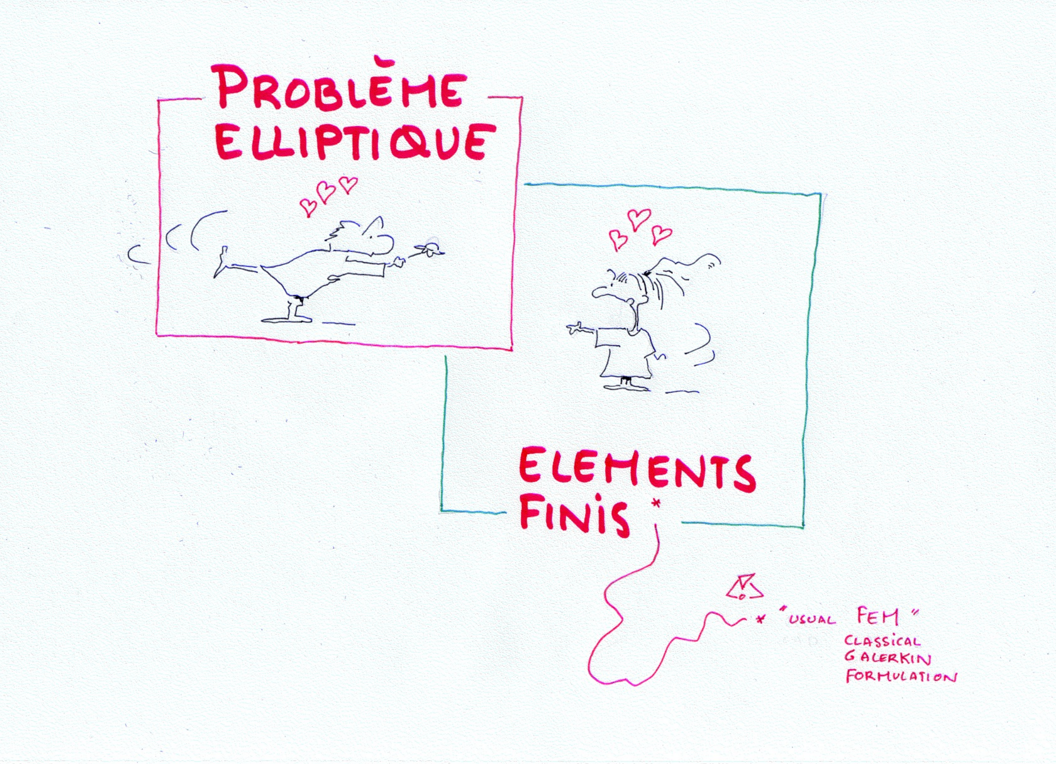
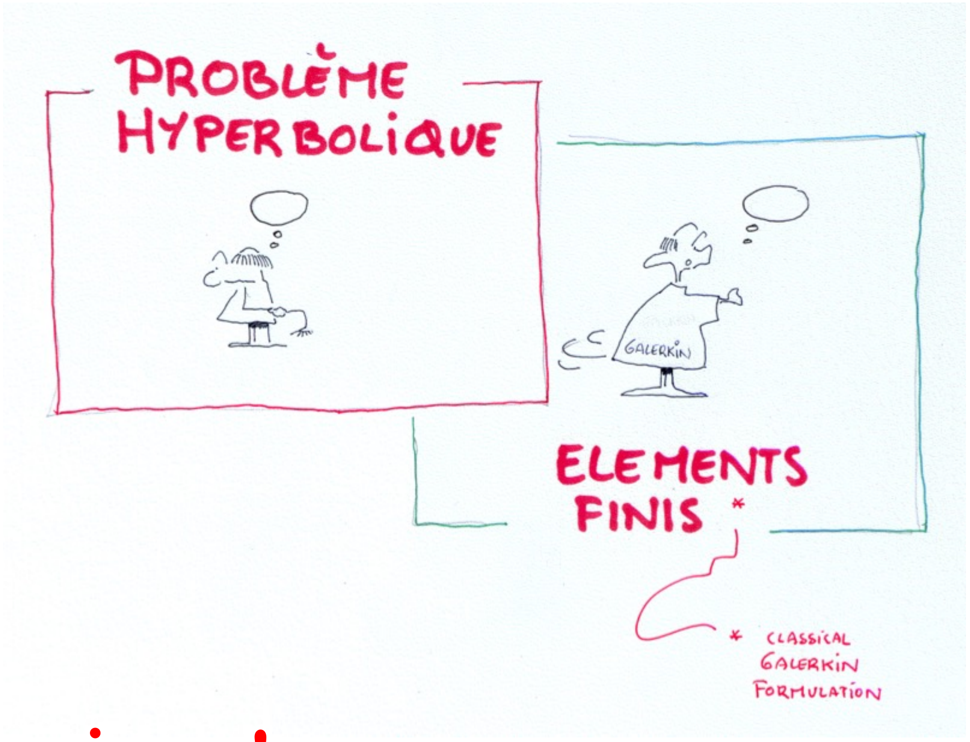
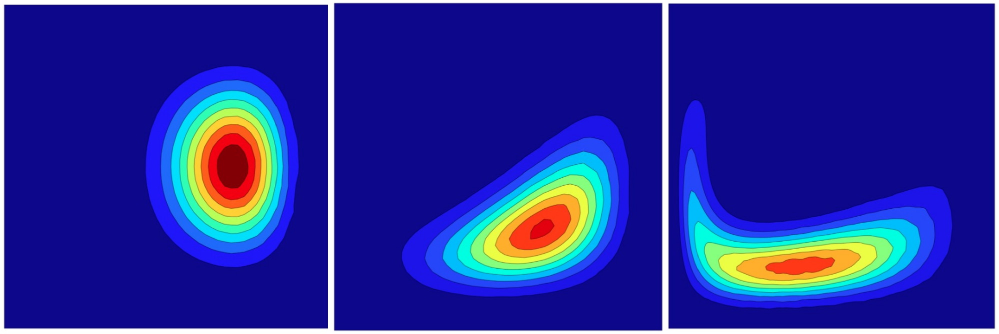


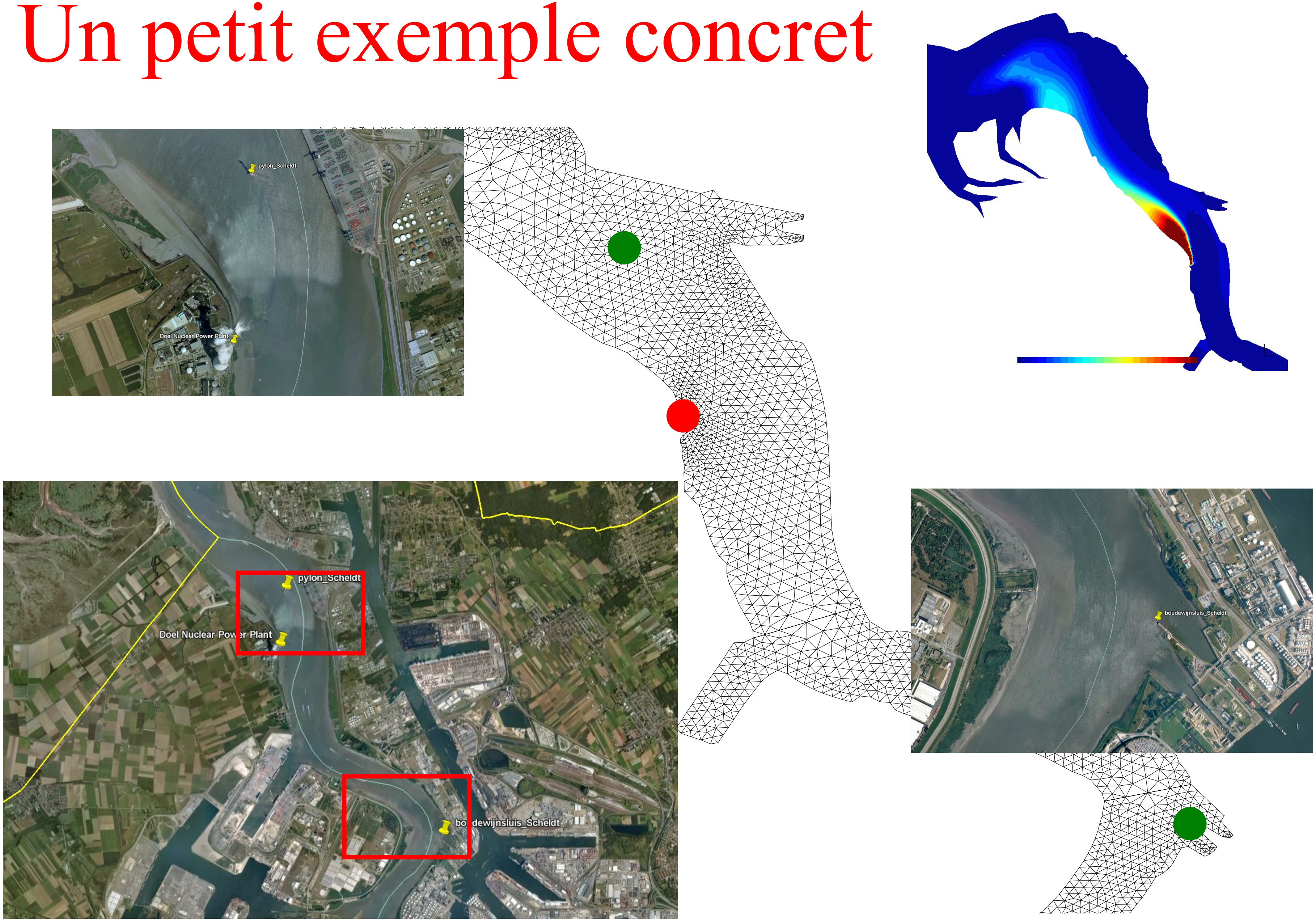
Galerkin, c'était donc optimal pour des équations elliptiques



Mais,
plus pour des
équations
d'advection diffusion !



Un petit exemple concret



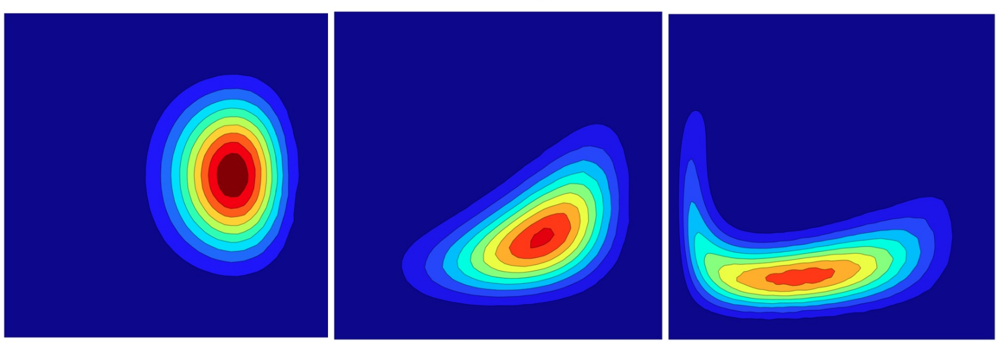
Diffusion et transport d'un traceur passif

MOYENNE DE LA CONCENTRATION SUR LA PROFONDEUR

COEFFICIENT DE DIFFUSION

$$\frac{\partial c}{\partial t} + \underbrace{\underset{\sim}{v} \cdot \underset{\sim}{\nabla} c}_{\text{TERME DE TRANSPORT}} = \underbrace{\underset{\sim}{\nabla} \cdot \left(D \, \underset{\sim}{\nabla} c \right)}_{\text{TERME DE DIFFUSION}} + S$$

$\underset{\sim}{v}$ = VITESSE HORIZONTALE



C'est une équation parabolique du second ordre !

Ce n'est pas elliptique !

$$\frac{\partial c}{\partial t} + v \frac{\partial c}{\partial x} = k \frac{\partial^2 c}{\partial x^2}$$

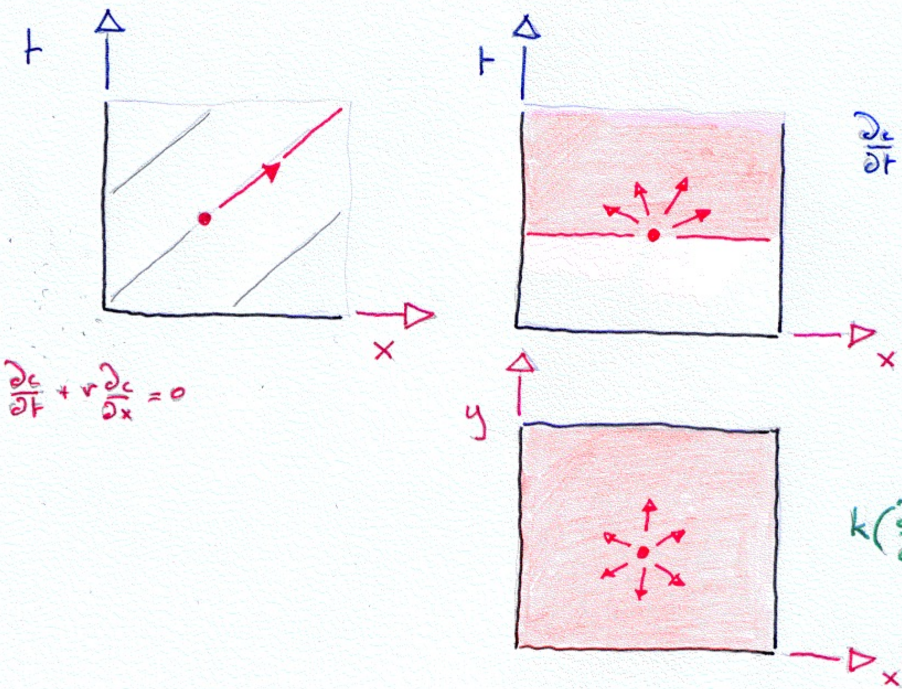
	STATIONNAIRE		INSTATIONNAIRE	
P_e PETIT	$0 = k \frac{\partial^2 c}{\partial x^2}$	ELL 2	$\frac{\partial c}{\partial t} = k \frac{\partial^2 c}{\partial x^2}$	PARA 2
	$v \frac{\partial c}{\partial x} = k \frac{\partial^2 c}{\partial x^2}$	ELL 2	$\frac{\partial c}{\partial t} + v \frac{\partial c}{\partial x} = k \frac{\partial^2 c}{\partial x^2}$	PARA 2
P_e GRAND	$v \frac{\partial c}{\partial x} = 0$	HYP 1	$\frac{\partial c}{\partial t} + v \frac{\partial c}{\partial x} = 0$	HYP 1

Le nombre de Péclet permet d'estimer l'importance du terme de transport par rapport à celui de la diffusion !

$$P_e = \frac{vL}{k}$$

PROBLEME BIEN POSE

→ CONDITIONS AUX LIMITES

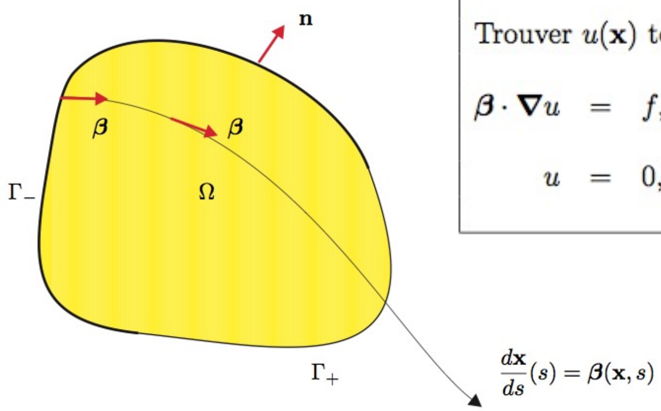


$$\frac{\partial c}{\partial t} + v\frac{\partial c}{\partial x} = 0$$

$$\frac{\partial c}{\partial t} = k\frac{\partial^2 c}{\partial x^2}$$

$$k\left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2}\right) = 0$$

Advection pure

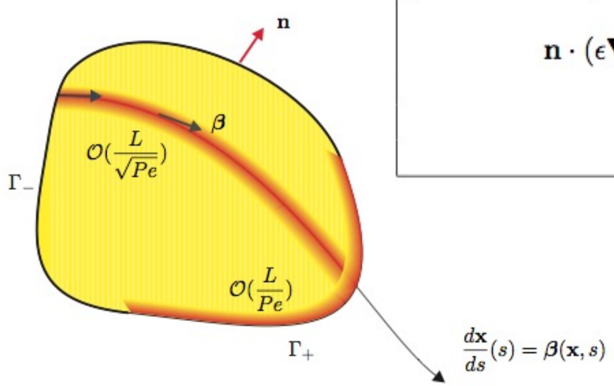


Trouver $u(\mathbf{x})$ tel que

$$\begin{aligned} \boldsymbol{\beta} \cdot \nabla u &= f, & \forall\, x \in \Omega, \\ u &= 0, & \forall\, x \in \Gamma_- \end{aligned}$$

Trouver $u(\mathbf{x}) \in \mathcal{U}_s$ tel que

$$\begin{aligned} \boldsymbol{\beta} \cdot \nabla u - \nabla \cdot (\epsilon \nabla u) &= f, & \forall \mathbf{x} \in \Omega, \\ \mathbf{n} \cdot (\epsilon \nabla u) &= g, & \forall \mathbf{x} \in \Gamma_N, \\ u &= t, & \forall \mathbf{x} \in \Gamma_D, \end{aligned}$$

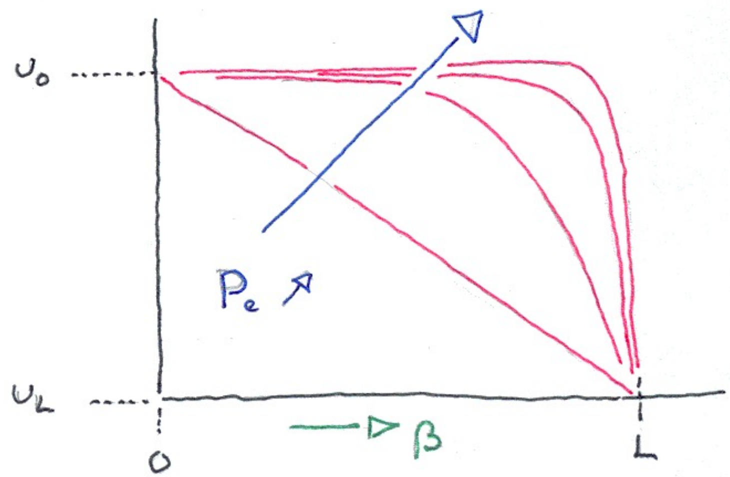


Advection-diffusion

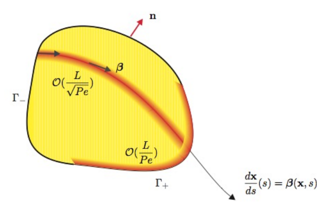
EQUATION D'ADVECTION - DIFFUSION

$$\boxed{\begin{gathered} \beta \frac{du}{dx} - \varepsilon \frac{d^2u}{dx^2} = 0 \\ u(0) = u_0 \\ u(L) = u_L \end{gathered}}$$

$$\boxed{P_e = \frac{\beta L}{\varepsilon}}$$



$$\frac{u - u_0}{u_L - u_0} = \frac{\exp\left(\overbrace{\beta x/\varepsilon}^{P_e\, x/L}\right) - 1}{\exp\left(\underbrace{\beta L/\varepsilon}_{P_e}\right) - 1}$$



Trouver $u(\mathbf{x}) \in \mathcal{U}_s$ tel que

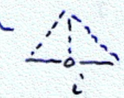
$$
\begin{aligned}
\boldsymbol{\beta} \cdot \boldsymbol{\nabla} u - \boldsymbol{\nabla} \cdot (\epsilon \boldsymbol{\nabla} u) &= f, & \forall \mathbf{x} \in \Omega,\\
\mathbf{n} \cdot (\epsilon \boldsymbol{\nabla} u) &= g, & \forall \mathbf{x} \in \Gamma_N,\\
u &= t, & \forall \mathbf{x} \in \Gamma_D,
\end{aligned}
$$

Advection-diffusion

EQUATION DE TRANSPORT

$$\boxed{\begin{aligned} \frac{du}{dx} &= f \\ u(0) &= u_0 \end{aligned}}$$

$$u \simeq u^h = \sum_i U_i \underbrace{\tau_i(x)}$$



GALERKIN

$$\boxed{\begin{gathered} ?\ U_i \ \text{TELS QUE} \\ \langle \tau_i \ r_h \rangle = 0 \\ \sum_j \overbrace{\langle \tau_i \ \tau_{j,x} \rangle}^{A_{ij}} U_j = \overbrace{\langle \tau_j \ f \rangle}^{B_i} \end{gathered}}$$

≠ MATRICE DEFINIE POSITIVE !

$$\boxed{\frac{du}{dx} = f}$$

GALERKIN

$$\boxed{\sum_j A_{ij} U_j = B_i}$$

BUBNOV - GALERKIN

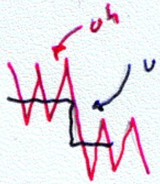
DIFFERENCES FINIES CENTREES

$$\frac{U_{j+1} - U_{j-1}}{2\Delta x} = B_j$$

PROBLEME DE MINIMISATION

$f(x) = \sin(x)$ ⟶ OK

$f(x) =$ ⟶ "Boum"



PETROV-GALERKIN

? U_j TELS QUE

$$\langle \tau_{j,x}\; r^h \rangle = 0$$

INTEGRATION LE LONG DES CARACTERISTIQUES

DIFFERENCES FINIES AMONT

$$\frac{U_{j+1} - U_j}{\Delta x} = B_j$$

$$\sum_j \underbrace{\langle \tau_{i,x}\; \tau_{j,x} \rangle}_{A_{ij}} U_j = \underbrace{\langle \tau_{i,x}\; f \rangle}_{B_i}$$

MATRICE DEFINIE POSITIVE

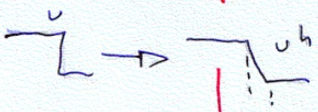
1 CONDITION AUX LIMITES !

$$\frac{du}{dx} = f$$

A ÉTÉ REMPLACÉ PAR $\dfrac{d^2u}{dx^2} = \dfrac{df}{dx}$

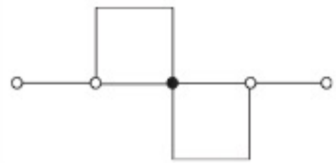
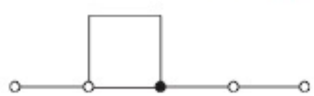
2 CONDITIONS AUX LIMITES !

HIC



En bref :-)

$$\boxed{\begin{aligned} \frac{du}{dx} &= f, \\ u(0) &= 0, \end{aligned}}$$

Galerkin $w_i = \tau_i$ 	*Différences finies centrées* Simple et donc tentant... Oscillations numériques si f n'est pas lisse ! $\frac{U_{i+1} - U_{i-1}}{2h} = \frac{F_{i+1} + 4F_i + F_{i-1}}{6},$
Petrov-Galerkin $w_i = \tau_{i,x}$ 	*Différences finies centrées d'ordre deux* Mathématiquement, tentant Condition frontière parasite ! $\frac{U_{i+1} - 2Ui + U_{i-1}}{h^2} = \frac{F_{i+1} - F_{i-1}}{2h},$
Petrov-Galerkin $w_i = \tau_{i-1}^{cst}$ 	*Différences finies amont* Quasiment optimal... Correspond à une intégration le long de la caractéristique, Pas d'oscillation numérique $\frac{U_i - U_{i-1}}{h} = \frac{F_{i+1} + F_{i-1}}{2},$

PETROV-GALERKIN

$$\langle \underbrace{(\tau_i + \zeta\, \tau_{i,x})}_{\hat{\tau}_i} \overbrace{(u^h_{,x} - \varepsilon\, u^h_{,xx} - f)}^{r^h} \rangle = 0$$

HOW TO SELECT ζ ?

2D

$$\hat{\tau}_i = \tau_i + \zeta\, \underset{\sim}{\beta} \cdot \underset{\sim}{\nabla} \tau_i$$

STREAMLINE UPWINDING

$$\sum_j \Big(\zeta \langle \tau_{i,x}\, \tau_{j,x} \rangle + \varepsilon \langle \tau_{i,x}\, \tau_{j,x} \rangle - \varepsilon\zeta \langle \tau_{i,x}\, \tau_{j,xx} \rangle + \langle \tau_i\, \tau_{j,x} \rangle \Big) U_j = \dots$$

DIFF. CENTRÉES

$$\beta \, \frac{U_{i+1} - U_{i-1}}{2h} = \varepsilon \, \frac{U_{i+1} - 2U_i + U_{i-1}}{h^2}$$

$$U_0 = u_0$$

$$U_N = u_L$$

EQUATION AUX RECURENCES !

$$U_i = A r^i + B$$

$$\cancel{A r^{i-1}} \, \underbrace{\frac{\beta h}{2\varepsilon}}_{} \, (r^2 - 1) = \cancel{A r^{i-1}} \, (r^2 - 2r + 1)$$

$$0 = \left(\frac{1 - P_e^h/2}{2}\right) r^2 - r + \left(\frac{1 + P_e^h/2}{2}\right)$$

$$r = \frac{1 \pm \sqrt{1 - (1 + P_e^h/2)(1 - P_e^h/2)}}{(1 - P_e^h/2)}$$

$$r = \frac{1 + P_e^h/2}{1 - P_e^h/2}$$

$$\triangleq \frac{P_e^h}{2}$$

PECLET DE MAILLE

REJECT $r = 1$ OF COURSE !

$$\frac{U_i - u_0}{u_L - u_0} = \frac{\left(\frac{1+P_e h/2}{1-P_e h/2}\right)^i - 1}{\underbrace{\left(\frac{1+P_e h/2}{1-P_e h/2}\right)^N}_{} - 1}$$

$$\simeq \exp(P_e h)^N$$

$$\simeq \exp\left(\underbrace{\frac{\beta h N}{\varepsilon}}_{P_e}\right)$$

[SCHAUM p 551]

$$\frac{x}{e^x - 1} \simeq 1 - x/2$$

$$(1 + x/2) = e^x (1 - x/2)$$

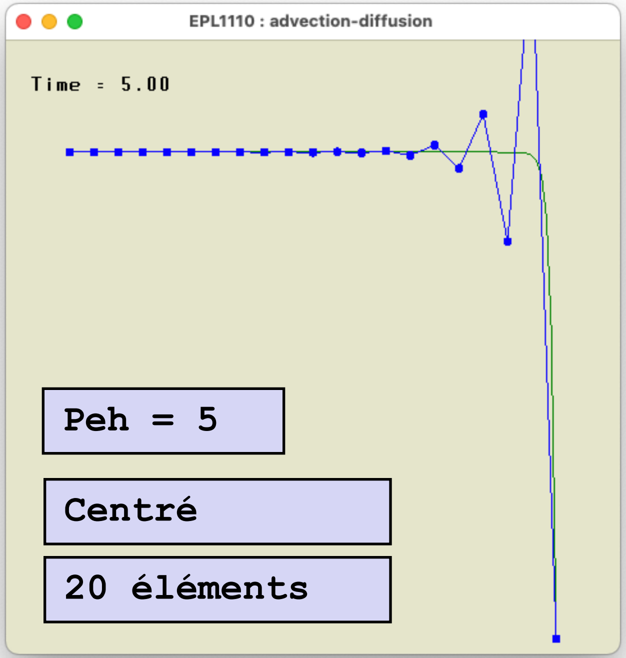
$$\frac{P_e h}{2} < 1$$

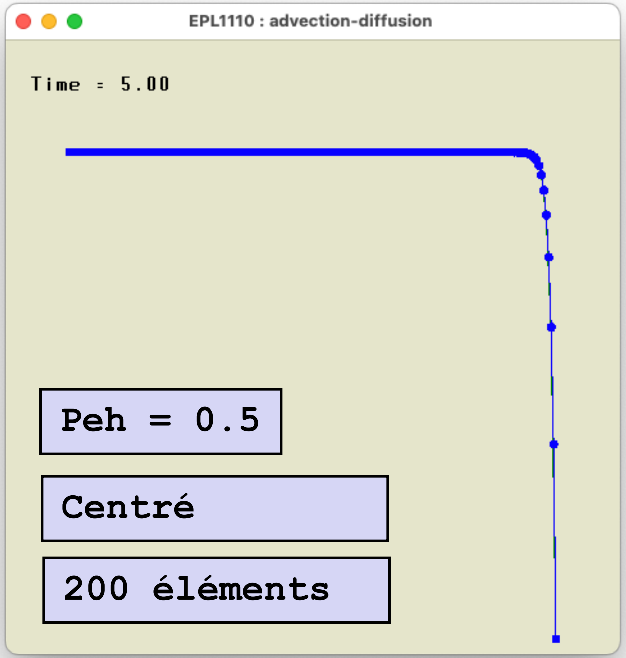
TO AVOID OSCILLATORY BEHAVIOUR

Galerkin converge si on raffine le maillage suffisament !

```
epsilon = 0.01;
beta = 1.0;
```

L'équation d'advection-diffusion est formellement une équation elliptique et donc c'était prévisible par la théorie !





UPWIND DIFF.

$$\boxed{\beta \frac{U_i - U_{i-1}}{h} = \varepsilon \frac{U_{i+1} - 2U_i + U_{i-1}}{h^2} \qquad U_0 = u_0 \qquad U_N = u_L}$$

$$\cancel{A r^{i-1}} \underbrace{\frac{\beta h}{\varepsilon}}_{\boxed{\hat{=}\, Pe^h}} (r-1) = \cancel{A r^{i-1}} (r^2 - 2r + 1)$$

$$0 = \frac{r^2}{2} - \left(1 + Pe^h/2\right) r + \frac{(1 + Pe^h)}{2}$$

$$r = (1 + Pe^h/2) \pm \sqrt{\underbrace{(1 + Pe^h/2)^2 - (1 + Pe^h)}_{Pe^h/2}}$$

$$= 1 + Pe^h$$

$$\frac{U_i - U_0}{U_L - U_0} = \frac{(1+P_e^h)^i - 1}{(1+P_e^h)^N - 1}$$

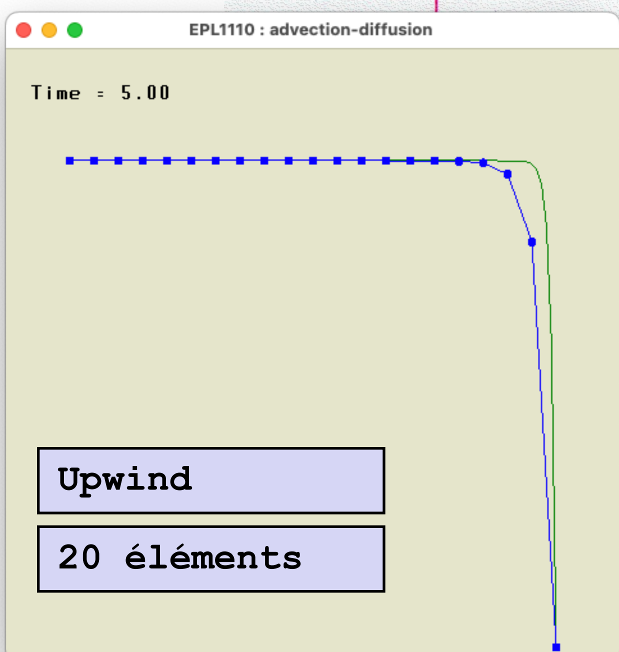
$$\boxed{(1+P_e^h) \underset{\forall h}{>} 0 \quad \text{NO OSCILLATIONS}}$$

BUT...

NUMERICAL DIFFUSION

$$\beta \, \frac{U_i - U_{i-1}}{h} = \varepsilon \, \frac{U_{i+1} - 2U_i + U_{i-1}}{h^2}$$

$$\beta \, \frac{U_{i+1} - U_{i-1}}{2h} - \underbrace{\frac{\beta h}{2}}_{\text{NUMERICAL DIFFUSIVITY}} \frac{U_{i+1} - 2U_i + U_{i-1}}{h^2}$$



HYBRID SCHEME

$$\boxed{(1-\zeta)\beta\,\frac{U_{i+1}-U_{i-1}}{2h} + \zeta\beta\,\frac{U_i-U_{i-1}}{2h} = \varepsilon\,\frac{U_{i+1}-2U_i+U_{i-1}}{h^2}}$$

$$0 = \left(1-\frac{(1-\zeta)P_e^h/2}{2}\right)r^2 - \left(1+\zeta P_e^h/2\right)r + \left(1+\frac{\overbrace{(1-\zeta)P_e^h/2+\zeta P_e^h}^{(1+\zeta)P_e^h/2}}{2}\right)$$

$$r = \frac{\left(1+\zeta P_e^h/2\right) \pm \sqrt{\left(1+\zeta P_e^h/2\right)^2 - \left(1-(1-\zeta)P_e^h/2\right)\cdot\left(1+(1+\zeta)P_e^h/2\right)}}{\left(1-(1-\zeta)P_e^h/2\right)}$$

$$\sqrt{\cdots} = P_e^h/2$$

$$\boxed{\pi = \frac{1 + (1+\zeta)\,Pe^h/2}{1 - (1-\zeta)\,Pe^h/2}}$$

How to select ζ?

$$U_i = u(ih)$$

$$\downarrow$$

$$\underbrace{\left(\frac{1+(1+\zeta)\,Pe^h/2}{1-(1-\zeta)\,Pe^h/2}\right)^i}_{\pi^i} = \underbrace{\exp\left(\frac{\beta h}{\varepsilon}\, i\right)}_{\left(\exp\left(Pe^h\right)\right)^i}$$

$$\downarrow$$

$$\left(1 + Pe^h/2\right) - \zeta\, Pe^h/2 = \exp(Pe)\left(\left(1 - Pe^h/2\right) + \zeta\, Pe^h/2\right)$$

$$\zeta \, \frac{P_e^h}{2}\left(1 - \exp\left(P_e\right)\right) = \exp\left(P_e^h\right)\left(1 - \frac{P_e^h}{2}\right) - \left(1 + \frac{P_e^h}{2}\right)$$

$$\Downarrow$$

$$\zeta = \frac{\exp\left(P_e^h\right)\left(\frac{2}{P_e^h} - 1\right) - \left(\frac{2}{P_e^h} + 1\right)}{\left(1 - \exp\left(P_e^h\right)\right)}$$

$$\Downarrow$$

$$= \underbrace{-\frac{\left(1 + \exp\left(P_e^h\right)\right)}{\left(1 - \exp\left(P_e^h\right)\right)}}_{\coth\left(P_e^h/2\right)} - \frac{2}{P_e^h}$$

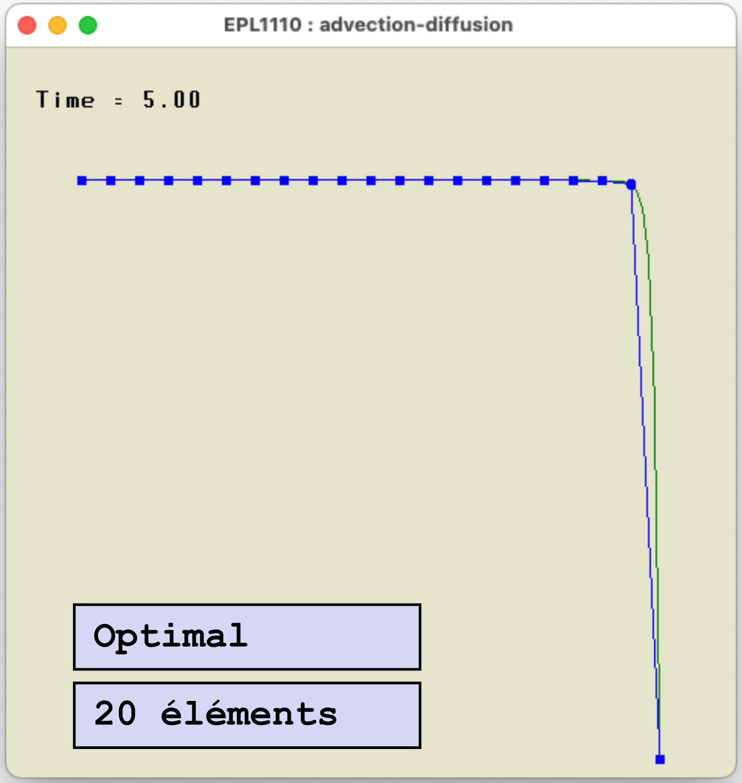
$$\boxed{\zeta = \coth\left(\frac{P_e^h}{2}\right) - \frac{2}{P_e^h}}$$

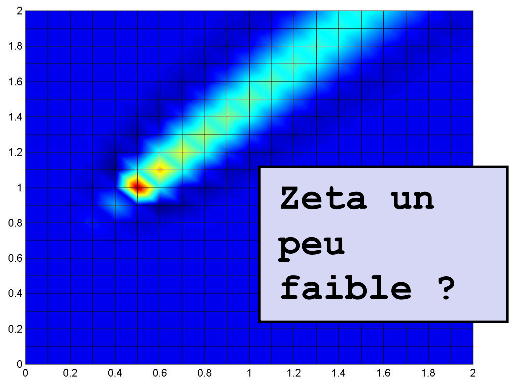
$\square$

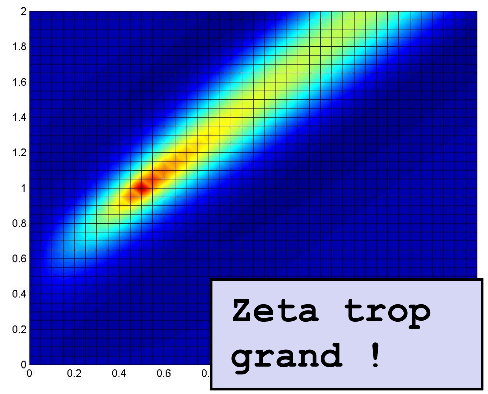
On a trouvé la méthode parfaite pour des équations unidimensionnelles !

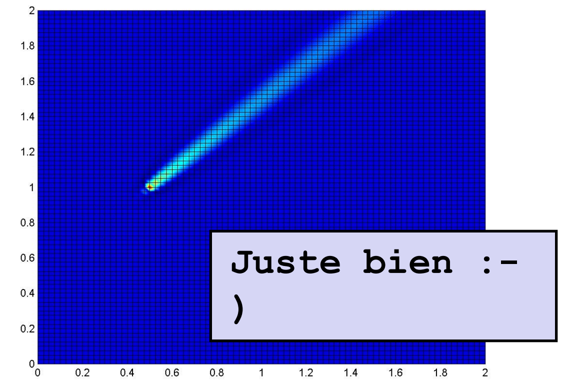
$$\begin{aligned} \beta \frac{du}{dx} - \epsilon \frac{d^2u}{dx^2} &= 0, \\ u(0) &= u_0, \\ u(L) &= u_L, \end{aligned}$$

$$\boxed{\zeta = \coth\left(\frac{Pe^h}{2}\right) - \frac{2}{Pe^h}}$$









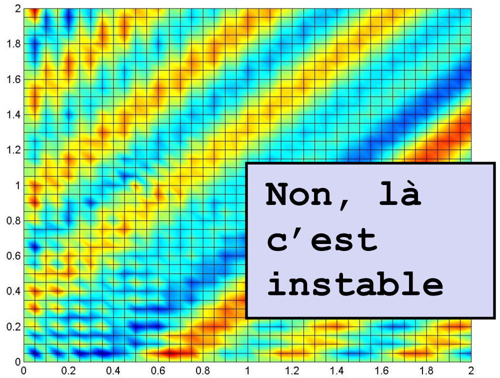
En extrapolant aux dimensions supérieures…

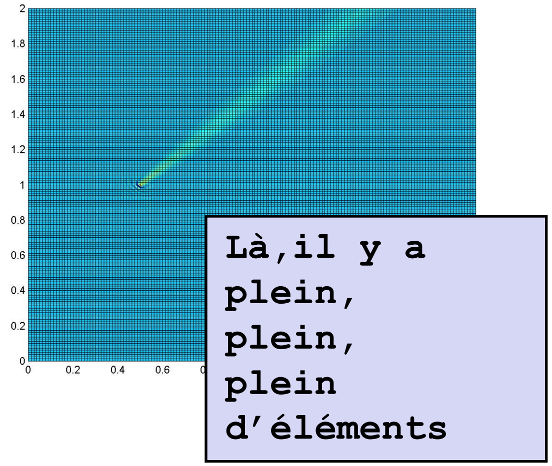
$$w_i = \tau_i + \zeta \boldsymbol{\beta} \cdot \boldsymbol{\nabla} \tau_i$$

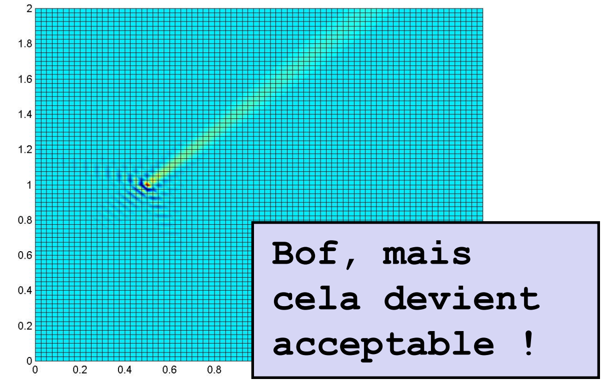
Facteur de stabilisation
Trop grand : diffusion numérique !
Trop petit : instable !

Trouver $U_j \in \mathbb{R}^n$ tel que

$$\sum_{j=1}^{n} \underbrace{< w_i \ \boldsymbol{\beta} \cdot \boldsymbol{\nabla} \tau_j + \epsilon \boldsymbol{\nabla} w_i \cdot \boldsymbol{\nabla} \tau_j >}_{A_{ij}} U_j \quad = \quad \underbrace{< w_i f > + \ll w_i g \gg_N}_{B_i} , \qquad i = 1, \ldots, n,$$







Et en payant le prix, Galerkin fonctionne !

$$w_i = \tau_i + \zeta \boldsymbol{\beta} \cdot \boldsymbol{\nabla} \tau_i$$

Pas de stabilisation !
Zeta = 0 !

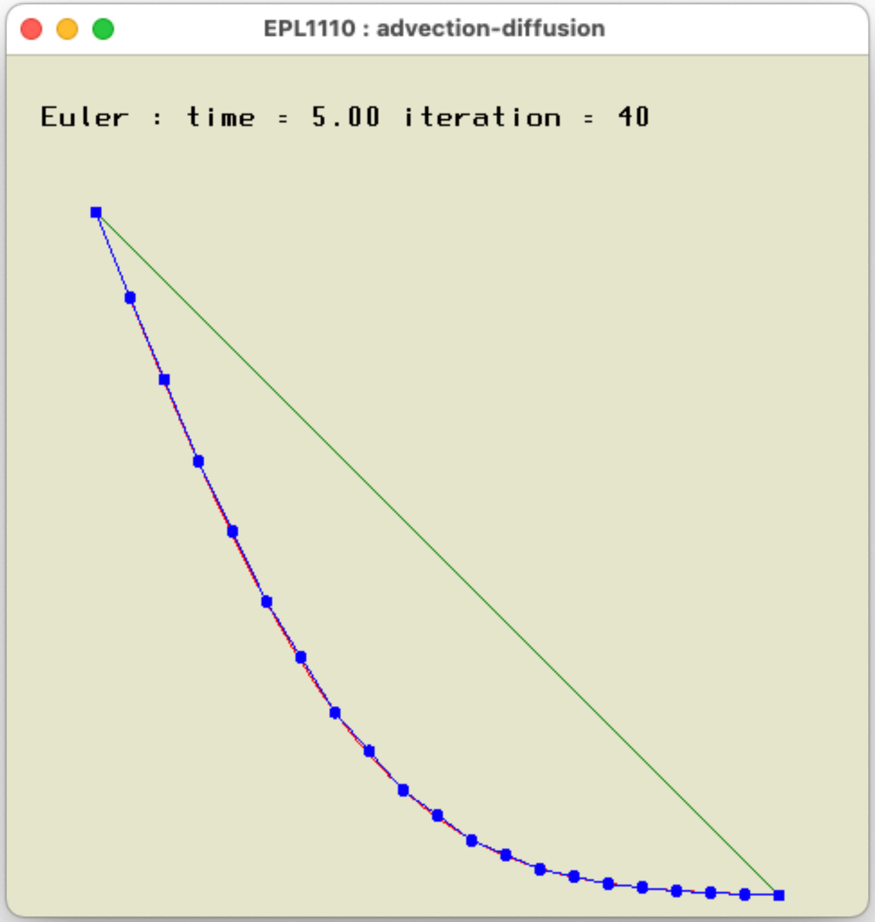
Trouver $U_j \in \mathbb{R}^n$ tel que

$$\sum_{j=1}^{n} \underbrace{< w_i \; \boldsymbol{\beta} \cdot \boldsymbol{\nabla} \tau_j + \epsilon \boldsymbol{\nabla} w_i \cdot \boldsymbol{\nabla} \tau_j >}_{A_{ij}} \; U_j \quad = \quad \underbrace{< w_i f > + \ll w_i g \gg_N}_{B_i} \;, \qquad i = 1, \ldots, n,$$

Et maintenant introduisons le temps…

$$\frac{\partial u}{\partial t} = \epsilon \frac{\partial^2 u}{\partial x^2}$$

$$u(0) = 1$$
$$u(1) = 0$$




```
epsilon = 0.01;
L = 1
```

Différences finies (espace)
Euler explicite (temps)

$$\left(\frac{U_i^{n+1} - U_i^n}{\Delta t}\right) = \epsilon \left(\frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{(\Delta x)^2}\right)$$

En définissant $b = \dfrac{\epsilon \Delta t}{(\Delta x)^2}$,

$$\boxed{U_i^{n+1} = U_i^n + b\Big(U_{i+1}^n + U_{i-1}^n - 2U_i^n\Big)}$$

C'est une itération pour un vecteur qui doit converger vers la solution de régime
C'est quelque chose qu'on a déjà rencontré…

On intègre un système linéaire…

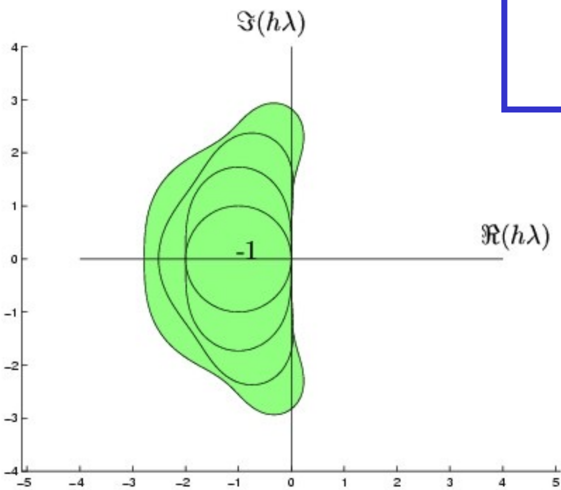
$$U_i^{n+1} \quad = \quad U_i^n + b\Big(U_{i+1}^n + U_{i-1}^n - 2U_i^n\Big)$$

En passant à une notation matricielle,

$$\begin{bmatrix} U_1^{n+1} \\ U_2^{n+1} \\ U_3^{n+1} \\ U_4^{n+1} \\ U_5^{n+1} \\ \\ U_m^{n+1} \end{bmatrix} = \begin{bmatrix} U_1^n \\ U_2^n \\ U_3^n \\ U_4^n \\ U_5^n \\ \\ U_m^n \end{bmatrix} + b \begin{bmatrix} -2 & 1 & & & & & \\ 1 & -2 & 1 & & & & \\ & 1 & -2 & 1 & & & \\ & & 1 & -2 & 1 & & \\ & & & 1 & -2 & 1 & \\ & & & & & & \\ & & & & & 1 & -2 \end{bmatrix} \begin{bmatrix} U_1^n \\ U_2^n \\ U_3^n \\ U_4^n \\ U_5^n \\ \\ U_m^n \end{bmatrix}$$

En définissant adéquatement u_n et A,

$$\mathbf{u}_{n+1} \quad = \quad \mathbf{u}_n + b\mathbf{A}\mathbf{u}_n$$



On résout le système linéaire défini par :

$$\mathbf{u}'(t) = \frac{\epsilon}{(\Delta x)^2}\mathbf{A}\mathbf{u}(t)$$

$$\Delta x = 0.1, \Delta t = 0.005$$

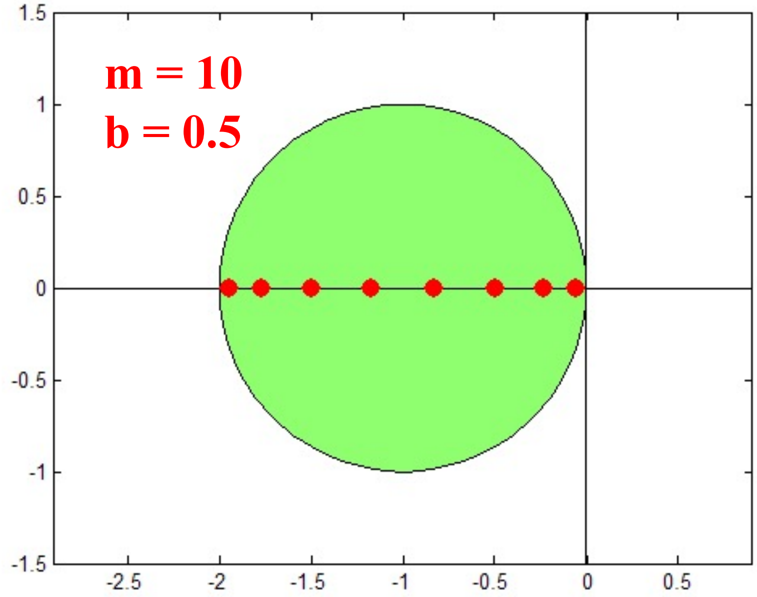
Euler explicite

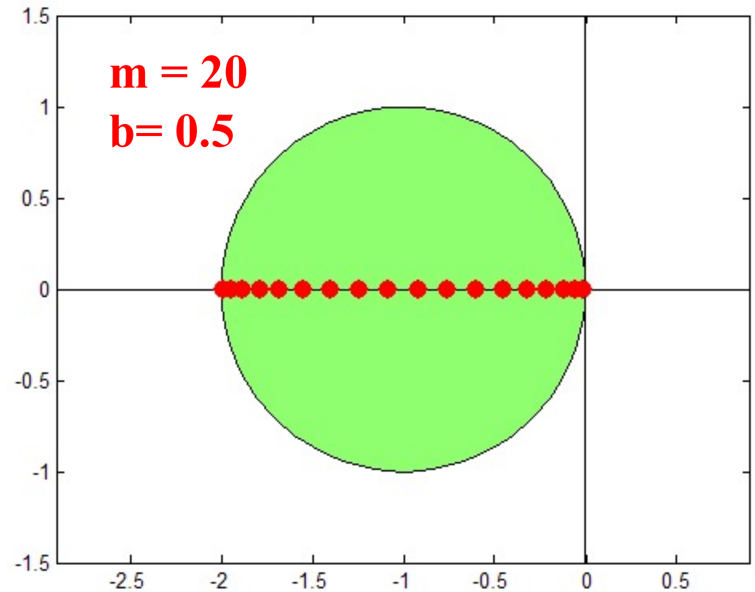
$$\mathbf{u}_{n+1} = \mathbf{u}_n + \underbrace{\frac{\epsilon \Delta t}{(\Delta x)^2}}_{b} \mathbf{A}\mathbf{u}_n$$

$$|1 + b\lambda_i| \leq 1$$

Valeurs propres de A

$$\Delta x = 0.05, \Delta t = 0.00125$$





Condition de stabilité pour la méthode d'Euler explicite….

$$\beta \quad = \quad \frac{\epsilon \Delta t}{(\Delta x)^2} \quad \leq \quad \frac{1}{2}$$

$$\Delta t \quad \leq \quad \frac{(\Delta x)^2}{2\epsilon}$$

Courant, Friedrichs et Lewy (1928)


```
epsilon = 0.01;
dt = h*h/(epsilon*1.94);
```

Condition de stabilité pour la méthode d'Euler explicite....

$$\beta = \frac{\epsilon \Delta t}{(\Delta x)^2} \leq \frac{1}{2}$$

$$\boxed{\Delta t \leq \frac{(\Delta x)^2}{2\epsilon}}$$

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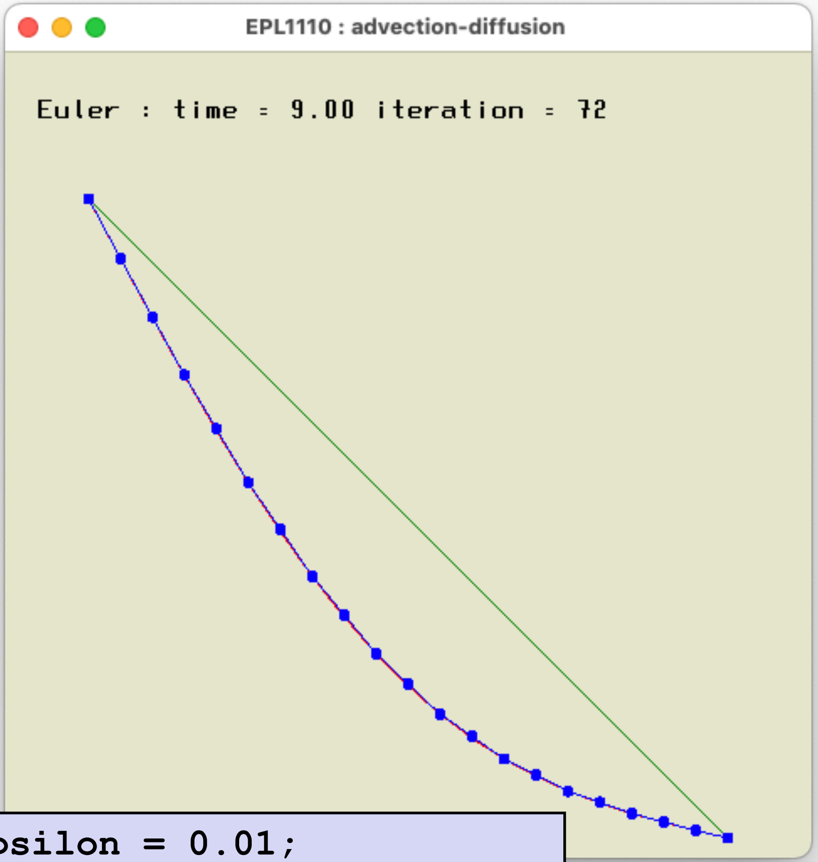
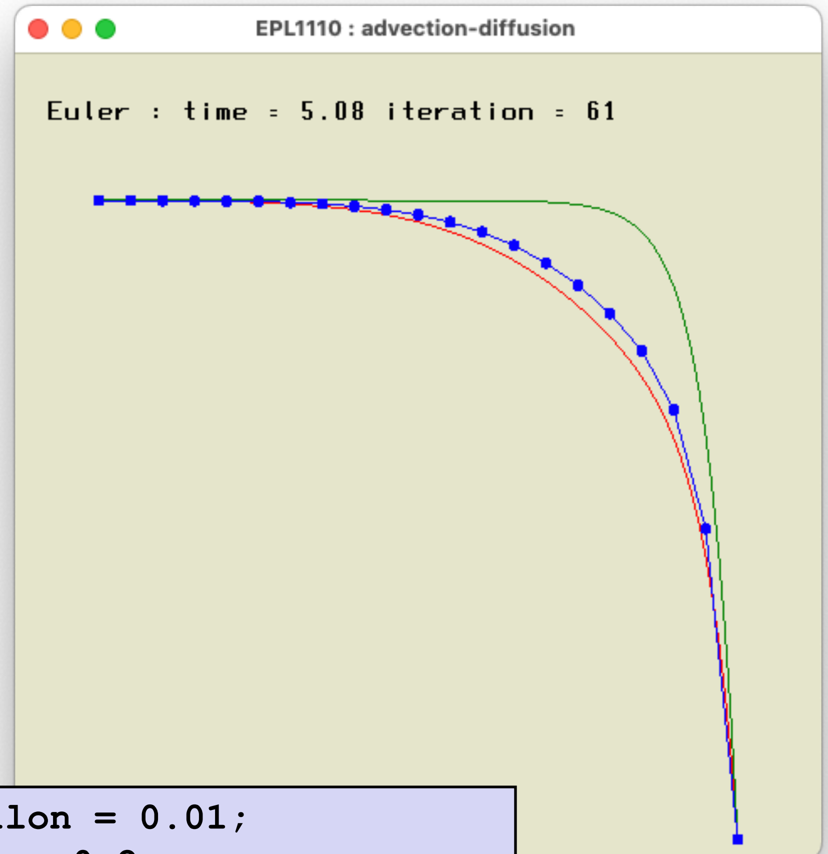



```
epsilon = 0.01;
dt = h*h/(epsilon*2.0);
```

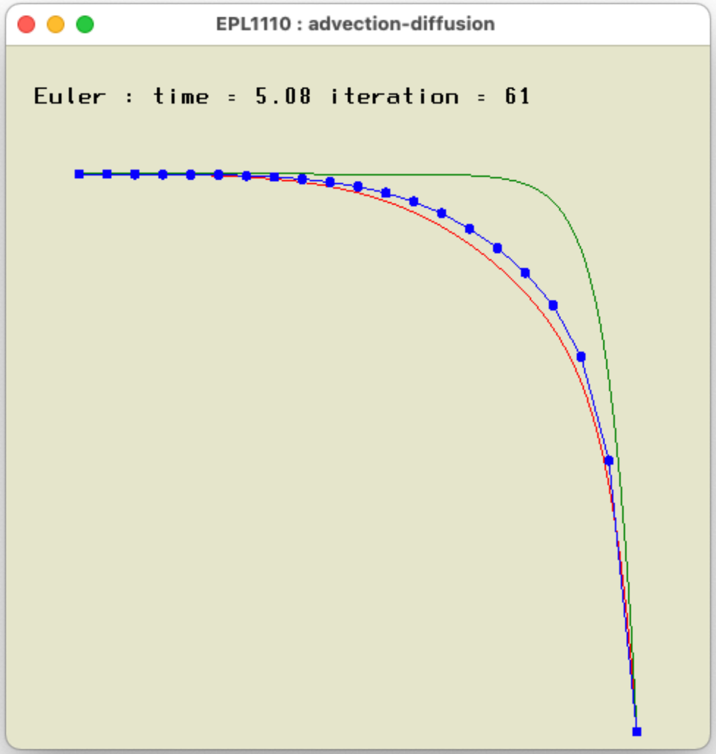
Et maintenant introduisons l'advection…

$$
\begin{aligned}
\frac{\partial u}{\partial t} + \beta \frac{\partial u}{\partial x} &= \epsilon \frac{\partial^2 u}{\partial x^2} \\
u(0) &= 1 \\
u(1) &= 0
\end{aligned}
$$




```
epsilon = 0.01;
beta = 0.2;
dt = h*h/(epsilon*3.0);
```

Et comment déduire le pas de temps ?



$$U_j^n = U^n e^{ikX_j}$$

Considérons une perturbation quelconque...

$$\begin{aligned}
U_j^{n+1} &= U_j^n + \Delta t\Big(\overbrace{\Big((\zeta - 1)\frac{\beta}{2h} + \frac{\epsilon}{h^2}\Big)}^{a} U_{j+1}^n + \overbrace{\Big(-\zeta\frac{\beta}{h} - 2\frac{\epsilon}{h^2}\Big)}^{b} U_j^n + \overbrace{\Big((\zeta + 1)\frac{\beta}{2h} + \frac{\epsilon}{h^2}\Big)}^{c} U_{j-1}^n \Big) \\
&= U_j^n \Big(1 + \Delta t\Big(a e^{ikh} + b + c e^{-ikh} \Big)\Big) \\
&= U_j^n \Big(1 + \Delta t\Big(\underbrace{(a + c)}_{-b} \cos kh + b + i \underbrace{(a - c)}_{-\beta/h} \sin kh \Big)\Big)
\end{aligned}$$

Il faut que le module du facteur d'amplification soit inférieur à l'unité :-)

$$\boxed{U = \left(1 + \Delta t\left(b - b\cos kh + b - i\frac{\beta}{h}\sin kh\right)\right)}$$

$$\left|\left(1 + \Delta t b - \Delta t b\cos(kh)\right) - i\Delta t\left(\tfrac{\beta}{h}\sin(kh)\right)\right| \leq 1$$

$$\downarrow$$

$$\begin{aligned}
1 + \Delta t^2 b^2(1-\cos(kh))^2 + 2b\Delta t(1-\cos(kh)) + \Delta t^2\tfrac{\beta^2}{h^2}\sin(kh)^2) &\leq 1\\
\Delta t^2 b^2(1-\cos(kh))^2 + 2b\Delta t(1-\cos(kh)) + \Delta t^2\tfrac{\beta^2}{h^2}(1-\cos(kh)^2) &\leq 0\\
\Delta t b^2(1-\cos(kh)) + 2b + \Delta t\tfrac{\beta^2}{h^2}(1+\cos(kh)) &\leq 0
\end{aligned}$$

On déduit finalement :

$$\Delta t \leq \frac{-2b}{(1-\cos(kh))b^2 + (1+\cos(kh))\frac{\beta^2}{h^2}}$$

$$\Delta t \leq \frac{2(\zeta\beta h + 2\epsilon)h^2}{(\zeta h\beta + 2\epsilon)^2 + \beta^2 h^2 + \cos(kh)(\beta^2 h^2 - (\zeta h\beta + 2\epsilon)^2)}$$

On conclut donc : $\boxed{\Delta t \;\leq\; \min\left(\frac{\zeta h\beta + 2\epsilon}{\beta^2}, \frac{h^2}{\zeta h\beta + 2\epsilon}\right)}$

Notons que l'on obtient les résultats habituels $\Delta t \leq \dfrac{h}{\beta}$ pour $\epsilon = 0, \zeta = 1$ et $\Delta t \leq \dfrac{h^2}{2\epsilon}$ pour $\beta = 0$.

Pratiquement…