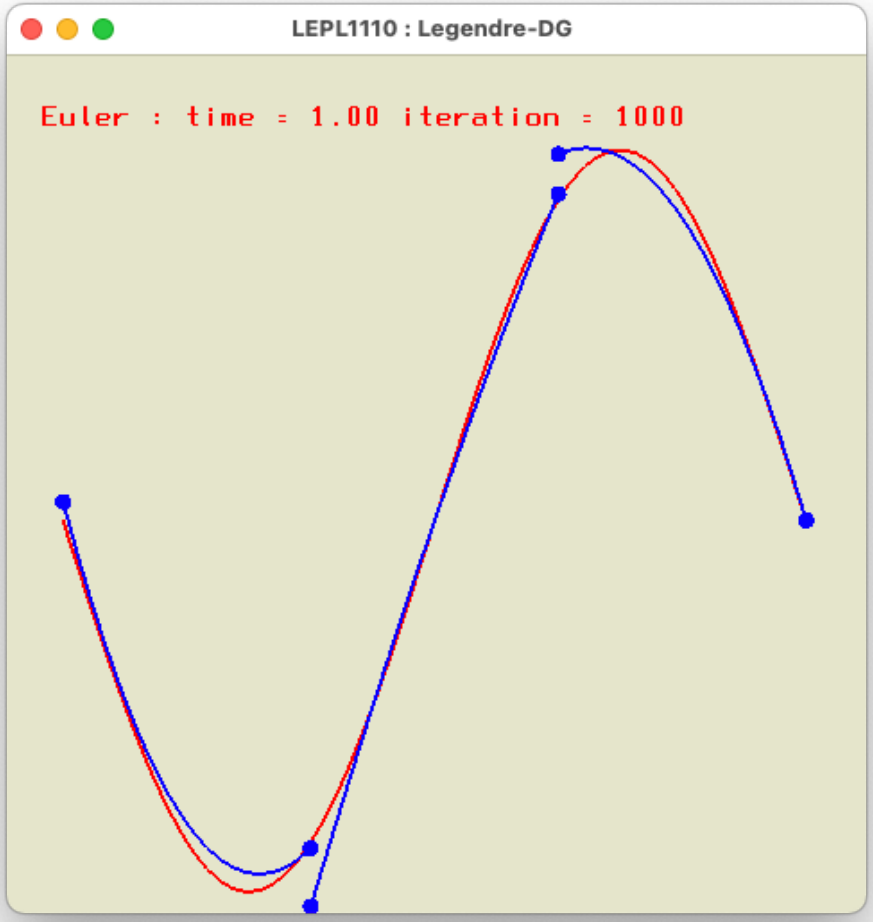
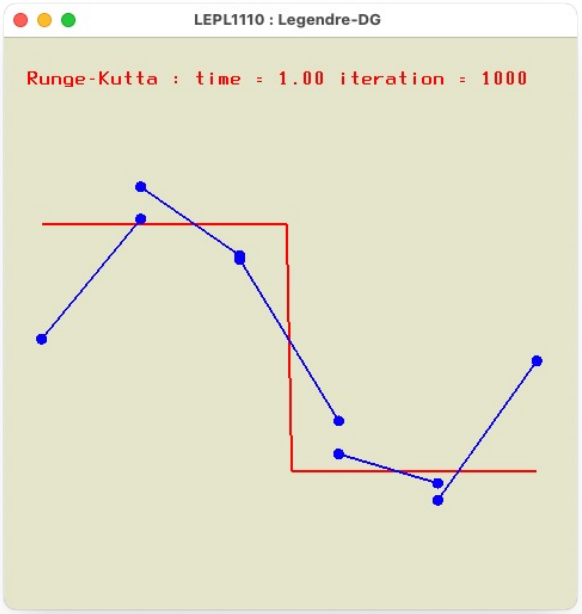
Discontinuous Galerkin Method



$$\begin{cases} \dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial x} &= 0 \\ \\ u(t, L/2) &= u(t, -L/2) \\ u(0, x) &= \overline{u}(x) \end{cases}$$

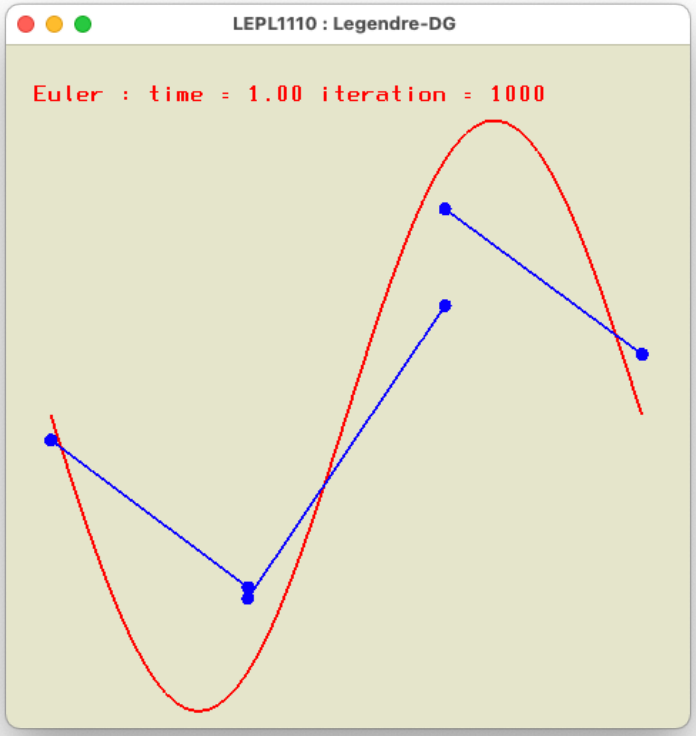
C’est quoi
physiquement
une équation
de transport ?

La solution exacte peut être discontinue !

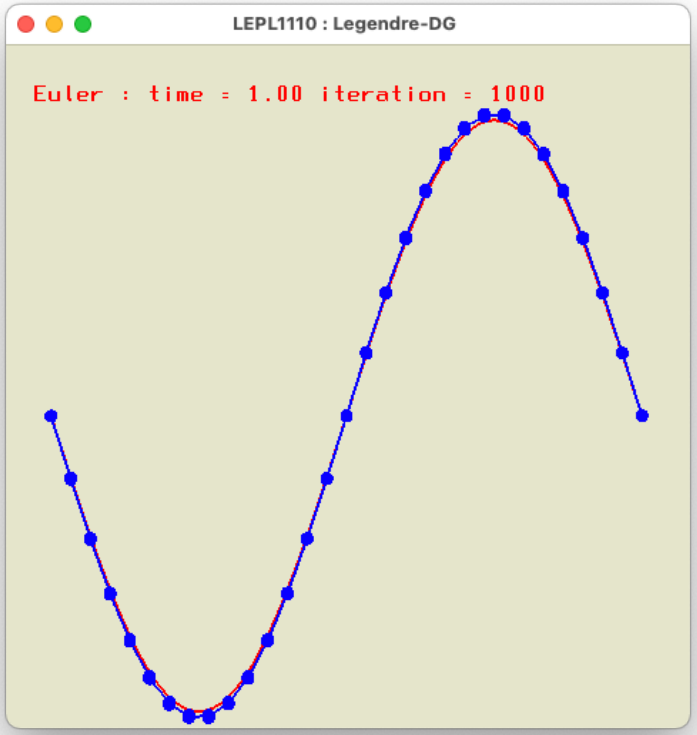


$$u(t,x) \approx u_e^h(t,x) = \sum_{i=0}^{p} U_i^e(t)\ \phi_i(x)$$

Avec peu d'éléments linéaires…



… ou beaucoup !



La méthode DG…

$$\sum_{j=0}^{p} < \phi_i \frac{\partial u_h}{\partial t} >_e + < \phi_i \frac{\partial c u_e^h}{\partial x} >_e = 0$$

En effectuant une intégration par partie,

$$\sum_{j=0}^{p} \underbrace{< \phi_i \phi_j >_e}_{A_{ij}^e} \frac{dU_j^e}{dt} = \overbrace{< \left(c \frac{d\phi_i}{dx} \right) u_e^h >_e - \left[\phi_i c u^* \right]_{X_{e-1}}^{X_e}}^{B_i^e(u^h)}$$

En multipliant par l'inverse de la matrice de masse,

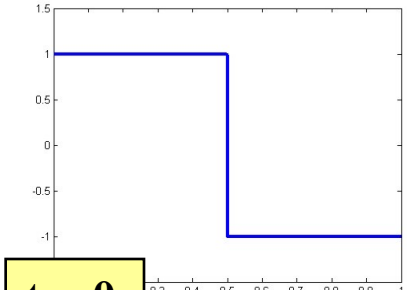
$$\frac{dU_i^e}{dt} = \underbrace{\sum_{j=0}^{p} A_{ij}^{-1} B_j^e(u^h)}_{F_i^e(u^h)}$$

L'idée originale de Lesaint-Raviart…

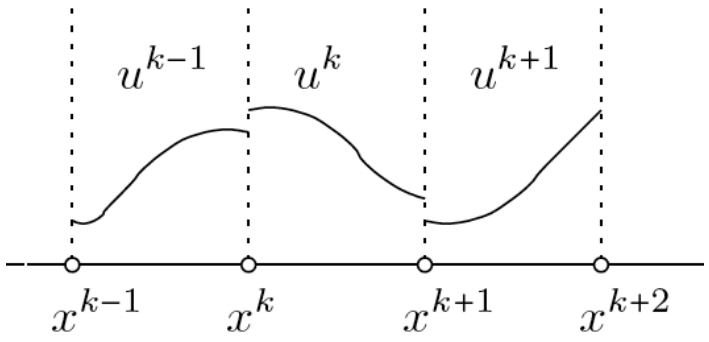
Lesaint, P., & Raviart, P. A. (1974).
On a finite element method for solving the neutron transport equation.
Publications mathématiques et informatique de Rennes, (S4), 1-40

The Discontinuous so-called Galerkin Method

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$$

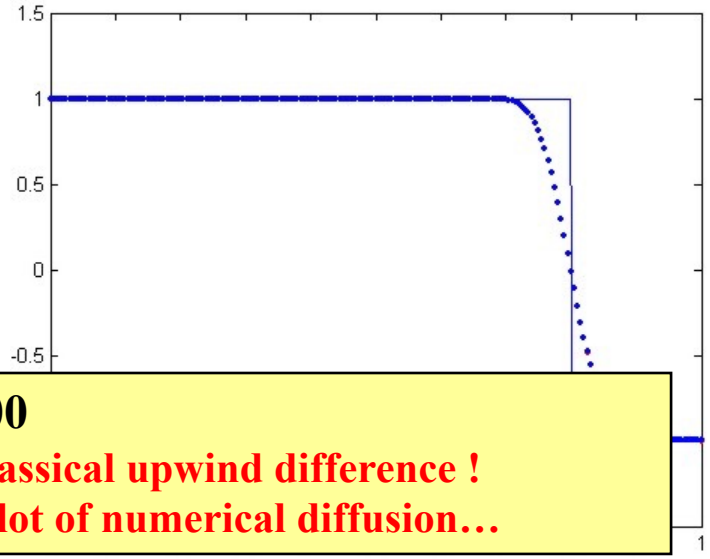


t = 0



L'approximation est discontinue !

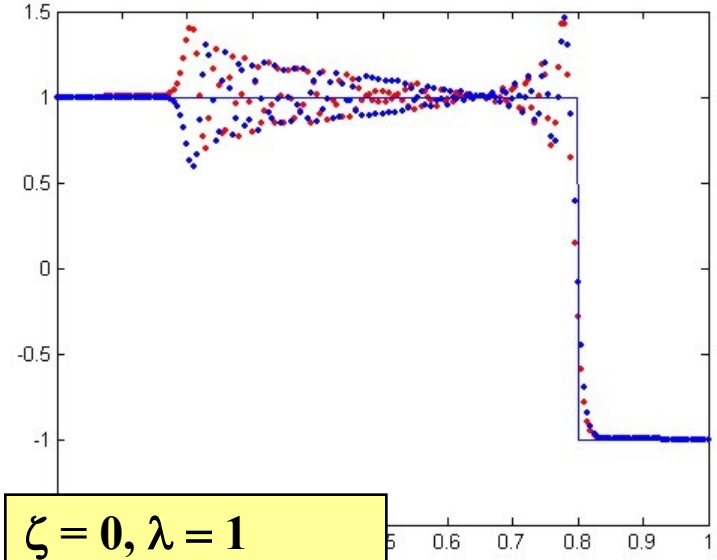
How to impose the continuity constraint ?



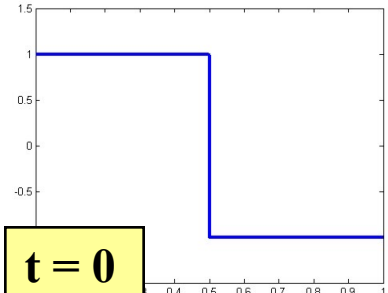
$\zeta = 1, \lambda = 100$

Almost the classical upwind difference !

Stable, but a lot of numerical diffusion…



$\zeta = 0, \lambda = 1$

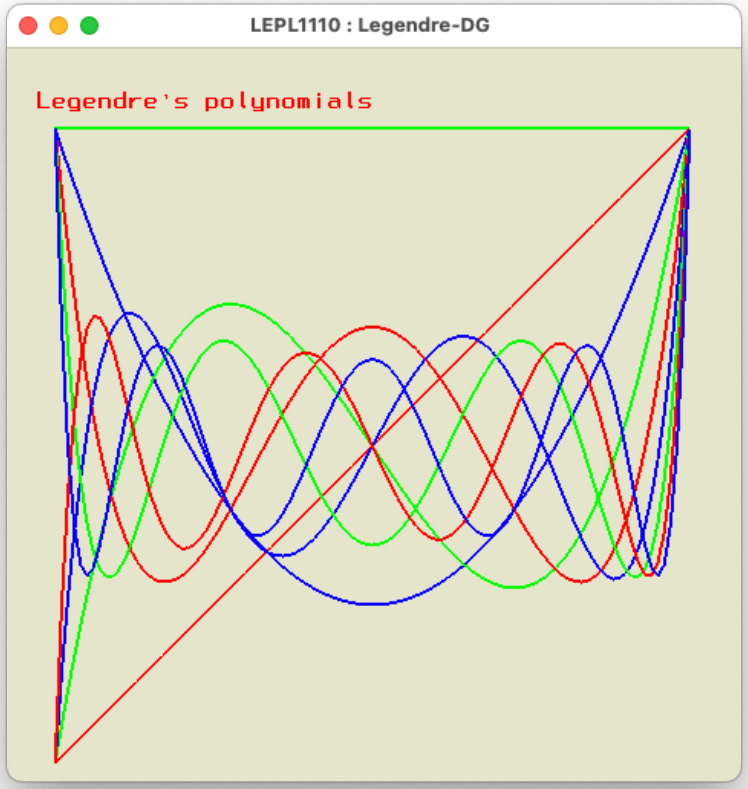


$t = 0$

$t = 0.3$

$\zeta = 1, \lambda = 1$

Des fonctions de base un fifrelin astucieuses…



$$\phi_0(x) = 1$$

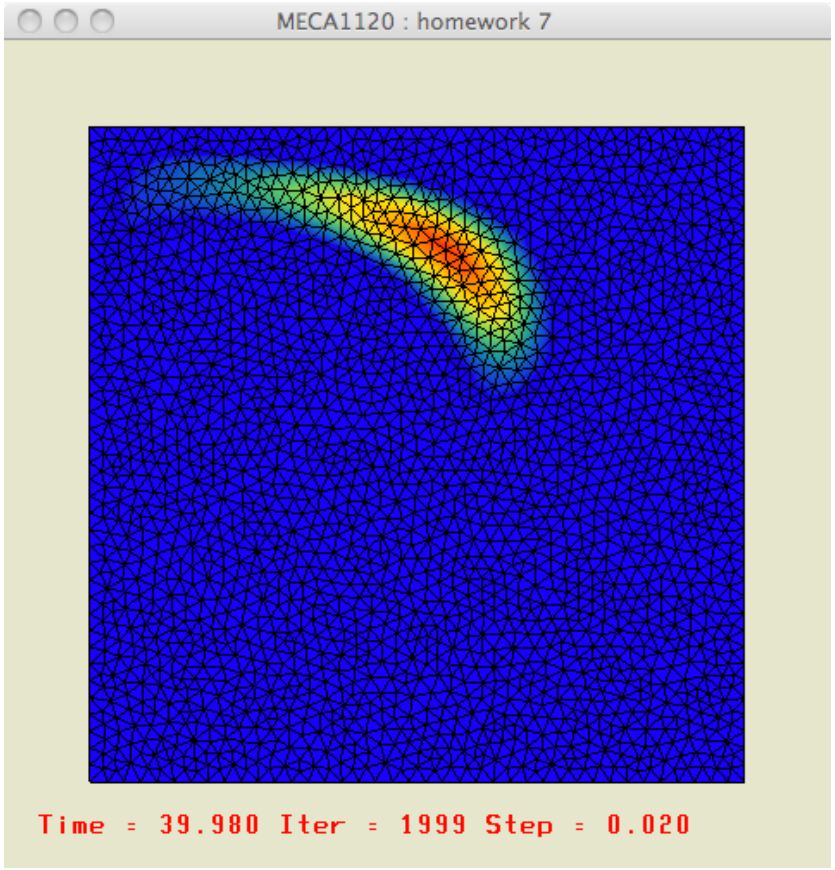
$$\phi_1(x) = x$$

$$\dots$$

$$n\ \phi_n(x) = (2n-1)\ x\ \phi_{n-1}(x) - (n-1)\ \phi_{n-2}(x)$$

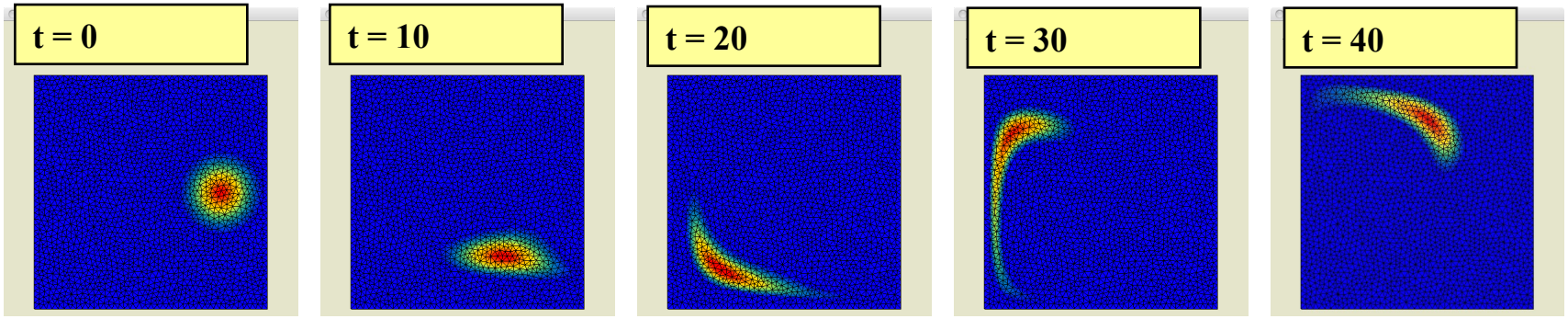
Et les deux matrices locales discrètes deviennent si simples !

Discontinuous Galerkin Method



Implémentation de la méthode DG…

$$\frac{\partial c}{\partial t} + \frac{\partial}{\partial x}(uc) + \frac{\partial}{\partial y}(vc) = 0$$



$$\frac{\partial c}{\partial t} + \frac{\partial}{\partial x}\left(uc\right) + \frac{\partial}{\partial y}\left(vc\right) \quad = \quad 0$$

$$c(t,x,y) \;\approx\; c_e^h(t,x,y) \;\approx\; \sum_{i=1}^{3} C_i^e(t)\;\phi_i(x,y)$$

$$\sum_{j=1}^{3} \underbrace{< \phi_i \phi_j >_e}_{A_{ij}^e = \hat{A}_{ij} J_e} \frac{dC_j^e}{dt} \quad = \quad \overbrace{< \left(u\frac{\partial \phi_i}{\partial x} + v\frac{\partial \phi_i}{\partial y}\right) c_e^h >_e + \ll \phi_i \beta c^* \gg_e}^{B_i^e(c^h)}$$

En multipliant par l'inverse de la matrice de masse,

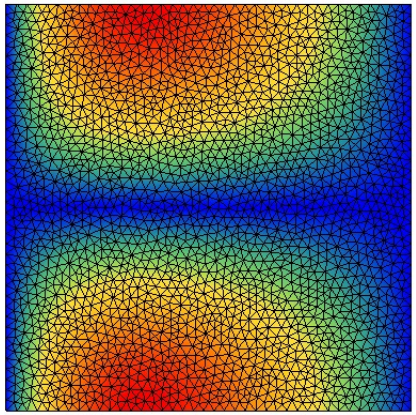
$$\frac{dC_i^e}{dt} \quad = \quad \underbrace{\sum_{j=1}^{3} \hat{A}_{ij}^{-1} B_j^e(c^h)/J_e}_{F_i^e(c^h)}$$

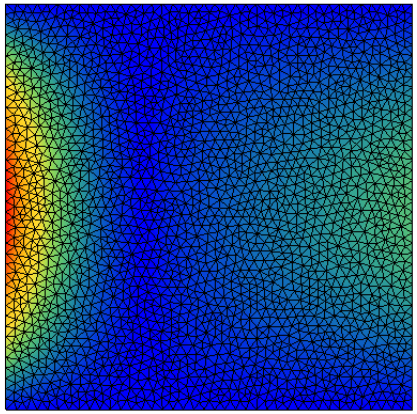
En fonction de la vitesse normale le long d'un segment, on cacule les flux à l'entrée du domaine :

$$\beta \;=\; (un_x + vn_y)$$

Equations discrètes

Un petit gyre océanique bien connu…





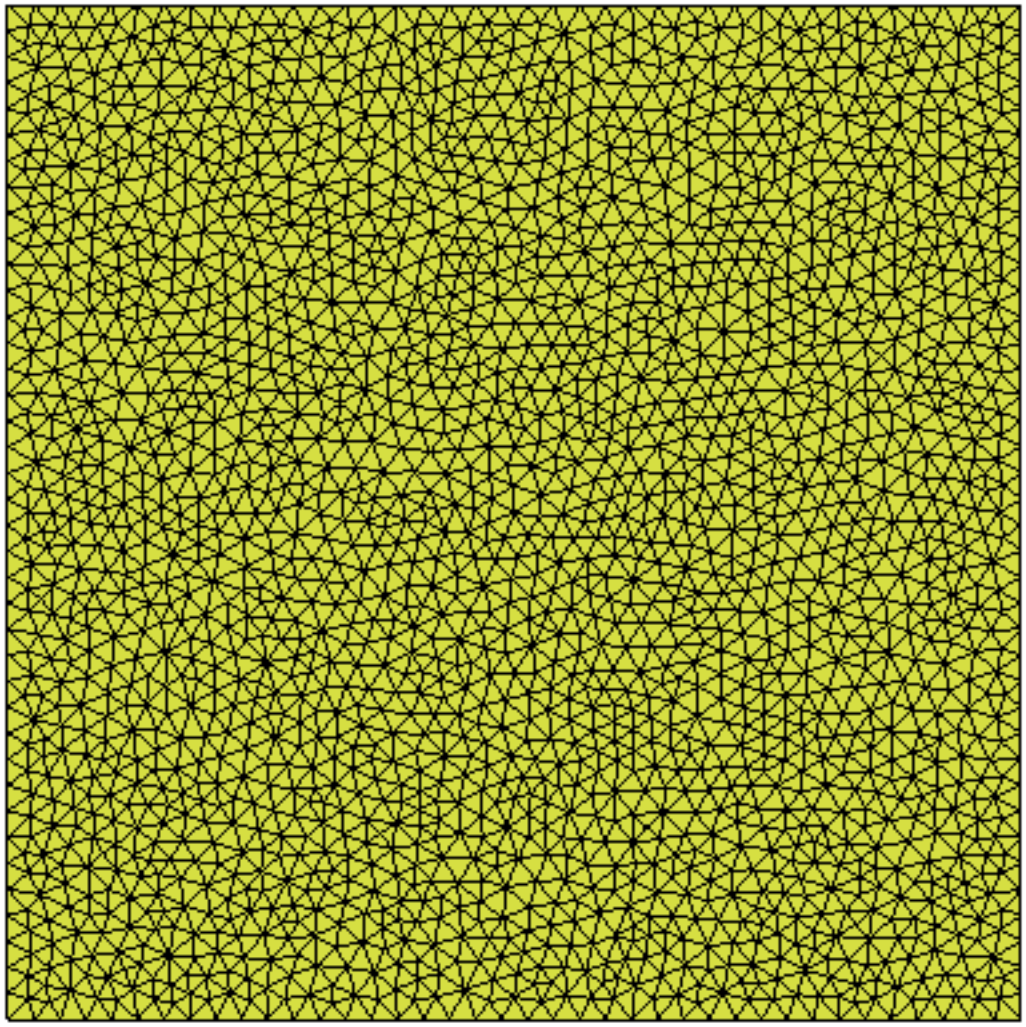
max(u) = 0.0548

max(v) = 0.1512

```
const double tau0 = 0.1;
const double L = 1e6;
const double gamm = 1e-6;
const double rho = 1000;
const double delta = 1;
const double g = 9.81;
const double h = 1000;
const double f0 = 1e-4;
const double beta = 0.5e-11;

double Y = y - 0.5;
double epsilon = gamm / (L * beta);
double Z1 = (-1 + sqrt(1 + (2 * M_PI * delta * epsilon) * (2 * M_PI * delta * epsilon))) / (2 * epsilon);
double Z2 = (-1 - sqrt(1 + (2 * M_PI * delta * epsilon) * (2 * M_PI * delta * epsilon))) / (2 * epsilon);
double D = ((exp(Z2) - 1) * Z1 + (1 - exp(Z1)) * Z2) / (exp(Z1) - exp(Z2));
double f1 = M_PI / D * (1 + ((exp(Z2) - 1) * exp(x * Z1) + (1 - exp(Z1)) * exp(x * Z2)) / (exp(Z1) - exp(Z2)));
double f2 = 1 / D* (((exp(Z2) - 1) * Z1 * exp(x * Z1) + (1 - exp(Z1)) * Z2 * exp(x * Z2)) / (exp(Z1) - exp(Z2)));

u[0] = D * tau0 / (M_PI * gamm * rho * h) * f1 * sin(M_PI * Y);
v[0] = D * tau0 / (M_PI * gamm * rho * delta * h) * f2 * cos(M_PI * Y);
```

Comme évaluer cela prend beaucoup de temps, on va calculer une approximation linéaire discontinue de la vitesse !

Un joli maillage…



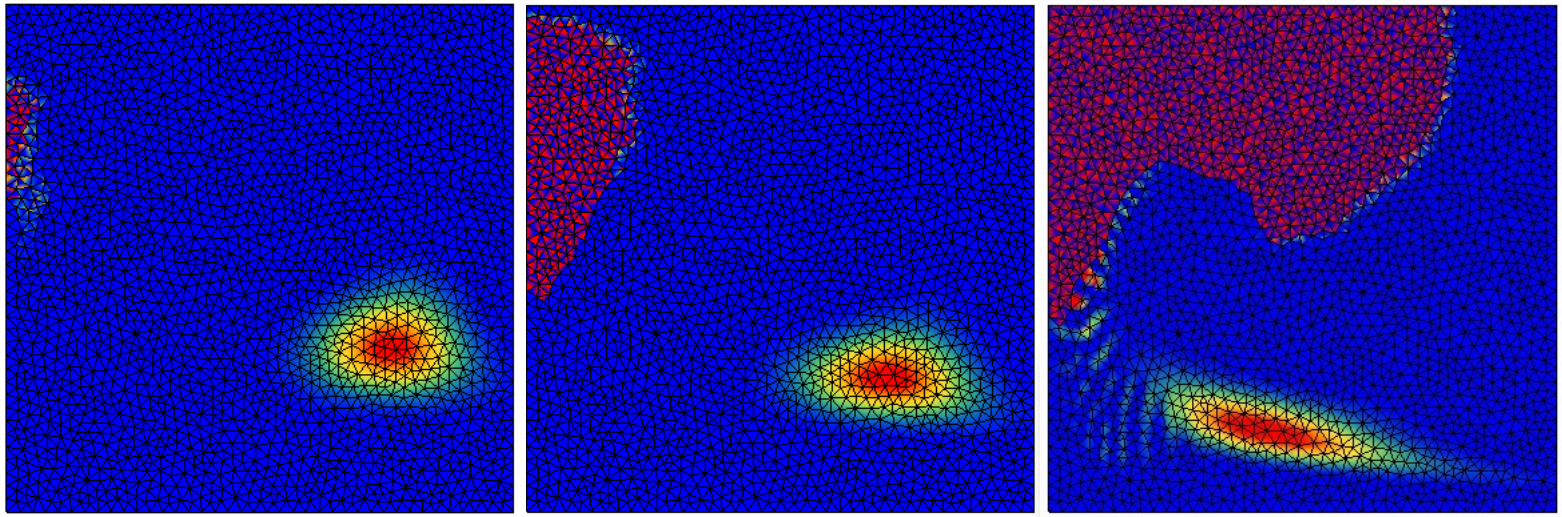
Taille typique des éléments : 0.016 … 0.032

Un pas de temps critique…

$$\Delta t \leq \frac{h}{\beta}$$

Pas de temps = 0.8 : Boum !

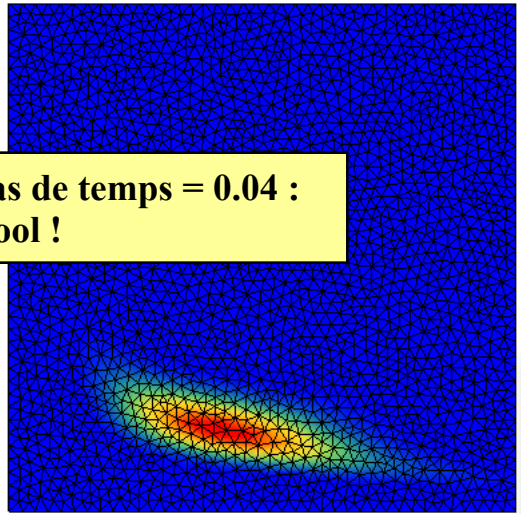
Endroit le plus critique !



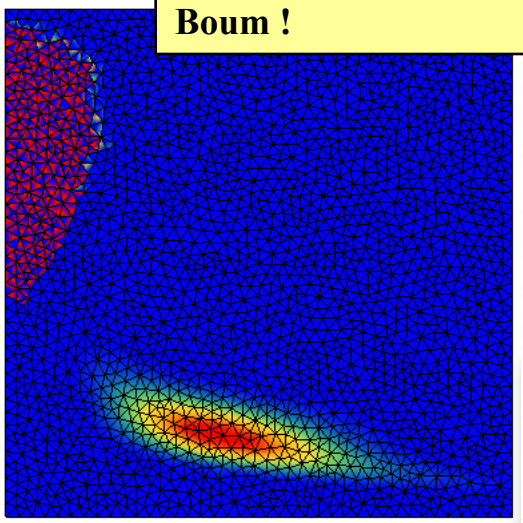
Plus petit élément : 0.016
Vitesse maximale : 0.152
Le plus petit pas de temps : 0.10 environ :-)

Diminuous le pas de temps…

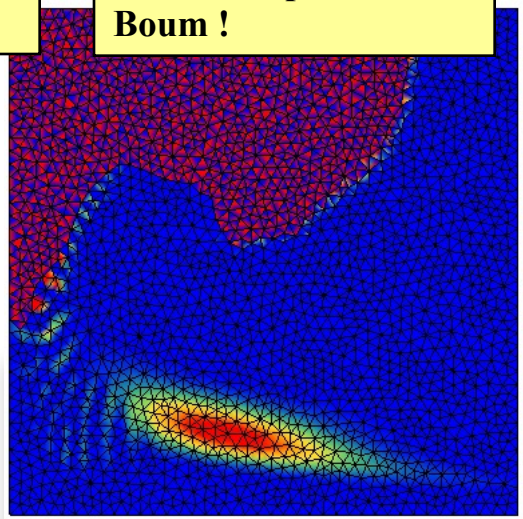
$$\Delta t \leq \frac{h}{\beta}$$



Pas de temps = 0.04 :
Cool !



Pas de temps = 0.06 :
Boum !



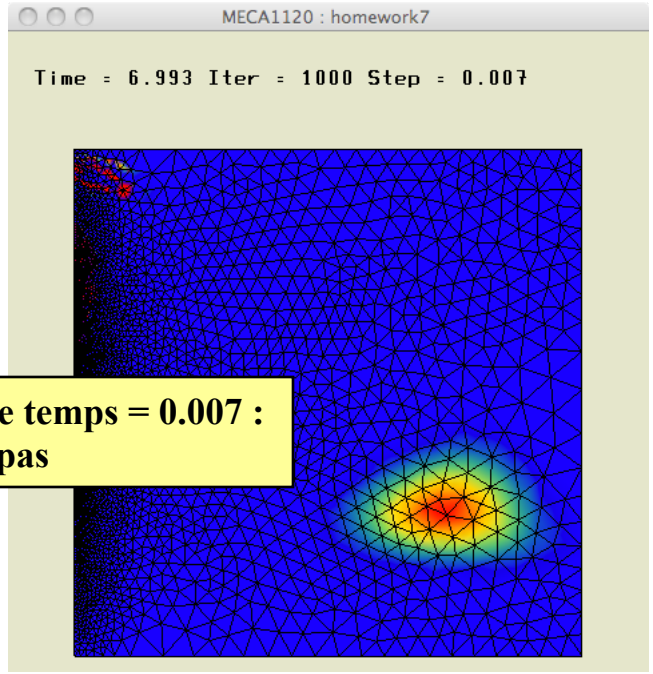
Pas de temps = 0.08 :
Boum !

Plus petit élément : 0.016
Vitesse maximale : 0.152
Le plus petit pas de temps : 0.10 environ :-)

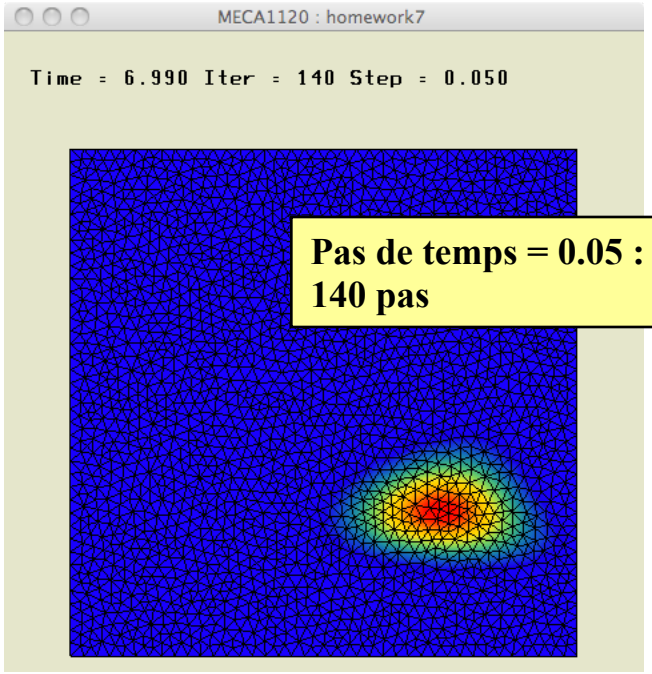
The time stepping issue

C'est le plus petit élément qui est contraignant !
Très très ennuyeux !

Endroit le plus critique !

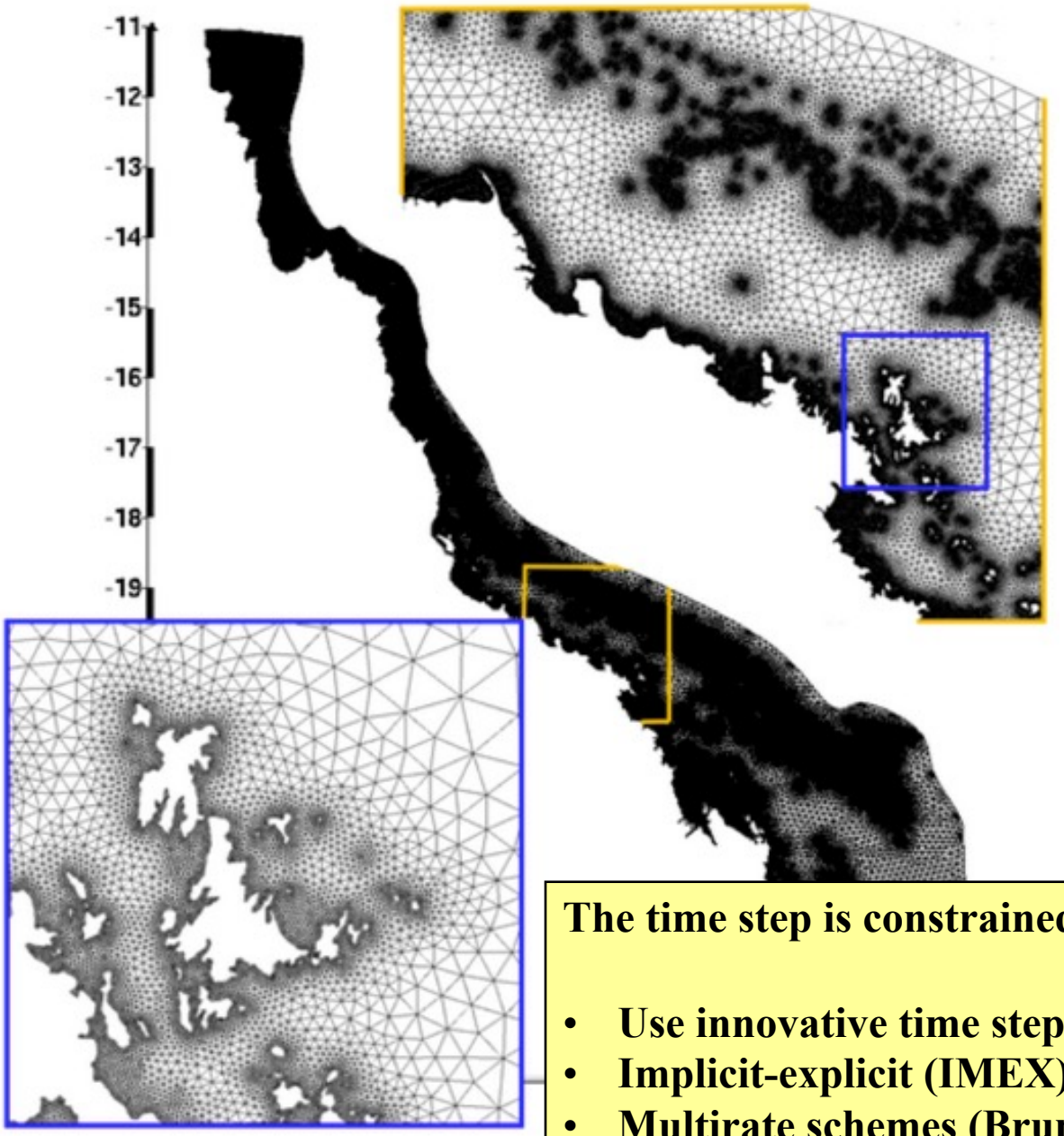


Pas de temps = 0.007 : 1000 pas



Pas de temps = 0.05 : 140 pas

The time stepping issue



- *890,000 triangles*
- *Smallest element : 7 m*
- *Largest element : 3,300 m*
- *99.9 % > 60m*

The time step is constrained by the smallest element .

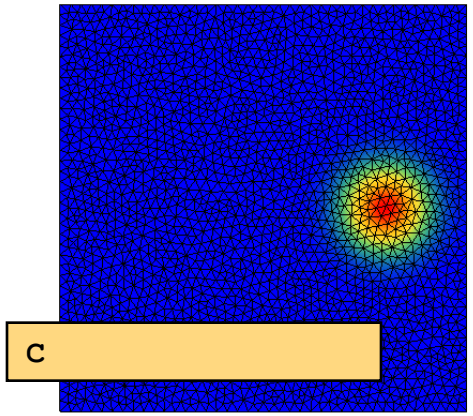
- **Use innovative time stepping procedures**
- **Implicit-explicit (IMEX) schemes**
- **Multirate schemes (Bruno Seny)**

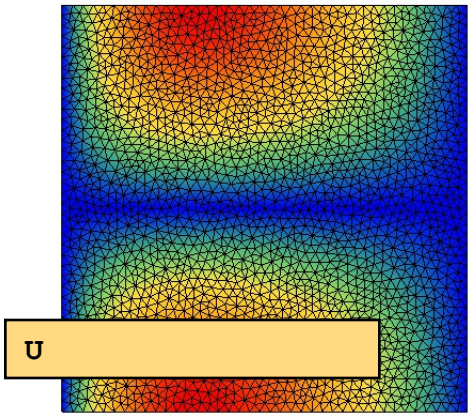
Notre problème…

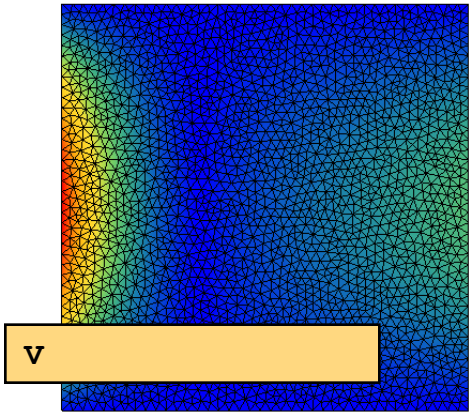
```
typedef struct {
    femMesh *mesh;
    femEdges *edges;
    ...
    int size;
    double *C;
    double *U;
    double *V;
    ...
} femAdvectionProblem;
```

Ces trois champs sont initialisés avec une apprimation au sens des moindres carrés L2

```
int myProblem->size = myProblem->mesh->nElem * nLocal + 1;
```







Et calculons aussi l'inverse de la matrice de masse

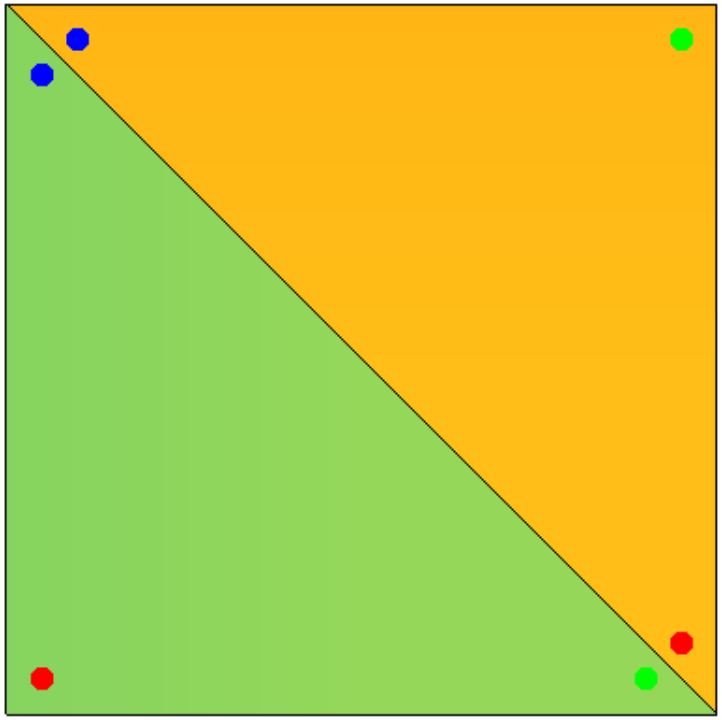
$\hat{A}_{ij}$

```
+8.3333333333e-02 +4.1666666667e-02 +4.1666666667e-02
+4.1666666667e-02 +8.3333333333e-02 +4.1666666667e-02
+4.1666666667e-02 +4.1666666667e-02 +8.3333333333e-02
```

$\hat{A}^{-1}_{ij}$

```
+1.8000000000e+01 -6.0000000000e+00 -6.0000000000e+00
-6.0000000000e+00 +1.8000000000e+01 -6.0000000000e+00
-6.0000000000e+00 -6.0000000000e+00 +1.8000000000e+01
```

Des triangles…



Nombre d'éléments : n = 2

```
Number of nodes 4
     0 :  0.0000000e+00  0.0000000e+00
     1 :  1.0000000e+00  0.0000000e+00
     2 :  0.0000000e+00  1.0000000e+00
     3 :  1.0000000e+00  1.0000000e+00
Number of triangles   2
     0 :       0       1       2
     1 :       1       3       2
```

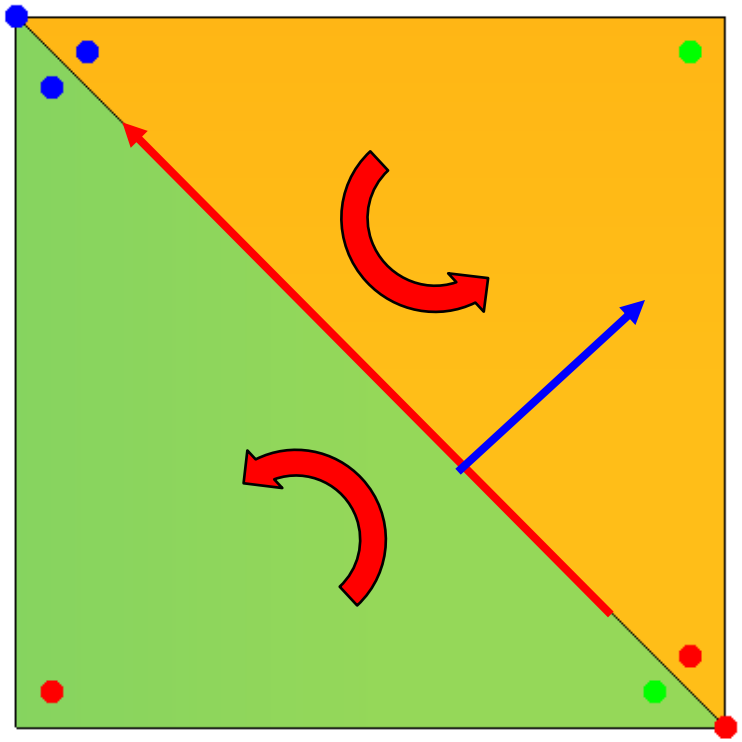
```
double *C = theProblem->soluce;
double a = 0.4;
double b = 0.7;

C[0] = a;
C[1] = a;
C[2] = a;
C[3] = b;
C[4] = b;
C[5] = b;
C[6] = 0;
```

Approximation linéaire discontinue

Nombre de degrés de liberté intérieur : 3n !
On ajoute un degré de liberté pour l'extérieur

On a un vecteur de taille 3n+1

…et des segments



```
Number of nodes 4
     0 :  0.0000000e+00  0.0000000e+00
     1 :  1.0000000e+00  0.0000000e+00
     2 :  0.0000000e+00  1.0000000e+00
     3 :  1.0000000e+00  1.0000000e+00
Number of triangles   2
     0 :       0       1       2
     1 :       1       3       2
```

0 :	3	2 :	1	-1
1 :	1	3 :	1	-1
2 :	1	2 :	0	1
3 :	2	0 :	0	-1
4 :	0	1 :	0	-1

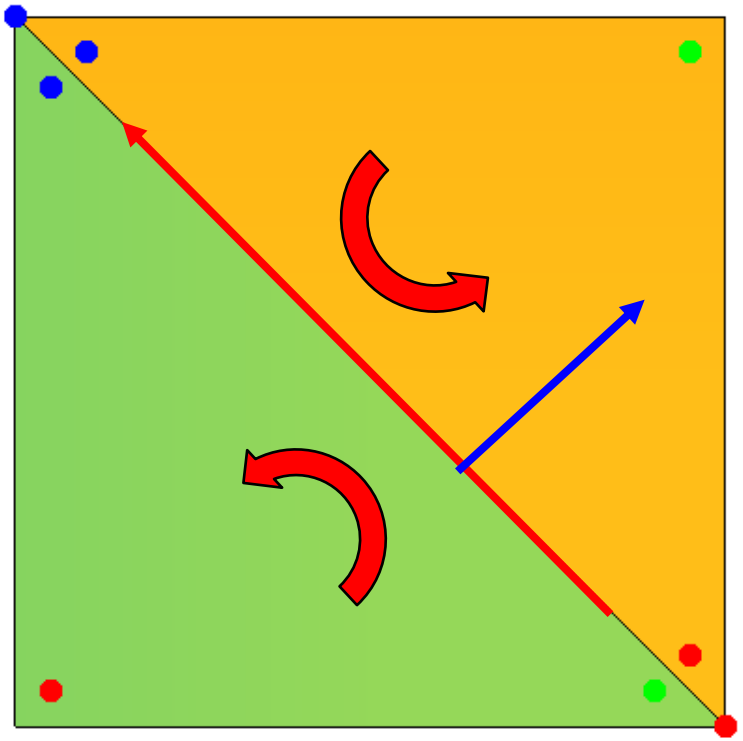
Sommets Eléments

L'orientation est en accord avec le premier élement et opposée au second !

Obtenir les indices des degrés de liberté

```
Number of nodes 4
     0 :  0.0000000e+00  0.0000000e+00
     1 :  1.0000000e+00  0.0000000e+00
     2 :  0.0000000e+00  1.0000000e+00
     3 :  1.0000000e+00  1.0000000e+00
Number of triangles   2
     0 :      0      1      2
     1 :      1      3      2
```

```
     0 :    3    2 :    1   -1
     1 :    1    3 :    1   -1
     2 :    1    2 :    0    1
     3 :    2    0 :    0   -1
     4 :    0    1 :    0   -1
```



```
femAdvTriangleMap(myProblem,1,map);
```

```
map[3] = {3,4,5};
```

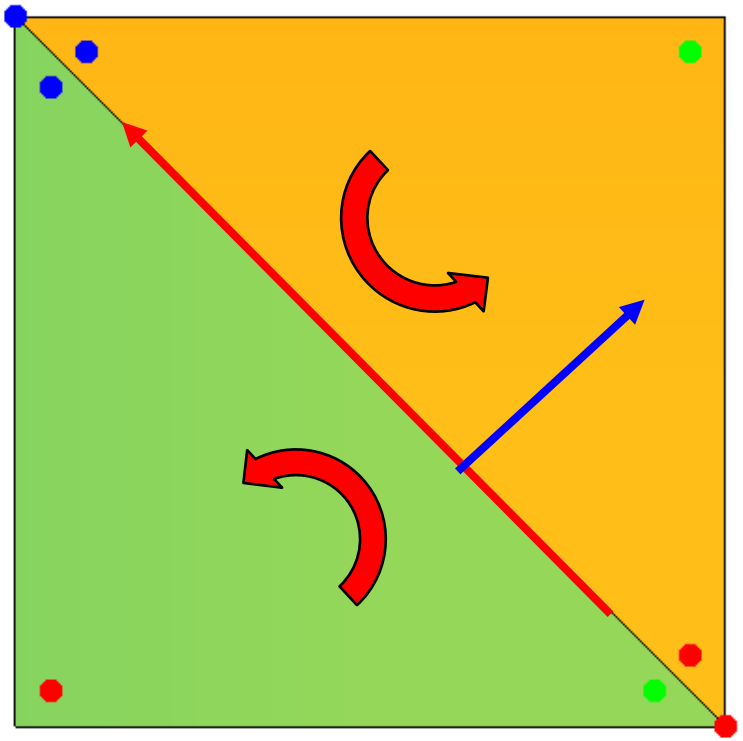
```
femAdvTriangleMap(myProblem,0,map);
```

```
map[3] = {0,1,2};
```

Obtenir les indices des degrés de liberté

```
Number of nodes 4
     0 :  0.0000000e+00  0.0000000e+00
     1 :  1.0000000e+00  0.0000000e+00
     2 :  0.0000000e+00  1.0000000e+00
     3 :  1.0000000e+00  1.0000000e+00
Number of triangles   2
     0 :      0      1      2
     1 :      1      3      2
```

```
     0 :    3    2 :    1   -1
     1 :    1    3 :    1   -1
     2 :    1    2 :    0    1
     3 :    2    0 :    0   -1
     4 :    0    1 :    0   -1
```



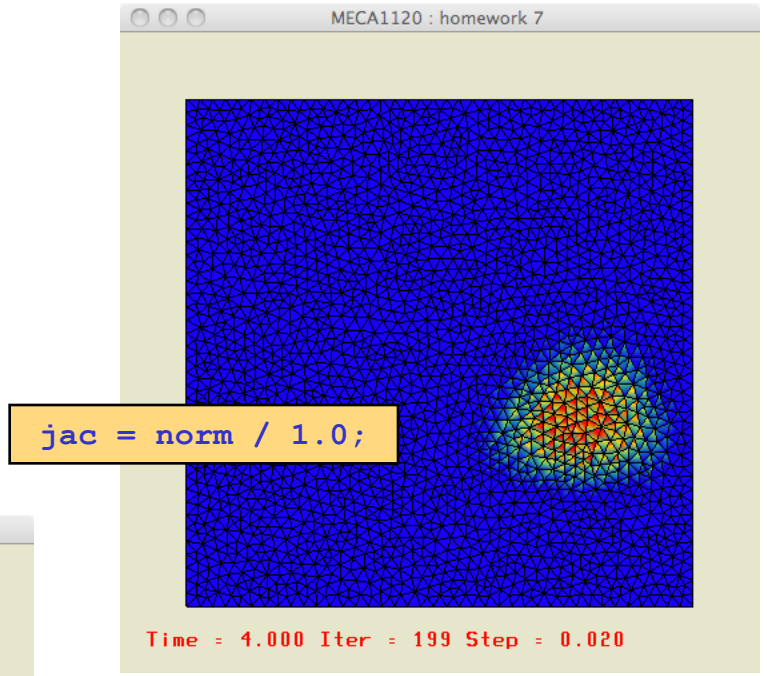
```
femAdvEdgeMap(myProblem,2,map);
```

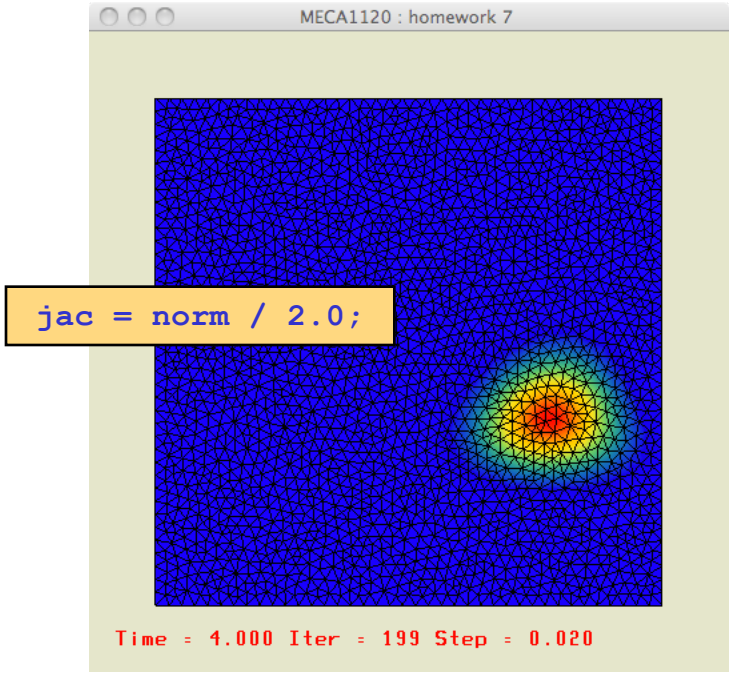
```
map[2][2] = {{1,2};{3,5}};
```

```
femAdvEdgeMap(myProblem,4,map);
```

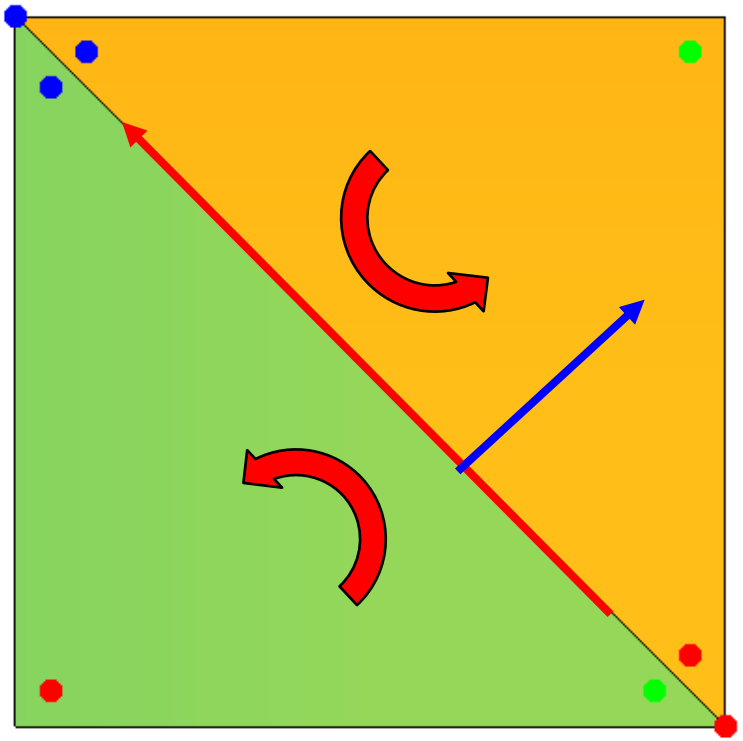
```
map[2][2] = {{0,1};{6,6}};
```

Help…
Professor
It blows up !





Commencer par le petit maillage !



```
Number of nodes 4
     0 :  0.0000000e+00  0.0000000e+00
     1 :  1.0000000e+00  0.0000000e+00
     2 :  0.0000000e+00  1.0000000e+00
     3 :  1.0000000e+00  1.0000000e+00
Number of triangles   2
     0 :      0      1      2
     1 :      1      3      2
```

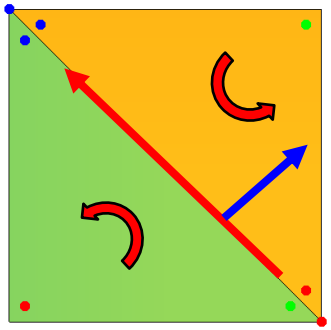
```
     0 :    3    2 :    1   -1
     1 :    1    3 :    1   -1
     2 :    1    2 :    0    1
     3 :    2    0 :    0   -1
     4 :    0    1 :    0   -1
```

```
map = dg.edgeMap(myProblem,2)
```

```
map = [[1,2],[3,5]]
```

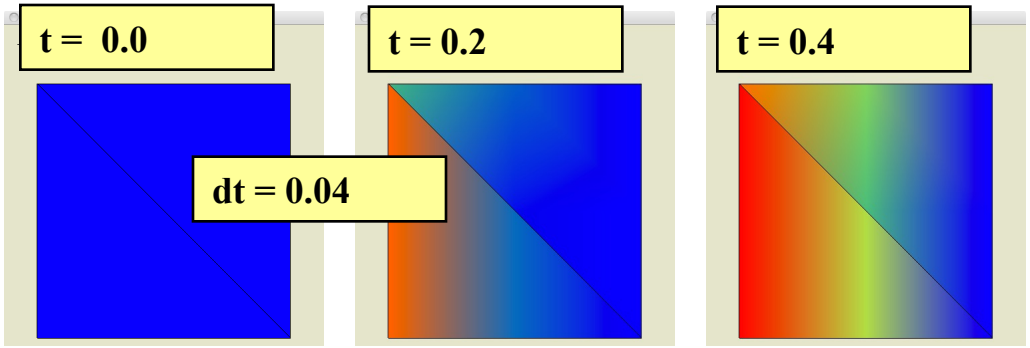
```
map = dg.edgeMap(myProblem,4)
```

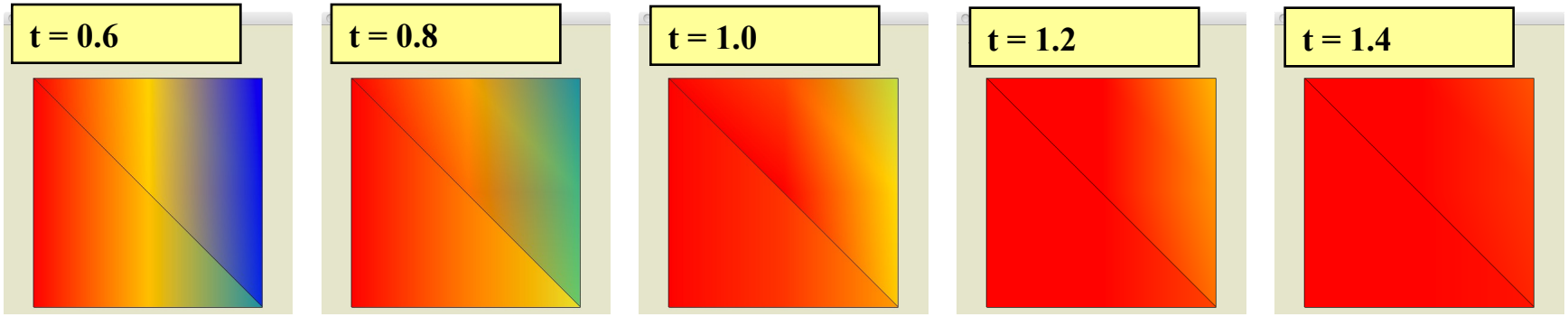
```
map = [[0,1],[]]
```

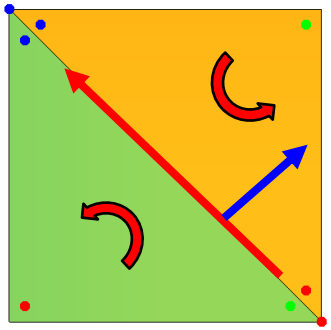


U = 1
V = 0
C = 0

Un problème tout simple tout d'abord



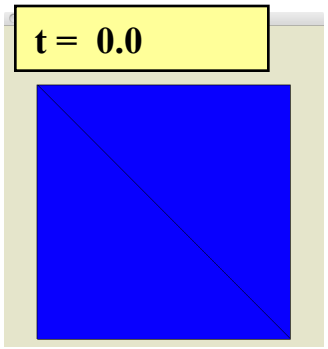


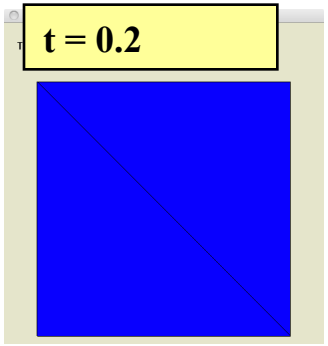


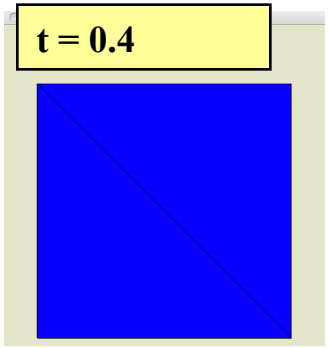
```
advectionAddIntegralsTriangles(theProblem);
advectionAddIntegralsEdges(theProblem);
advectionMultiplyInverseMatrix(theProblem);
```

U = 1
V = 0
C = 0

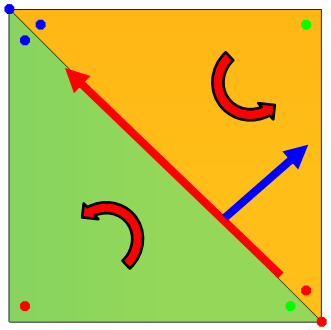
```
 +0.0e+00
 +0.0e+00
 +0.0e+00
 +0.0e+00
 +0.0e+00
 +0.0e+00
 +0.0e+00
-- after triangles
 +0.0e+00
 +0.0e+00
 +0.0e+00
 +0.0e+00
 +0.0e+00
 +0.0e+00
 +0.0e+00
 +0.0e+00
-- after edges
```



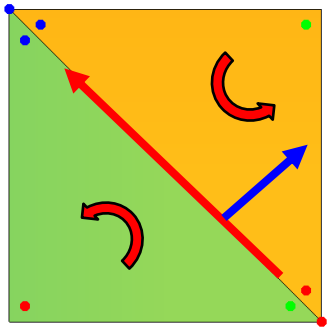




Sanity check #1



Comment imposer un flux de matière sur le coté gauche ?

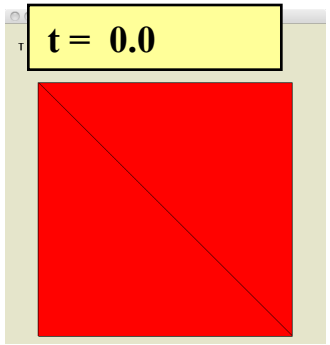


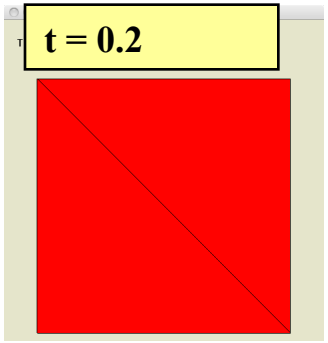
```
advectionAddIntegralsTriangles(theProblem);
advectionAddIntegralsEdges(theProblem);
advectionMultiplyInverseMatrix(theProblem);
```

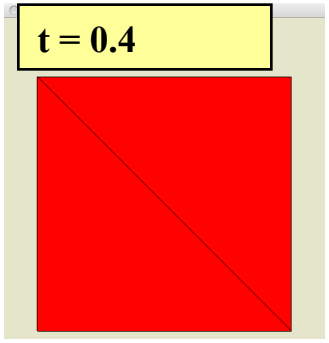
U = 1
V = 0
C = 1

Flux à gauche = 1

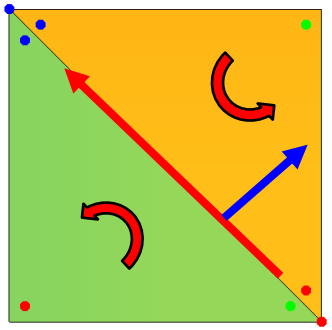
```
 +5.0e-01
 -5.0e-01
 -2.8e-17
 +2.8e-17
 -5.0e-01
 +5.0e-01
 +4.4e-16
-- after edges
 -1.1e-15
 +1.0e-15
 -2.8e-17
 +2.8e-17
 +9.2e-16
 -1.1e-15
 +4.4e-16
-- after triangles
```

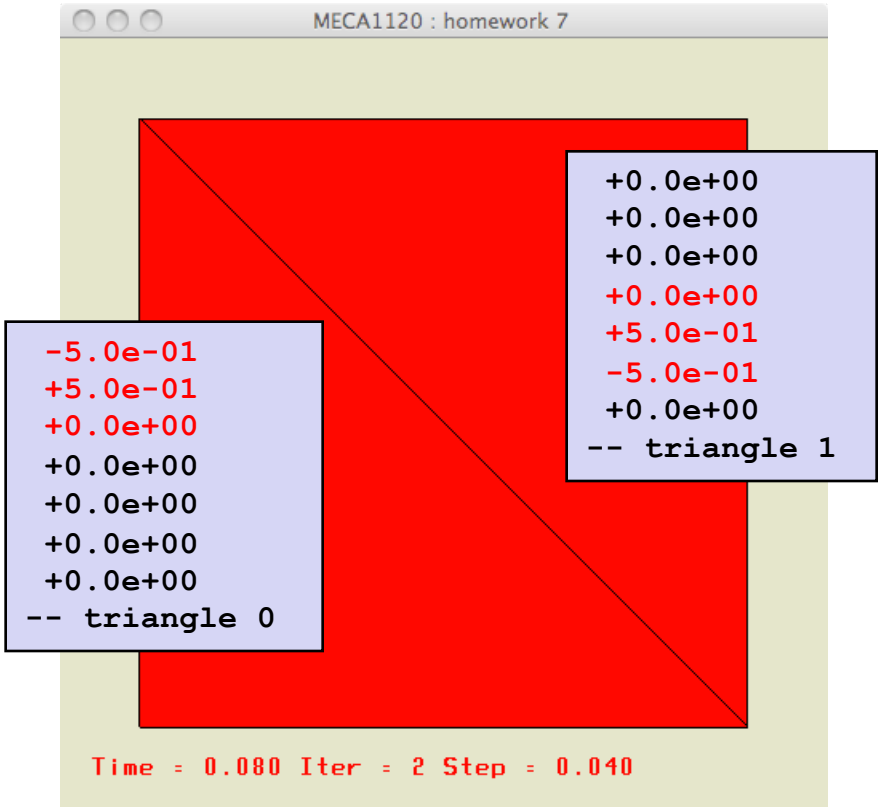






Sanity check #2

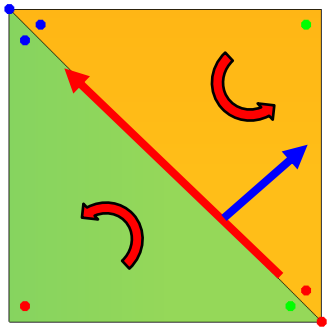


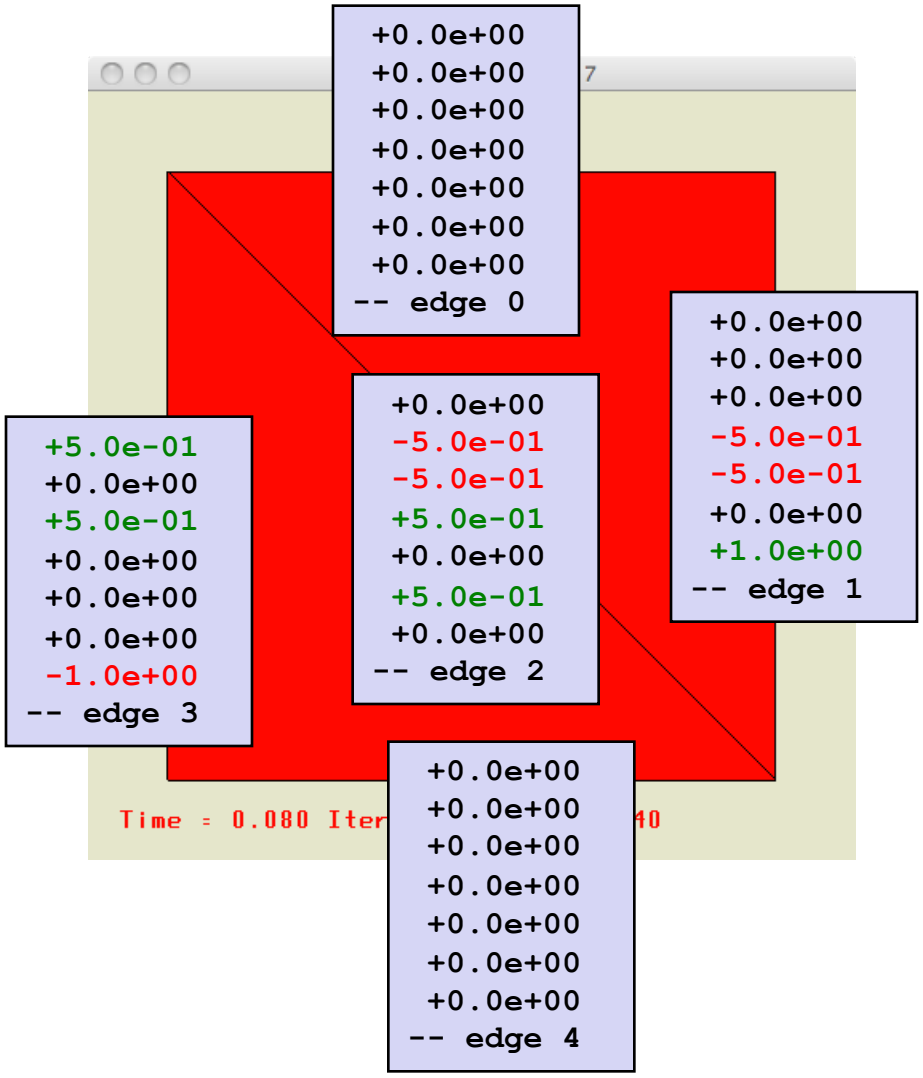



```
Number of nodes 4
     0 :  0.0000000e+00  0.0000000e+00
     1 :  1.0000000e+00  0.0000000e+00
     2 :  0.0000000e+00  1.0000000e+00
     3 :  1.0000000e+00  1.0000000e+00
Number of triangles   2
     0 :      0      1      2
     1 :      1      3      2
```

```
     0 :    3    2 :    1   -1
     1 :    1    3 :    1   -1
     2 :    1    2 :    0    1
     3 :    2    0 :    0   -1
     4 :    0    1 :    0   -1
```

Sanity check #2

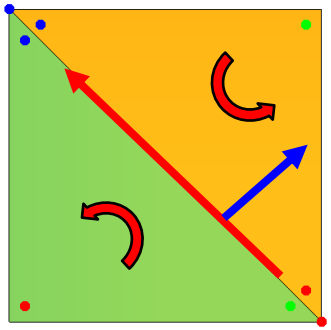





```
Number of nodes 4
     0 :  0.0000000e+00  0.0000000e+00
     1 :  1.0000000e+00  0.0000000e+00
     2 :  0.0000000e+00  1.0000000e+00
     3 :  1.0000000e+00  1.0000000e+00
Number of triangles   2
     0 :       0       1       2
     1 :       1       3       2
```

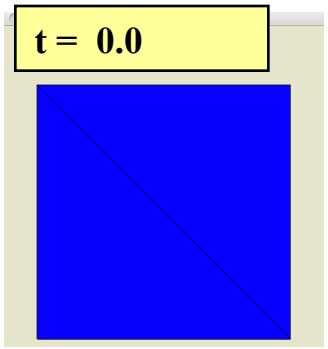
```
     0 :    3    2 :    1   -1
     1 :    1    3 :    1   -1
     2 :    1    2 :    0    1
     3 :    2    0 :    0   -1
     4 :    0    1 :    0   -1
```

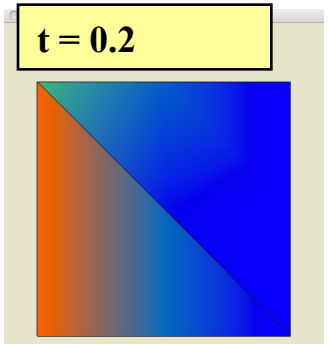
Sanity check #2

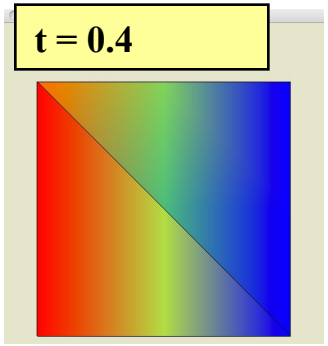


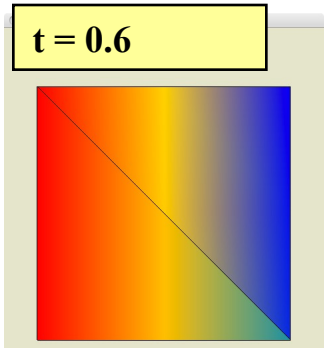
U = 1
V = 0
C = 0

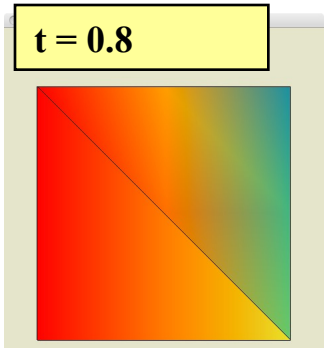
And now and only now

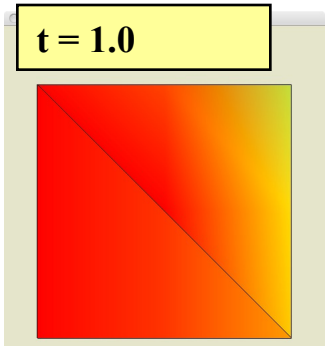


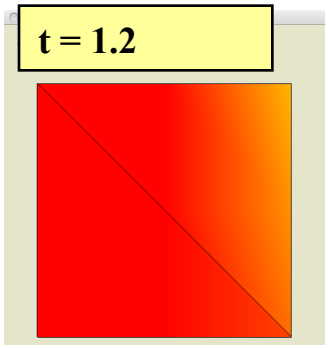


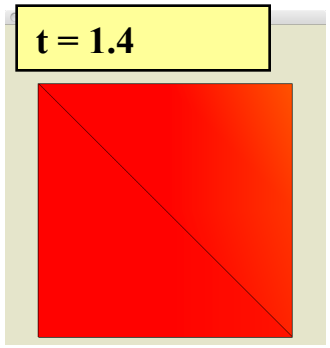




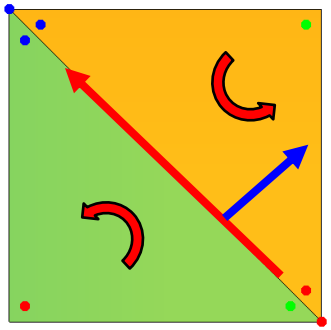




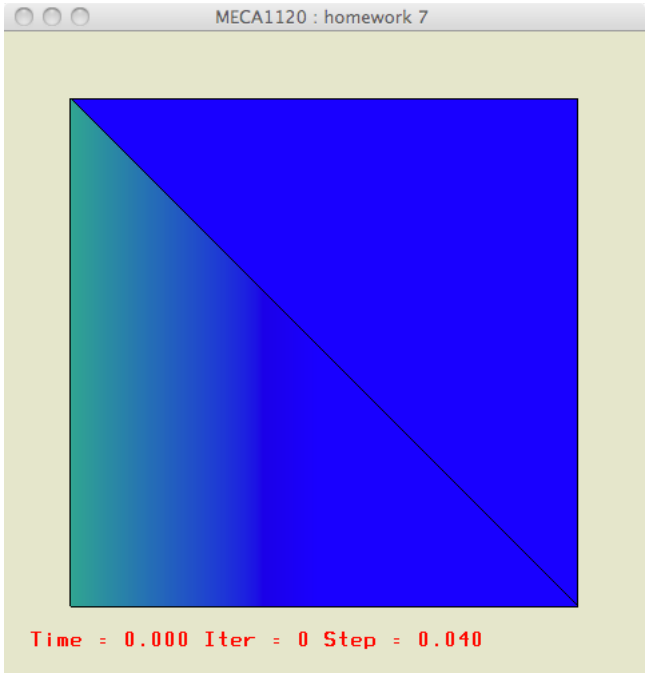




Itération one :-)



```
advectionAddIntegralsTriangles(theProblem);
advectionAddIntegralsEdges(theProblem);
advectionMultiplyInverseMatrix(theProblem);
```

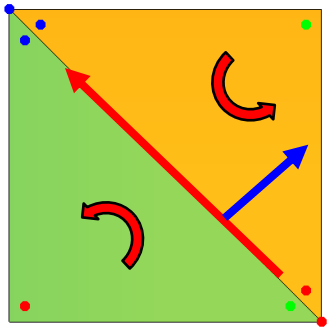


$$< \left(u \frac{\partial \phi_i}{\partial x} + v \frac{\partial \phi_i}{\partial y} \right) c_e^h >_e$$

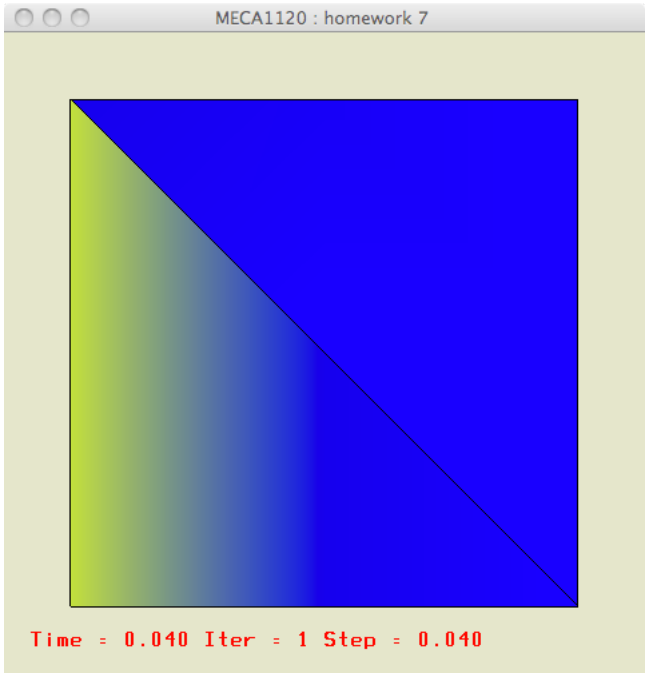
$$\ll \phi_i \beta c^* \gg_e$$

```
 +0.0e+00
 +0.0e+00
 +0.0e+00
 +0.0e+00
 +0.0e+00
 +0.0e+00
 +0.0e+00
-- after triangles
 +5.0e-01
 +0.0e+00
 +5.0e-01
 +0.0e+00
 +0.0e+00
 +0.0e+00
 -1.0e+00
-- after edges
```

Itération two :-)



```
advectionAddIntegralsTriangles(theProblem);
advectionAddIntegralsEdges(theProblem);
advectionMultiplyInverseMatrix(theProblem);
```

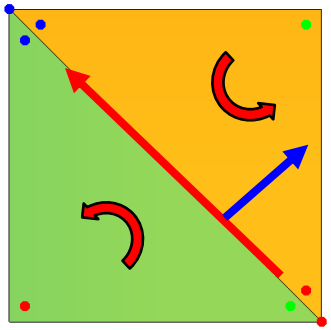


$$< \left(u \frac{\partial \phi_i}{\partial x} + v \frac{\partial \phi_i}{\partial y} \right) c_e^h >_e$$

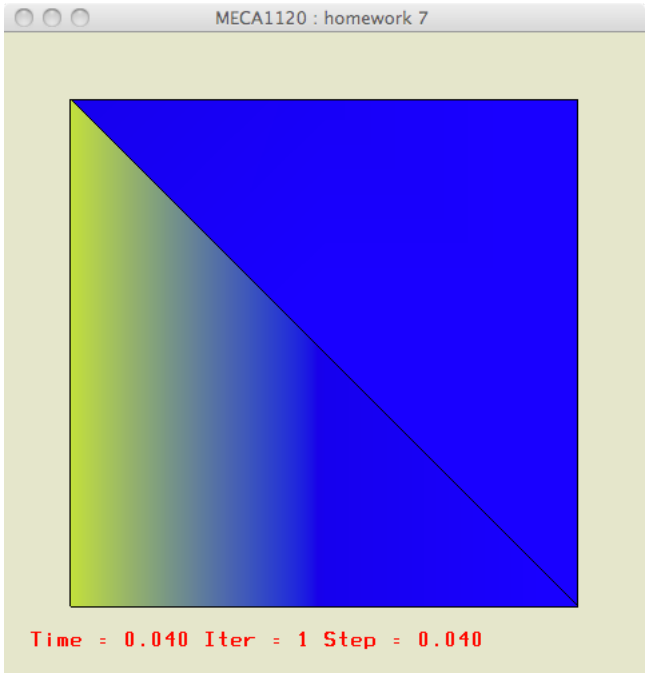
$$\ll \phi_i \beta c^* \gg_e$$

```
 -4.0e-02
 +4.0e-02
 +0.0e+00
 +0.0e+00
 +0.0e+00
 +0.0e+00
 +0.0e+00
 +0.0e+00
-- after triangles
 +4.6e-01
 +8.0e-02
 +4.6e-01
 -4.0e-02
 +0.0e+00
 +4.0e-02
 -1.0e+00
-- after edges
```

Itération two :-)



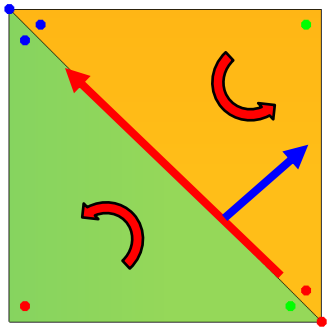
```
advectionAddIntegralsTriangles(theProblem);
advectionAddIntegralsEdges(theProblem);
advectionMultiplyInverseMatrix(theProblem);
```



$$\ll \phi_i \beta c^* \gg_e$$

$$< \left(u \frac{\partial \phi_i}{\partial x} + v \frac{\partial \phi_i}{\partial y} \right) c_e^h >_e$$

```
 +5.0e-01
 +4.0e-02
 +5.0e-01-0.4e-02
 -4.0e-02
 +0.0e+00
 +4.0e-02
 -1.0e+00
-- after edges
 +4.6e-01
 +8.0e-02
 +4.6e-01
 -4.0e-02
 +0.0e+00
 +4.0e-02
 -1.0e+00
-- after triangles
```



C’est quoi la solution ?