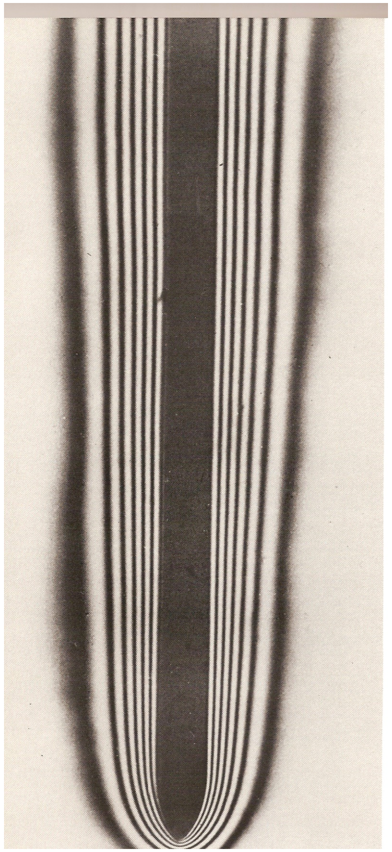
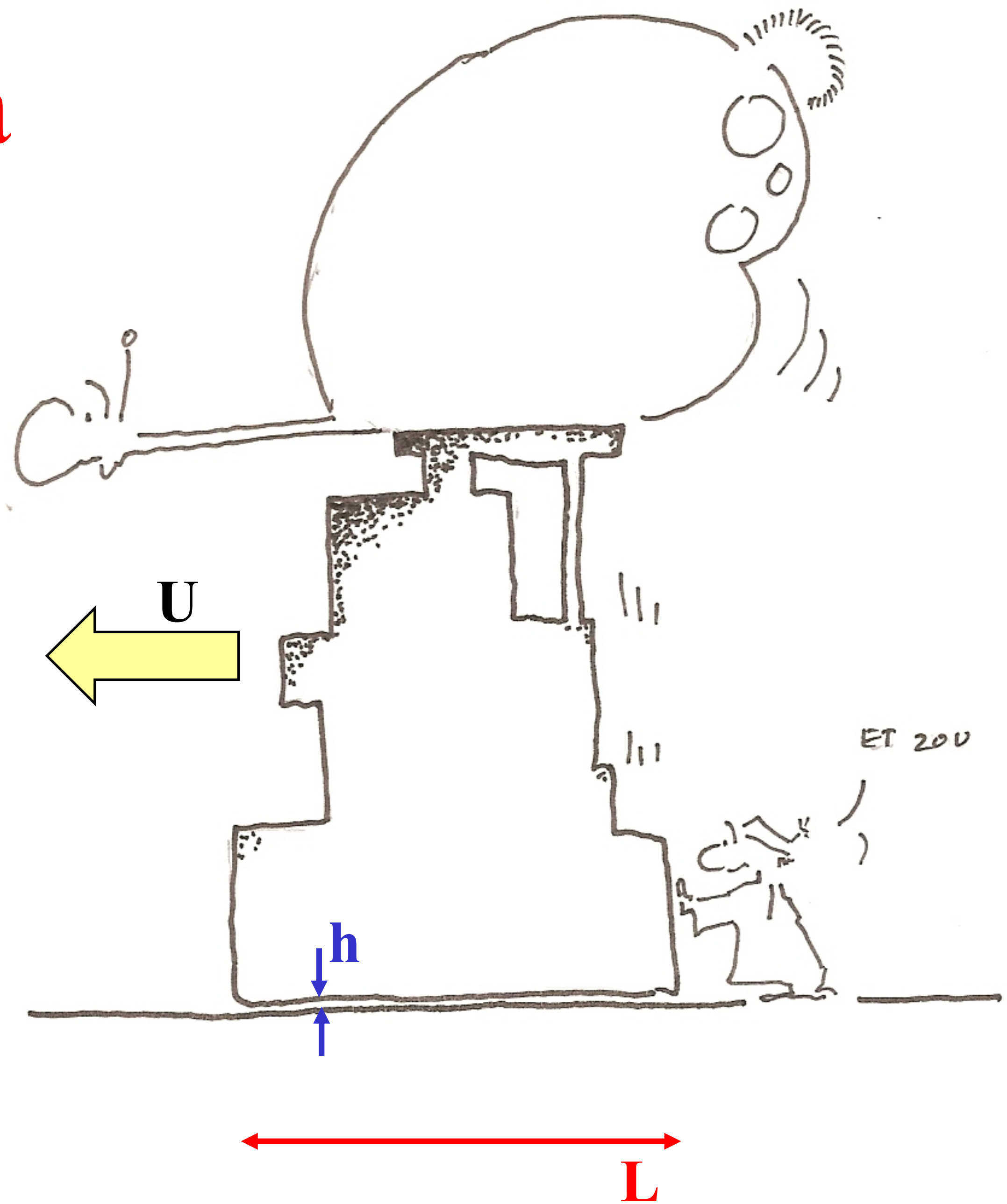
Mais que faire pour des écoulements avec deux échelles spatiales ?



Convection naturelle le long d'une plaque verticale : écoulement laminaire permanent

Lubrification et convoyage hydraulique : butée Michell

Théorie de la lubrification

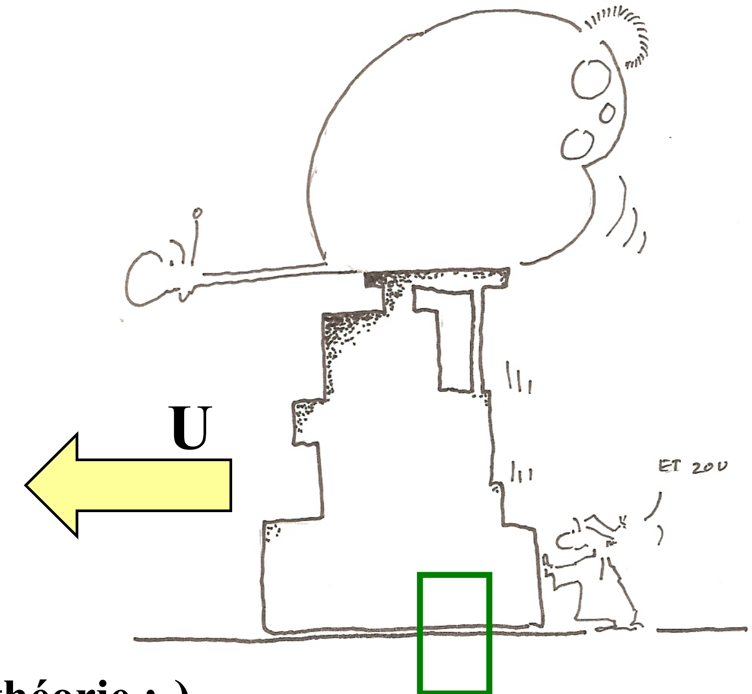


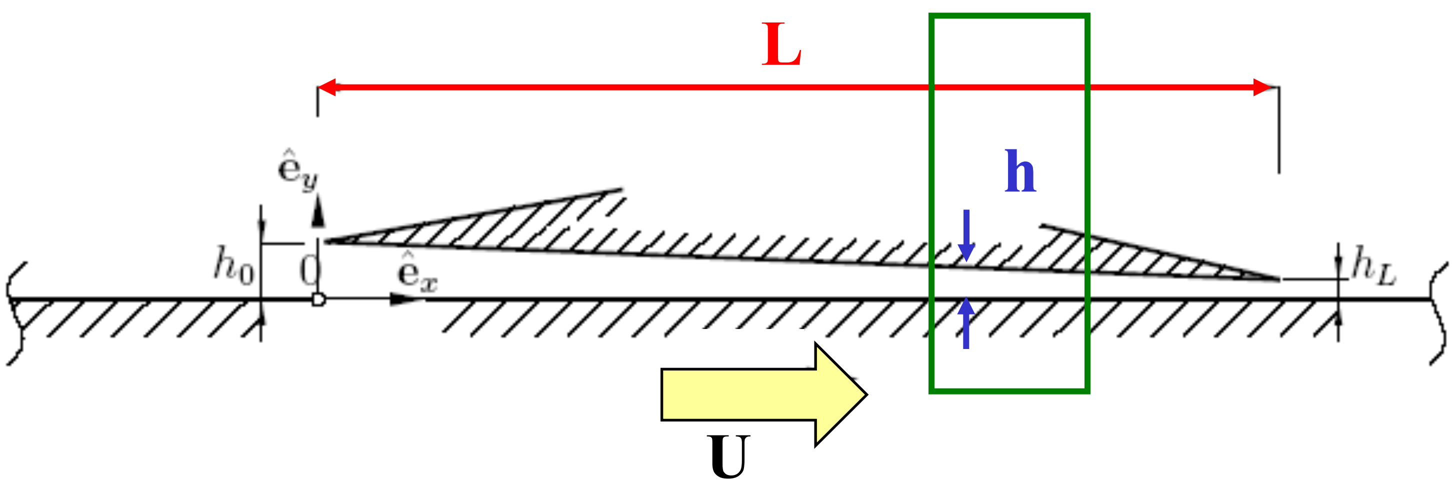
Convoyage hydraulique de charges très importantes :
- ***turbines hydroélectriques***
- ***applications marines***
- ***butées hydrauliques***

Théorie de la lubrification

$$\boxed{h \ll L}$$

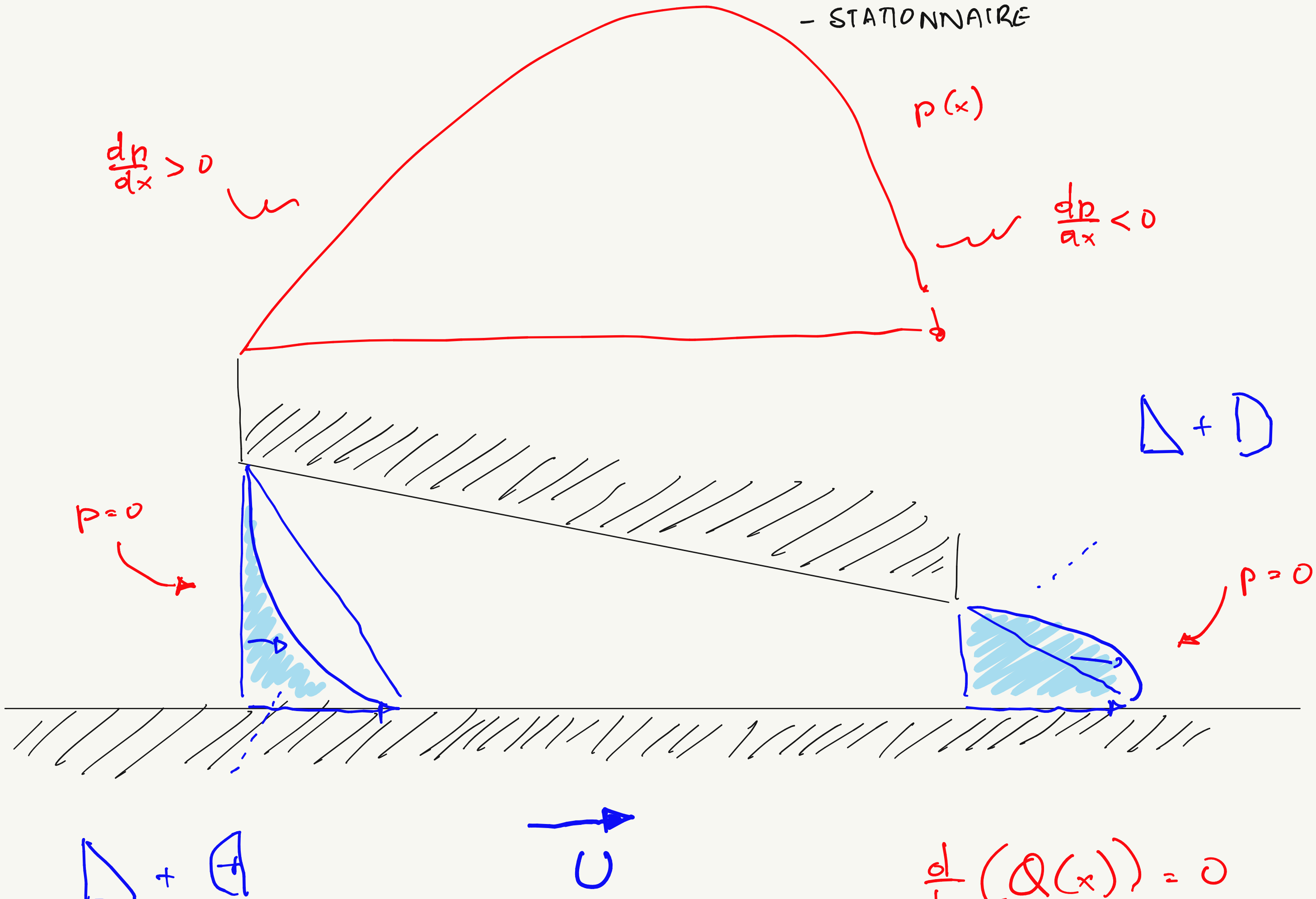
Hypothèse géométrique de base
Valable dans la zone centrale uniquement en théorie :-)





1 INTUITIVEMENT ?

- ECOULEMENT INCOMPRESSIBLE
- STATIONNAIRE



$$\frac{d}{dx}\Big(\underbrace{Q(x)}_{\text{DEBIT}}\Big) = 0$$

Ecoulements incompressibles plans stationnaires

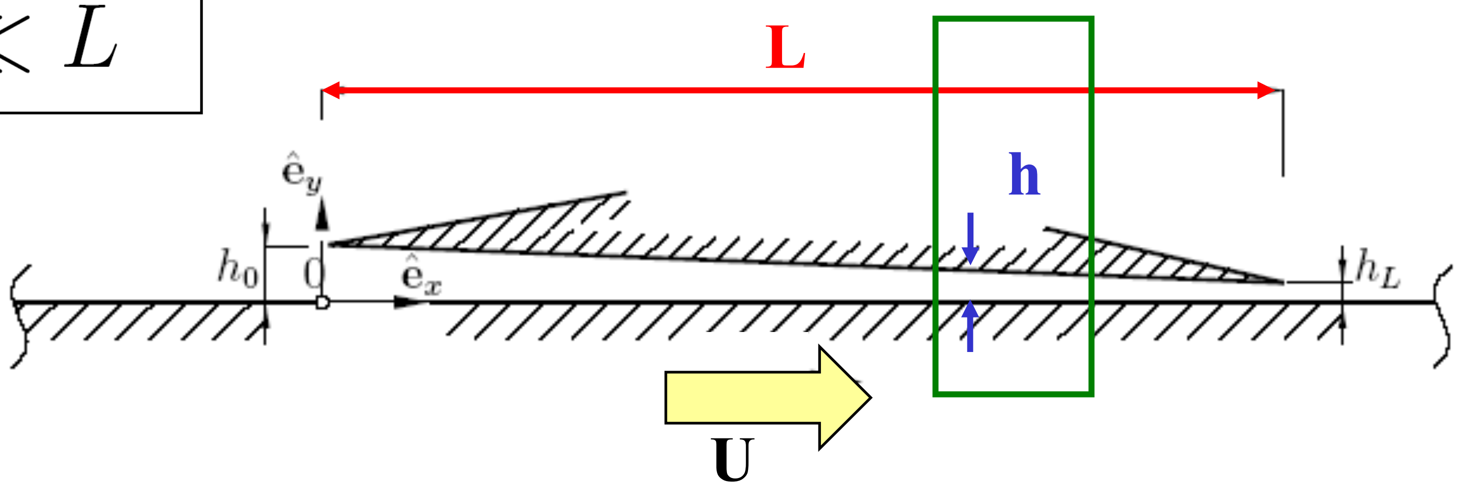
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\rho\, u \frac{\partial u}{\partial x} + \rho\, v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial y^2}$$

$$\rho\, u \frac{\partial v}{\partial x} + \rho\, v \frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \mu \frac{\partial^2 v}{\partial x^2} + \mu \frac{\partial^2 v}{\partial y^2}$$

Que deviennent ces équations ?

$$h \ll L$$



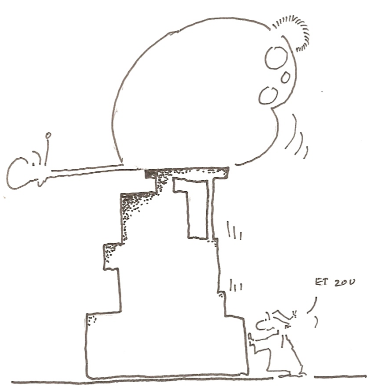
$$h \ll L$$

$$\begin{aligned}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \\
\rho\, u \frac{\partial u}{\partial x} + \rho\, v \frac{\partial u}{\partial y} &= -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial y^2} \\
\rho\, u \frac{\partial v}{\partial x} + \rho\, v \frac{\partial v}{\partial y} &= -\frac{\partial p}{\partial y} + \mu \frac{\partial^2 v}{\partial x^2} + \mu \frac{\partial^2 v}{\partial y^2}
\end{aligned}$$

Longueur horizontale caractéristique : L

Longueur verticale caractéristique : h

Vitesse horizontale caractéristique : U



Comment choisir une vitesse verticale caractéristique ?

$\boxed{2}$

DEFINIR
UNE VITESSE
VERTICALE
CARACTERISTIQUE :-)

JE L'AI DEDUIT
DE $h \ll L$

$$\boxed{\frac{\partial u}{\partial x}} + \boxed{\frac{\partial v}{\partial y}} = 0$$

$$V \ll U$$

CECI
N'EST PAS
UNE
HYPOTHESE

$$\frac{U}{L} = \frac{V}{h}$$

$$\boxed{V = \underbrace{\frac{h}{L}}_{\ll 1} U \ll U}$$

$$\underbrace{\frac{\partial u}{\partial x}}_{\mathcal{O}(U/L)} + \underbrace{\frac{\partial v}{\partial y}}_{\mathcal{O}(V/h)} = 0$$

Il ne faut pas définir de vitesse caractéristique verticale !

$$\boxed{V = \frac{Uh}{L} \ll U}$$

$\boxed{3}$ SIMPLIFIONS !

$$\cancel{\underbrace{\rho u \frac{\partial u}{\partial x}}_{\partial\left(\rho \frac{U^2}{L}\right)}} + \cancel{\underbrace{\rho v \frac{\partial u}{\partial y}}_{\partial\left(\rho \frac{VU}{h}\right)}} = -\frac{\partial p}{\partial x} + \cancel{\underbrace{\mu \frac{\partial^2 u}{\partial x^2}}_{\partial\left(\mu \frac{U}{L^2}\right)}} + \underbrace{\mu \frac{\partial^2 u}{\partial y^2}}_{\partial\left(\mu \frac{U}{h^2}\right)}$$

$$\partial\left(\mu \frac{U}{L^2}\right) \ll \partial\left(\mu \frac{U}{h^2}\right)$$

$$V = \frac{Uh}{L}$$

$$\partial\left(\rho \frac{U\cancel{h}}{L} \frac{U}{\cancel{h}}\right)$$

$$\frac{\text{INERTIE}}{\text{DIFFUSION}} = \frac{\rho U^2/L}{\mu U/h^2} = \underbrace{\frac{\rho U L}{\mu}}_{Re} \overbrace{\frac{h^2}{L^2}}^{\ll 1}$$

EN SUPPOSANT QUE LE Re RESTE ACCEPTABLE

Quand peut-on négliger les termes d'inertie ?

$$\overset{\mathcal{O}(\rho U^2/L)}{\boxed{\cancel{\rho\, u \frac{\partial u}{\partial x}}}} + \underset{\mathcal{O}(\rho V U/h)}{\overset{\mathcal{O}(\rho U^2/L)}{\boxed{\cancel{\rho\, v \frac{\partial u}{\partial y}}}}} = -\frac{\partial p}{\partial x} + \boxed{\cancel{\mu \frac{\partial^2 u}{\partial x^2}}} + \boxed{\mu \frac{\partial^2 u}{\partial y^2}}$$

$$\mathcal{O}(\mu U/L^2) \ll \mathcal{O}(\mu U/h^2)$$

Hypothèse de lubrification :
Écoulements rampants

$$\frac{\boxed{\text{Forces d'inertie}}}{\boxed{\text{Forces visqueuses}}} = \frac{\rho U^2/L}{\mu U/h^2} = \underbrace{\frac{\rho U L}{\mu}}_{Re_L}\,\frac{h^2}{L^2} \ll 1$$

Et l'autre équation ?

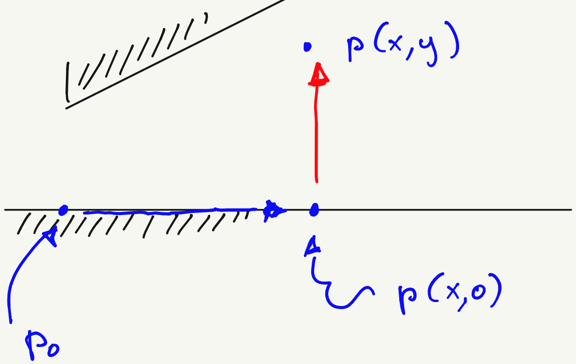
$$\underbrace{\boxed{\cancel{\rho\, u \frac{\partial v}{\partial x}}}}_{}^{\mathcal{O}(\rho U^2 h/L^2)} + \overbrace{\boxed{\cancel{\rho\, v \frac{\partial v}{\partial y}}}}^{\mathcal{O}(\rho U^2 h/L^2)} = -\frac{\partial p}{\partial y} + \boxed{\cancel{\mu \frac{\partial^2 v}{\partial x^2}}} + \boxed{\mu \frac{\partial^2 v}{\partial y^2}}$$

$$\mathcal{O}(\mu U h/L^3) \ll \mathcal{O}(\mu U/Lh)$$

On obtient la
<u>même condition</u>…

$$\frac{\boxed{\text{Forces d'inertie}}}{\boxed{\text{Forces visqueuses}}} = \frac{\rho U^2 h/L^2}{\mu U/Lh} = \underbrace{\frac{\rho U L}{\mu}}_{Re_L} \frac{h^2}{L^2} \ll 1$$

4 ET LA PRESSION ?



$\partial(h)$

$\partial\left(\mu \frac{V}{h^2}\right) \Big\} \; \partial\left(\cancel{h}\mu \frac{U\cancel{h}}{L} \frac{1}{\cancel{h^2}}\right)$

$$p(x,y) - p_0 = \boxed{p(x,0) - p_0} + \boxed{y \left.\frac{\partial p}{\partial y}\right|_{y=0}} + \underbrace{y^2 \dots}$$

C'EST PETIT CAR $y^2 \ll y$:-)

$x \frac{\partial p}{\partial x}$

$\partial(L)$ $\quad$ $\partial\left(\frac{\mu U}{h^2}\right)$

$\partial(h)$

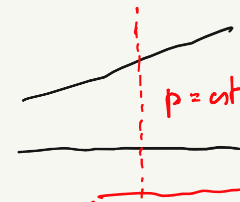
$\partial\left(\mu \frac{V}{h^2}\right)$ $\Big\}$ $\partial\left(\cancel{h}\mu \frac{U\cancel{h}}{L}\frac{1}{\cancel{h^2}}\right)$

$$p(x,y) - p_0 = \boxed{p(x,0) - p_0} + \cancel{\boxed{y \left.\frac{\partial p}{\partial y}\right|_{y=0}}} + \underbrace{y^2 \ldots}$$

C'EST PETIT CAR $y^2 \ll y$:-)

$x \frac{\partial p}{\partial x}$

$\partial(L)$ $\qquad$ $\partial\left(\frac{\mu U}{h^2}\right)$



$\boxed{p(x) \text{ :-)}}$

LA PRESSION NE DEPEND PAS DE y :-)

$$\frac{\mu U L}{h^2} \gg \frac{\mu U}{L} = \frac{\mu U L}{L^2}$$

$$\mu U L \gg \mu U L \frac{h^2}{L^2}$$

REYNOLDS

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{dp}{dx} = \mu \frac{\partial^2 u}{\partial y^2}$$

THEORIE DE LA LUBRIFICATION

$$\boxed{\cancel{\rho\, u \frac{\partial v}{\partial x}}} + \boxed{\cancel{\rho\, v \frac{\partial v}{\partial y}}} = -\frac{\partial p}{\partial y} + \boxed{\cancel{\mu \frac{\partial^2 v}{\partial x^2}}} + \boxed{\mu \frac{\partial^2 v}{\partial y^2}}$$

$$\mathcal{O}(\mu U/Lh)$$

Et la pression ?

$$p(x,y) - p_0 = \boxed{p(x,0) - p_0} + \boxed{y\, \cancel{\left.\frac{\partial p}{\partial y}\right|_{y=0}}}$$

$$\mathcal{O}(\mu UL/h^2) \gg \mathcal{O}(\mu UL/L^2)$$

$p(x)$:-)

$$\mathcal{O}(\mu U/h^2)$$

$$\boxed{\cancel{\rho\, u \frac{\partial u}{\partial x}}} + \boxed{\cancel{\rho\, v \frac{\partial u}{\partial y}}} = -\frac{\partial p}{\partial x} + \boxed{\cancel{\mu \frac{\partial^2 u}{\partial x^2}}} + \boxed{\mu \frac{\partial^2 u}{\partial y^2}}$$

Equations de Reynolds (1889)

Théorie de la lubrification

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$0 = -\frac{dp}{dx} + \mu \frac{\partial^2 u}{\partial y^2}$$

Film fluide mince

$$h \ll L$$

Hypothèse de lubrification :
Écoulements rampants

$$\underbrace{\frac{\rho U L}{\mu}}_{Re_L} \frac{h^2}{L^2} \ll 1$$

Est-ce que l'hypothèse de lubrification est réaliste ?

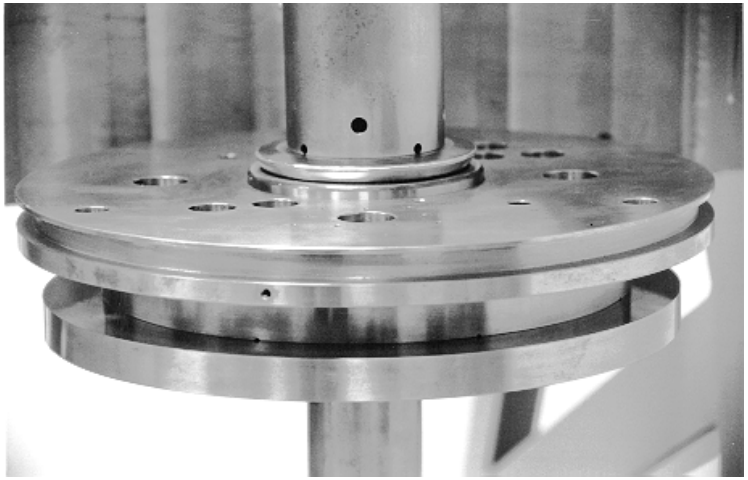
0.0375

$$\underbrace{\frac{\rho U L}{\mu}}_{Re_L} \frac{h^2}{L^2} \ll 1$$

$$\begin{aligned} L &= 10\ cm \\ h &= 0.5\ mm \\ U &= 1\ m/s \end{aligned}$$

$$\begin{aligned} \rho &= 900\ kg/m^3 \\ \mu &= 60\ 10^{-3}\ Ns/m^2 \end{aligned}$$

Huile SAE50 à 60 degrés



Huile SAE 50
C'est quoi ?

Transport maritime

Marine LCX

Une huile formulée spécialement pour la lubrification des gros moteurs diesel marins à crosse. Elle lubrifie les cylindres grâce à un indice de basicité très élevé de 70 et un grade SAE* 50.

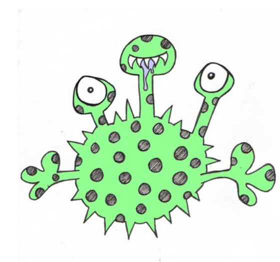
Grades offerts :
SAE 50

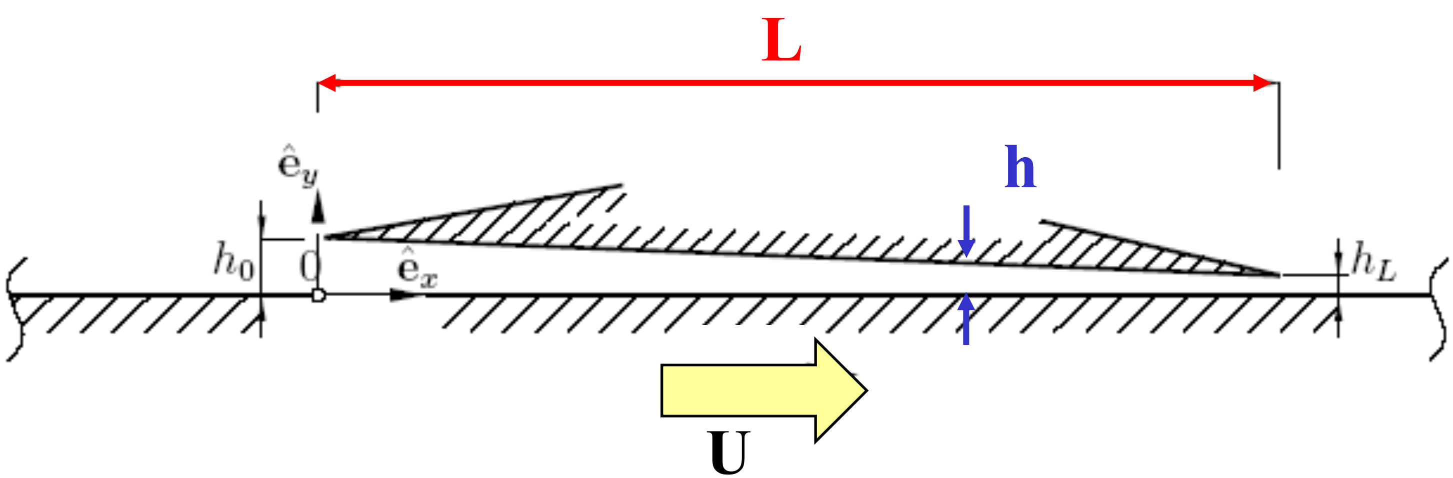
Fiche technique
Fiche signalétique

-i- calcul de u(x,y)

$$\left\{\begin{array}{rcl} \dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y} &=& 0 \\[2ex] -\dfrac{dp}{dx}+\mu\,\dfrac{\partial^2 u}{\partial y^2} &=& 0 \end{array}\right.$$

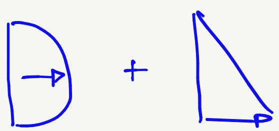
$$\boxed{u(x,y) = -\frac{dp}{dx}\,\frac{h^2}{2\mu}\,\frac{y}{h}\left(1-\frac{y}{h}\right)+U\left(1-\frac{y}{h}\right)}$$





5 CALCUL DU PROFIL DE VITESSE :-)

$$\frac{dp}{dx} = \mu \frac{\partial^2 u}{\partial y^2}$$



$$\boxed{u(x,y) = -\underbrace{\frac{dp}{dx}}\,\underbrace{\frac{h^2}{2\mu}}\left(1-\frac{y}{h}\right)\frac{y}{h} + U\left(1-\frac{y}{h}\right)}$$

$$\left[\frac{\cancel{N}}{\cancel{m^3}}\right]\left[\frac{m^{\cancel{2}}}{\cancel{N}\,s/\cancel{m^2}}\right] = \left[\frac{m}{s}\right] \quad :-)$$

$$\tau = 2\mu d$$

$$\left[\frac{N}{m^2}\right] \qquad \left[\frac{1}{s}\right]$$

$\boxed{6}$ CALCUL DE LA PRESSION

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$0 = \int_0^h \frac{\partial u}{\partial x}\,dy + \int_0^h \frac{\partial v}{\partial y}\,dy$$

$$= \underbrace{\frac{d}{dx}\int_0^h u\,dy}_{Q(x)} + \underbrace{\Big[v\Big]_0^h}_{=0}$$

P

$$\boxed{u(x,y) = -\frac{dp}{dx}\frac{h^2}{2\mu}\left(1-\frac{y}{h}\right)\frac{y}{h} + U\left(1-\frac{y}{h}\right)}$$

$$0 = \frac{d}{dx}\left[-\frac{dp}{dx}(x)\,\frac{h^2(x)}{2\mu}\underbrace{\int_0^{h(x)}\left(1-\frac{y}{h(x)}\right)\frac{y}{h(x)}\,dy}_{=\frac{h(x)}{6}\ \ :-)\ \ (*)} + U\,\frac{h(x)}{2}\right]$$

(*) $$\int_0^h \frac{y}{h} - \frac{y^2}{h^2}\,dy$$

$$= \left[\frac{y^2}{2h} - \frac{y^3}{3h^2}\right]_0^h = \frac{h}{2} - \frac{h}{3}$$

$$= \frac{3h-2h}{6} = \frac{h}{6} \quad \square \quad :-)$$

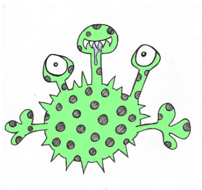
-ii- calcul de p(x)

$$\left\{\begin{array}{rcl} \boxed{-\dfrac{dp}{dx}+\mu\dfrac{\partial^2 u}{\partial y^2} = 0} \\ \boxed{\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y} = 0} \end{array}\right.$$

$$0 = \int_0^h \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}dy$$

$$0 = \frac{d}{dx}\overbrace{\int_0^h u(x,y)dy}^{Q(x)} + \cancel{\Big[v(x,y)\Big]_0^h}$$

En utilisant l'expression de u(x,y)

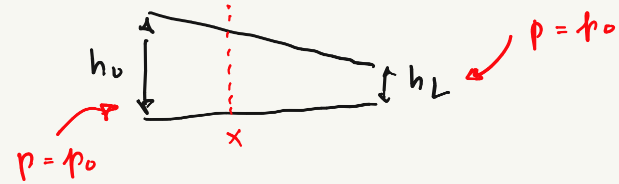


Equation classique de Reynolds (1889)

$$\boxed{0 = \frac{d}{dx}\left(-\frac{dp}{dx}\frac{h^3}{12\mu}+\frac{Uh}{2}\right)}$$

$$0 = \frac{d}{dx}\left[-\frac{dp}{dx}(x)\,\frac{h^2(x)}{2\mu} \underbrace{\int_0^{h(x)} \left(1-\frac{y}{h(x)}\right)\frac{y}{h(x)}\,dy}_{h(x)/6} + U\,\frac{h(x)}{2}\right]$$

$$\frac{dh}{dx} = \frac{h_L - h_0}{L}$$



$$0 = \frac{d}{dx}\left[-\frac{dp}{dx}(x)\,\frac{h^3(x)}{12\mu} + U\,\frac{h(x)}{2}\right]$$

$$\frac{d}{dx}\left[\frac{dp}{dx}(x)\,h^3(x)\right] = 6\mu\,\frac{dh}{dx}\,U$$

$$\left(\frac{dh}{dx}\right)^2 \frac{d}{dh}\left[\frac{dp}{dh}(h)\,h^3\right] = 6\mu\,\frac{dh}{dx}\,U$$

$$-\frac{d}{dh}\left[\frac{dp}{dh}(h)\,h^3\right] = \underbrace{\frac{6\mu U L}{h_0 - h_L}}_{C}$$

$$-\frac{dp}{dh}\,h^3 = C\left[h + A\right]$$

$$-\frac{dp}{dh} = C\left[\frac{1}{h^2} + \frac{A}{h^3}\right]$$

$$p(h) = C\left[\frac{1}{h} + \frac{A}{2h^2} + B\right]$$

$$0 = \frac{d}{dx}\left(-\frac{dp}{dx}\frac{h^3}{12\mu} + \frac{Uh}{2}\right)$$

$$\frac{d}{dx}\left(h^3(x)\,\frac{dp}{dx}(x)\right) = 6\mu U \boxed{\frac{dh}{dx}}$$

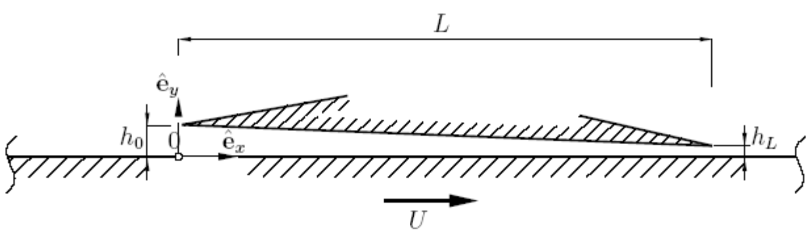
$$-\frac{d}{dh}\left(h^3\frac{dp}{dh}(h)\right) = \frac{6\mu U L}{h_0 - h_L}$$

$$-h^3\frac{dp}{dh}(h) = \frac{6\mu U L}{h_0 - h_L}(h + A)$$

$$-\frac{dp}{dh}(h) = \frac{6\mu U L}{h_0 - h_L}\left(\frac{1}{h^2} + \frac{A}{h^3}\right)$$

$$p(h) = \frac{6\mu U L}{h_0 - h_L}\left(B + \frac{1}{h} + \frac{A}{2h^2}\right)$$

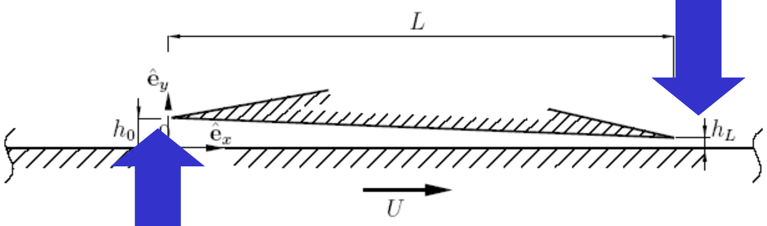
Palier plat



$$\frac{x}{L} = \frac{h_0 - h(x)}{h_0 - h_L}$$

$$\frac{dh}{dx} = -\frac{h_0 - h_L}{L}$$

$$p(h_L) = p_0$$



$$p(h_0) = p_0$$

Deux conditions aux limites

Deux constantes

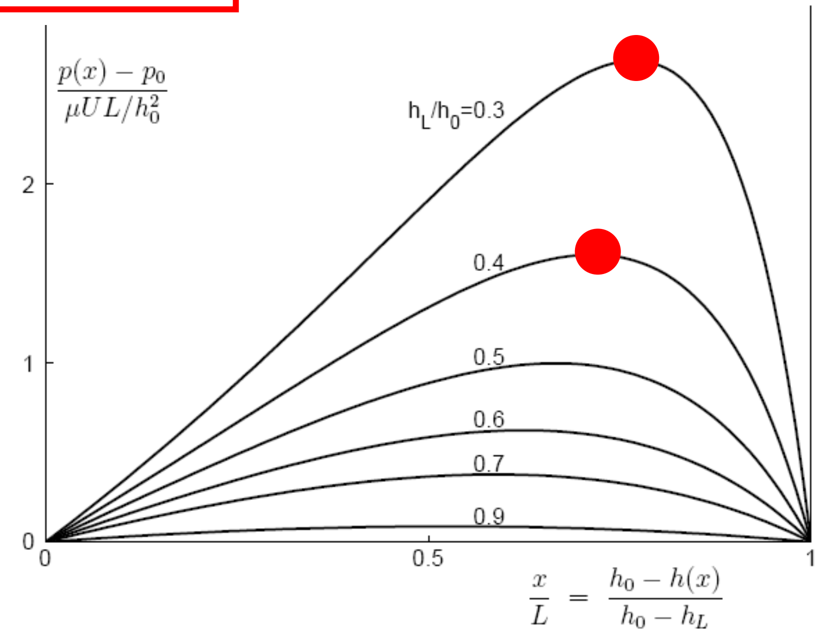
$$p(h) = \frac{6\mu U L}{h_0 - h_L}\left(\boxed{B} + \frac{1}{h} + \frac{\boxed{A}}{2h^2}\right)$$

$$\downarrow$$

$$p(h) - p_0 = \frac{6\mu U L (h_0 - h)(h - h_L)}{(h_0^2 - h_L^2)h^2}$$

Où la pression est-elle maximale ?

$$\boxed{h = \frac{2\,h_0\,h_L}{(h_0 + h_L)}}$$

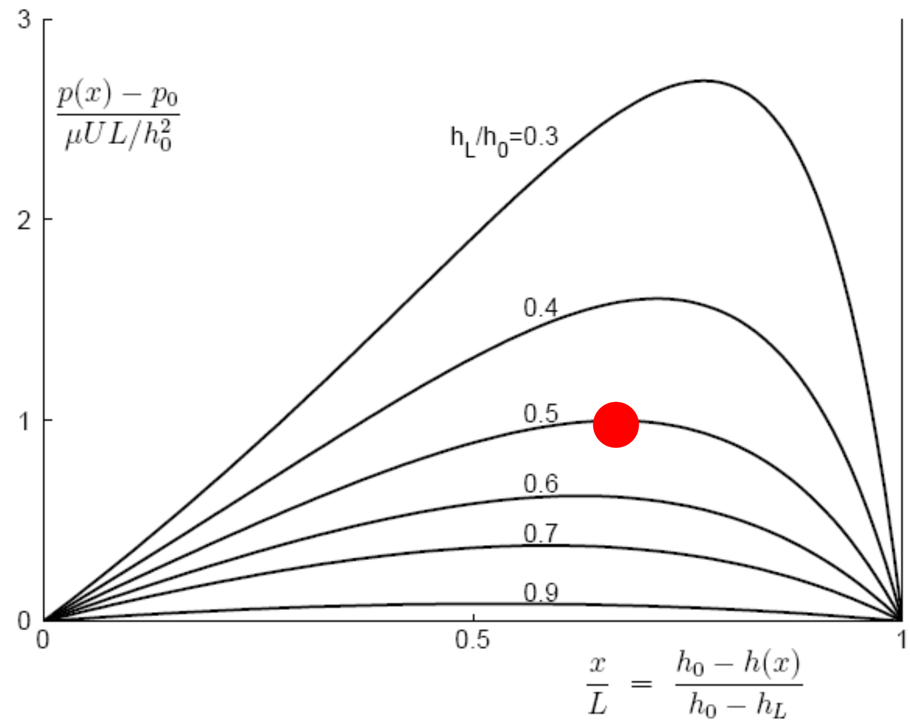


Cette pression peut être énorme !

$$
\begin{aligned}
L &= 10\ cm \\
h_0 &= 0.1\ mm \\
h_L &= 0.05\ mm \\
U &= 10\ m/s
\end{aligned}
$$

$$
\begin{aligned}
\rho &= 900\ kg/m^3 \\
\mu &= 0.1\ Ns/m^2
\end{aligned}
$$

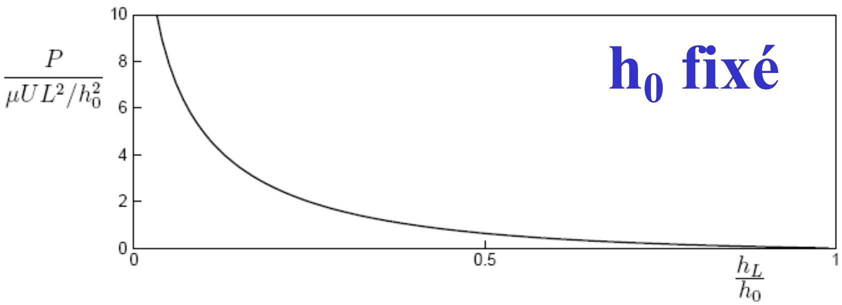
Huile SAE50 à 50 degrés

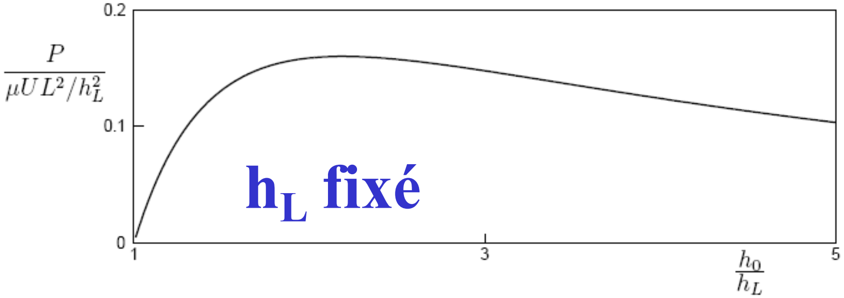


$$p_{\max} - p_0 = \boxed{\frac{3}{2} \frac{\mu\, U\, L}{h_0\, h_L} \frac{(h_0 - h_L)}{(h_0 + h_L)}}$$

10^7 Pascal

Charge utile



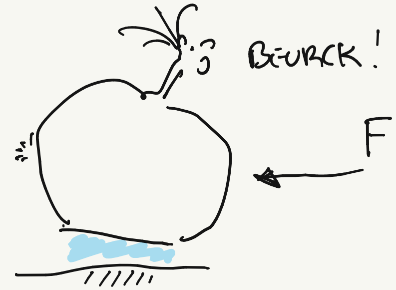


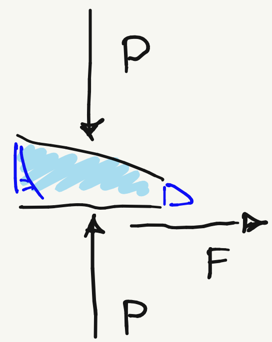
$$
\begin{aligned}
P &= \int_0^L (p(x) - p_0)\, dx \\
&= -\frac{L}{(h_0 - h_L)} \int_{h_0}^{h_L} (p(h) - p_0)\, dh \\
&= -\frac{L}{(h_0 - h_L)} \frac{6\,\mu\, U\, L}{(h_0^2 - h_L^2)} \int_{h_0}^{h_L} \left[(h_0 + h_L)\frac{1}{h} - \frac{h_0\, h_L}{h^2} - 1\right] dh \\
&= -6\,\mu\, U\, L^2 \left[\frac{1}{(h_0 - h_L)^2} \log\left(\frac{h_L}{h_0}\right) + \frac{2}{(h_0^2 - h_L^2)}\right]
\end{aligned}
$$

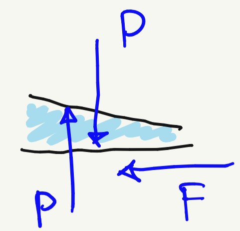
$\boxed{7}$

CHARGE UTILE

$$P = \int_0^L p(x) - p_0 \, dx$$



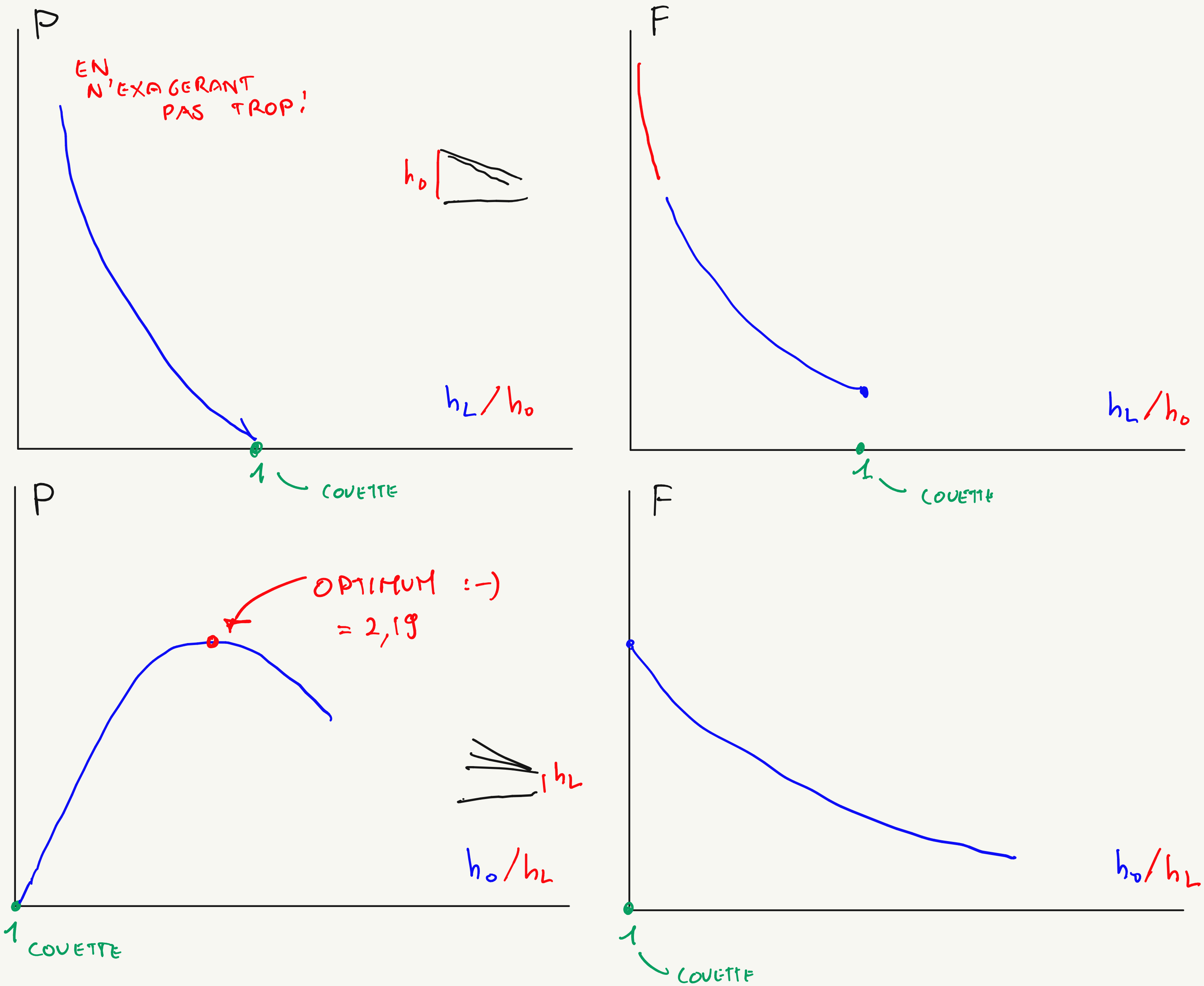


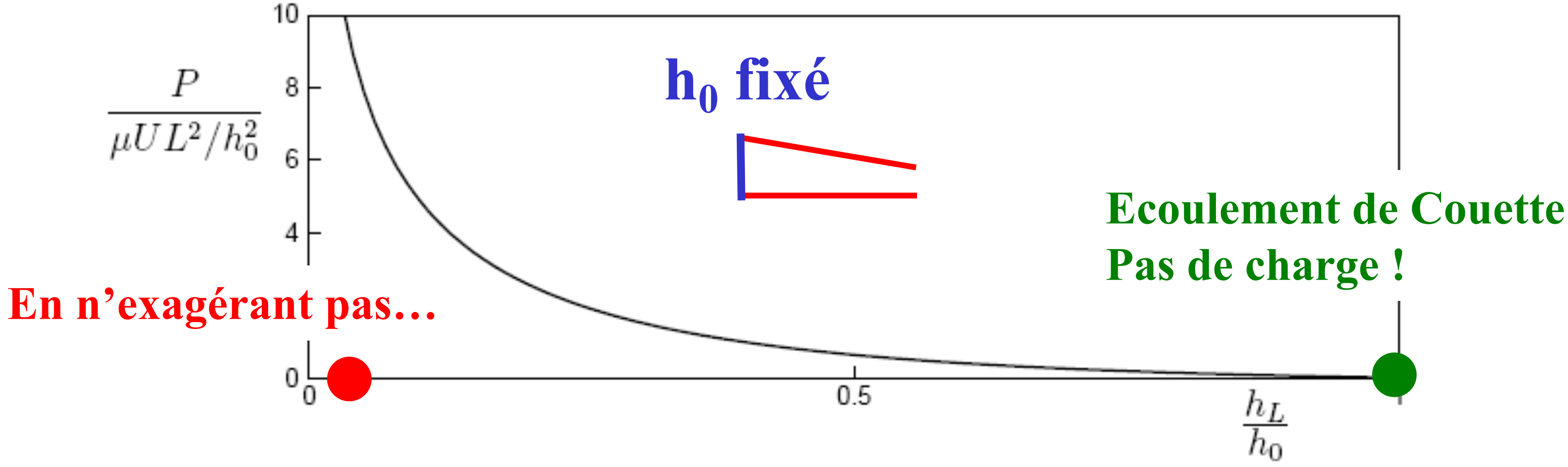


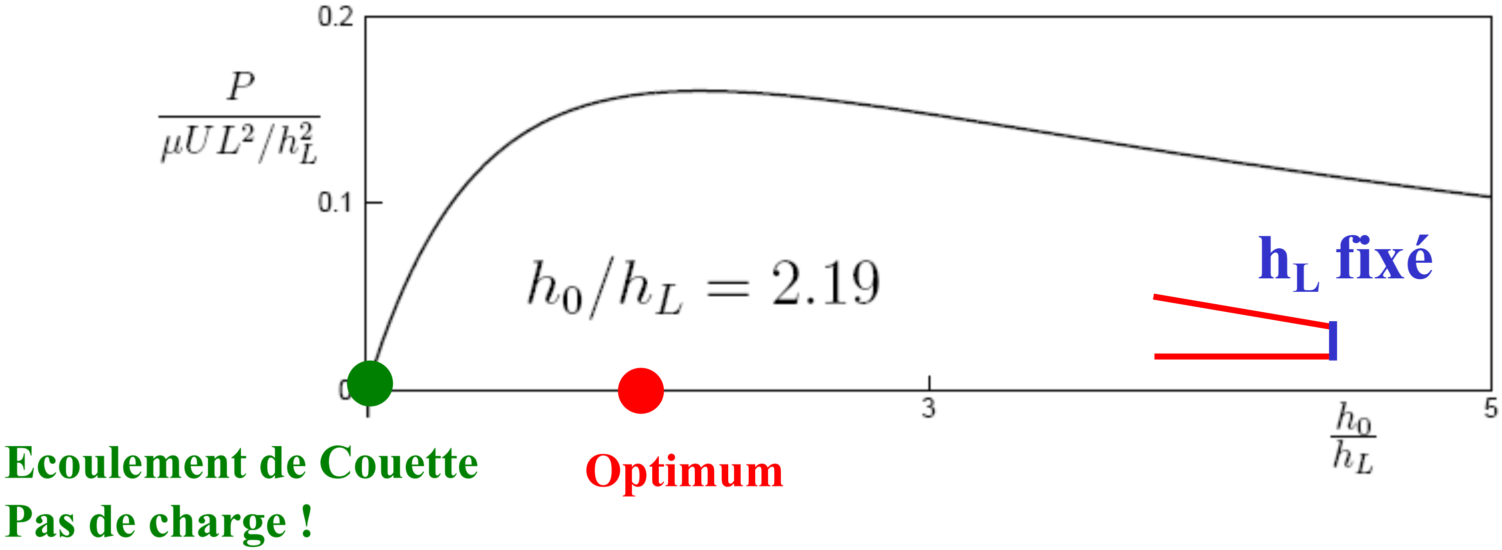
FORCE DE POUSSÉE

$$F = -\mu \int_0^L \frac{\partial u}{\partial y} \, dx$$

PUISSANCE DISSIPEE $= F . U$

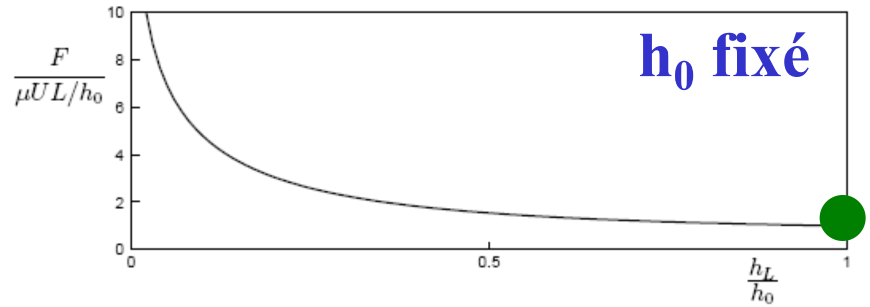

P
EN N'EXAGERANT PAS TROP!
h0
hL / h0
1
COUETTE
F
hL / h0
1
COUETTE
P
OPTIMUM :-)
= 2,19
hL
h0 / hL
1
COUETTE
F
h0 / hL
1
COUETTE

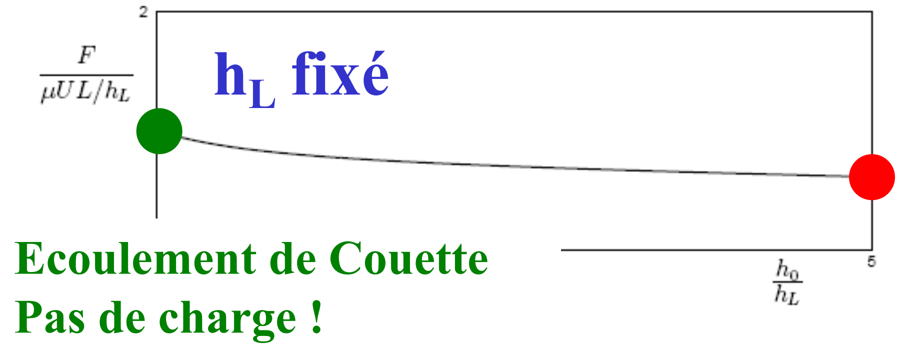




Rapport optimal…

Force exercée par le fluide sur la partie mobile





$$
\begin{aligned}
F &= -\int_0^L \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} dx \\
&= \frac{\mu U L}{(h_0 - h_L)} \int_{h_0}^{h_L} \left[\frac{6}{h^2}\frac{h_0\, h_L}{(h_0 + h_L)} - \frac{4}{h}\right] dh \\
&= -\mu U L \left[\frac{6}{(h_0 + h_L)} + \frac{4}{(h_0 - h_L)} \log\left(\frac{h_L}{h_0}\right)\right]
\end{aligned}
$$

La force diminue de façon monotone lorsque le rapport augmente…

La puissance consommée est dissipée…

$$F\,U = -\frac{\mu\,U^2\,L}{h_0}\left[\frac{6}{(1+h_L/h_0)} + \frac{4}{(1-h_L/h_0)}\log\left(\frac{h_L}{h_0}\right)\right]$$

Embêtant…

S'assurer que l'huile est bien refroidie car la viscosité (et donc la charge utile) décroît rapidement avec la température…

…en chaleur !

A propos de la viscosité de notre huile SAE 50

Transport maritime

Marine LCX

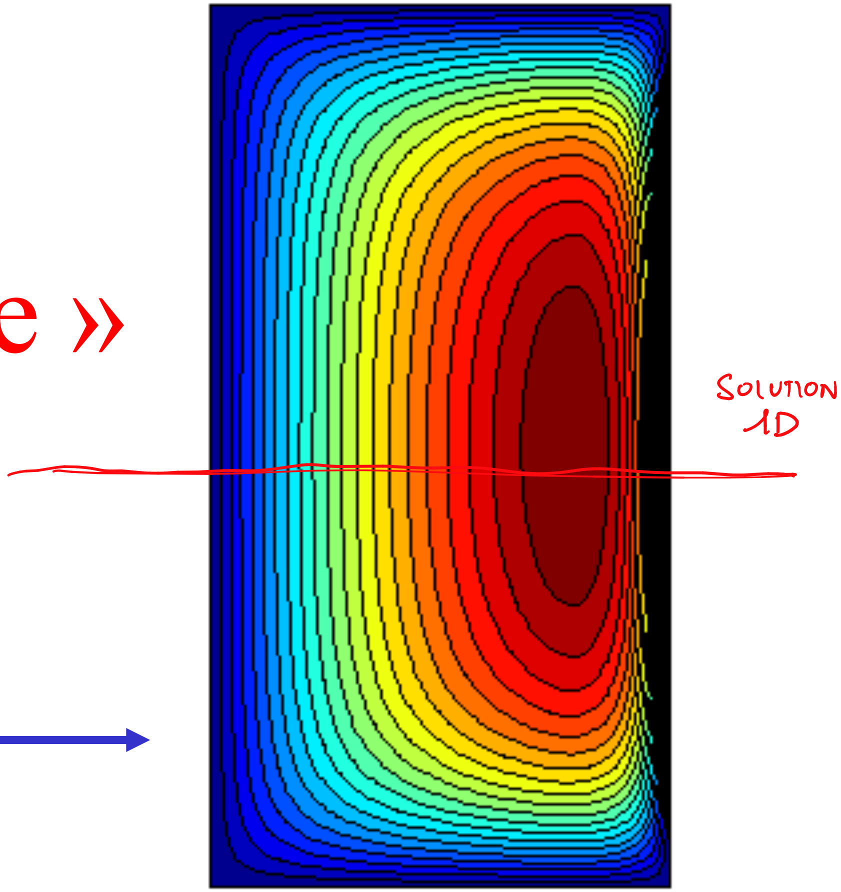
Une huile formulée spécialement pour la lubrification des gros moteurs diesel marins à crosse. Elle lubrifie les cylindres grâce à un indice de basicité très élevé de 70 et un grade SAE* 50.

Grades offerts :
SAE 50

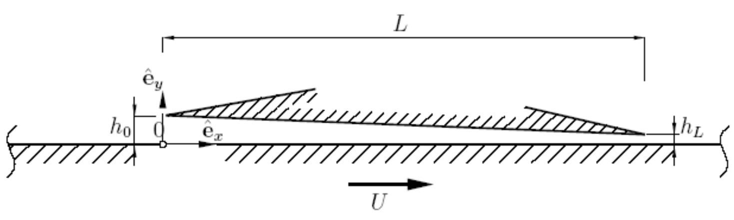
Fiche techniqu
Fiche signalét

$$
\begin{aligned}
T &= 20^0 C & \mu &= 1.100\ Ns/m^2 \\
T &= 40^0 C & \mu &= 0.210\ Ns/m^2 \\
T &= 50^0 C & \mu &= 0.100\ Ns/m^2 \\
T &= 60^0 C & \mu &= 0.060\ Ns/m^2 \\
T &= 80^0 C & \mu &= 0.025\ Ns/m^2 \\
T &= 100^0 C & \mu &= 0.013\ Ns/m^2
\end{aligned}
$$

Analyse « tridimensionnelle » du palier plat



pression sous un palier dont la largeur vaut le double de la longueur



Lubrification 2D ½

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial z^2}$$

$$\frac{\partial p}{\partial y} = \mu \frac{\partial^2 v}{\partial z^2}$$

$$\frac{\partial p}{\partial z} = 0$$

Théorie de la lubrification

Film fluide mince

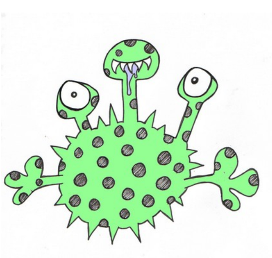
$$h \ll L$$

Hypothèse de lubrification : Écoulements rampants

$$\underbrace{\frac{\rho U L}{\mu}}_{Re_L} \frac{h^2}{L^2} \ll 1$$

$$\left\{\begin{array}{rcl} \boxed{\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}+\dfrac{\partial w}{\partial z} \;=\; 0} \\ -\dfrac{\partial p}{\partial x}+\mu\,\dfrac{\partial^2 u}{\partial z^2} &=& 0 \\ -\dfrac{\partial p}{\partial y}+\mu\,\dfrac{\partial^2 v}{\partial z^2} &=& 0 \end{array}\right.$$

-i- calcul de u(x,y,z) et de v(x,y,z)



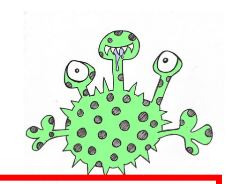
$$\begin{aligned} u(x,y,z) &= -\frac{\partial p}{\partial x}\,\frac{h^2}{2\mu}\,\frac{z}{h}\left(1-\frac{z}{h}\right)+U\left(1-\frac{z}{h}\right) \\ v(x,y,z) &= -\frac{\partial p}{\partial y}\,\frac{h^2}{2\mu}\,\frac{z}{h}\left(1-\frac{z}{h}\right) \end{aligned}$$

-ii- calcul de p(x,y)

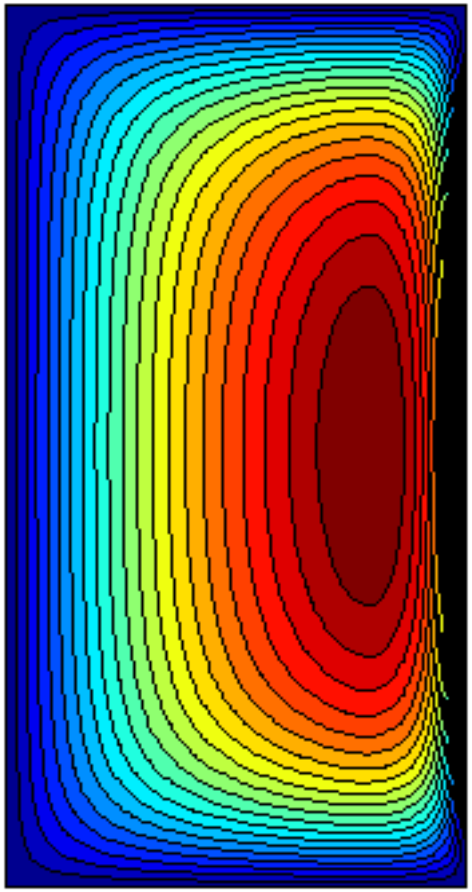
$$\left\{\begin{array}{rcl} \boxed{\begin{array}{rcl} -\dfrac{\partial p}{\partial x} + \mu\,\dfrac{\partial^2 u}{\partial z^2} & = & 0 \\[2ex] -\dfrac{\partial p}{\partial y} + \mu\,\dfrac{\partial^2 v}{\partial z^2} & = & 0 \end{array}} \\[4ex] \boxed{\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} \quad = \quad 0} \end{array}\right.$$

$$\int_0^h \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\, dz \quad = \quad 0$$

$$\frac{\partial}{\partial x}\int_0^h u(x,y,z)dz + \frac{\partial}{\partial y}\int_0^h v(x,y,z)dz + \Big[\, \cancel{w(x,y,z)}\, \Big]_0^h \quad = \quad 0$$



$$\boxed{\frac{\partial}{\partial x}\left(h^3\frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial y}\left(h^3\frac{\partial p}{\partial y}\right) \quad = \quad 6\mu U\frac{dh}{dx}}$$



-iiii- calcul numérique par différences finies de p(x,y)

$$\underbrace{h^3 \left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} \right)}_{\text{DIFFUSION}} + 3h^2 \overbrace{\left(\frac{h_L - h_0}{L} \right) \frac{\partial p}{\partial x}}^{\text{TRANSPORT}} = 6\mu U \left(\frac{h_L - h_0}{L} \right)$$