

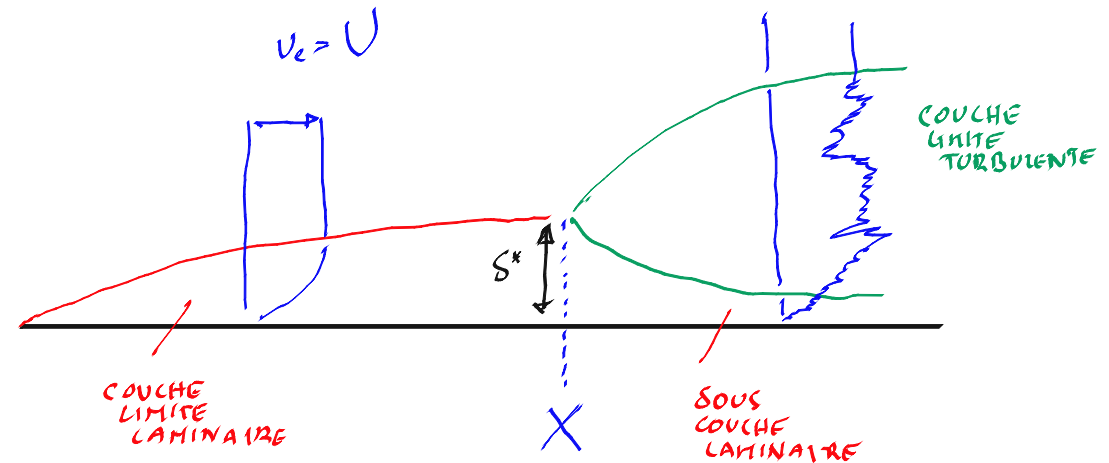
Et soudain  
cela devient  
turbulent

# Instabilité de la couche limite

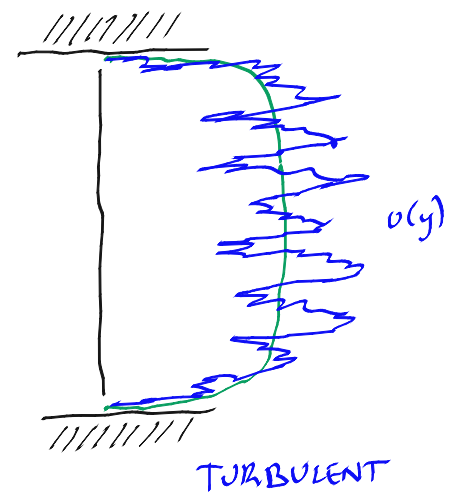
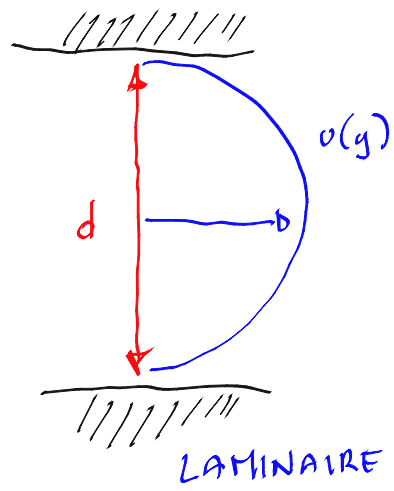
$$\underbrace{\frac{UX}{15}}_{400} \frac{\delta^*}{X} \approx 1,72 \sqrt{\underbrace{\frac{15}{UX} \frac{U^2 X^2}{15^2}}_{\sqrt{\frac{UX}{15}}}}$$

$$\frac{\rho \delta^* U}{\mu} \approx 400$$

$$\frac{\rho XU}{\mu} \approx 54000$$

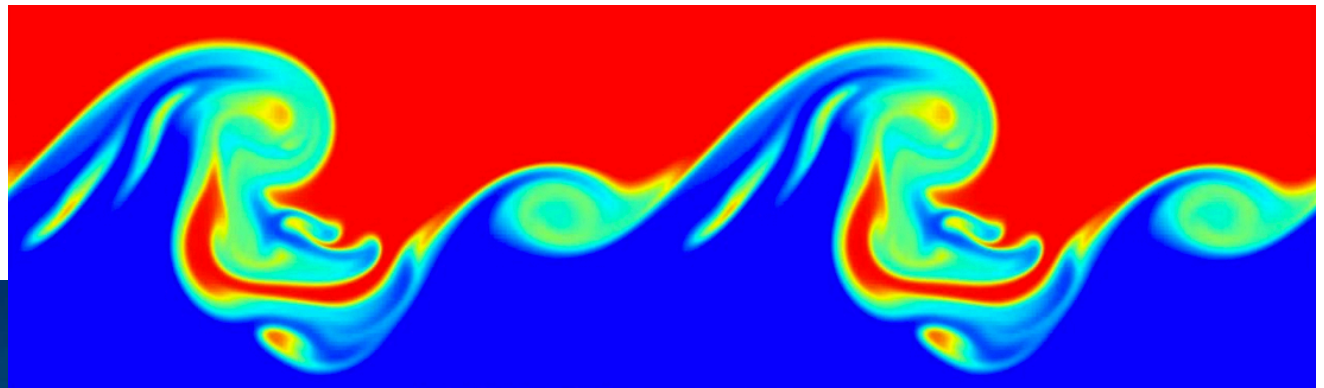


# Instabilité du Poiseuille plan

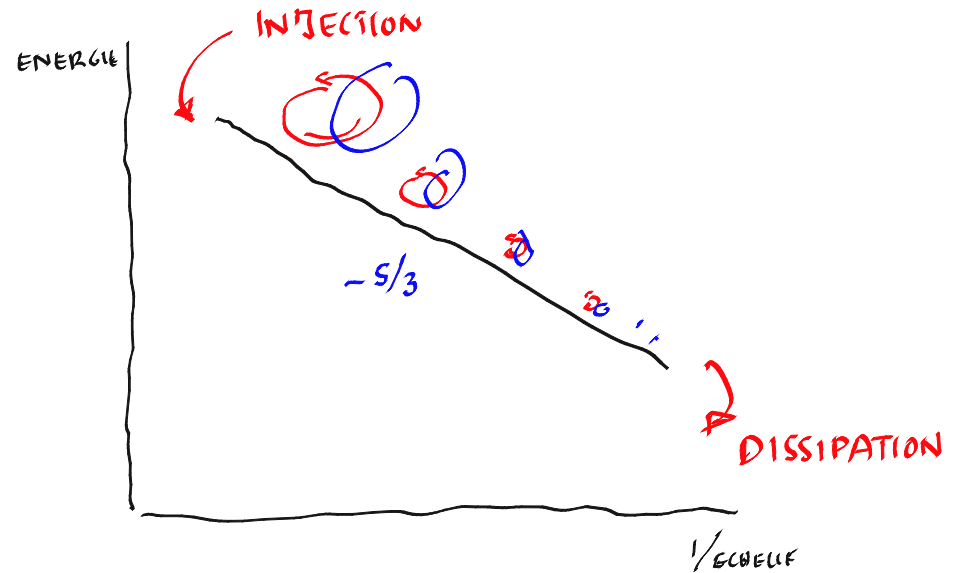


$$\frac{\rho u_m}{\mu} \approx 7690$$

# Instabilité du Kelvin-Helmholtz



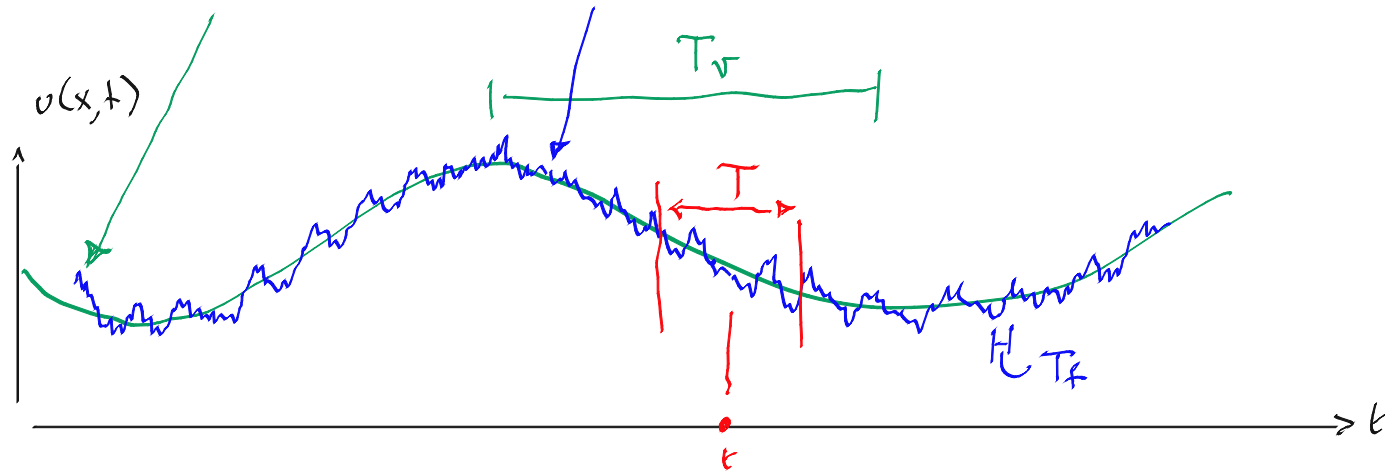
# Caractéristiques de la turbulence



Fluctuations temporelles et spatiales  
Tourbillons de tailles différentes interagissant entre eux  
Le spectre d'énergie tend vers zéros aux petites échelles  
Persistance de la turbulence  
Transfert de chaleur augmenté par la turbulence

# Moyenne et fluctuation

$$u(x, t) = \overline{u}(x, t) + u'(x, t)$$



Écoulement  
avec deux échelles de temps  
bien distinctes

# Variance Covariance

$$u(x, t) = \bar{u}(x, t) + \overbrace{u'(x, t)}^{\text{FLUCTUATIONS}}$$

$$\underbrace{\bar{u}(x, t)}_{\text{VITESSE MOYENNE}} = \frac{1}{T} \int_{t-T/2}^{t+T/2} u(x, \zeta) d\zeta$$

$$0 = \overline{u'(x, t)} = \frac{1}{T} \int_{t-T/2}^{t+T/2} u'(x, \zeta) d\zeta$$

$$\text{var}(u(x, t)) \triangleq \overline{u'(x, t) u'(x, t)} = \frac{1}{T} \int_{t-T/2}^{t+T/2} u'(x, \zeta) u'(x, \zeta) d\zeta$$

$$\text{covar}(u(x, t), v(x, t)) \triangleq \overline{u'(x, t) v'(x, t)} = \frac{1}{T} \int_{t-T/2}^{t+T/2} u'(x, \zeta) v'(x, \zeta) d\zeta$$

# Equation de Burgers

$$\rho \frac{\partial u}{\partial t} + \underbrace{\rho \frac{\partial}{\partial x} (u^2)}_{\text{TRANSPORT}} = \underbrace{\mu \frac{\partial^2 u}{\partial x^2}}_{\text{DIFFUSION}}$$

En appliquant l'opérateur de moyenne,

$$\overline{\rho \frac{\partial u}{\partial t}} + \overline{\rho \frac{\partial}{\partial x} (u^2)} = \overline{\mu \frac{\partial^2 u}{\partial x^2}}$$

Clairement, les opérateurs de moyenne et dérivation spatiale commutent,

$$\rho \frac{\partial \bar{u}}{\partial t} + \rho \frac{\partial}{\partial x} (\bar{u}^2) = \mu \frac{\partial^2 \bar{u}}{\partial x^2}$$

En supposant que les échelles temporelles soient bien distinctes  $T_f \ll T \ll T_v$

$$\rho \frac{\partial \bar{u}}{\partial t} + \rho \frac{\partial}{\partial x} (\bar{u}^2) = \mu \frac{\partial^2 \bar{u}}{\partial x^2}$$



$$\rho \frac{\partial \bar{u}}{\partial t} + \rho \frac{\partial}{\partial x} \overline{(u^2)} = \mu \frac{\partial^2 \bar{u}}{\partial x^2}$$

$$\rho \frac{\partial \bar{u}}{\partial t} + \rho \frac{\partial}{\partial x} \overline{(\bar{u} + u')(\bar{u} + u')} = \mu \frac{\partial^2 \bar{u}}{\partial x^2}$$

$$\rho \frac{\partial \bar{u}}{\partial t} + \rho \frac{\partial}{\partial x} \left( \bar{u} \bar{u} + \underbrace{2 \overline{\bar{u} u'}}_{=0} + \overline{u' u'} \right) = \mu \frac{\partial^2 \bar{u}}{\partial x^2}$$

En définissant le tenseur de Reynolds

$$\rho \frac{\partial \bar{u}}{\partial t} + \rho \frac{\partial}{\partial x} (\bar{u} \bar{u}) = \mu \frac{\partial^2 \bar{u}}{\partial x^2} - \rho \frac{\partial}{\partial x} \left( \underbrace{\overline{u' u'}}_{\triangleq \bar{\sigma}^t} \right)$$

**La moyenne d'un produit  
n'est pas le produit des moyennes !**

$$\frac{\partial v_j}{\partial x_j} = 0$$

$$\rho \left[ \frac{\partial v_i}{\partial t} + \frac{\partial}{\partial x_j} (v_i v_j) \right] = -\frac{\partial p}{\partial x_i} + \rho g_i + \frac{\partial \tau_{ij}}{\partial x_j}$$

$$\rho c \left[ \frac{\partial T}{\partial t} + \frac{\partial}{\partial x_j} (T v_j) \right] = \tau_{ij} d_{ij} - \frac{\partial q_j}{\partial x_j}$$

# Navier-Stokes

$$\frac{\partial \bar{v}_j}{\partial x_j} = 0$$

$$\rho \left[ \frac{\partial \bar{v}_i}{\partial t} + \frac{\partial}{\partial x_j} (\bar{v}_i \bar{v}_j + \overline{v'_i v'_j}) \right] = -\frac{\partial \bar{p}}{\partial x_i} + \rho g_i + \frac{\partial \bar{\tau}_{ij}}{\partial x_j}$$

$$\rho c \left[ \frac{\partial \bar{T}}{\partial t} + \frac{\partial}{\partial x_j} (\bar{T} \bar{v}_j + \overline{T' v'_j}) \right] = \bar{\tau}_{ij} \bar{d}_{ij} + \overline{\tau'_{ij} d'_{ij}} - \frac{\partial \bar{q}_j}{\partial x_j}$$

**Navier-Stokes moyennées !**

TENSEURS  
DES CONTRAINTES  
TURBULENCE

$$\rho \left[ \frac{\partial \bar{v}_i}{\partial t} + \frac{\partial}{\partial x_j} (\bar{v}_i \bar{v}_j) \right] = -\frac{\partial \bar{p}}{\partial x_i} + \rho g_i + \frac{\partial \bar{\tau}_{ij}}{\partial x_j} + \frac{\partial}{\partial x_j} \underbrace{(-\rho \overline{v'_i v'_j})}_{\triangleq \bar{\sigma}_{ij}^t}$$

$$\rho c \left[ \frac{\partial \bar{T}}{\partial t} + \frac{\partial}{\partial x_j} (\bar{T} \bar{v}_j) \right] = \bar{\tau}_{ij} \bar{d}_{ij} + \underbrace{\overline{\tau'_{ij} d'_{ij}}}_{\triangleq \rho \bar{\epsilon}} - \frac{\partial \bar{q}_j}{\partial x_j} - \frac{\partial}{\partial x_j} \underbrace{(\rho c \overline{T'_i v'_j})}_{\triangleq \bar{q}_j^t}$$

TAUX DE  
DISSIPATION TURB.

FLUX  
DE CHALEUR  
TURBULENT

# Navier-Stokes moyennées !

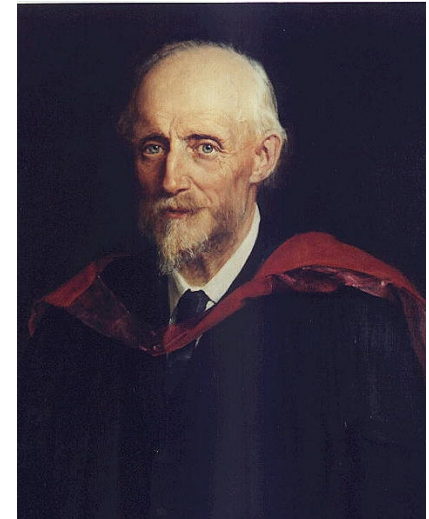
$$\overline{\sigma}_{ij}^t = \frac{1}{3} \overline{\sigma}_{kk}^t \delta_{ij} + \overline{\tau}_{ij}^t$$

TENSEUR  
DES  
EXTRA  
CONTRAINTES  
TURBULENTES

En définissant l'énergie cinétique des fluctuations  $\overline{k} \triangleq \frac{\overline{v'_k v'_k}}{2}$

$$\overline{\sigma}_{ij}^t = -\frac{2}{3} \rho \overline{k} \delta_{ij} + \overline{\tau}_{ij}^t$$

Décomposons  
le tenseur de Reynolds !



$$\frac{\partial \bar{v}_j}{\partial x_j} = 0$$

$$\rho \left[ \frac{\partial \bar{v}_i}{\partial t} + \frac{\partial}{\partial x_j} (\bar{v}_i \bar{v}_j) \right] = -\frac{\partial \bar{p}}{\partial x_i} - \boxed{\frac{2}{3}\rho \frac{\partial \bar{k}}{\partial x_j}} + \rho g_i + \frac{\partial \bar{\tau}_{ij}}{\partial x_j} + \boxed{\frac{\partial \bar{\tau}_{ij}^t}{\partial x_j}}$$

$$\rho c \left[ \frac{\partial \bar{T}}{\partial t} + \frac{\partial}{\partial x_j} (\bar{T} \bar{v}_j) \right] = \bar{\tau}_{ij} \bar{d}_{ij} + \boxed{\rho \bar{\epsilon}} - \frac{\partial \bar{q}_j}{\partial x_j} - \boxed{\frac{\partial \bar{q}_j^t}{\partial x_j}}$$

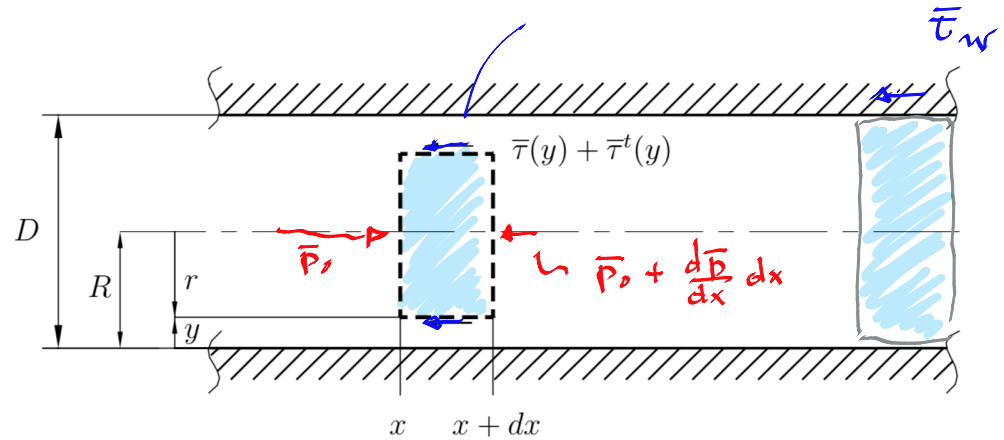
Quatre nouvelles variables

# Fermeture des équations

$$\bar{\tau}_{ij}^t \stackrel{\text{mod}}{=} 2 \mu_t \bar{d}_{ij}$$

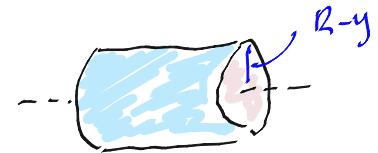
$$\bar{q}_j^t \stackrel{\text{mod}}{=} -k_t \frac{\partial \bar{T}}{\partial x_j}$$

# Écoulements turbulents en conduite



2  $\tau_w$

$$[\bar{\tau}(y) + \bar{\tau}^t(y)] 2\pi (R-y) dx + \frac{d\bar{p}}{dx} dx \pi (R-y)^2 = 0$$

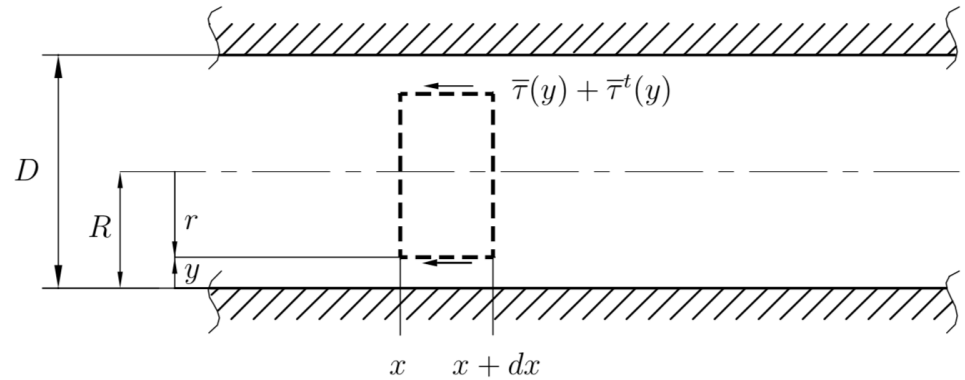


$$2 [\bar{\tau}(y) + \bar{\tau}^t(y)] = - \frac{d\bar{p}}{dx} (R-y)$$

$$2 \bar{\tau}_w = - \frac{d\bar{p}}{dx} R$$



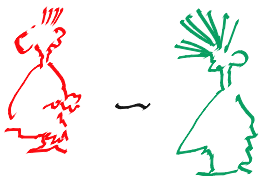
# Écoulements turbulents en conduite



$$2 \left( \bar{\tau}(y) + \bar{\tau}_t(y) \right) = -\frac{d\bar{p}}{dx} (R - y)$$



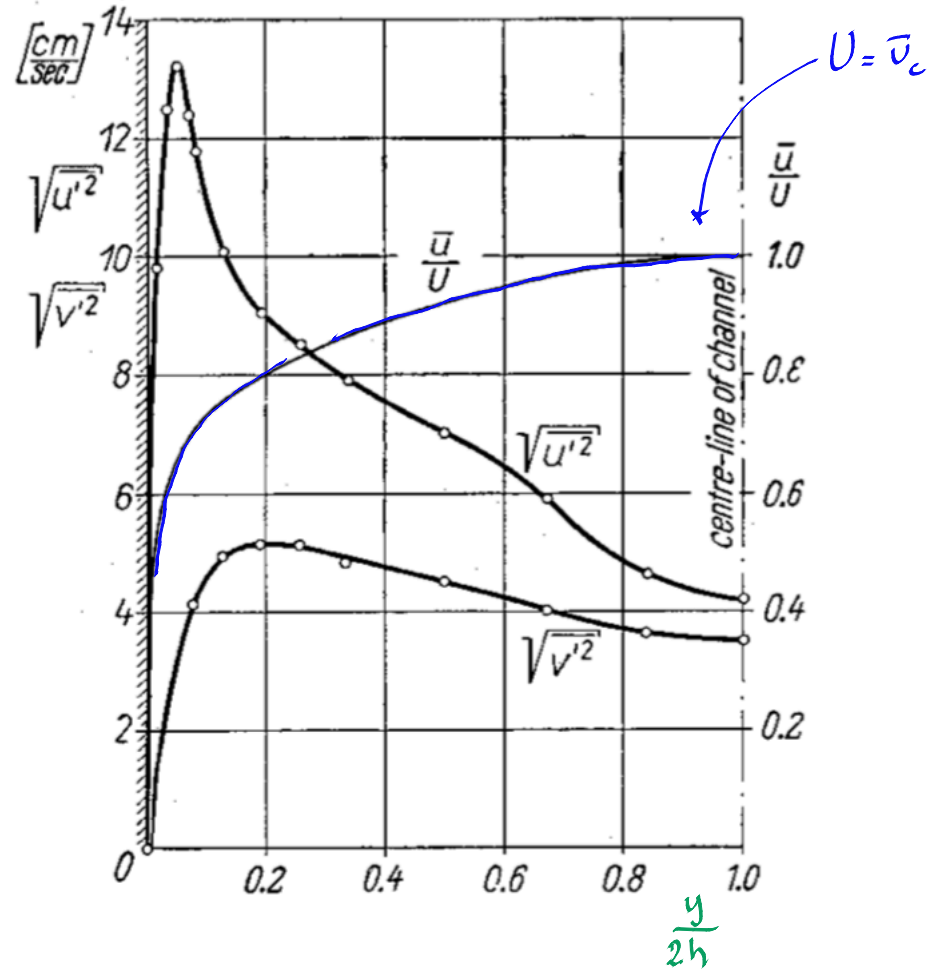
$$2 \tau_w = -\frac{d\bar{p}}{dx} R$$



$$\bar{\tau}(y) + \bar{\tau}_t(y) = \tau_w \left( 1 - \frac{y}{R} \right)$$

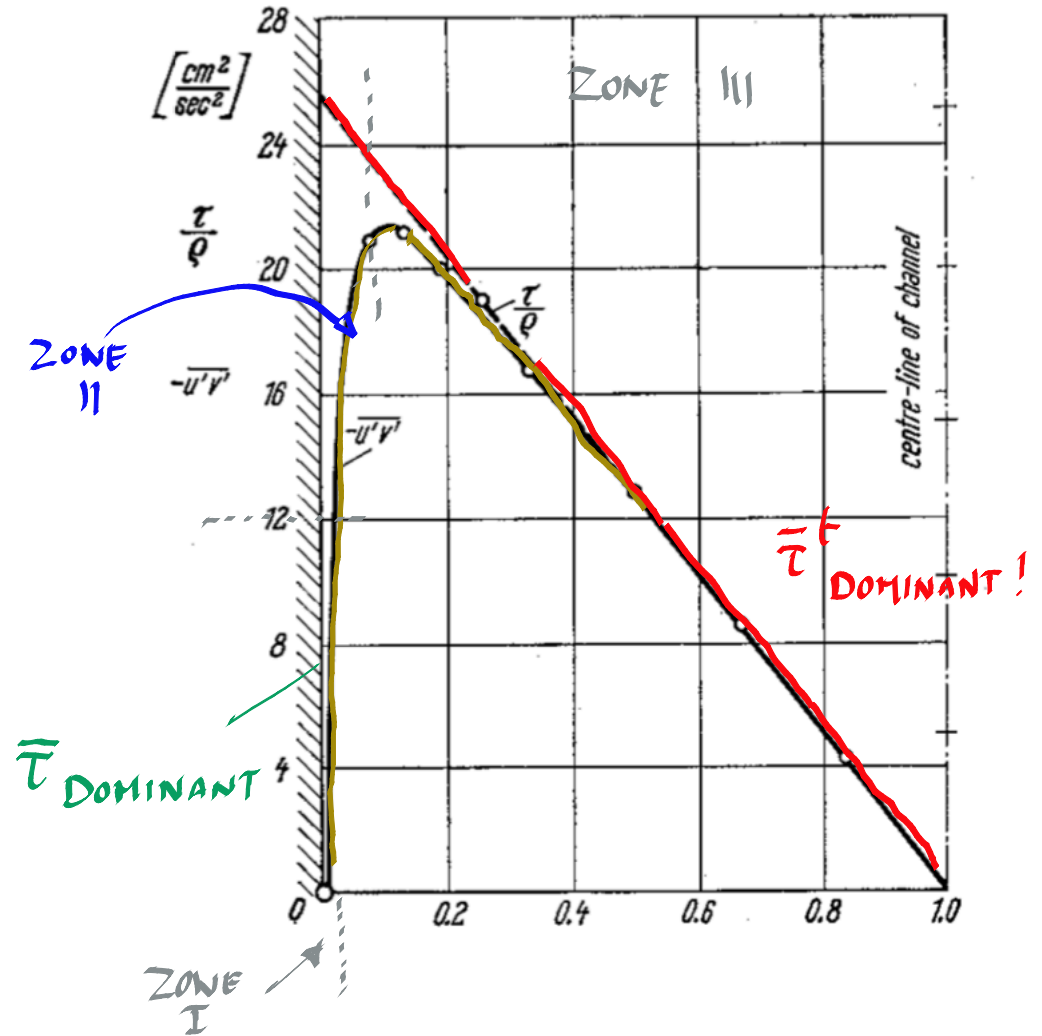
$$\mu \frac{d\bar{u}}{dy} - \rho \overline{u'v'} = \mu \frac{d\bar{u}}{dy} + \mu_t \frac{d\bar{u}}{dy} = \tau_w \left( 1 - \frac{y}{R} \right)$$

# Mesurons



$$\mu \frac{d\bar{u}}{dy} - \rho \overline{u'v'} = \mu \frac{d\bar{u}}{dy} + \mu_t \frac{d\bar{u}}{dy} = \tau_w \left(1 - \frac{y}{R} \right) h$$

Trois zones  
bien  
distinctes



$$\mu \frac{d\bar{u}}{dy} - \rho \overline{u'v'} = \mu \frac{d\bar{u}}{dy} + \mu_t \frac{d\bar{u}}{dy} = \tau_w \left( 1 - \frac{y}{R} \right)$$

# Viscosité effective de turbulence

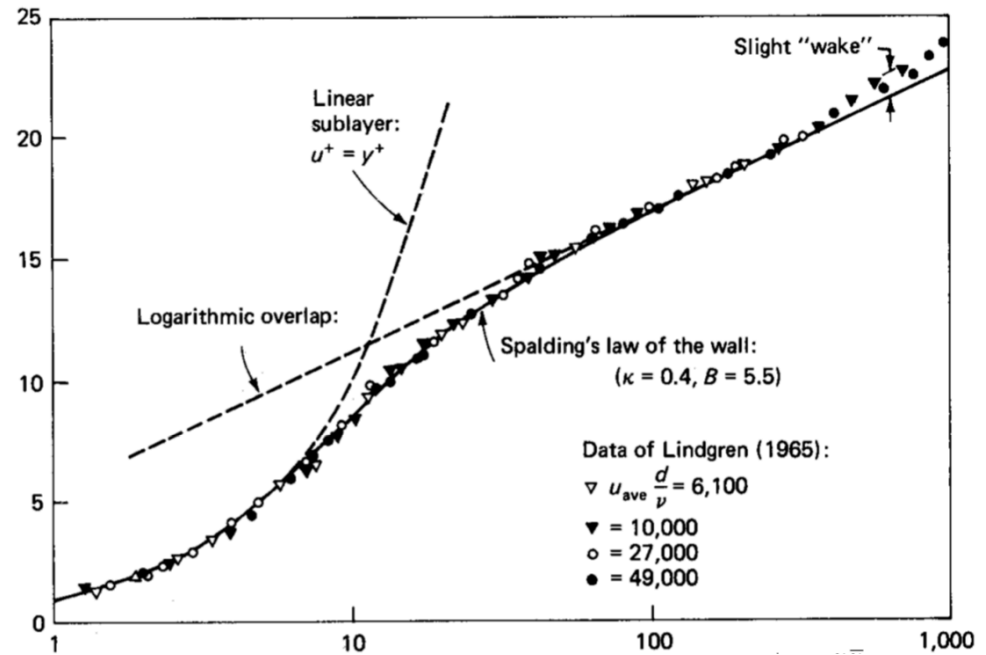
$$\underbrace{-\rho \overline{u'v'}}_{\substack{\text{CONTRAINTE} \\ \text{TURBULENTE} \\ \tau^t}} = \mu_t \underbrace{\frac{d\bar{u}}{dy}}_{\substack{\text{CISAILLEMENT} \\ \text{DE LA VITESSE} \\ \text{MOYENNE}}}$$

$$\nu_t \triangleq -\frac{\overline{u'v'}}{d\bar{u}/dy}$$

On peut déduire le profil de viscosité effective à partir de mesures expérimentales des profils de vitesse moyenne et des corrélations entre les fluctuations turbulentes.

Cela caractérise aussi la contrainte effective de turbulence !

# Profil universel dans la zone proche paroi



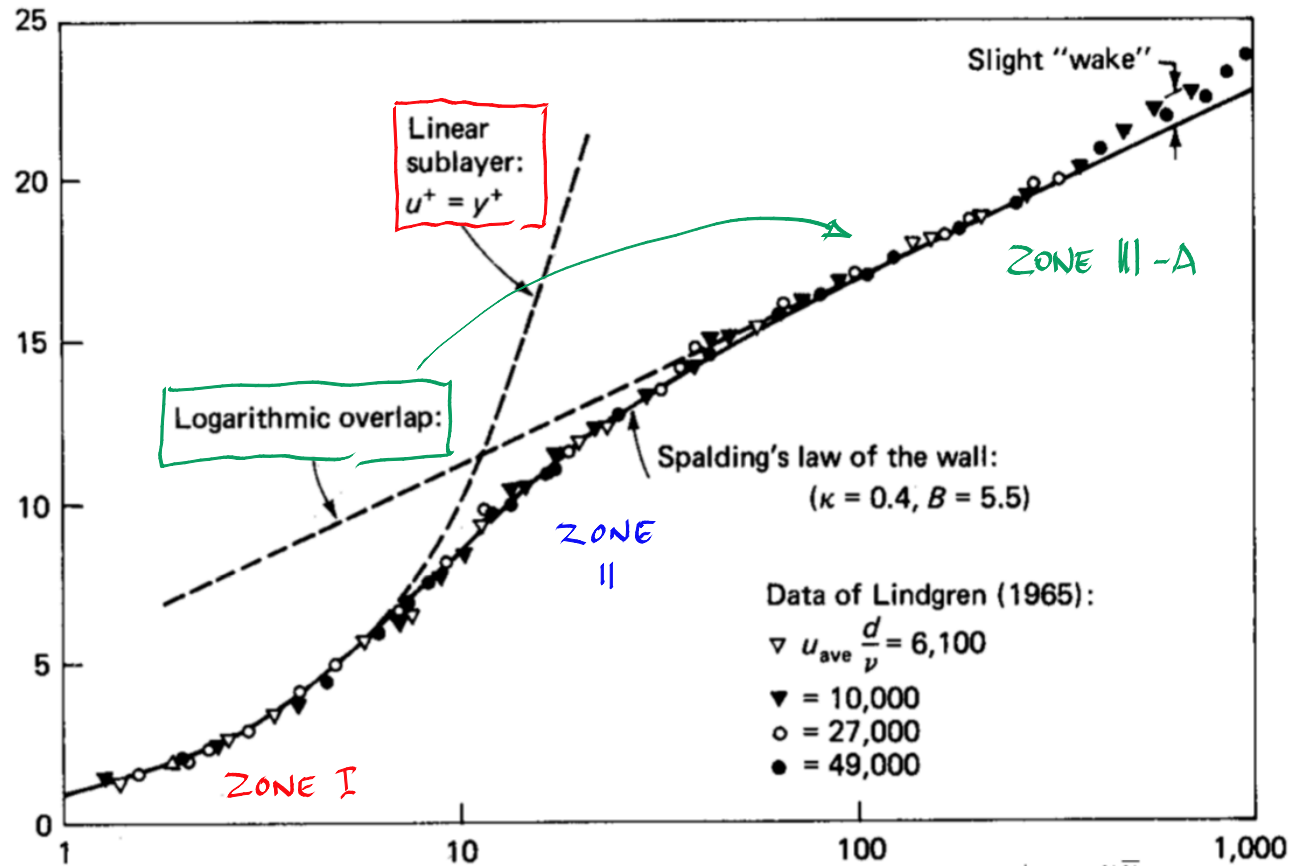
$$\nu \frac{d\bar{u}}{dy} + \nu_t \frac{d\bar{u}}{dy} = \underbrace{\frac{\tau_w}{\rho}}_{\equiv \bar{u}_\tau^2} \left(1 - \frac{y}{R}\right)$$

$$\equiv \bar{u}_\tau^2$$

$\bar{u}_\tau$  VITESSE  
DE FROTTEMENT

$$\bar{u}^+ \triangleq \frac{\bar{u}}{\bar{u}_\tau}$$

VITESSE  
PROCHE  
PAROI



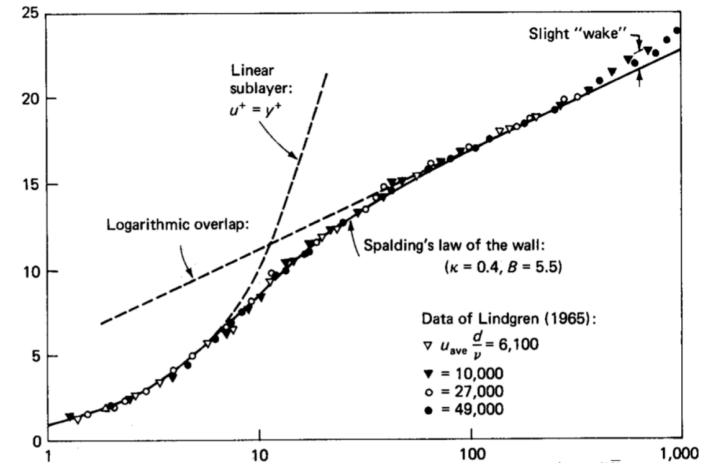
$$y^+ \triangleq \frac{y\bar{u}_\tau}{\nu}$$

ABSCISSE  
PROCHE  
PAROI

Il faut bien  
définir les axes !

# Modèles simplifiés

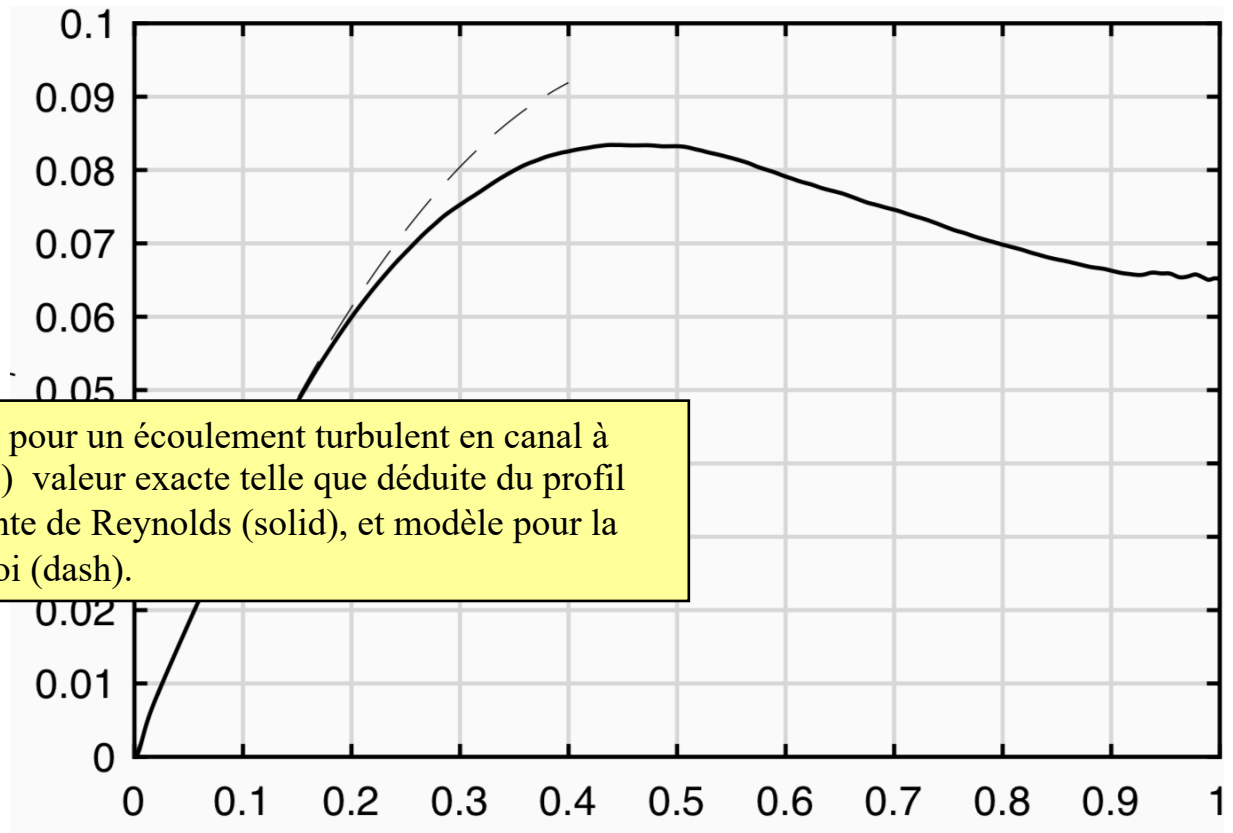
$$\bar{u}^+ \triangleq \frac{\bar{u}}{\bar{u}_\tau}$$



$$y^+ \triangleq \frac{y \bar{u}_\tau}{\nu}$$

Zone I	$\bar{u}^+(y^+) = y^+$
Zone IIIa	$\bar{u}^+(y^+) = \frac{1}{\kappa} \log y^+ + C$
Zone I-II-IIIa	$\bar{u}^+(y^+) = \frac{1}{\kappa} \log (1 + \kappa y^+) + 7.8 \left( 1 - \exp \left( -\frac{y^+}{11} \right) - \frac{y^+}{11} \exp \left( -\frac{y^+}{3} \right) \right) .$

# Résultats numériques



Viscosité effective de turbulence pour un écoulement turbulent en canal à grand Reynolds (avec  $h^+ = 5200$ ) valeur exacte telle que déduite du profil de vitesse et du profil de contrainte de Reynolds (solid), et modèle pour la zone turbulente proche de la paroi (dash).