\frac{c}{3} - cx^2 = 0 \rightarrow x = \pm\frac{\sqrt{3}}{3}$$

$$? \; A, B \text{ such that } G(A,B) = \min_{\tilde{A}\tilde{B}} \int_{-1}^{1} (\tilde{A} + \tilde{B}x - a - bx - cx^2)^2 \, dx$$

$$\rightarrow \begin{cases} B \int_{-1}^{1} (A + Bx - a - bx - cx^2)\, x \, dx = 0 \\ A \int_{-1}^{1} (A + Bx - a - bx - cx^2) \, dx = 0 \end{cases}$$

$$\begin{cases} B^2 \frac{2}{3} - Bb\frac{2}{3} = 0 & \longrightarrow B = b \\ 2A^2 - 2Aa - Ac\frac{2}{3} = 0 & \longrightarrow A = a + \frac{c}{3} \end{cases}$$

LAX-MILGRAM

Si $\mathcal{U}$ ESPACE D'HILBERT

a FORME BILINEAIRE CONTINUE COERCIVE

b FORME LINEAIRE CONTINUE

ALORS

$$\boxed{\begin{array}{l}\text{TROUVER } u \in \mathcal{U} \text{ TEL QUE} \\ a(\hat{v}, u) = b(\hat{v}) \\ \qquad \forall \hat{v} \in \hat{\mathcal{U}}\end{array}}$$

A UNE SOLUTION UNIQUE

QUI DEPEND CONTINUMENT DE b

$$\|u\| \leqslant \frac{1}{\alpha} \underbrace{\sup_{v \in \mathcal{U}} \frac{|b(v)|}{\|v\|}}_{\triangleq \|b\|}$$

ELLIPTIC PROBLEMS

? $u^h \in \mathcal{U}^h$ SUCH THAT

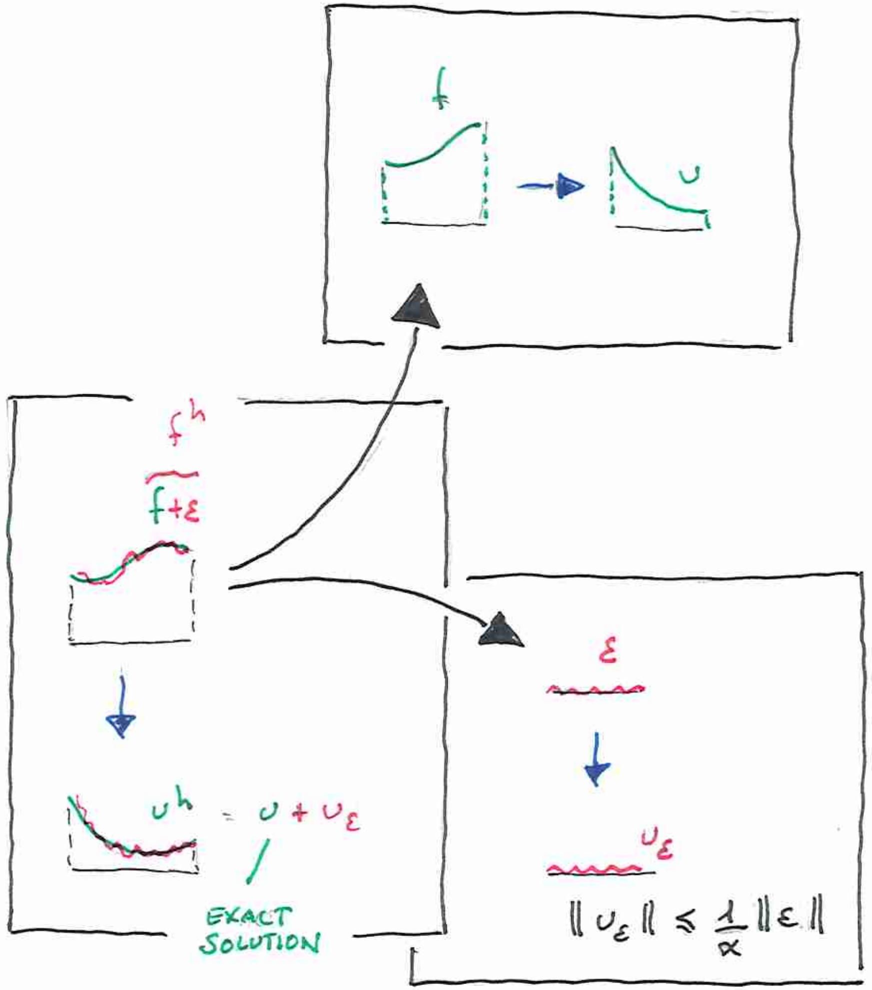
$$a(u^h, \hat{u}^h) = f(\hat{u}^h) \quad \forall \hat{u}^h \in \hat{\mathcal{U}}^h$$

COERCIVE
CONTINUE
BILINEAR

LAX
MILGRAM

$$\boxed{\|u^h\| \leq \frac{1}{\alpha} \|f^h\|}$$

HAVE A NATURAL
STABILITY
INEQUALITY



SOLUTION
DEPENDS CONTINUOUSLY
ON DATA !

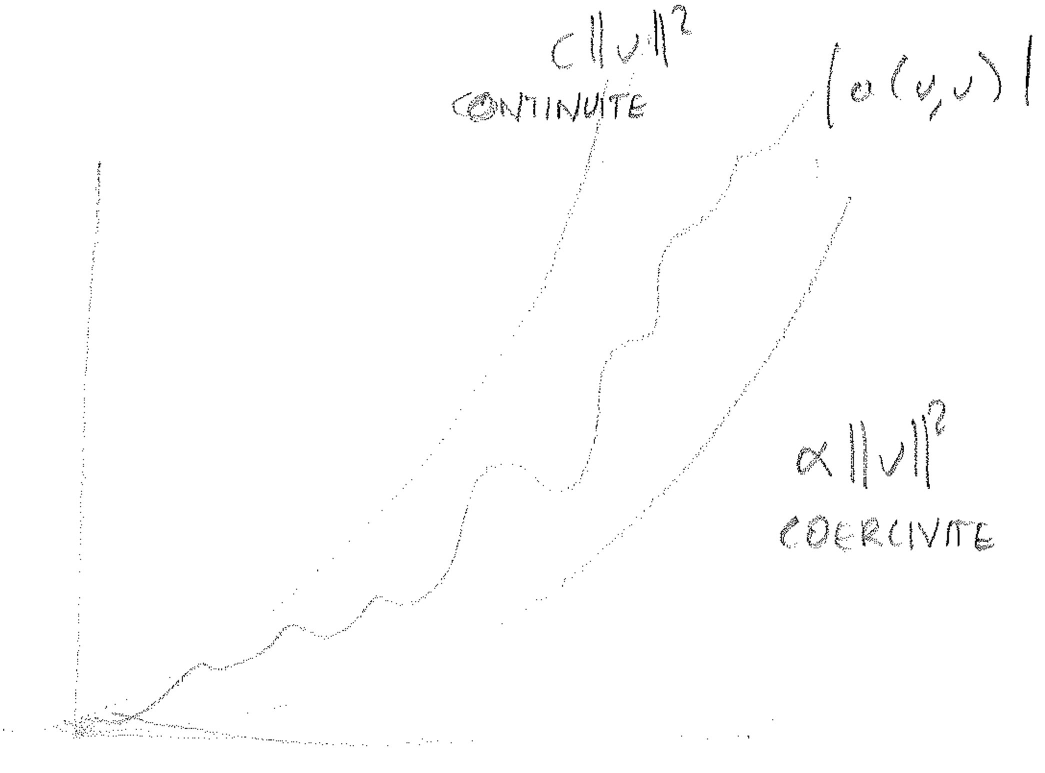
Si a linéaire

CONTINUE

$\exists\, c > 0$ tel que $|a(u,v)| \leq c\|u\|\|v\|$

COERCIVE

$\exists\, \alpha > 0$ tel que $|a(u,u)| \geq \alpha\|u\|^2$



CAUCHY INEQUALITY

$$0 \leq \langle \underbrace{u + \lambda v}_{\in \mathcal{U}},\; u + \lambda v \rangle \qquad \in \mathbb{R}$$

$$\leq \underbrace{\langle u, u \rangle + 2\lambda \langle u, v \rangle + \lambda^2 \langle v, v \rangle}$$

SECOND ORDER POLYNOMIAL $c + 2b\lambda + \lambda^2 a$

POSITIVE FOR ALL λ $\quad b^2 - ac < 0$

$$\langle u, v \rangle^2 \leq \langle u, u \rangle \langle v, v \rangle$$

$$\boxed{\langle u, v \rangle \leq \underbrace{\sqrt{\langle u, u \rangle}}_{\|u\|} \; \underbrace{\sqrt{\langle v, v \rangle}}_{\|v\|}}$$

$\square$

$$? \, u \in \overbrace{\{ w \in H^1(\Omega) \quad w = 0 \text{ on } \partial\Omega \}}^{\triangleq\, \mathcal{U}}$$

$$\underbrace{\langle \underline{\nabla} \hat{u} \cdot k \, \underline{\nabla} u \rangle}_{\triangleq\, a(\hat{u}, u)} = \underbrace{\langle f \hat{u} \rangle}_{\triangleq\, b(\hat{u})} \qquad \forall \hat{u} \in \underbrace{\hat{\mathcal{U}}}_{=\, \mathcal{U}}$$

CONTINUITÉ ?

COERCIVITÉ ?

$$a(\hat{u}, u) \leq \underbrace{\| \underline{\nabla} \hat{u} \|_0}_{\leq\, \| \hat{u} \|_1} \underbrace{\| \underline{\nabla} u \|_0}_{\leq\, \| u \|_1}$$

$$\leq \| \hat{u} \|_1 \, \| u \|_1 \qquad \square$$

$\Gamma_D \neq \emptyset$

$$v(x) - \underbrace{v(0)}_{=0} = \int_0^x \frac{dv}{dx}(t)\, dt$$

$(ab)^2 \leqslant a^2 b^2$

CAUCHY INEQUALITY

$(\langle f\ 1 \rangle)^2 \leqslant \|1\|^2 \|f\|^2$

$$\Downarrow$$

$$(v(x))^2 \leqslant \underbrace{\int_0^x dt}_{C_0} \; \underbrace{\int_0^x \left(\frac{dv}{dx}\right)^2 dt}_{\leqslant \left\|\frac{dv}{dx}\right\|_0^2}$$

EN INTEGRANT SUR Ω

$$\Downarrow$$

$$\|v\|_0^2 \leqslant C_1 \left\|\frac{dv}{dx}\right\|_0^2$$

$$\Downarrow$$

$$C_2 \left(\|v\|_0^2 + \left\|\frac{dv}{dx}\right\|_0^2 \right) \leqslant \underbrace{k \left\|\frac{dv}{dx}\right\|_0^2}_{a(v,v)}$$

$\square$

$$\overbrace{u - u^h}^{e} \leqslant \overbrace{u - \tilde{u}^h}^{\tilde{e}} \leqslant C\, h^{p+1}$$

THEOREME DE LA MEILLEURE APPROXIMATION

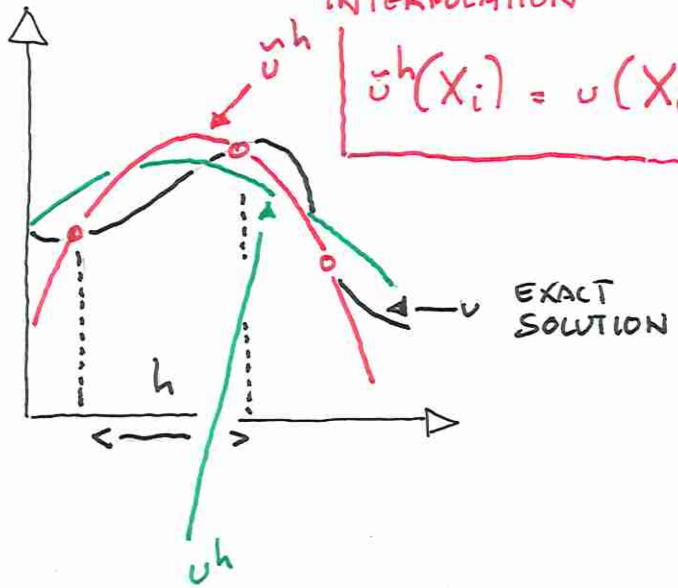
LEMME DE CEA

THEORIE DE L'INTERPOLATION

LEMME DE BRAMBLE HILBERT

POLYNOMIAL INTERPOLATION

$$\tilde{u}^h(X_i) = u(X_i)$$

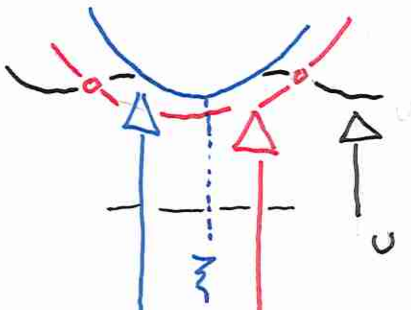


APPROXIMATION

$$a(\hat{u}^h, u^h) = b(\hat{u}^h)$$
$$\forall \hat{u} \in \hat{\mathcal{U}}^h$$

BRAMBLE - HILBERT LEMMA

$$\frac{d^i}{d\xi^i}\overbrace{\left(u - \tilde{u}^h\right)}^{\tilde{e}^h} = \frac{d^i u}{d\xi^i} - \frac{d^i u^t}{d\xi^i} + \underbrace{\frac{d^i u^t}{d\xi^i}}_{= \frac{d^i \tilde{u}^t}{d\xi^i} = \sum_{j=0}^{p} u^t(\Xi_j)\frac{d^i\phi_j}{d\xi^i}} - \underbrace{\frac{d^i \tilde{u}^h}{d\xi^i}}_{= \sum_{j=0}^{p} u(\Xi_j)\frac{d^i\phi_j}{d\xi^i}}$$



$\tilde{u}^h$

POLYNOMIAL INTERPOLATION OF ORDER p

u^t

TAYLOR DEVELOPMENT OF ORDER p AROUND ξ

$$\left|\frac{d^i \tilde{e}}{d\xi^i}\right| \leqslant \underbrace{\left|\frac{d^i u}{d\xi^i} - \frac{d^i u^t}{d\xi^i}\right|}_{\leqslant C \max\left|\frac{d^{p+1}u}{d\xi^{p+1}}\right|} + \sum_{j=0}^{p} \underbrace{\left|\left(u^t(\Xi_j) - u(\Xi_j)\right)\right|}_{\leqslant C \max\left|\frac{d^{p+1}u}{d\xi^{p+1}}\right|} \overbrace{\left|\frac{d^i\phi}{d\xi^i}\right|}^{\leqslant C}$$

$$\frac{d^{p+1}}{d\xi^{p+1}}(u - u^h) = \underbrace{\frac{d^{p+1}u}{d\xi^{p+1}}}$$

$$\left|\ \right| \leq C^{max} \left|\frac{d^{p+1}u}{d\xi^{p+1}}\right|$$

LEMME

ok pour $i+1 \Rightarrow i$

$$\frac{d}{d\xi^i}(u-u^h) = \int_{-1}^{\xi} \underbrace{\frac{d}{d\xi}\frac{d}{d\xi^i}(u-u^h)}\, d\xi$$

$$\left|\ \right| \leq \int_{-1}^{\xi} \underbrace{\left|\frac{d}{d\xi^{i+1}}(u-u^h)\right|}\, d\xi$$

$$\leq C^{max}\left|\frac{d^{p+1}u}{d\xi^{p+1}}\right|$$

$$\leq \tilde{C}^{max}\left|\frac{d^{p+1}u}{d\xi^{p+1}}\right|$$

$\square$

$$\|\tilde{e}\|_m^2 = \sum_{e=1}^{n} \sum_{i=0}^{m} \underbrace{\int_{\Omega_e} \left(\frac{d^i \tilde{e}}{dx^i} \right)^2 dx}_{= \left(\frac{h}{2}\right)^{1-2i} \int_{-1}^{1} \left(\frac{d^i \tilde{e}}{d\xi^i} \right)^2 d\xi}$$

$$\underbrace{\int_{-1}^{1} \left(\frac{d^i \tilde{e}}{d\xi^i} \right)^2 d\xi}_{\leq C^2 \int_{-1}^{1} \max \left| \frac{d^{p+1} u}{d\xi^{p+1}} \right|^2 d\xi}$$

$$\leq C^2 \left(\frac{2}{h}\right)^{1-2(p+1)} \underbrace{\int_{\Omega_e} \max \left| \frac{d^{p+1} u}{dx^{p+1}} \right|^2 dx}_{\leq C \|u\|_{p+1}^2}$$

$$\leq C^2 \sum_{e=1}^{n} \sum_{i=0}^{m} \left(\frac{h}{2}\right)^{2(p+1-i)} \|u\|_{p+1}^2$$

ASYMPTOTIC INTERPOLATION ERROR

$h \to 0$

$$\boxed{\|\tilde{e}\|_m \leq C\, h^{(p+1-m)} \|u\|_{p+1}}$$

$p=1 \longrightarrow$

$$\|\tilde{e}\|_0 \leq C_0\, h^2 \|u\|_2$$
$$\|\tilde{e}\|_1 \leq C_1\, h \|u\|_2$$
$$\|\tilde{e}\|_* \leq C_1\, h \|u\|_2$$

BEST ENERGY APPROXIMATION THEOREM

u EXACT SOLUTION

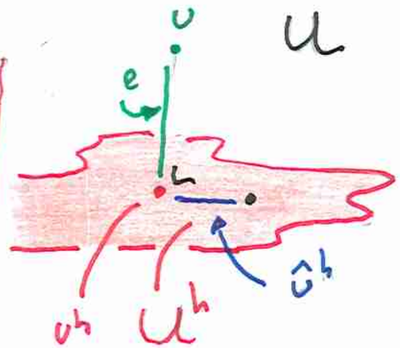
$$a(u, \hat{u}) = b(\hat{u}) \qquad \forall \hat{u} \in \hat{\mathcal{U}}$$

$$a(u, \hat{u}^h) = b(\hat{u}^h) \qquad \forall \hat{u}^h \in \hat{\mathcal{U}}^h \subset \hat{\mathcal{U}}$$

$$a(u^h, \hat{u}^h) = b(\hat{u}^h) \qquad \forall \hat{u}^h \in \hat{\mathcal{U}}^h$$

u^h DISCRETE SOLUTION

$$a(\underbrace{u - u^h}_{e}, \hat{u}^h) = 0 \qquad \forall \hat{u}^h \in \hat{\mathcal{U}}^h$$

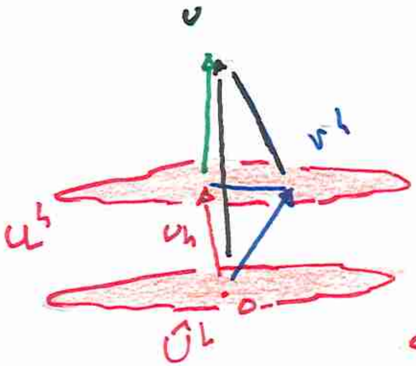


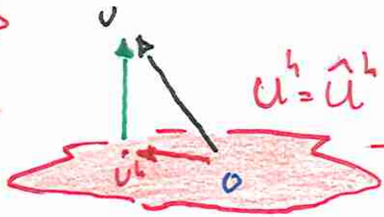
$$a(u-\underbrace{v^h}_{\in U^h}, u-v^h) = a(\underbrace{u-u^h}_{e}+u^h-v^h, \overbrace{u-u^h}^{e}+u^h-v^h)$$

$$= a(e,e) + \underbrace{a(u^h-v^h, u^h-v^h)}_{\geq 0} + \overbrace{2a(e, \underbrace{u^h-v^h}_{\in \hat{U}^h})}^{=0}$$

BEST ENERGY APPROXIMATION THEOREM

$$\|u-v^h\|_* \geq \|e\|_* \quad \forall v^h \in U^h$$





$$\|u-v^h\|_*^2 = \|e\|_*^2 + \|u^h-v^h\|_*^2$$

$$\|u\|_*^2 = \|e\|_*^2 + \|u^h\|_*^2 \quad \text{IF } \hat{U}^h = U^h$$

CEA'S LEMMA

CONTINUITY

$\exists c \quad |a(v,v)| \leqslant c\|v\|^2$

$$\|e\|^2 \leqslant \frac{1}{\alpha}\|e\|_*^2 \leqslant \frac{1}{\alpha}\|\tilde{e}\|_*^2 \leqslant \frac{c}{\alpha}\|\tilde{e}\|^2$$

COERCIVITY

$\exists \alpha \quad \alpha\|v\|^2 \leqslant a(v,v)$

BEST ENERGY APPROXIMATION

$$\boxed{\|e\|^2 \leqslant \frac{c}{\alpha}\|\tilde{e}\|^2}$$

POLYNOMIAL PIECE WISE APPROXIMATIONS

BRAMBLE - HILBERT LEMMA

COERCIVITY CONTINUITY OF a

BEST ENERGY APPROXIMATION THEOREM

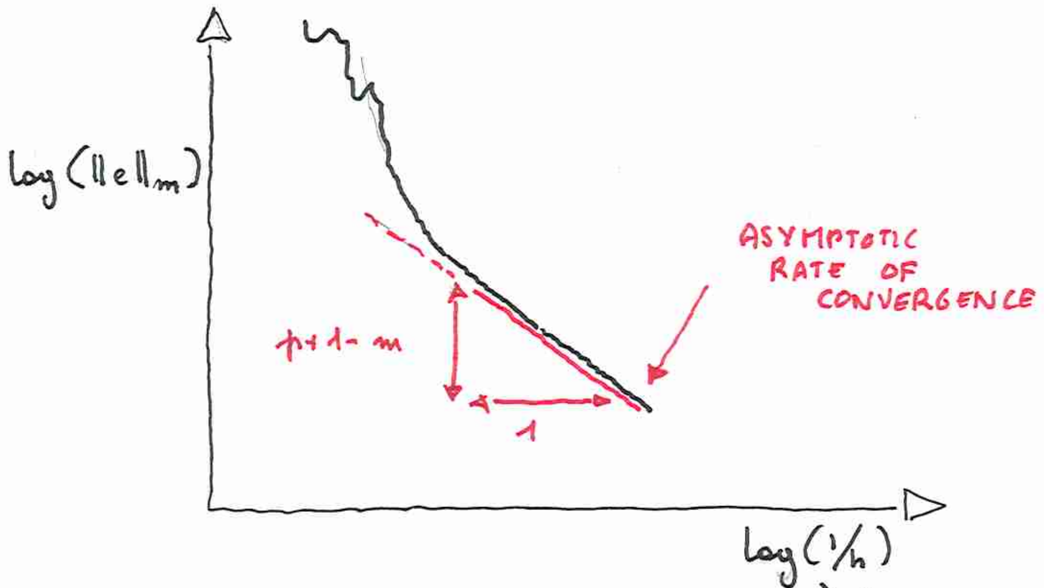
CEA LEMMA

A PRIORI ERROR ESTIMATE

$$\|e\|_m^2 \leq \frac{c}{\alpha} C^2 h^{2(p+1-m)} \|u\|_{p+1}^2$$

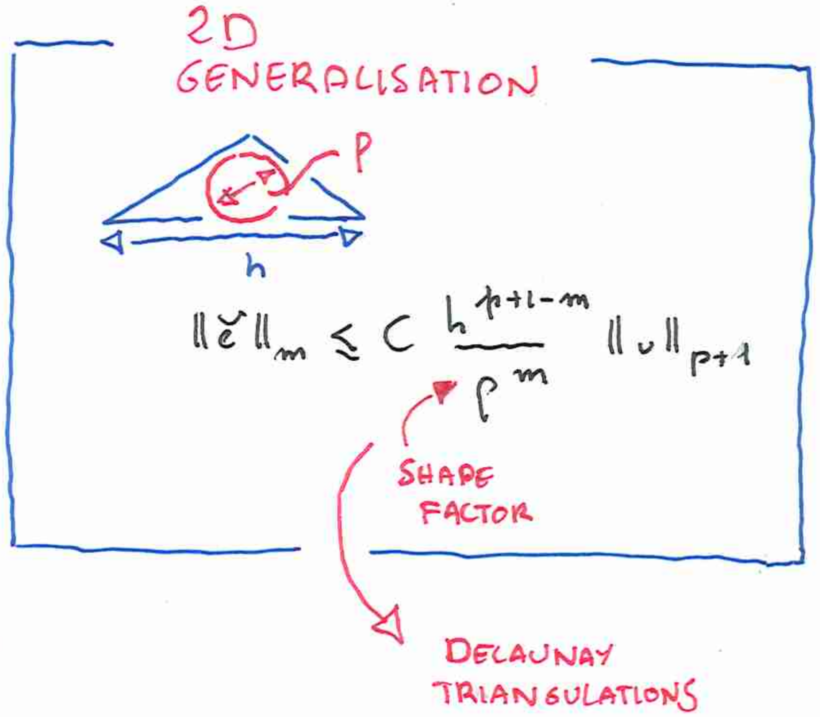
- $u \notin H^{p+1}$

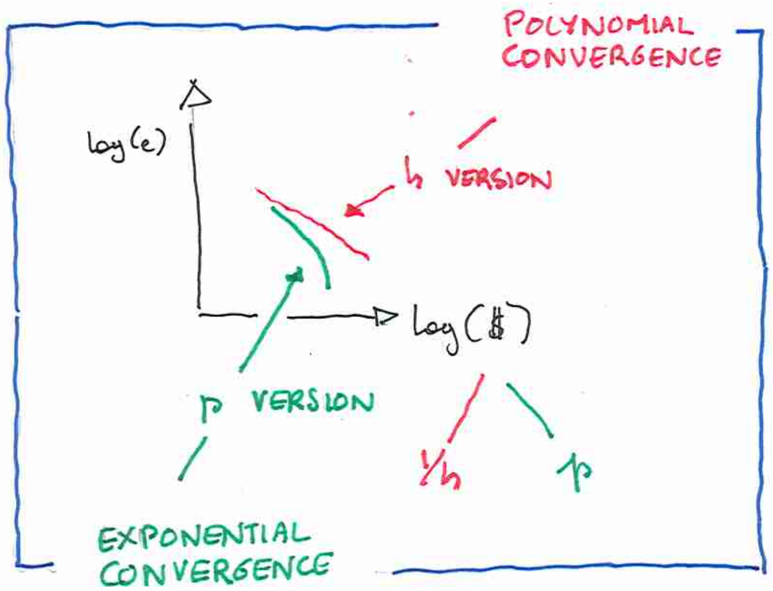
$$\|e\|_m^2 \leq \frac{c}{\alpha} C^2 h^{2(\min(p+1,\, r)-m)} \|u\|_r^2$$



$\propto$ NUMBER OF DEGREES OF FREEDOM

CPU
MEMORY
$



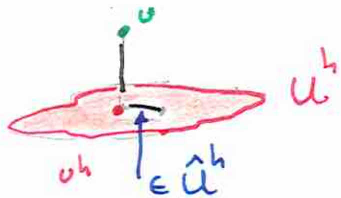


$$a(u, \hat{u}^h) = b(\hat{u}^h) \qquad \forall \hat{u}^h \in \hat{U}^h$$
$$a(u^h, \hat{u}^h) = b(\hat{u}^h) \qquad \forall \hat{u}^h \in \hat{U}^h$$

$$a(\underbrace{u - u^h}_{e}, \hat{u}^h) = 0$$

ORTHOGONALITY OF e WITH $\hat{u}^h$

BEST APPROXIMATION THEORY



BUT

$$a(e, \hat{u}) \neq 0$$
$$= \underbrace{a(u, \hat{u})}_{b(\hat{u})} - a(u^h, \hat{u}) \qquad \forall \hat{u} \in \hat{U}$$

$$\boxed{\underbrace{a(e,e)}_{\|e\|_*^2} \neq 0}$$

HOW TO MEASURE THE ERROR ?

ENERGY NORM

$$\underbrace{\| \overbrace{u - u^h}^{e} \|_*^2}_{\text{GLOBAL ERROR NORM } \Theta^2} = a(e, e)$$

$$= \sum_{e=1}^{N} \underbrace{\int_{\Omega_e} (\nabla e) \cdot (k \nabla e) \, d\Omega}_{\| e \|^2_{*, \Omega_e}}$$

$\| e \|^2_{*, \Omega_e}$ — LOCAL ERROR NORM Θ_e^2

THIS IS IN FACT A SEMI-NORM, ONLY

ASSUME

$a(u,v) = \langle \nabla u \cdot k \nabla v \rangle$

... WITHOUT ANY LOSS OF GENERALITY

HOW TO ESTIMATE THE ERROR ?

SHAPE FUNCTIONS OF u^h

$$e \simeq e^{h+} = \sum_{i=1}^{n+} E_i^+ \tau_i^+$$

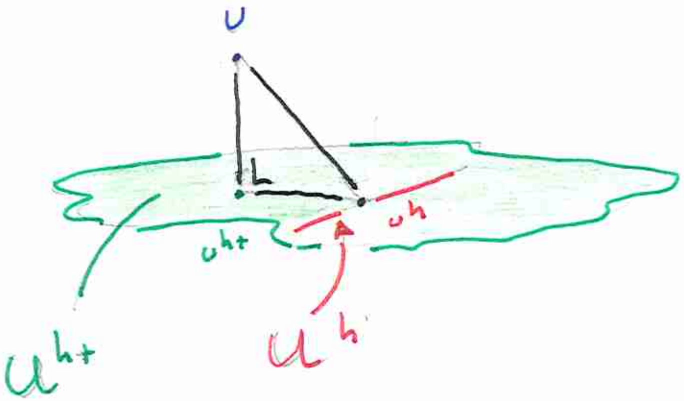
$$\in \mathcal{U}^{h+}$$

$$\left[\tau_1 \dots \tau_n, \tau_{n+1} \dots \tau_{n+}\right]$$

ADDITIONAL SHAPE FUNCTIONS

TYPICALLY, BUBBLE FUNCTIONS 

$$\underbrace{\|u^{h+} - u^h\|_*^2}_{(\Theta^h)^2} + \underbrace{\|u - u^{h+}\|_*^2}_{\to 0 \text{ IF } h^+ \to 0} = \underbrace{\|u - u^h\|_*^2}_{\Theta^2}$$



$$\Downarrow$$

$$(\Theta^h)^2 \longrightarrow \Theta^2$$

IF $h^+ \to 0$

RESIDUAL ERROR'S

? e^{h+} SUCH THAT

$$a(e^{h+}, \tau_i^+) = b(\tau_i^+) - a(u^h, \tau_i^+)$$

$$\forall \tau_i^+$$

MESH
REFINEMENT
ANALYSIS

VERY
EXPENSIVE

PROBLEM

BUT 

$$\tau_i^+ = [\tau_1 \dots \tau_m \ \tau^+]$$

DEFINE ONLY
1 BUBBLE FUNCTION PER ELEMENT

AND ASSUME THAT $\underbrace{E_1^+ = E_2^+ \dots = E_m^+}_{?\ \text{SMALL COMPARED TO } E^+} = 0$

$u^h \not\subset u^{h+}$

→ THIS IS
WRONG !

BUT IT IS AN EASY
AND RELATIVELY EFFICIENT
APPROXIMATION

ELEMENT-WISE LOCAL PROBLEM

? E_e^+ SUCH THAT

$$E_e^+ a(\phi_e^+, \phi_e^+) = b(\phi_e^+) - a(u^h, \phi_e^+)$$

$$\phi_e^+ = \frac{(1-\xi)(1+\xi)}{2}$$

$$\Theta_e^h = \sqrt{a(E_e^+\phi_e^+, E_e^+\phi_e^+)}$$

$$= E_e^+ \sqrt{a(\phi_e^+, \phi_e^+)}$$

$$= \frac{b(\phi_e^+) - a(u^h, \phi_e^+)}{\sqrt{a(\phi_e^+, \phi_e^+)}}$$

NOTHING TO SOLVE !

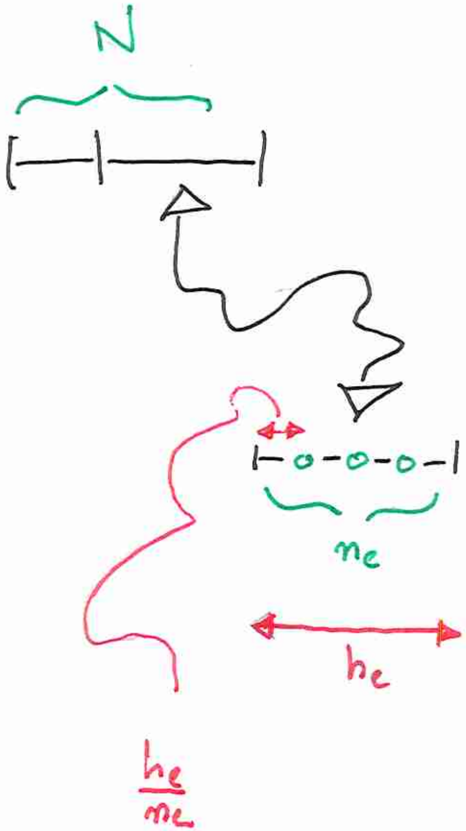
Θ^h $\partial(n)$

VALUE TO OBTAIN

REQUIRED GLOBAL ERROR

$$\Theta_e^2(n_e) = \frac{\Theta^2}{N} \simeq C^2 \left(\frac{h_e}{n_e}\right)^{2\alpha}$$

$$\Theta_e^2(1) \simeq (\Theta_e^h)^2 \simeq C^2 (h_e)^{2\alpha}$$



$$\boxed{e \leq C h^\alpha}$$

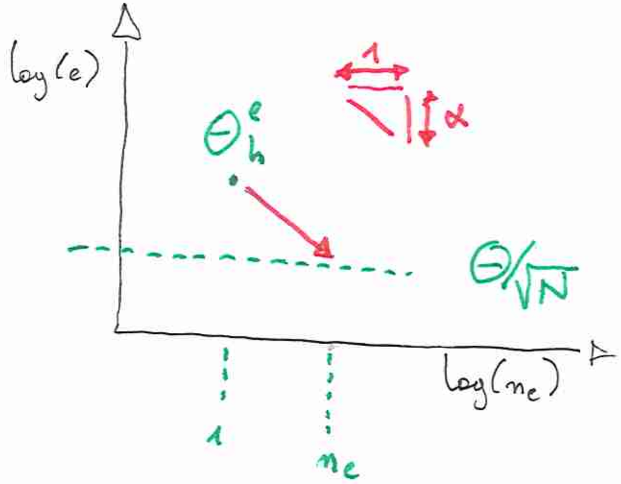
A PRIORI ERROR ESTIMATE

$$(\Theta_e^h)^2 \simeq C^2 h_e^{2\alpha}$$

$$C^2 h_e^{2\alpha} = (\Theta_e^h)^2$$

$$C^2 h_e^{2\alpha} = \Theta^2 \frac{n_e^{2\alpha}}{N}$$

$$\left(\frac{\Theta}{N}\right)^2 \simeq C^2 \left(\frac{h_e}{n_e}\right)^{2\alpha}$$



$$\boxed{\frac{n_e^{2\alpha}}{N} = \left(\frac{\Theta_e^h}{\Theta}\right)^2}$$

ADAPTIVE STRATEGY

MIXED FINITE ELEMENT METHODS

$$\underline{\underline{\sigma}} = \underbrace{\frac{E}{(1+\nu)}}_{2\mu} \underline{\underline{\varepsilon}} + \underbrace{\frac{E\nu}{(1+\nu)(1-2\nu)}}_{\lambda} \operatorname{tr}(\underline{\underline{\varepsilon}}) \, \underline{\underline{S}}$$

SHEAR MODULUS

$\lambda \to \infty$

IF $\nu \to 1/2$

POISSON'S COEFFICIENT $\nu \in \,]-1 \;\; 1/2[$

INCOMPRESSIBLE MATERIAL

STRANGE MATERIALS

USUAL BEHAVIOUR

COMPRESSIBILITY MODULUS

$$\underbrace{k}_{} \triangleq \frac{tr(\underline{\underline{\sigma}})/3}{tr(\underline{\underline{\varepsilon}})} = \frac{\overbrace{(2\mu + 3\lambda)\, \cancel{tr(\underline{\underline{\varepsilon}})}/3}^{\frac{2}{3}\mu + \lambda}}{\cancel{tr(\underline{\underline{\varepsilon}})}}$$

COMPRESSIBILITY MODULUS

SHEAR MODULUS

$$\frac{k}{\mu} = \frac{\frac{E\nu}{(1+\nu)(1-2\nu)} + \frac{1}{3}\frac{E}{(1+\nu)}}{\frac{1}{2}\frac{E}{(1+\nu)}}$$

$$= \frac{(3\nu + 1 - 2\nu)}{(1-2\nu)}\frac{2}{3} = \frac{2(1+\nu)}{3(1-2\nu)}$$

IF $\nu \to 1/2$

$\to \infty$

$$\underline{\underline{\sigma}} = 2\mu\,\underline{\underline{\varepsilon}} + \lambda\,\underbrace{\mathrm{tr}\,(\underline{\underline{\varepsilon}})}_{\to\, 0 \text{ IF } \nu \to 1/2}\,\underline{\underline{\delta}}$$

$$= 2\mu\,\underline{\underline{\varepsilon}} - p\,\underline{\underline{\delta}}$$

$$\boxed{\underset{\sim}{\nabla} \cdot \underset{\sim}{u} = 0}$$

CINEMATIC CONSTRAIN

PRESSURE

= "LAGRANGE MULTIPLIER"

ISOTROPIC INCOMPRESSIBLE ELASTICITY

? $(\underline{u}, p)$ SUCH THAT

$$\begin{cases} \underline{\nabla} \cdot \underline{\underline{\sigma}}(\underline{u}, p) + \underline{f} = 0 \\ \underline{\nabla} \cdot \underline{u} = 0 \end{cases} \qquad x \in \Omega$$

$$\underline{\underline{\sigma}}(\underline{u}, p) \cdot \underline{n} = \underline{g} \qquad x \in \Gamma_N$$

$$\underline{u} = \underline{u}^* \qquad x \in \Gamma_D$$

$$\ll \underline{u}^* \cdot \underline{n} \gg = 0 \qquad \text{IF } \Gamma_N = \emptyset$$

$$< \underline{\nabla} \cdot \underline{u} > = 0 \; !$$

$$\langle \hat{\underline{u}} \cdot (\underline{\nabla} \cdot \underline{\underline{\sigma}}) \rangle = -\langle \underline{\nabla}\, \hat{\underline{u}} : \underline{\underline{\sigma}} \rangle + \langle\!\langle \hat{\underline{u}} \cdot \overbrace{\underline{n} \cdot \underline{\underline{\sigma}}}^{g} \rangle\!\rangle_N$$

$$= \left\langle \left(\underbrace{\frac{\underline{\nabla}\hat{\underline{u}} + \underline{\nabla}\hat{\underline{u}}^T}{2}}_{\underline{\underline{\varepsilon}}(\hat{\underline{u}})} + \frac{\underline{\nabla}\hat{\underline{u}} - \underline{\nabla}\hat{\underline{u}}^T}{2} \right) : \underline{\underline{\sigma}}(\underline{u}, p) \right\rangle$$

$$= \langle \underline{\underline{\varepsilon}}(\hat{\underline{u}}) : (2\mu\, \underline{\underline{\varepsilon}}(\underline{u}) - p\, \underline{\underline{\delta}}) \rangle$$

$$= \langle \underline{\underline{\varepsilon}}(\hat{\underline{u}}) : 2\mu\, \underline{\underline{\varepsilon}}(\underline{u}) \rangle - \langle p\, (\underline{\nabla} \cdot \hat{\underline{u}}) \rangle$$

.... USUAL ALGEBRA

MIXED FORMULATION

$$? \ (\underset{\sim}{u}, p) \in \mathcal{U} \times P$$

$$\underbrace{\langle \underset{=}{\varepsilon}(\underset{\sim}{\hat{u}}) : 2\mu \, \underset{=}{\varepsilon}(\underset{\sim}{u}) \rangle}^{a(\underset{\sim}{\hat{u}}, \underset{\sim}{u})} - \overbrace{\langle p \ \underset{\sim}{\nabla} \cdot \underset{\sim}{\hat{u}} \rangle}^{b(p, \underset{\sim}{\hat{u}})} = \overbrace{\langle \underset{\sim}{\hat{u}} \cdot \underset{\sim}{f} \rangle + \langle\langle \underset{\sim}{\hat{u}} \cdot g \rangle\rangle_N}^{f(\underset{\sim}{\hat{u}})}$$

$$\underbrace{-\langle \hat{p} \ \underset{\sim}{\nabla} \cdot \underset{\sim}{u} \rangle}_{b(\hat{p}, \underset{\sim}{u})} = 0$$

$$\forall (\underset{\sim}{\hat{u}}, \hat{p}) \in \hat{\mathcal{U}} \times \hat{P}$$

$$\boxed{\begin{cases} \displaystyle\sum_{j=1}^{n} \underset{=}{A}_{ij} \cdot \underset{\sim}{U}_j + \sum_{k=1}^{m} \underset{\sim}{B}_{ik} \, P_k = \underset{\sim}{F}_i \\ \displaystyle\sum_{j=1}^{n} \underset{\sim}{B}_{jk} \cdot \underset{\sim}{U}_j = 0 \end{cases} \qquad \begin{matrix} i = 1 \dots n \\ k = 1 \dots m \end{matrix}}$$

MINIMUM PROBLEM

? U_i SUCH THAT

$$\sum_{j=1}^{n} A_{ij} U_j = F_i$$

DEF. POS. SYM.

$i = 1 \dots n$

$\Updownarrow$

? U_i SUCH THAT

$$J(U_i) = \min_{V_k} \underbrace{\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} V_i A_{ij} V_j - \sum_{i=1}^{n} U_i F_i}_{J(V_k)}$$

$+$

CONSTRAIN

$$\sum_{j=1}^{n} U_j B_{jk} = 0$$

$k = 1 \dots m$

CONSTRAINED MINIMUM PROBLEM

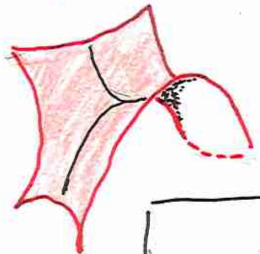
LAGRANGE MULTIPLIERS

$$\underset{\uparrow}{L}\left(U_i, \overset{\downarrow}{\Lambda_k}\right)$$

LAGRANGIAN

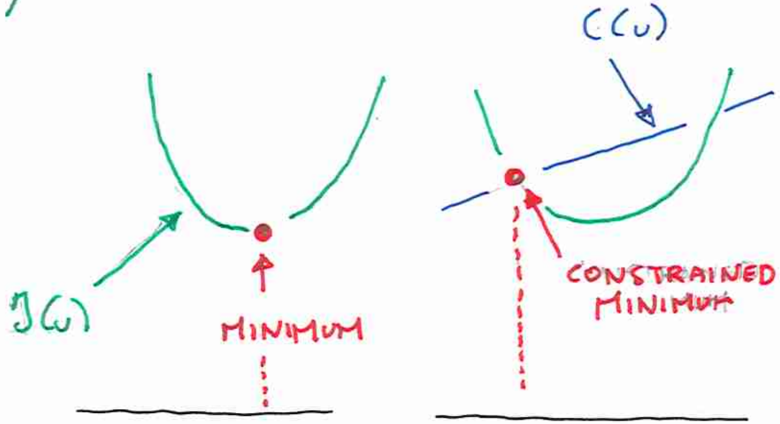
$$\underbrace{\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{2} U_i A_{ij} U_j - \sum_{i=1}^{n} U_i F_i}_{J(U_i)} + \sum_{k=1}^{m}\sum_{j=1}^{n} \overbrace{U_j B_{jk}}^{C_k(U_j)} \Lambda_k$$

SADDLE POINT PROBLEM !!



$$\downarrow$$

$$\boxed{\begin{gathered} \frac{\partial L}{\partial U_i} = 0 \\[1em] \frac{\partial L}{\partial \Lambda_k} = 0 \end{gathered}}$$



PENALTY FORMULATION

$\varepsilon \to 0$

CONSTRAINS WILL BE PERFECTLY SATISFIED !

$$J_\varepsilon(U_i) = J(U_i) + \frac{1}{2\varepsilon}\sum_{k=1}^{m}\underbrace{\left(\sum_{j=1}^{n} U_j B_{jk}\right)^2}_{(C(U_j))^2}$$

? U_i SUCH THAT

$$\sum_{j=1}^{n} A_{ij} U_i + \sum_{k=1}^{m} B_{ik} \underbrace{\sum_{j=1}^{n} \frac{1}{\varepsilon} U_j B_{jk}} = F_i$$

$$P_k = \frac{\sum_j U_j B_{jk}}{\varepsilon}$$

USELESS FROM A NUMERICAL POINT OF VIEW !

MIXED PROBLEMS

$? \; (\underline{u}^h, p^h) \in \mathcal{U}^h \times P^h$ SUCH THAT

$$a(\underline{u}^h, \hat{\underline{u}}^h) + b(p^h, \hat{\underline{u}}^h) = f(\hat{\underline{u}}^h)$$

$$b(\hat{p}^h, \underline{u}^h) = 0 \qquad \forall (\hat{\underline{u}}^h, \hat{p}^h) \in \hat{\mathcal{U}}^h \times \hat{P}^h$$

COERCIVE
CONTINUE
BILINEAR

$\Longrightarrow$ ~~LAX-MILGRAM~~

NO NATURAL STABILITY CONDITION

$\mathcal{U}^h$ AND P^h HAVE TO BE SELECTED IN A CAREFUL WAY !

WHAT IS NEEDED....

$$\underbrace{\|\underset{\sim}{u}^h\|_1}_{\sqrt{\|u\|_1^2+\|v\|_1^2+\|w\|_1^2}} + \|p^h\|_0 \leq C \underbrace{\|f\|_{-1}}_{\sup_{v\in\mathcal{U}} \frac{|f(v)|}{\|v\|_1}}$$

$\underset{\sim}{u} = (u, v, w)$

LADYSENSKAYA BREZZI BABUSKA

OK IF LBB CONDITION IS SATISFIED

$\exists\ C_{LBB} > 0$ SUCH THAT

$$\inf_{q^h\in P^h} \sup_{v^h\in\mathcal{U}^h} \frac{|b(q^h, v^h)|}{\|\underset{\sim}{v}^h\|_1 \|q^h\|_0} \geq C_{LBB}$$

$$\boxed{\begin{aligned}\hat{u}^h &= u^h \\ \hat{p}^h &= p^h\end{aligned}}$$

FOR SIMPLICITY !

$$|a(u^h, u^h)| = |f(u^h)|$$

$$\leq \|f\|_{-1} \, \|u^h\|_1$$

CONTINUITY of f

$$\exists c \quad |f(v)| \leq c \, \|u^h\|_1$$

$$c = \sup_{v \in V^h} \frac{|f(v)|}{\|u^h\|_1}$$

$$\alpha \, \|u\|_1^2 \leq$$

COERCIVITY of a

$$\exists \alpha \quad \alpha \, \|u\|_1^2 \leq |a(u, u)|$$

$$\boxed{\|u_1\|_1 \leq \frac{1}{\alpha} \, \|f\|_{-1}}$$

$$b(p^h\ \underline{u}^h) = a(\underline{u}^h\ \underline{u}^h) - f(\underline{u}^h)$$

$$|b(p^h\ \underline{u}^h)| \leqslant |a(\underline{u}^h\ \underline{u}^h)| + |f(\underline{u}^h)|$$

$$\leqslant \|f\|_{-1}\ \|\underline{u}^h\|_1$$

CONTINUITY OF F

$$\leqslant c\ \|\underline{u}^h\|_1^2$$

CONTINUITY OF P

$$\leqslant c\ \underbrace{\|\underline{u}^h\|_1}_{\leqslant \frac{1}{\alpha}\|f\|_{-1}}\ \|\underline{u}^h\|_1$$

$$C_{LBB}\ \|p^h\|_0\ \|\underline{u}^h\|_1 \leqslant$$

IS NEEDED !

$$\boxed{C_{LBB}\ \|p^h\|_0 \leqslant \left(\frac{c}{\alpha} + 1\right) \|f\|_{-1}}$$

$$\exists\, C_{LBB} \quad C_{LBB} \|p^h\|_0 \|\underline{u}^h\|_1 \leq |b(p^h, \underline{u}^h)| \quad \forall\, \underline{u}^h, p^h$$

$$\Downarrow$$

$$\exists\, C_{LBB} \quad C_{LBB} \|p^h\|_0 \leq \frac{|b(p^h, \underline{u}^h)|}{\|\underline{u}^h\|_1} \quad \forall\, \underline{u}^h, p^h$$

$$\Downarrow$$

$$\exists\, C_{LBB} \quad C_{LBB} \|p^h\|_0 \leq \sup_{\underline{v} \in U^h} \frac{|b(p^h, \underline{u}^h)|}{\|\underline{u}^h\|_1} \quad \forall\, p^h$$

$$\Downarrow$$

$$\boxed{\exists\, C_{LBB} \quad C_{LBB} \leq \inf_{p^h \in P^h} \sup_{\underline{v}^h \in U^h} \frac{|b(p^h, \underline{u}^h)|}{\|\underline{u}^h\|_1 \|p^h\|_0}}$$

$\square$

ACCURACY OF MIXED FORMULATIONS

$$\underbrace{\|\underline{u} - \underline{u}^h\|_1 + \|p - p^h\|_0}_{} \leq \underbrace{C_{LBB}}_{\text{MUST BE AS SMALL AS POSSIBLE !}} \; C \left(\inf_{\underline{v}^h \in U_h} \|\underline{u} - \underline{v}^h\|_1 + \inf_{q^h \in P^h} \|p - q^h\|_0 \right)$$

$$\boxed{\exists\ C_{LBB} \qquad C_{LBB} \leqslant \inf_{p^h \in P^h} \sup_{v^h \in V^h} \frac{\| b(p^h, v^h) \|}{\| v^h \|_1 \, \| p^h \|_0}}$$

p^h CHOISI $\qquad u^h \nearrow \;\longrightarrow\; \nearrow$

$\qquad\qquad\qquad \hookrightarrow C_{LBB} \nearrow$

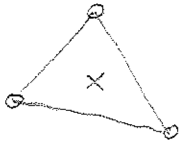
u^h CHOISI $\qquad p^h \nearrow \;\longrightarrow\; \searrow$

$\qquad\qquad\qquad \hookrightarrow C_{LBB} \searrow$

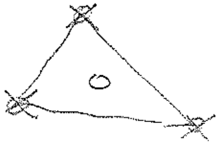
C_{LBB} OPTIMAL

C_{LBB} PETIT $\longrightarrow$ ACCURACY

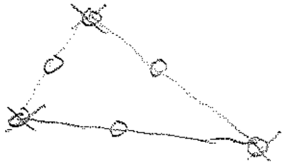
$C_{LBB} > 0 \longrightarrow$ STABILITY



$P_1 - P_0$ LBB KO

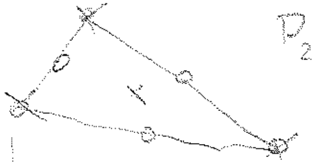


$P_1^+ - P_1$ "MINI ELEMENT" LBB OK



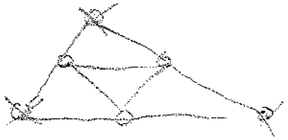
$P_2 - P_1$ LBB OK

AN EARLY FAVORITE



$P_2 - P_1^+$ LBB OK

MASS BALANCE BETTER !

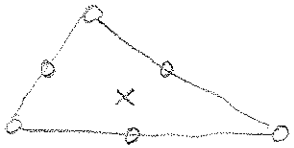


$P_{1_{2\times 2}} - P_1$ LBB OK

BEST ELEMENT WITH LINEAR VELOCITY

$P_2 - P_{-1}$

LBB KO.



$P_2 - P_0$ SIMPLE BUT INEFFICIENT

$P_2^+ - P_{-1}$

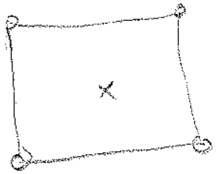
LBB OK

CROUZEIX - RAVIART

$P_2^+ - P_1$ LBB OK

GOOD ELEMENT

CUBIC BUBBLE FUNCTION

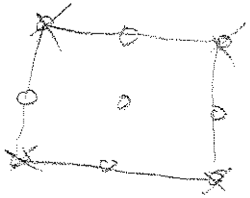


$Q_1 - Q_0$

LBB KO

MOST CONTROVERSIAL
LEAST WELL UNDERSTOOD

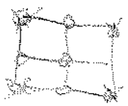
"NOTHING REPLACES EXPERIENCE ACCUMULATED OVER THE YEARS BY ENGINEERS"



$Q_2 - Q_1$

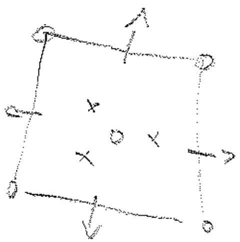
LBB OK

SIMPLEST
C_0 PRESSURE



$Q_{1_{2\times 2}} - Q_1$

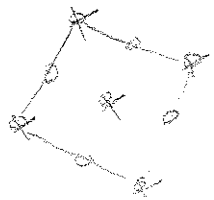
$\nabla \cdot u$ NOT VERY STRONGLY IMPOSED



MINI ELEMENT

$\tilde{Q}_2 - P_{-1}$

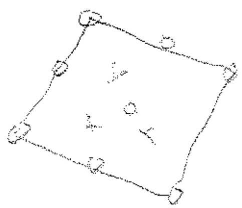
LBB OK



$Q_2 - Q_1^{+}$

LBB OK

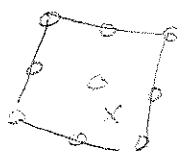
BETTER MASS BALANCE



$Q_2 - P_{-1}$

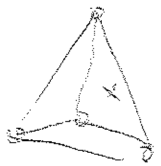
LBB OK

PROBABLY THE MOST ACCURATE



$Q_2 - Q_0$

LBB OK
FEW ADVANTAGES
EXCEPT STABILITY



$P_1 - P_0$ LBB KO

$P_2 - P_1$

$P_1^{+} - P_1$

MINI
ELEMENT

LBB OK
STABLE, BUT
NOT VERY ACCURATE

$P_1^{++} - P_{-1}$

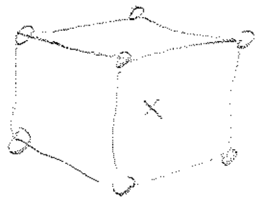
BEST
LINEAR
TETRAHEDRAL
ELEMENT

LBB OK

$P_2^{+} - P_{-1}$

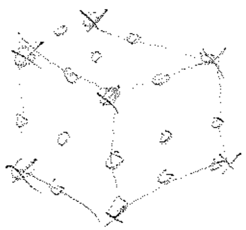
CROUZEIX
RAVIART

COST EFFECTIVE
LBB OK



LBB KO

USED BY POLYFLOW



$Q_1 - Q_1$

LBB OK

COSTLY !

$Q_2^- - Q_1$

LBB ?

USED BY POLYFLOW

PROBLÈME
HYPERBOLIQUE

ELEMENTS
FINIS *

* CLASSICAL
GALERKIN
FORMULATION

EQUATION DE TRANSPORT

$$\frac{du}{dx} = f$$

$$u(0) = u_0$$

$$u \simeq u^h = \sum_i U_i \underbrace{\tau_i(x)}$$

GALERKIN

? U_i TELS QUE

$$\langle \tau_i \; r_h \rangle = 0$$

$$\sum_j \underbrace{\langle \tau_i \; \tau_{j,x} \rangle}_{A_{ij}} U_j = \underbrace{\langle \tau_i \; f \rangle}_{B_i}$$

$\neq$ MATRICE DEFINIE POSITIVE !

$$\boxed{\frac{du}{dx} = f}$$

GALERKIN

$$\boxed{\sum_j A_{ij} U_j = B_i}$$

BUBNOV - GALERKIN

DIFFERENCES FINIES CENTREES

$$\frac{U_{j+1} - U_{j-1}}{2\Delta x} = B_j$$

$\not\Leftrightarrow$ **PROBLEME DE MINIMISATION**

$f(x) = \sin(x)$ ⇝ OK

$f(x) =$ ⇝ "Boum."

PETROV-GALERKIN

? U_j TELS QUE

$$\langle \tau_{j,x} \; r^h \rangle = 0$$

INTEGRATION LE LONG DES CARACTERISTIQUES

≃

DIFFERENCES FINIES AMONT

$$\frac{U_{j+1} - U_j}{\Delta x} = B_j$$

$$\sum_j \underbrace{\langle \tau_{i,x} \; \tau_{j,x} \rangle}_{A_{ij}} U_j = \underbrace{\langle \tau_{i,x} \; f \rangle}_{B_i}$$

MATRICE DEFINIE POSITIVE

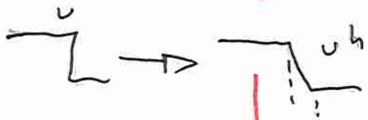
1 CONDITION AUX LIMITES !

$$\frac{du}{dx} = f$$

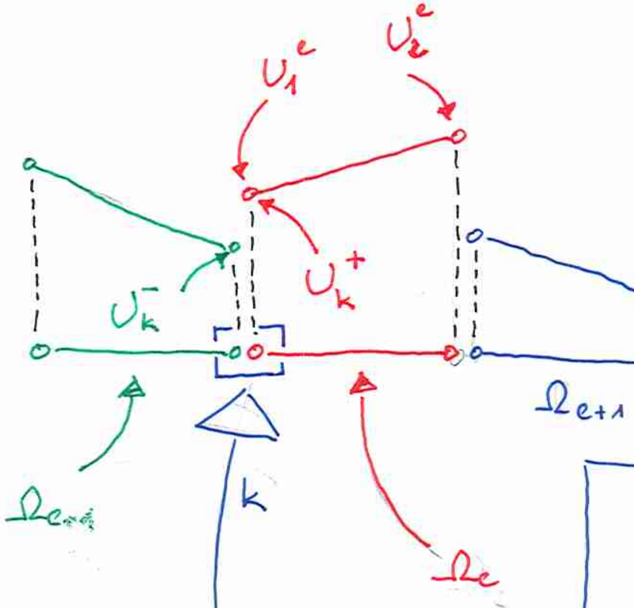
A ÉTÉ REMPLACÉ PAR $\frac{d^2u}{dx^2} = \frac{df}{dx}$

2 CONDITIONS AUX LIMITES !

HIC



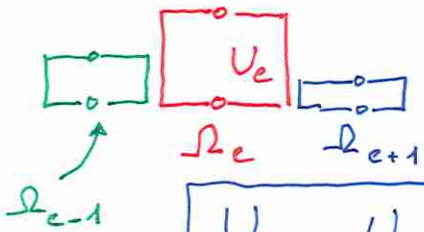
GALERKIN DISCONTINU



$$\boxed{\frac{du}{dx} = f}$$

APPROXIMATION LINEAIRE DISCONTINUE

APPROXIMATION CONSTANTE DISCONTINUE



$$\boxed{U_e - U_{e-1} = \langle f \rangle_{\Omega_e}}$$

OU $\tau_{j,x}$?

$$\sum_{j=1}^{2} \langle \tau_i \, \tau_{j,x} \rangle_{\Omega_e} U_j^e = \langle \tau_i \, f \rangle_{\Omega_e}$$

$$+ \left[\underset{U_1^e}{U_k^+} - \underset{U_2^{e-1}}{U_k^-} \right] \underbrace{\tau_1(\xi = -1)}_{\text{CECI EST UNE SORTE D'UPWINDING !}}$$

$\forall \Omega_e$

$\forall i = 1,2$

LE TERME NON LINEAIRE D'ADVECTION DU PROJET ....

$$\boxed{(\underset{\sim}{u}\cdot\underset{\sim}{\nabla})\,u}$$

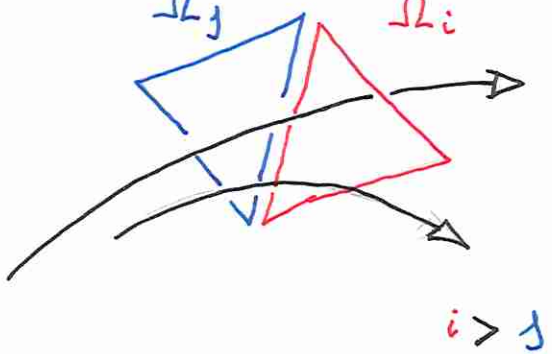
VITESSE VECTORIELLE (u, v)

LA COMPOSANTE HORIZONTALE DE VITESSE

$$\nabla$$

$$\sum_i \int_{\Omega_i} (\underset{\sim}{u}\cdot\underset{\sim}{\nabla})\,u\,\hat{u}\;d\Omega$$

FONCTIONS TEST DEFINIES SUR Ω_i !



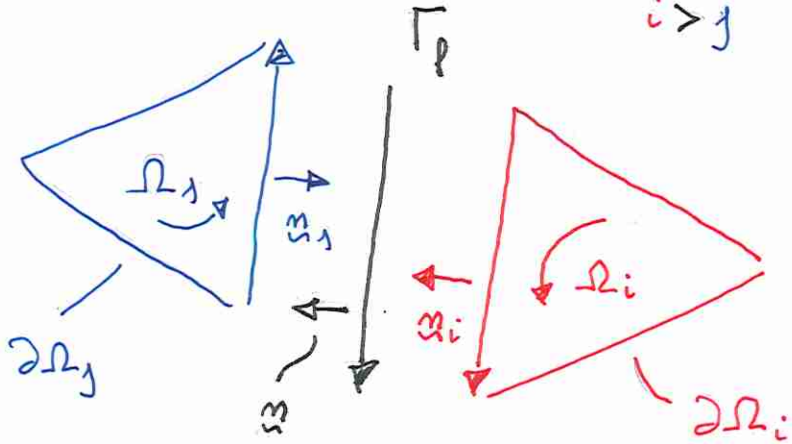
$\underset{\sim}{\nabla}\,\underset{\sim}{u}$ IS NOT DEFINED ALONG THE INTERFACES !

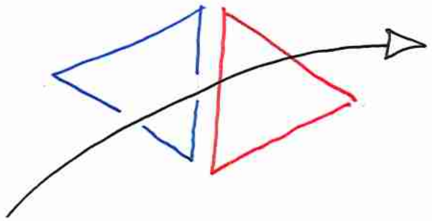
→ ON NE PEUT PAS ECRIRE FORMELLEMENT $\int_\Omega$ ....

\+ TERMES LE LONG DES FRONTIERES POUR TENIR COMPTE DES RACCORDS ENTRE ELEMENTS

LES TERMES „FRONTIERE"

$i > j$





$$\left[\overbrace{\sum_p \int_{\Gamma_p} [\,u\,]\,[\,a(\hat{u})\,]\; ds}^{\text{TERME MAGIQUE .....}}\right]$$

$[\;] \triangleq u|_{\Omega_i} - u|_{\Omega_j}$

sur Ω_i

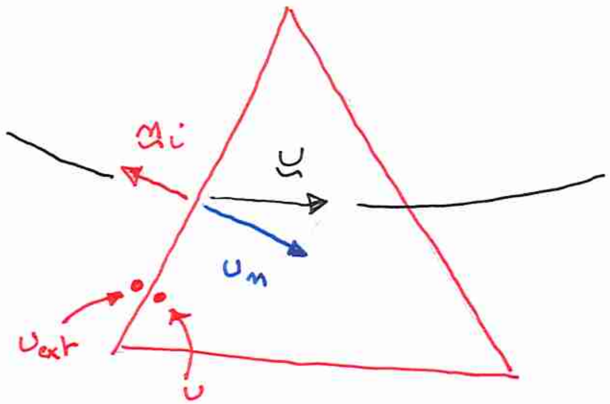
$$\frac{1}{2}\,\underbrace{u \cdot n}_{\substack{=0 \\ \text{SUR LA} \\ \text{FRONTIERE } \partial\Omega\ !}}\,\big(\text{sign}\,(u \cdot n) \mp 1\big)\,\hat{u}$$

sur Ω_j

$$\sum_i \int_{\partial\Omega_i} \underbrace{[u]_i}_{\triangleq\, u - u_{ext}} \; \underbrace{\tfrac{1}{2}\, \overbrace{\underline{u}\cdot\underline{n}_i}^{\begin{array}{c}-U_n\\+U_n\end{array}} \Big(\overbrace{\underbrace{\mathrm{sign}(\underline{u}\cdot\underline{n}_i)}_{\begin{array}{ll}-1 & \text{INLET}\\ 1 & \text{OUTLET}\end{array}} - 1}^{\begin{array}{ll}-2 & \text{INLET}\\ 0 & \text{OUTLET}\end{array}} \Big)}_{\begin{array}{ll}U_n & \text{INLET}\\ 0 & \text{OUTLET}\end{array}} \; \hat{u} \; ds$$

$$\Downarrow$$

$$\sum_i \int_{\partial\Omega_i} [u]_i \; \begin{array}{ll}U_n & \leftarrow \text{INLET}\\ 0 & \leftarrow \text{OUTLET}\end{array} \; \hat{u} \; ds$$



WEAKLY IMPOSED BOUNDARY CONDITION AT INLETS !

$$\sum_i \int_{\Omega_i} \underline{u}\cdot(\nabla \underline{u})\,\hat{u}\,d\Omega \;-\; \sum_i \int_{\partial\Omega_i} \underbrace{[u]_i}_{(u\,-\,u_{ext})} (\underline{u}\cdot\underline{n}_i)\,\hat{u}\,ds$$

EXISTE UNIQUEMENT POUR LES "INLET"

ch $\nabla\cdot\underline{u} = 0$

$$= \sum_i \int_{\Omega_i} \nabla\cdot(\underline{u}\,u)\,\hat{u}\,d\Omega$$

$$= \sum_i \int_{\Omega_i} (\underline{u}\,u)\cdot\nabla\hat{u}\,d\Omega + \sum_i \int_{\partial\Omega_i} (\underline{u}\cdot u)\cdot\underline{n}_i\,\hat{u}\,ds$$

$$+ \sum_i \int_{\partial\Omega_i} u\,(\underline{u}\cdot\underline{n}_i)\,\hat{u}\,ds$$

$\Longleftrightarrow$

$$\boxed{\sum_i \int_{\Omega_i} (\underline{u}\,u)\cdot\nabla\hat{u}\,d\Omega + \sum_i \int_{\partial\Omega_i} \begin{matrix} u_{ext} \\ u \end{matrix}\;\underline{u}\cdot\underline{n}_i\,\hat{u}\,ds}$$

INLET

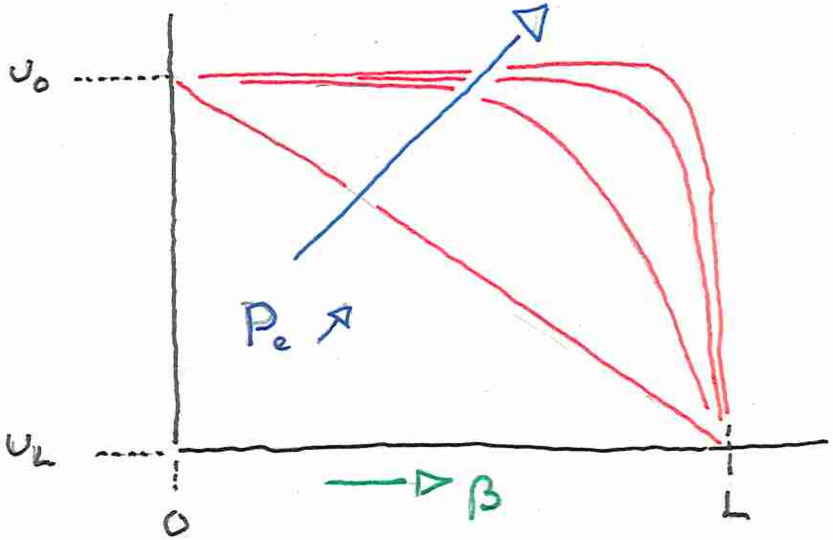
OUTLET

EQUATION D'ADVECTION - DIFFUSION

$$\beta \frac{du}{dx} - \varepsilon \frac{d^2u}{dx^2} = 0$$

$$u(0) = u_0$$
$$u(L) = u_L$$

$$P_e = \frac{\beta L}{\varepsilon}$$



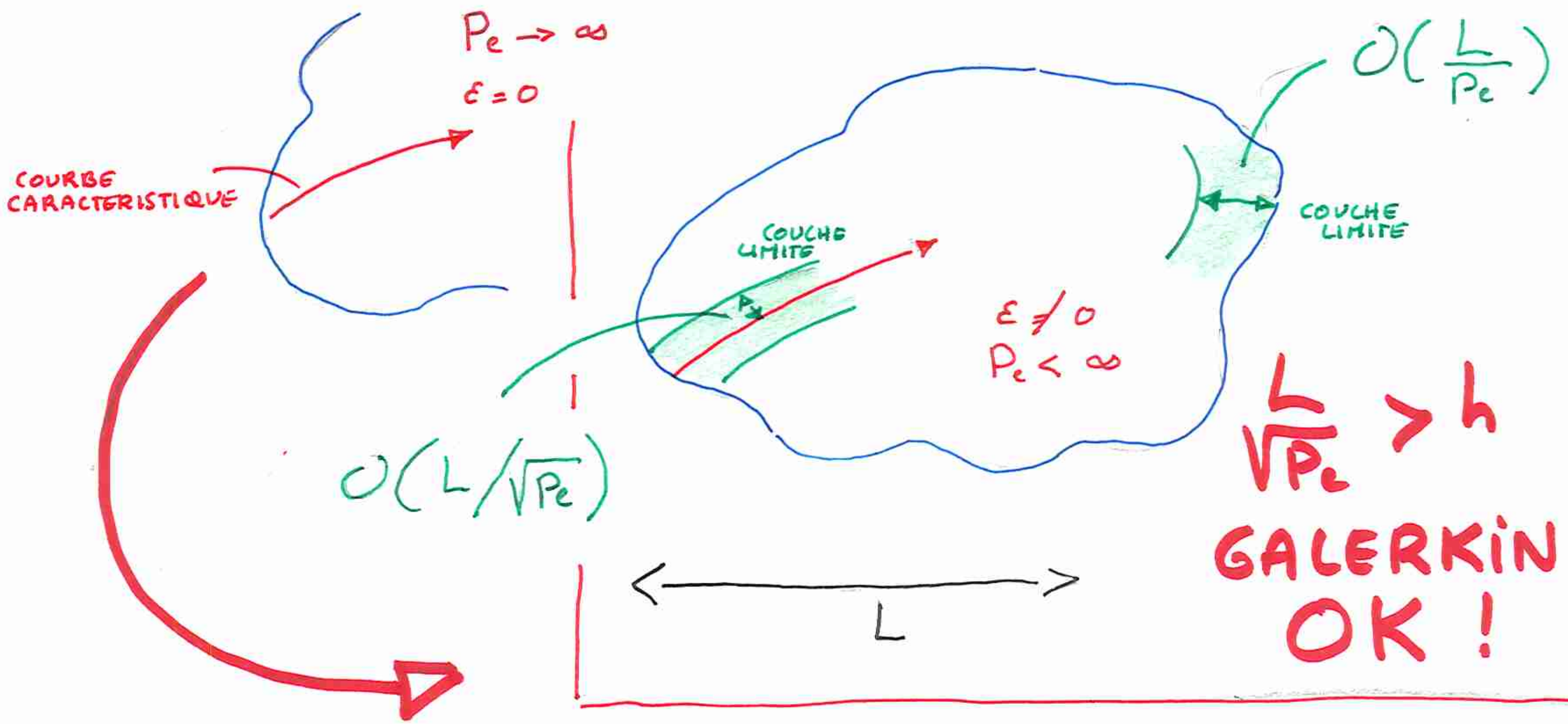
$$\frac{u - u_0}{u_L - u_0} = \frac{\exp\big(\overbrace{\beta x/\varepsilon}^{P_e\, x/L}\big) - 1}{\exp\big(\underbrace{\beta L/\varepsilon}_{P_e}\big) - 1}$$

COUCHES LIMITES

$$\underset{\sim}{\beta} \cdot \underset{\sim}{\nabla} u + \varepsilon \underset{\sim}{\nabla}^2 u = f$$

$$P_e = \frac{\beta L}{\varepsilon}$$

$$\beta = \sqrt{\beta_x^2 + \beta_y^2 + \beta_z^2}$$



2D

$$\hat{\tau}_i = \tau_i + \zeta \, \underline{\beta} \cdot \underline{\nabla} \tau_i$$

STREAMLINE UPWINDING.

PETROV-GALERKIN

$$\langle \underbrace{(\tau_i + \zeta \tau_{i,x})}_{\hat{\tau}_i} \overbrace{(u^h_{,x} + \varepsilon u^h_{,xx} - f)}^{r^h} \rangle = 0$$

HOW TO SELECT ζ ?

$$\sum_j \Big(- \zeta \langle \tau_{i,x} \, \tau_{j,x} \rangle - \varepsilon \langle \tau_{i,x} \, \tau_{j,x} \rangle + \varepsilon \zeta \langle \tau_{i,x} \, \tau_{j,xx} \rangle + \langle \tau_i \, \tau_{j,x} \rangle \Big) U_j = \ldots$$

DIFF. CENTRÉES

$$\beta \frac{U_{i+1} - U_{i-1}}{2h} = \varepsilon \frac{U_{i+1} - 2U_i + U_{i-1}}{h^2}$$

$U_0 = u_0$

$U_N = u_L$

EQUATION AUX RECURENCES !

$$\boxed{U_i = A r^i + B}$$

$$\cancel{A r^{i-1}} \underbrace{\frac{\beta h}{2\varepsilon}}_{} (r^2 - 1) = \cancel{A r^{i-1}} (r^2 - 2r + 1)$$

$$0 = \left(\frac{1 - P_e^h/2}{2}\right) r^2 - r + \left(\frac{1 + P_e^h/2}{2}\right)$$

$$r = \frac{1 \pm \sqrt{1 - (1 + P_e^h/2)(1 - P_e^h/2)}}{(1 - P_e^h/2)}$$

$$r = \frac{1 + P_e^h/2}{1 - P_e^h/2}$$

$\triangleq \dfrac{P_e^h}{2}$

PECLET DE MAILLE

REJECT $r = 1$ OF COURSE !

$$\frac{U_i - u_0}{u_L - u_0} = \frac{\left(\frac{1+P_e^h/2}{1-P_e^h/2}\right)^i - 1}{\left(\frac{1+P_e^h/2}{1-P_e^h/2}\right)^N - 1}$$

$$\left(\frac{1+P_e^h/2}{1-P_e^h/2}\right)^N \simeq \exp\left(P_e^h\right)^N \simeq \exp\left(\underbrace{\frac{\beta h N}{\varepsilon}}_{P_e}\right)$$

[SCHAUM p 551]

$$\frac{x}{e^x - 1} \simeq 1 - x/2$$

$$\left(1 + x/2\right) = e^x \left(1 - x/2\right)$$

$$\frac{P_e^h}{2} < 1$$

TO AVOID OSCILLATORY BEHAVIOUR

UPWIND DIFF.

$$\boxed{\begin{gathered}\beta\,\frac{U_i - U_{i-1}}{h} = \varepsilon\,\frac{U_{i+1} - 2U_i + U_{i-1}}{h^2}\\ U_0 = u_0\\ U_N = u_L\end{gathered}}$$

$$\cancel{A}\,\cancel{r}^{i-1}\,\underbrace{\frac{\beta h}{\varepsilon}}_{\boxed{\triangleq Pe^h}}\,(r-1) = \cancel{A}\,\cancel{r}^{i-1}\,(r^2 - 2r + 1)$$

$$0 = \frac{r^2}{2} - \left(1 + Pe^h/2\right) r + \frac{\left(1 + Pe^h\right)}{2}$$

$$r = \left(1 + Pe^h/2\right) \pm \sqrt{\underbrace{\left(1 + Pe^h/2\right)^2 - \left(1 + Pe^h\right)}_{Pe^h/2}}$$

$$= 1 + Pe^h$$

$$\frac{U_i - U_0}{U_L - U_0} = \frac{(1 + Pe^h)^i - 1}{(1 + Pe^h)^N - 1}$$

$$\boxed{(1 + Pe^h) > 0 \quad \forall h}$$

NO OSCILLATIONS

BUT...

NUMERICAL DIFFUSION

$$\beta \frac{U_i - U_{i-1}}{h} = \varepsilon \frac{U_{i+1} - 2U_i + U_{i-1}}{h^2}$$

$$\beta \frac{U_{i+1} - U_{i-1}}{2h} - \underbrace{\frac{\beta h}{2}}_{\text{NUMERICAL DIFFUSIVITY}} \frac{U_{i+1} - 2U_i + U_{i-1}}{h^2}$$

HYBRID SCHEME

$$\boxed{(1-\zeta)\beta\,\frac{U_{i+1}-U_{i-1}}{2h} + \zeta\beta\,\frac{U_i-U_{i-1}}{2h} = \varepsilon\,\frac{U_{i+1}-2U_i+U_{i-1}}{h^2}}$$

$$0 = \left(1 - \frac{(1-\zeta)Pe^h}{2}\right) r^2 - \left(1 + \zeta \frac{Pe^h}{2}\right) r + \left(1 + \overbrace{\frac{(1-\zeta)Pe^h}{2} + \zeta Pe^h}^{(1+\zeta)Pe^h/2}\right)$$

$$r = \frac{\left(1+\zeta \frac{Pe^h}{2}\right) \pm \sqrt{\underbrace{\left(1+\zeta \frac{Pe^h}{2}\right)^2 - \left(1-(1-\zeta)\frac{Pe^h}{2}\right)\cdot\left(1+(1+\zeta)\frac{Pe^h}{2}\right)}_{= Pe^h/2}}}{\left(1-(1-\zeta)\frac{Pe^h}{2}\right)}$$

$$r = \frac{1 + (1+\zeta)\, P_e^h/2}{1 - (1-\zeta)\, P_e^h/2}$$

HOW TO SELECT ζ?

$$U_i = u(ih)$$

$$\underbrace{\left(\frac{1+(1+\zeta)\, P_e^h/2}{1-(1-\zeta)\, P_e^h/2}\right)^i}_{r^i} = \underbrace{\exp\left(\frac{\beta h}{\varepsilon}\, i\right)}_{\left(\exp(P_e^h)\right)^i}$$

$$(1 + P_e^h/2) - \zeta\, P_e^h/2 = \exp(P_e)\left((1 - P_e^h/2) + \zeta\, P_e^h/2\right)$$

$$\zeta \, \frac{P_e^h}{2}\left(1 - \exp(P_e)\right) = \exp(P_e^h)\left(1 - \frac{P_e^h}{2}\right) - \left(1 + \frac{P_e^h}{2}\right)$$

$$\zeta = \frac{\exp(P_e^h)\left(\frac{2}{P_e^h} - 1\right) - \left(\frac{2}{P_e^h} + 1\right)}{\left(1 - \exp(P_e^h)\right)}$$

$$= \underbrace{- \frac{\left(1 + \exp(P_e^h)\right)}{\left(1 - \exp(P_e^h)\right)}}_{\coth(P_e^h/2)} - \frac{2}{P_e^h}$$

$$\boxed{\zeta = \coth\left(\frac{P_e^h}{2}\right) - \frac{2}{P_e^h}}$$

$\square$

LE S^T GRAAL !

$$\rho(\underset{\sim}{v}\cdot\underset{\sim}{\nabla})\underset{\sim}{v} = \mu \underset{\sim}{\nabla}^2\underset{\sim}{v} - \underset{\sim}{\nabla} p + \rho \underset{\sim}{g}$$

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{v} = 0$$

"EQUATION D' ADVECTION-DIFFUSION NON-LINÉAIRE
SOUS LA CONTRAINTE
D' INCOMPRESSIBILITY"


BEURCK !


BEURCK

LBB ?

COUCHES LIMITES $\mathcal{O}(1/\sqrt{Re})$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