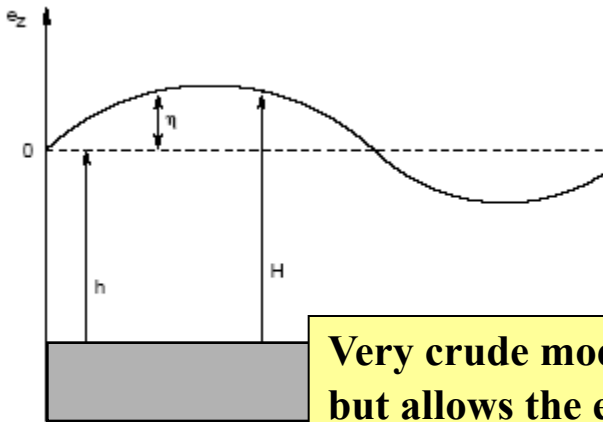Et maintenant,
les équations du tsunami !

$$
\begin{cases}
\dfrac{\partial \eta}{\partial t} + \dfrac{\partial}{\partial x}\left(hu\right) + \dfrac{\partial}{\partial y}\left(hv\right) &= 0 \\[2ex]
\dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}\left(g\eta\right) &= -\gamma u + fv + \dfrac{\tau}{\rho h} \\[2ex]
\dfrac{\partial v}{\partial t} + \dfrac{\partial}{\partial y}\left(g\eta\right) &= -\gamma v - fu
\end{cases}
$$

The so-called Shallow Water Equations

Very crude model for geophysical flows, but allows the existence of inertia-gravity waves

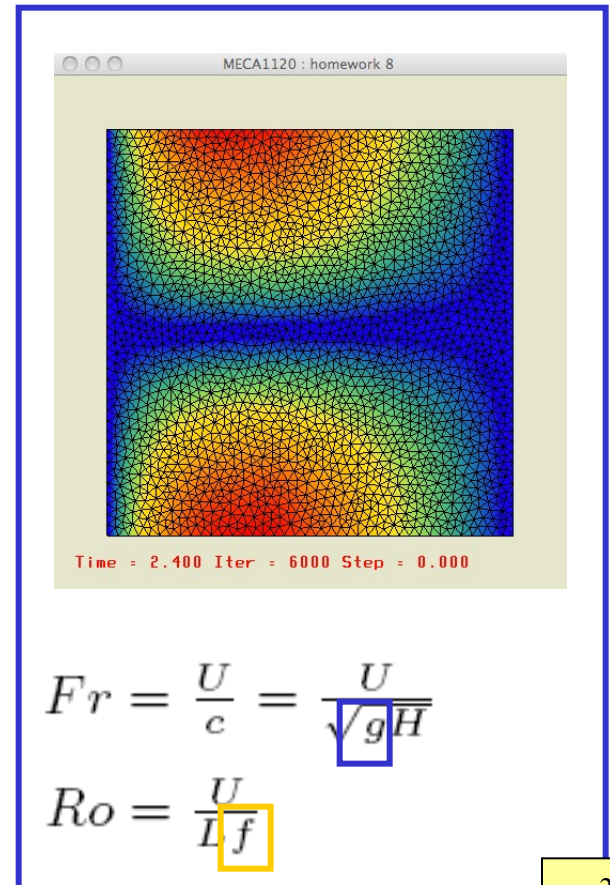# An analytical problem as a numerical validation : Stommel :-)

$$\begin{cases} \dfrac{\partial \eta}{\partial t} + \dfrac{\partial}{\partial x}(hu) + \dfrac{\partial}{\partial y}(hv) = 0 \\[2ex] \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}(g\eta) = -\gamma u + fv + \dfrac{\tau}{\rho h} \\[2ex] \dfrac{\partial v}{\partial t} + \dfrac{\partial}{\partial y}(g\eta) = -\gamma v - fu \end{cases}$$
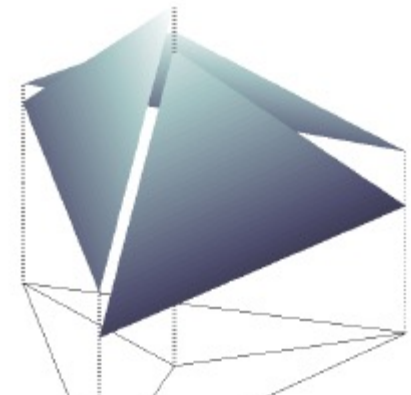
Forcing wind term [N m⁻²]

Gravity [m s⁻²]

Coriolis factor [s⁻¹]

Dissipation coefficient [s⁻¹]

MECA1120 : homework 8

Time = 2.400   Iter = 6000   Step = 0.000

$$Fr = \frac{U}{c} = \frac{U}{\sqrt{gH}}$$

$$Ro = \frac{U}{Lf}$$

3

Un modèle unidimensionnel le long de l'interface…

$$\begin{cases} \dfrac{\partial \eta}{\partial t} + h\dfrac{\partial u}{\partial x} & = & 0 \\[2em] \dfrac{\partial u}{\partial t} + g\dfrac{\partial \eta}{\partial x} & = & 0 \end{cases}$$
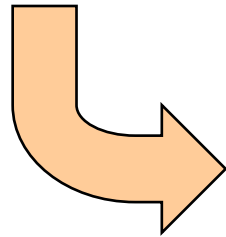
Discontinuous Galerkin

Comment calculer les flux de masse et de quantité de mouvement aux interfaces ?

Un solveur de Riemann…

$$\left\{ \begin{array}{rcl} \dfrac{\partial \eta}{\partial t} + h\dfrac{\partial u}{\partial x} & = & 0 \\[2ex] \dfrac{\partial u}{\partial t} + g\dfrac{\partial \eta}{\partial x} & = & 0 \end{array} \right.$$

$$\begin{bmatrix} \dfrac{\partial \eta}{\partial t} \\[2ex] \dfrac{\partial u}{\partial t} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -h \\[1ex] -g & 0 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} \dfrac{\partial \eta}{\partial x} \\[2ex] \dfrac{\partial u}{\partial x} \end{bmatrix}$$

Discontinuous Galerkin

Comment calculer les flux de masse et de quantité de mouvement aux interfaces ?

Calculons les valeurs propres de A pour découpler les deux équations…

$$\begin{bmatrix} \dfrac{\partial \eta}{\partial t} \\[2mm] \dfrac{\partial u}{\partial t} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -h \\[2mm] -g & 0 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} \dfrac{\partial \eta}{\partial x} \\[2mm] \dfrac{\partial u}{\partial x} \end{bmatrix}$$

$$\lambda = \pm\sqrt{gh}$$

Deux valeurs propres

Deux vecteurs propres

$$\mathbf{v} = \begin{bmatrix} 1 \\[2mm] \pm\sqrt{\dfrac{g}{h}} \end{bmatrix}$$

Effectuons un changement de variables…

$$\begin{bmatrix} r \\ s \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{2} & \frac{1}{2}\sqrt{\frac{h}{g}} \\ \frac{1}{2} & -\frac{1}{2}\sqrt{\frac{h}{g}} \end{bmatrix}}_{\mathbf{R}^{-1}} \begin{bmatrix} \eta \\ u \end{bmatrix}$$

r et s sont appelées
les invariants de Riemann :-)

Matrice des deux vecteurs propres

$$\mathbf{R} = \begin{bmatrix} 1 & 1 \\ \sqrt{\frac{g}{h}} & -\sqrt{\frac{g}{h}} \end{bmatrix}$$

Et on obtient…

$$\left(\frac{\partial \eta}{\partial t} + h\frac{\partial u}{\partial x}\right) + \sqrt{\frac{h}{g}}\left(\frac{\partial u}{\partial t} + g\frac{\partial \eta}{\partial x}\right) = 0$$

$$\frac{\partial}{\partial t}\left(\underbrace{\eta + \sqrt{\frac{h}{g}}\,u}_{r}\right) + \sqrt{gh}\,\frac{\partial}{\partial x}\left(\underbrace{\eta + \sqrt{\frac{h}{g}}\,u}_{r}\right) = 0$$

$$\left(\frac{\partial \eta}{\partial t} + h\frac{\partial u}{\partial x}\right) - \sqrt{\frac{h}{g}}\left(\frac{\partial u}{\partial t} + g\frac{\partial \eta}{\partial x}\right) = 0$$

$$\frac{\partial}{\partial t}\left(\underbrace{\eta - \sqrt{\frac{h}{g}}\,u}_{s}\right) - \sqrt{gh}\,\frac{\partial}{\partial x}\left(\underbrace{\eta - \sqrt{\frac{h}{g}}\,u}_{s}\right) = 0$$

… deux équations de transport découplées !

Les invariants
de Riemann sont
constants le long
des courbes
caractéristiques !

$$x = \sqrt{gh}\, t$$
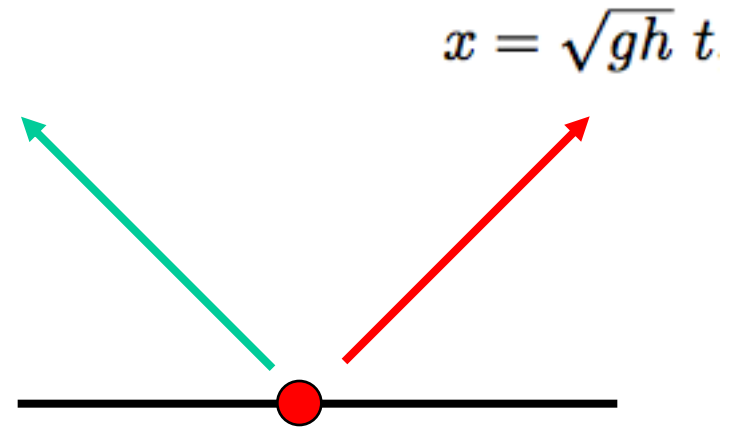
$$\frac{\partial r}{\partial t} + \sqrt{gh}\, \frac{\partial r}{\partial x} = 0$$

Car $\sqrt{gh} = \frac{dx}{dt}$ sur la courbe caractéristique

$$\underbrace{\frac{\partial r}{\partial t} + \frac{dx}{dt}\, \frac{\partial r}{\partial x}}_{\frac{dr}{dt}} = 0$$

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Et on sait
ce qu'il faut
faire pour
une équation
de transport pur !

$$x = \sqrt{gh}\, t$$
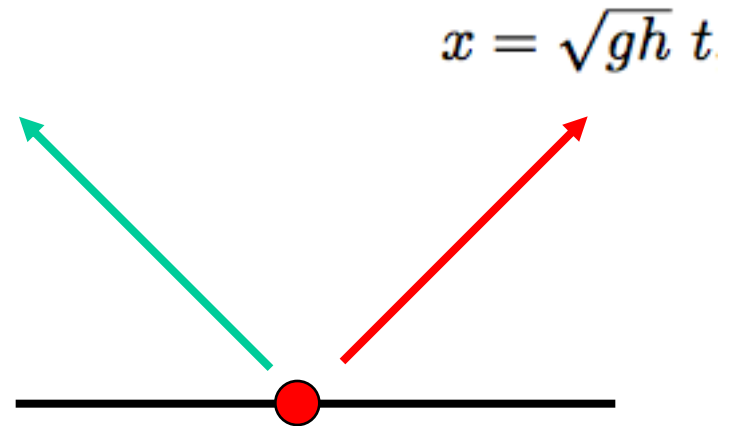
$$\left(\eta + \sqrt{\tfrac{h}{g}}\, u\right)^* \;=\; r^* \;=\; r_L \;=\; \left(\eta_L + \sqrt{\tfrac{h}{g}}\, u_L\right)$$

$$\left(\eta - \sqrt{\tfrac{h}{g}}\, u\right)^* \;=\; s^* \;=\; s_R \;=\; \left(\eta_R - \sqrt{\tfrac{h}{g}}\, u_R\right)$$
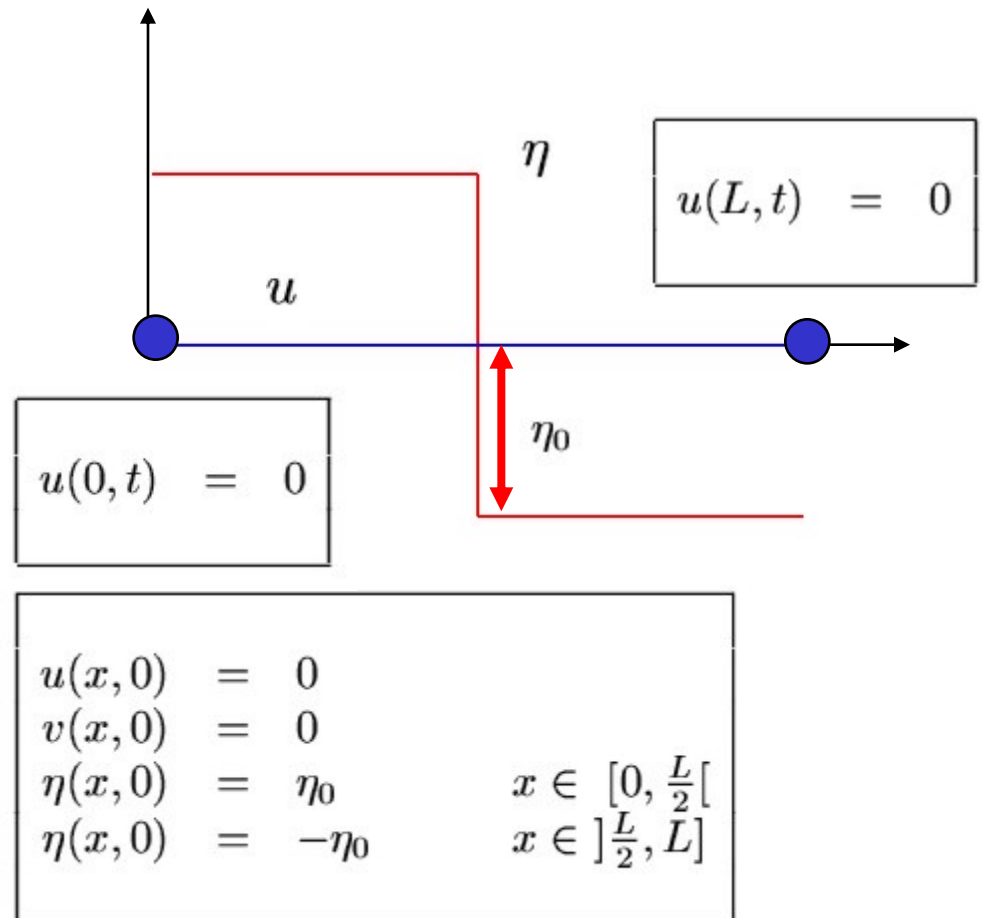
Le solveur dit de Riemann :-)

Et en termes de vitesses et d'élévation

$$x = \sqrt{gh}\, t$$

$$\left(\eta + \sqrt{\tfrac{h}{g}}\, u\right)^* = r^* = r_L = \left(\eta_L + \sqrt{\tfrac{h}{g}}\, u_L\right)$$

$$\left(\eta - \sqrt{\tfrac{h}{g}}\, u\right)^* = s^* = s_R = \left(\eta_R - \sqrt{\tfrac{h}{g}}\, u_R\right)$$
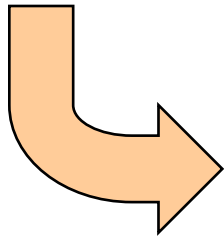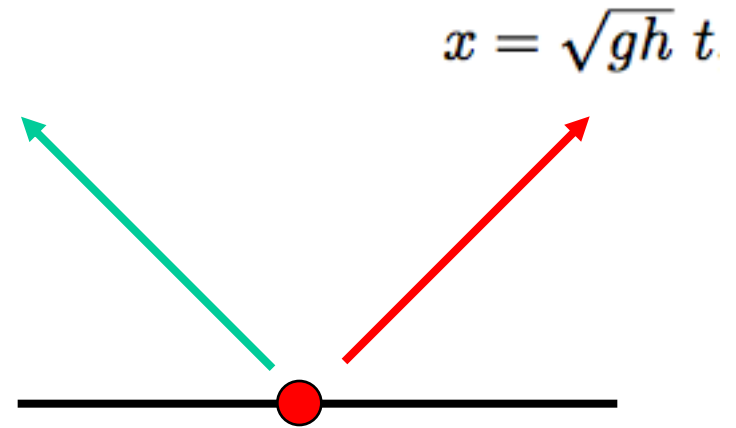
$$\eta^* = \left(\frac{r_L + s_R}{2}\right) = \left(\frac{\eta_L + \eta_R}{2}\right) + \sqrt{\frac{h}{g}}\left(\frac{u_L - u_R}{2}\right)$$

$$u^* = \sqrt{\frac{g}{h}}\left(\frac{r_L - s_R}{2}\right) = \left(\frac{u_L + u_R}{2}\right) + \sqrt{\frac{g}{h}}\left(\frac{\eta_L - \eta_R}{2}\right)$$

Le solveur dit de Riemann :-)

A 1D sharp simplified problem in a finite domain

$$\frac{\partial \eta}{\partial t} + H \frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} - fv + g\frac{\partial \eta}{\partial x} = 0$$

$$\frac{\partial v}{\partial t} + fu = 0$$

$$u(L, t) = 0$$
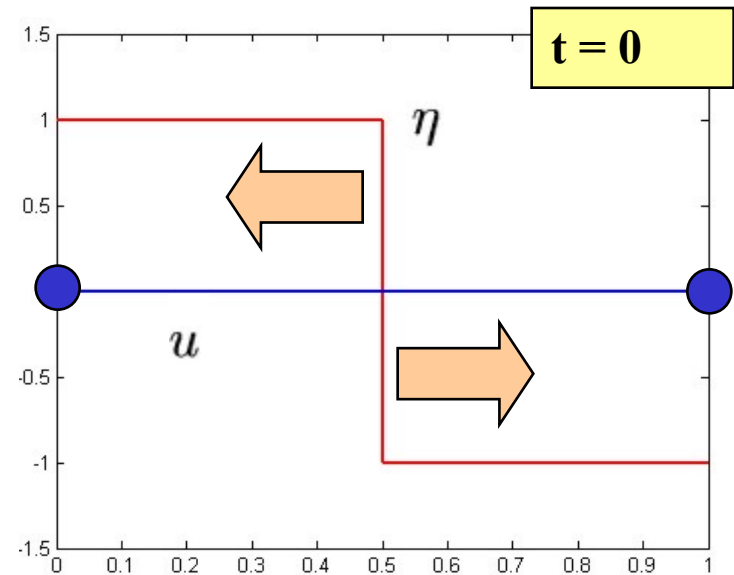
$$u(0, t) = 0$$

η

u

η_0

$$
\begin{aligned}
u(x, 0) &= 0 \\
v(x, 0) &= 0 \\
\eta(x, 0) &= \eta_0 & x \in [0, \tfrac{L}{2}[\\
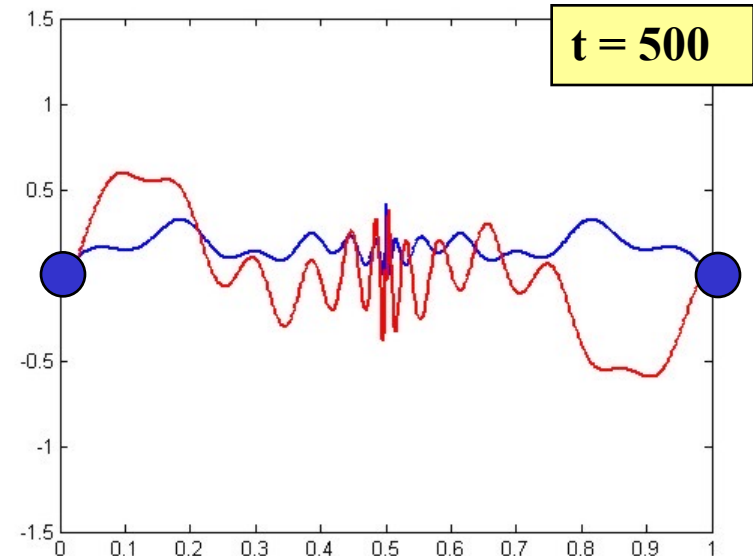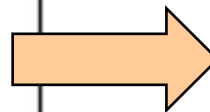\eta(x, 0) &= -\eta_0 & x \in]\tfrac{L}{2}, L]
\end{aligned}
$$

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What is the solution ?

$$\frac{\partial \eta'}{\partial t'} + \frac{\partial u'}{\partial x'} = 0$$

$$\frac{\partial u'}{\partial t'} - v' + \alpha^2 \frac{\partial \eta'}{\partial x'} = 0$$
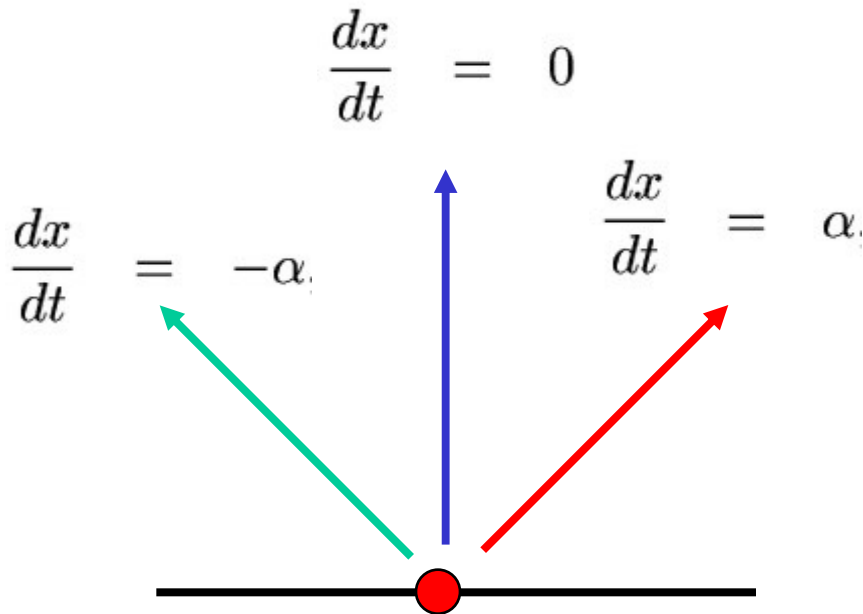
$$\frac{\partial v'}{\partial t'} + u' = 0$$

$$x' = \frac{x}{L}, \quad t' = f\, t, \quad \eta' = \frac{\eta}{\eta_0}, \quad u' = \frac{H u}{Lf\eta_0},$$

$$\alpha = \boxed{\frac{\sqrt{gH}}{f}} \frac{1}{L}$$

Rossby's radius

t = 0

t = 1
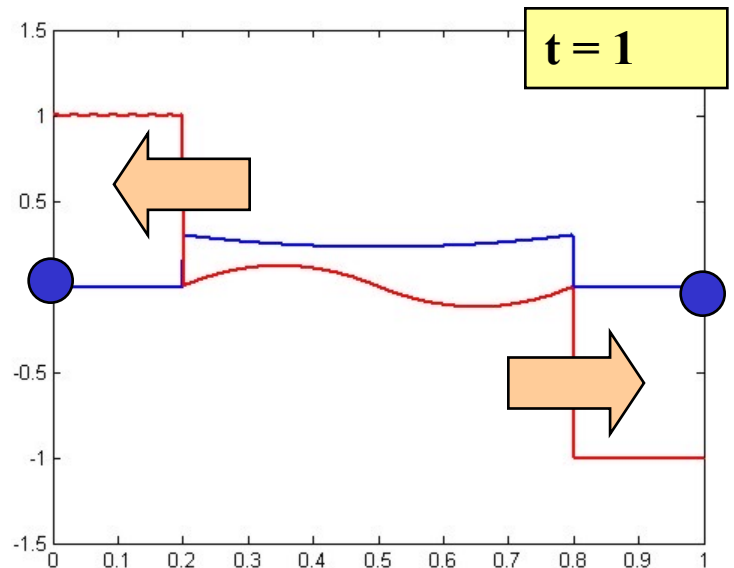
A more and more complex and interesting solution…

t = 2

$$\alpha = \frac{\sqrt{gH}}{f} \frac{1}{L} = \frac{\sqrt{10}}{10} = 0.3162$$

$$
\begin{aligned}
f &= 10^{-4}\ 1/s \\
L &= 1000\ Km \\
H &= 100\ m \\
g &= 10\ m/s^2
\end{aligned}
$$

t = 33
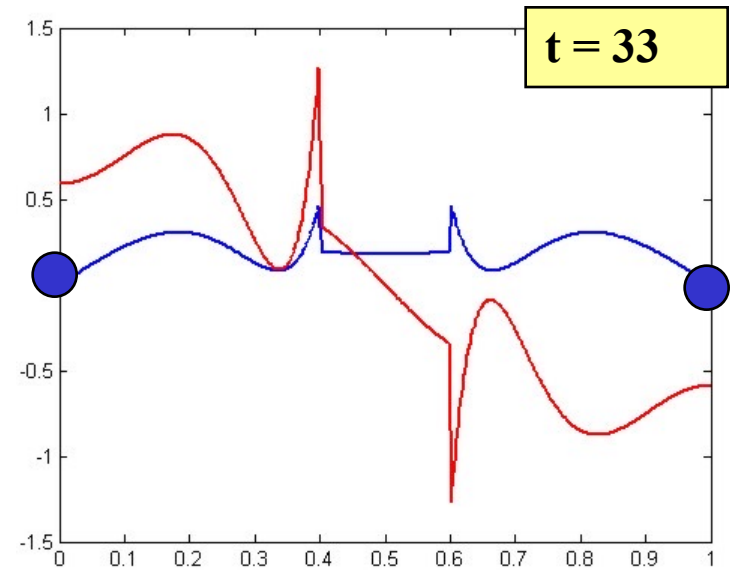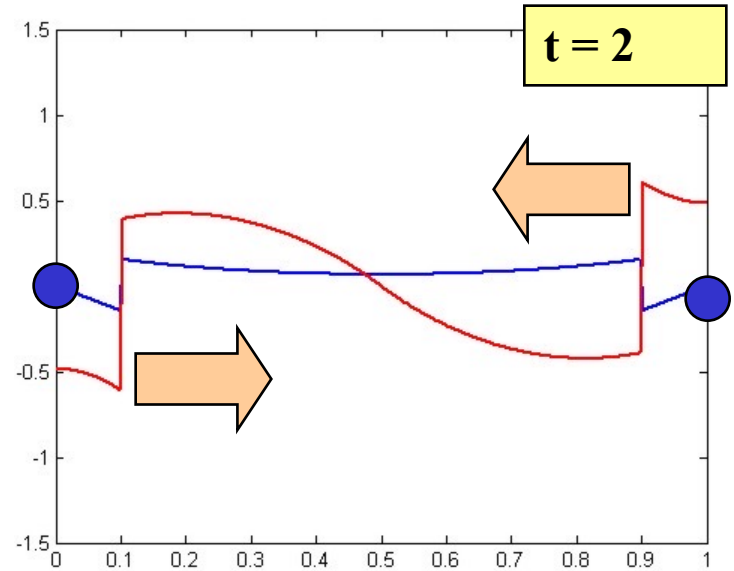
What are the equations ?

t = 500

$$\frac{\partial \eta}{\partial t} + \frac{\partial u}{\partial x} = 0$$

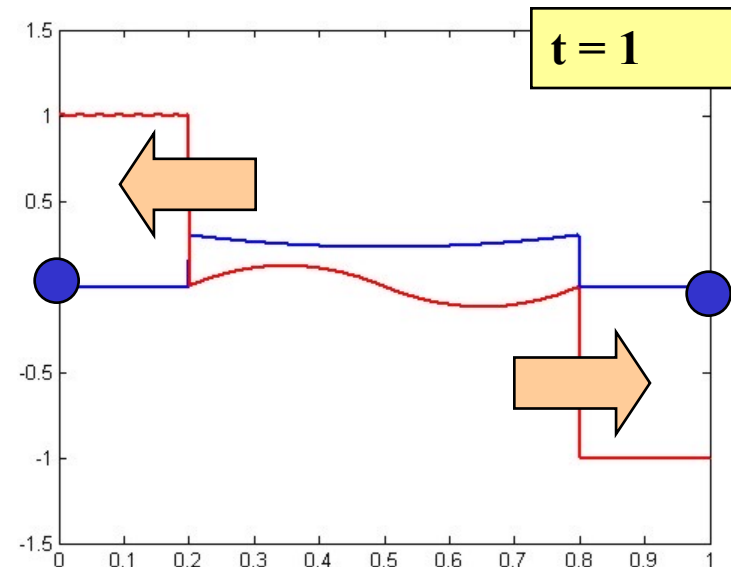$$\frac{\partial u}{\partial t} - v + \alpha^2 \frac{\partial \eta}{\partial x} = 0$$

$$\frac{\partial v}{\partial t} + u = 0$$

$$\frac{\partial^2 u}{\partial t^2} - \underbrace{\frac{\partial v}{\partial t}}_{-u} + \alpha^2 \underbrace{\frac{\partial^2 \eta}{\partial t \partial x}}_{-\frac{\partial^2 u}{\partial x^2}} = 0$$

Helmholtz's Equation
Forced Wave Equation

$$\frac{\partial^2 u}{\partial t^2} + u - \alpha^2 \frac{\partial^2 u}{\partial x^2} = 0$$

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How does information propagate ?
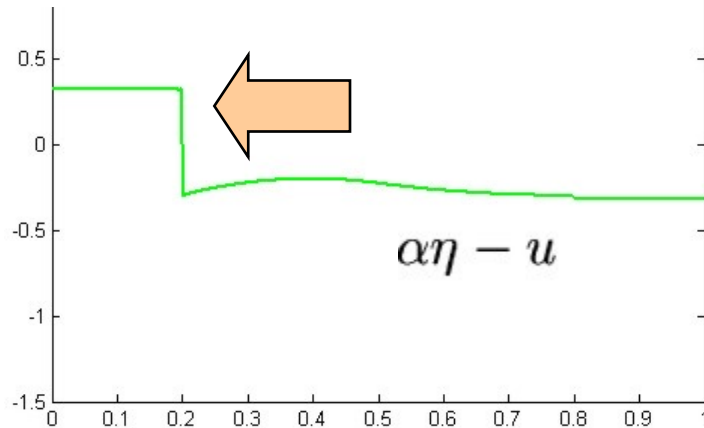
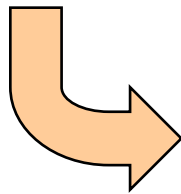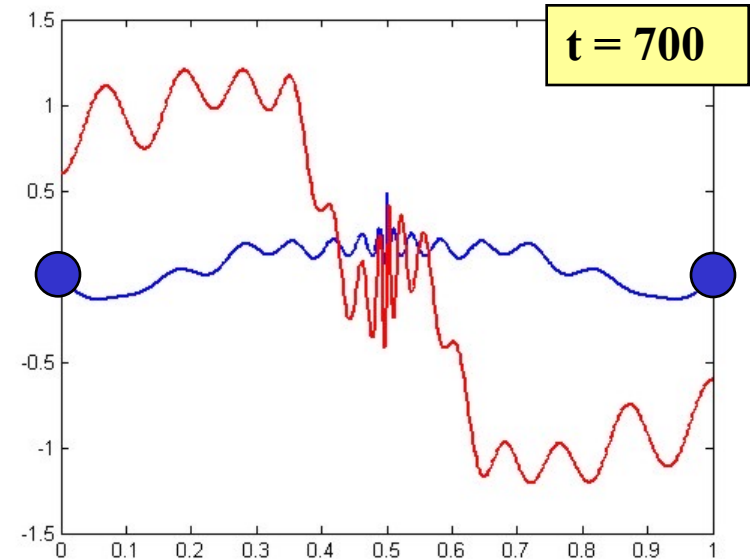$$\frac{d}{dt}\boxed{(\alpha\eta - u)} = -v \qquad \text{on } \frac{dx}{dt} = -\alpha,$$

$$\frac{d}{dt}\boxed{(\alpha\eta + u)} = v \qquad \text{on } \frac{dx}{dt} = \alpha,$$

$$\frac{dv}{dt} = -u \qquad \text{on } \frac{dx}{dt} = 0,$$

Riemann's Invariants

$$\frac{dx}{dt} = 0$$

$$\frac{dx}{dt} = -\alpha$$

$$\frac{dx}{dt} = \alpha$$

t = 1

Two distinct waves…

$$\frac{d}{dt}\boxed{(\alpha\eta - u)} = -v \quad \text{on } \frac{dx}{dt} = -\alpha,$$

$$\frac{d}{dt}\boxed{(\alpha\eta + u)} = v \quad \text{on } \frac{dx}{dt} = \alpha,$$

$$\frac{dv}{dt} = -u \quad \text{on } \frac{dx}{dt} = 0,$$

t = 1

u

η

$\alpha\eta - u$

$\alpha\eta + u$
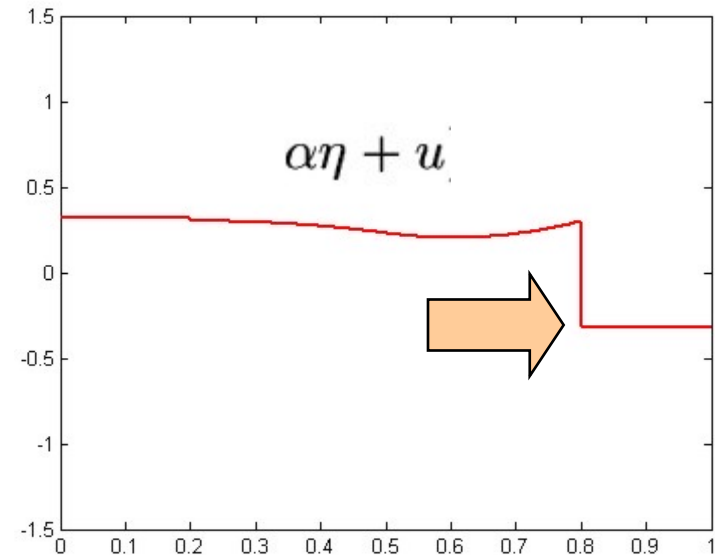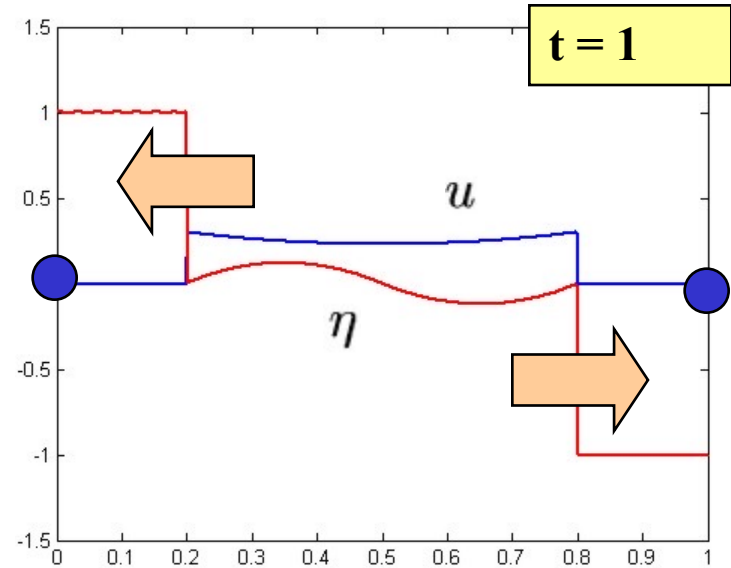
An analytical solution exists !

$$\frac{\partial^2 u}{\partial t^2} + u - \alpha^2 \frac{\partial^2 u}{\partial x^2} = 0$$

t = 700

Separation of the Classical Equations with the boundary conditions

$$u(x,t) = T(t)f(x)$$

$$\frac{T''}{T} = \alpha^2 \frac{f''}{f} - 1$$

$$u(x,t) = \sum_{i=1}^{\infty} \frac{4\alpha^2 (-1)^{i+1}}{\omega_i} \sin(\omega_i t) \sin(k_i x)$$

$$k_i = (2i-1)\pi$$

$$\omega_i = \sqrt{1 + \alpha^2 k_i^2}$$

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Analytical solution for any initial elevation data

$$u(x,t) = \sum_{i=1}^{\infty} A_i \frac{\alpha^2 k_i}{\omega_i} \sin(\omega_i t) \sin(k_i x)$$

$$v(x,t) = \sum_{i=1}^{\infty} A_i \frac{\alpha^2 k_i}{\omega_i^2} \left(\cos(\omega_i t) - 1 \right) \sin(k_i x)$$
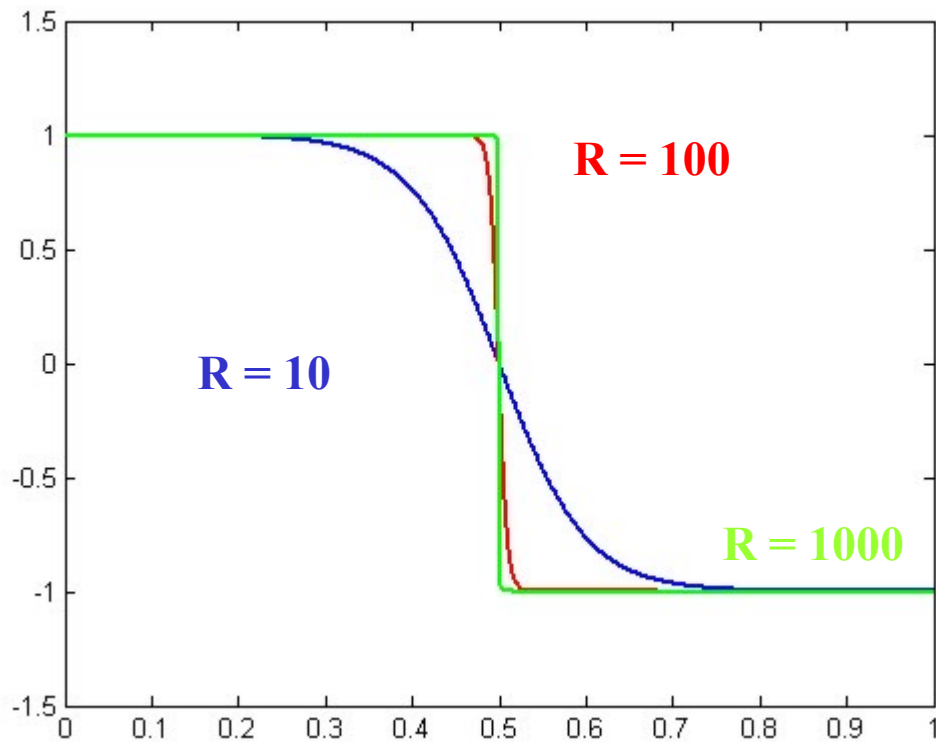
$$\eta(x,t) = \sum_{i=1}^{\infty} A_i \left(1 - \frac{\alpha^2 k_i^2}{\omega_i^2} \left(1 - \cos(\omega_i t) \right) \right) \cos(k_i x)$$

$$k_i = (2i - 1)\pi$$

$$\omega_i = \sqrt{1 + \alpha^2 k_i^2}$$

$$A_i = 2 \int_0^1 \eta(x,0) \cos(k_i x) \, dx$$

A family of initial conditions…

R = 100

R = 10

R = 1000

Stiffness factor

$$\eta(x,0) = -\frac{\tanh\left(R(x-0.5)\right)}{\tanh\left(0.5R\right)}$$
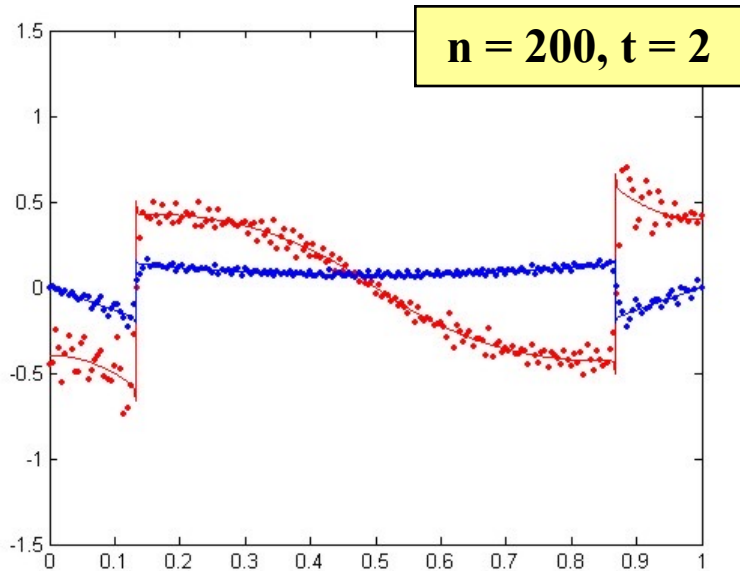
The Continuous Galerkin Method

Find $\eta^h \in \mathcal{E}^h$ and $(u^h, v^h) \in \mathcal{U}^h \times \mathcal{U}^h$ such that

$$\int_\Omega \left(\frac{\partial \eta^h}{\partial t} \hat\eta^h + \frac{\partial u^h}{\partial x} \hat\eta^h \right) dx = 0 \qquad \forall \hat\eta^h \in \mathcal{E}^h,$$

$$\int_\Omega \left(\frac{\partial u^h}{\partial t} \hat u^h + v^h \hat u^h + \alpha^2 \frac{\partial \eta^h}{\partial x} \hat u^h \right) dx = 0 \qquad \forall \hat u^h \in \mathcal{U}^h,$$

$$\int_\Omega \left(\frac{\partial v^h}{\partial t} \hat v^h - u^h \hat v^h \right) dx = 0 \qquad \forall \hat v^h \in \mathcal{U}^h,$$
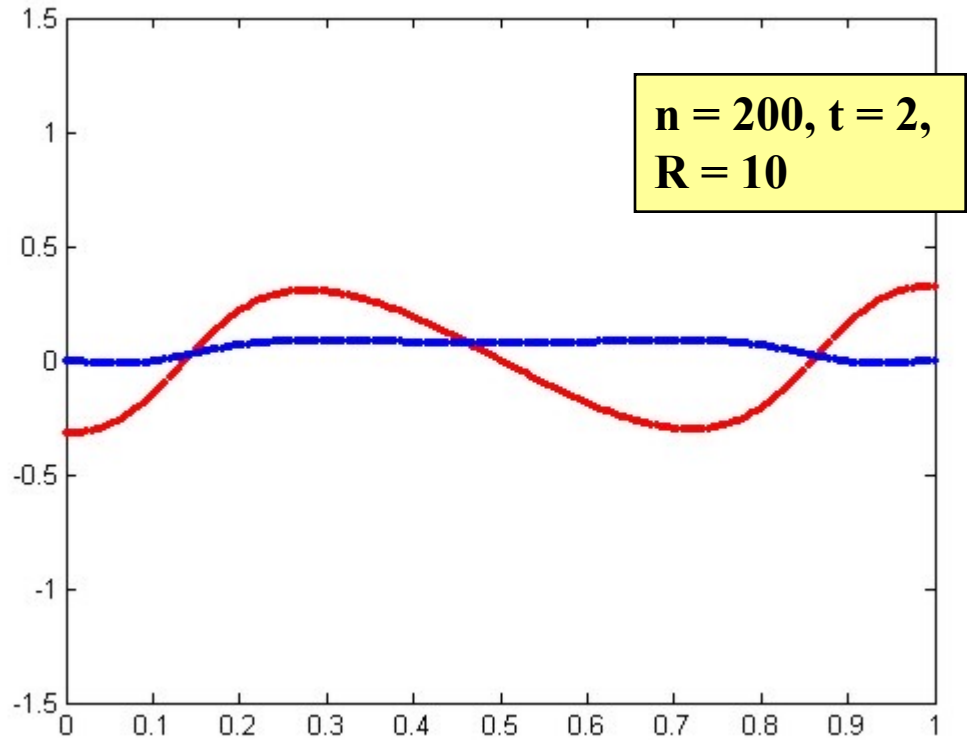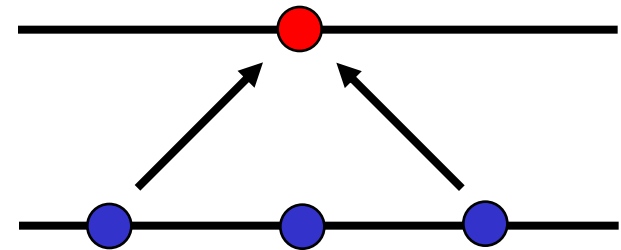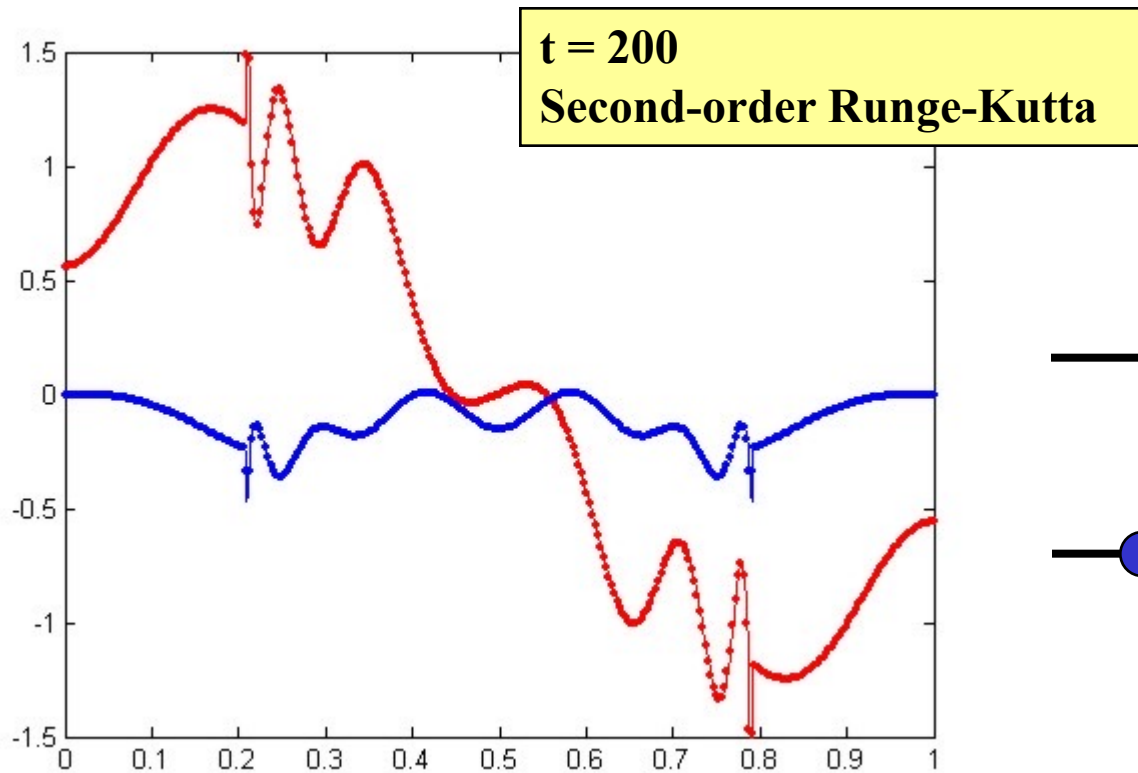
The Continuous Galerkin Method
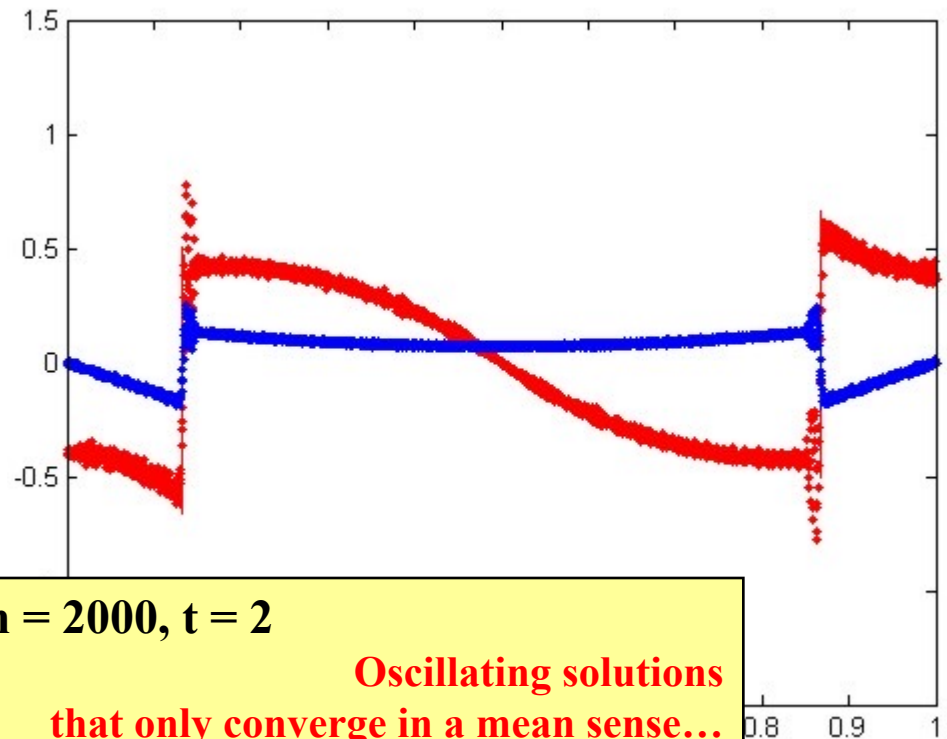
n = 200, t = 2

n = 2000, t = 2

**Oscillating solutions
that only converge in a mean sense…**

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For smooth solutions, it works !
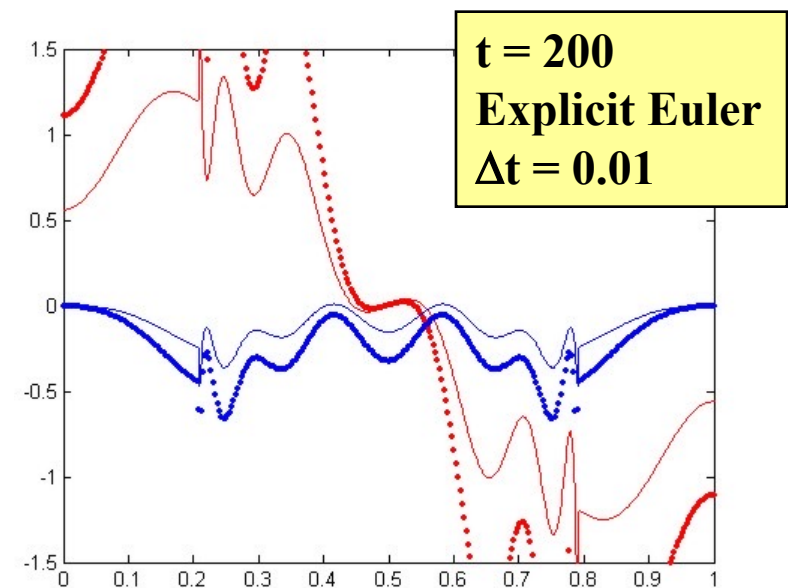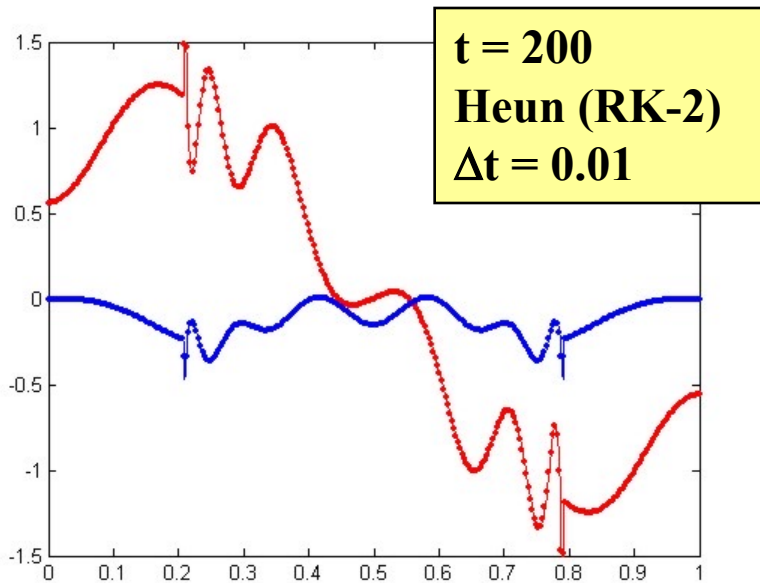
n = 200, t = 2, R = 10
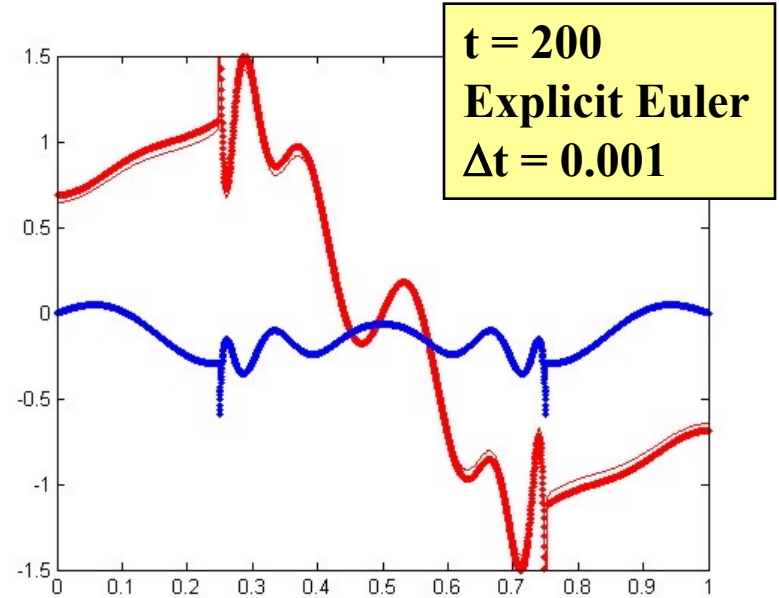
n = 200, t = 2, R = 100
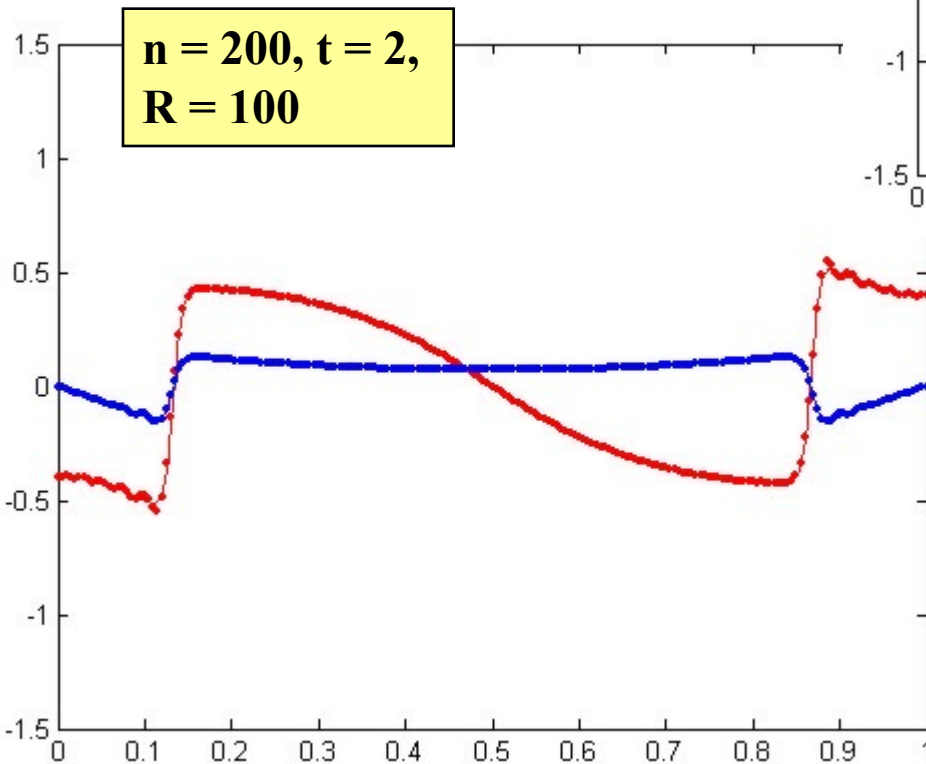
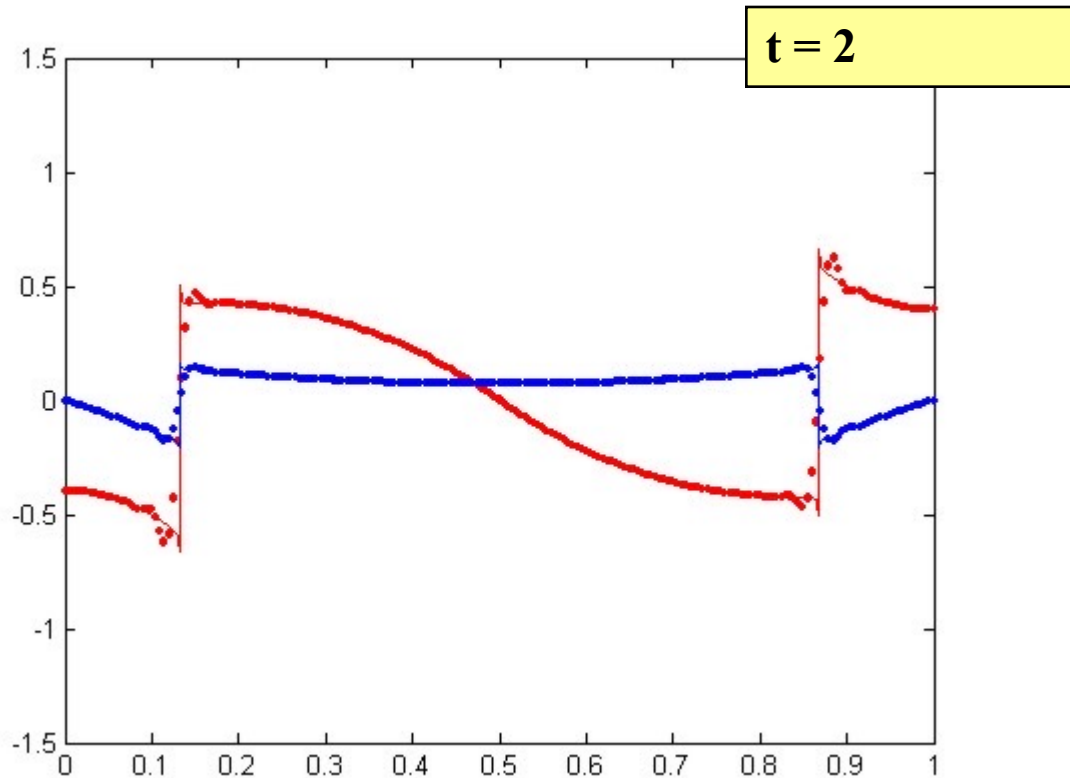$$\eta(x, 0) = -\frac{\tanh\left(R(x - 0.5)\right)}{\tanh\left(0.5R\right)}$$

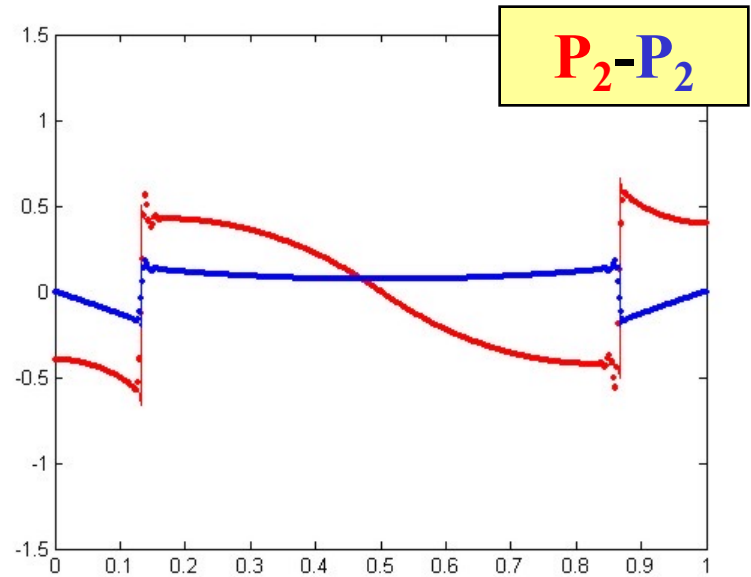# The Optimal Technique : Integrating along characteristics

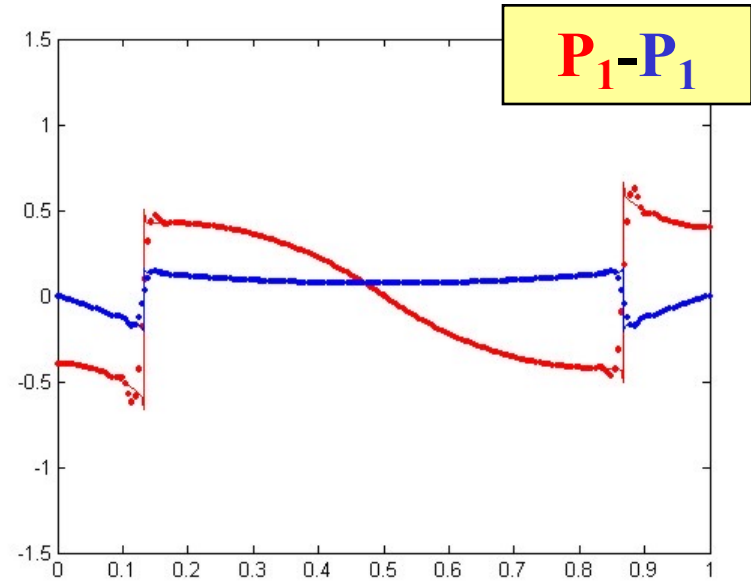t = 200
Second-order Runge-Kutta

Time integration has to be accurately performed…

t = 200
Explicit Euler
Δt = 0.001

t = 200
Heun (RK-2)
Δt = 0.01

t = 200
Explicit Euler
Δt = 0.01

The Discontinuous Riemann-Galerkin Method

t = 2

Increasing the order of shape functions…

P_1-P_1

P_2-P_2

The jumps at discontinuities are proportional to the local error

How to estimate the local error ?

$$j \cong e \cong h^{p+1}$$

The local error are also proportional to h^{p+1} where p is the order of elements and h the characteristic size.
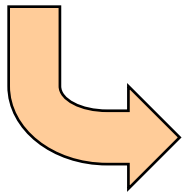
The Discontinuous Galerkin Method provides an efficient and simple error estimator !

Adaptive strategy

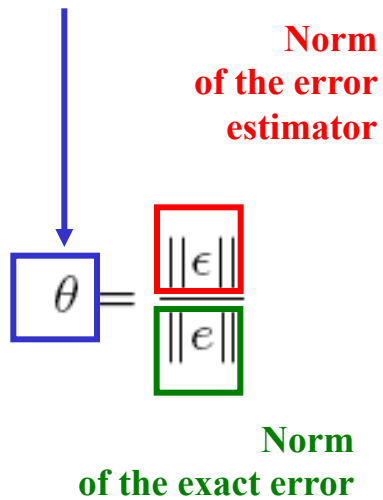$$h_{n+1} = \sqrt[p+1]{\frac{e_t}{j_n}} \, h_n$$

Target error

New requested mesh size field from the error estimator

From this new mesh size field, we can create a new adapted mesh.

How to evaluate the error estimator ?

The error estimator slightly overestimates the error, but converges to the true error

Effectivity index

Norm of the error estimator

Norm of the exact error

$$\theta = \frac{\|\epsilon\|}{\|e\|}$$

Asymptotic behaviour