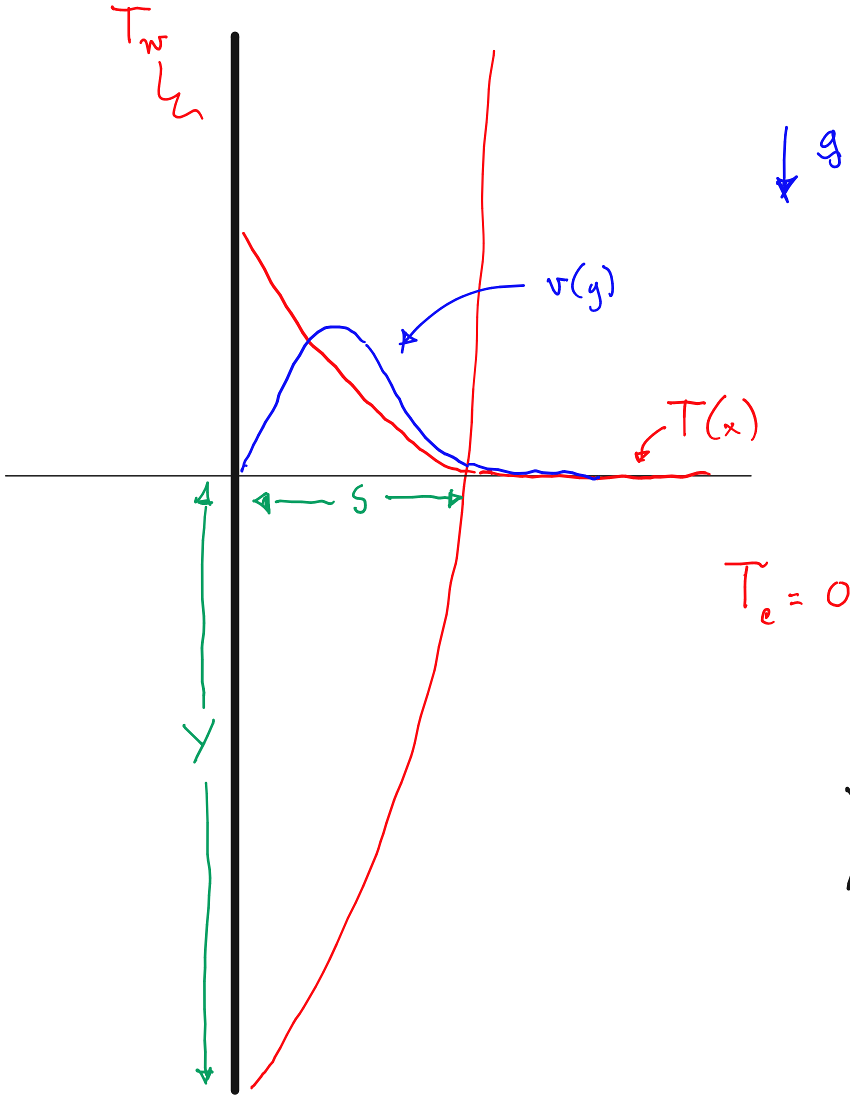
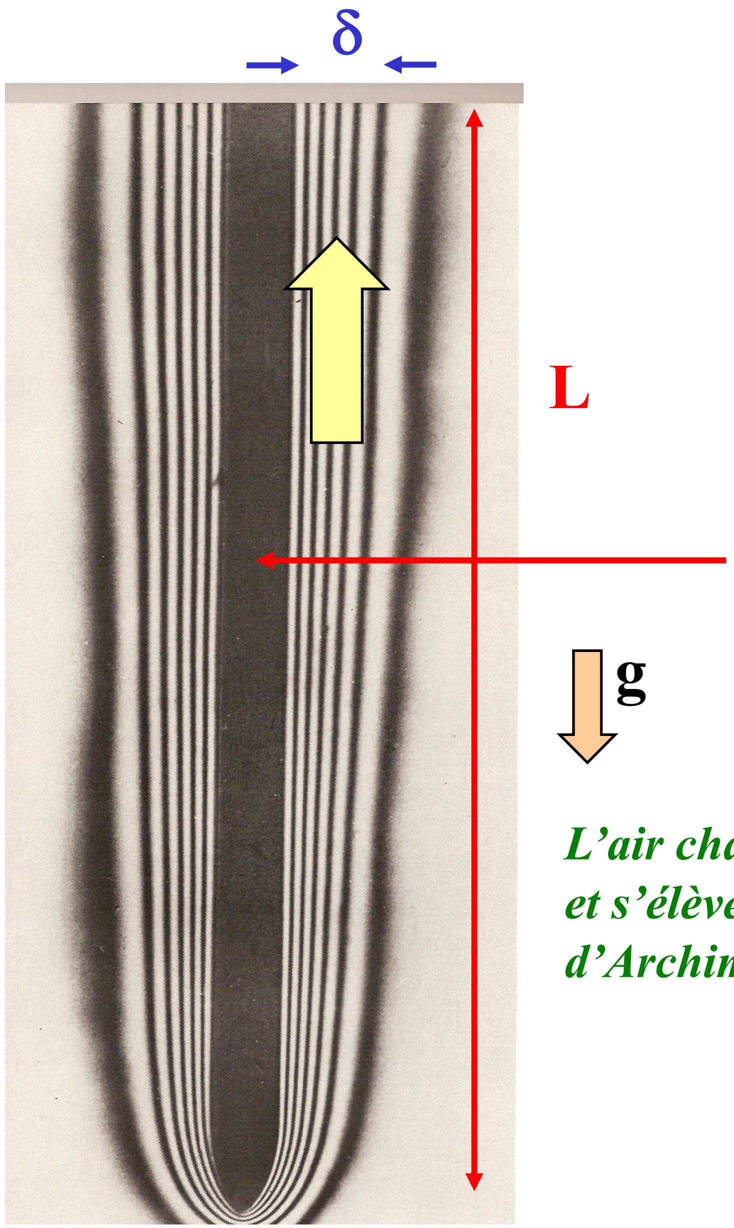
Et enfin, un peu de convection naturelle



$Y \gg \delta$

EPAISSEUR DE LA COUCHE LIMITE

Convection naturelle le long d'une plaque suspendue dans l'air



L'air chaud près de la plaque devient plus léger et s'élève naturellement sous l'effet de la force d'Archimède (flottabilité)(buoyancy)

La photo a été dilatée d'un facteur six dans le long de l'axe horizontal !
(Gebhart, University of Pennsylvania)

$$\rho_0(1 - \cancel{\underbrace{\beta(T - T_0)}_{\ll 1}}) \left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = \cancel{\underbrace{-\frac{\partial p}{\partial y}}_{\rho_0 g}} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - \cancel{\rho_0(1} - \beta(T - T_0))g$$

$$\downarrow$$

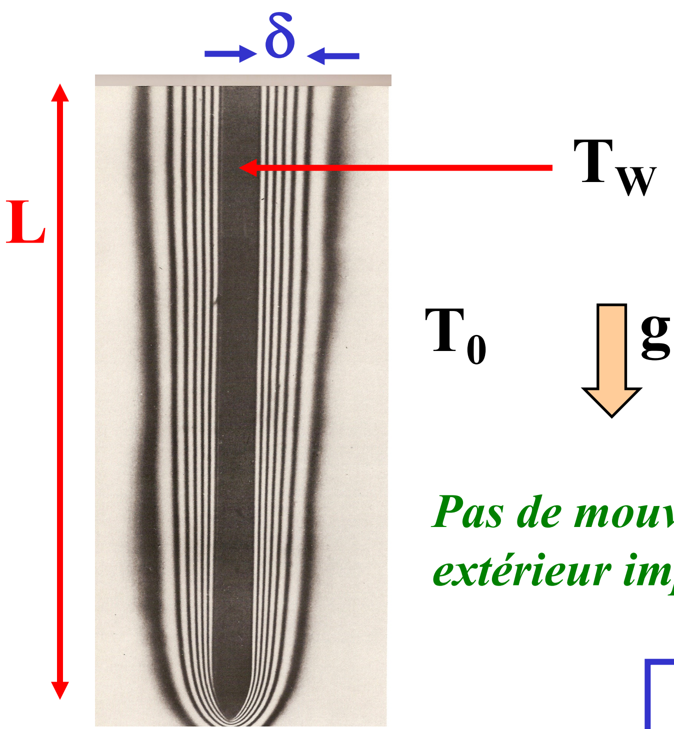
$$\rho_0 \left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \rho_0\beta(T - T_0)g$$

Approximation de Boussinesq

On néglige les variations de masse volumique dans tous les termes sauf dans la poussée d'Archimède…

****Au passage, on observe que cette hypothèse n'a vraiment du sens que dans le cas où la pression est hydrostatique … En d'autres mots, on introduit soit l'approximation hydrostatique, soit l'approximation de Boussinesq, mais on doit introduire en tous cas une approximation :-)***

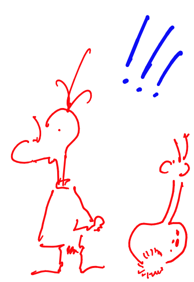
Reprenons le problème de convection naturelle



Pas de mouvement extérieur imposé

$$\delta \ll Y$$

$$\delta_T \ll Y$$



$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \boxed{\beta g(T - T_0)} + \nu\frac{\partial^2 v}{\partial x^2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial x^2}$$

$$\boxed{u\frac{\partial T}{\partial x}} + \boxed{v\frac{\partial T}{\partial y}} = \boxed{\alpha\frac{\partial^2 T}{\partial x^2}} + \cancel{\boxed{\alpha\frac{\partial^2 T}{\partial y^2}}}$$

$$\underbrace{\frac{U\Delta T}{S} \qquad \frac{V\Delta T}{Y}}_{\frac{V\Delta T}{Y}} \qquad \alpha\frac{\Delta T}{S^2} \qquad \cancel{\alpha\frac{\Delta T}{Y^2}}$$

MAIS....

$$\frac{U}{S} = \frac{V}{Y}$$

$$1 = \frac{\blacksquare}{\blacksquare} = \frac{V\cancel{\Delta T}}{Y}\,\frac{S^2}{\alpha\cancel{\Delta T}}$$

$$\frac{S^2}{Y^2} = \frac{\alpha}{VY}$$

$$\boxed{\frac{S}{Y} = \sqrt{\frac{\alpha}{VY}} = \sqrt{\frac{1}{Pe}}}$$

Couche limite thermique, on connaît :-)

Lieu où l'ordre de la convection et de la conduction sont identiques

$$\frac{\boxed{\text{Convection}}}{\boxed{\text{Conduction}}} = \frac{V\Delta T/Y}{\alpha\Delta T/\delta_T^2} = \underbrace{\frac{VY}{\alpha}}_{Pe_Y} \frac{\delta_T^2}{Y^2} = 1$$

Couche limite thermique, on connaît :-)

$$\frac{\delta_T}{Y} = \sqrt{\frac{1}{Pe_Y}} = \sqrt{\frac{1}{Pr\,Re_Y}}$$

Problème : on <u>ne connaît pas</u> V :-(

$$\delta \ll Y$$
$$\delta_T \ll Y$$

$$
\begin{aligned}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \\
u\,\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial y} &= \beta g(T - T_0) + \nu\,\frac{\partial^2 v}{\partial x^2} \\
u\,\frac{\partial T}{\partial x} + v\,\frac{\partial T}{\partial y} &= \alpha\,\frac{\partial^2 T}{\partial x^2}
\end{aligned}
$$

Longueur verticale caractéristique : Y
Longueur horizontale caractéristique : δ
Vitesse horizontale caractéristique : U = V δ / Y (incompressibilité :-)
Ecart de température caractéristique : $T_w - T_0$

Donc, comment choisir la vitesse verticale caractéristique V ?

Buoyancy balanced by friction

$$\cancel{u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}} = \boxed{\beta g (T - T_0)} + \boxed{\nu \frac{\partial^2 v}{\partial x^2}}$$

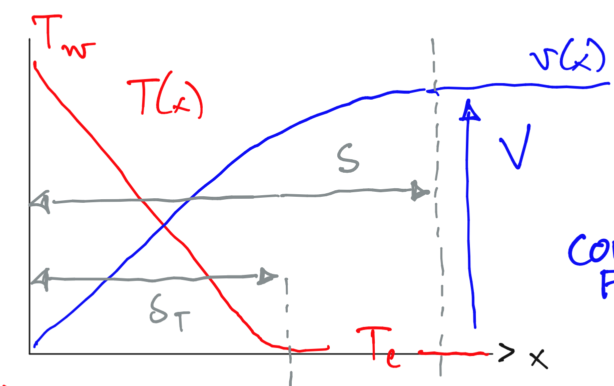
$$\beta g \Delta T \simeq \nu \frac{V}{\delta_T^2}$$

$$V = \frac{\beta g \Delta T \delta_T^2}{\nu}$$

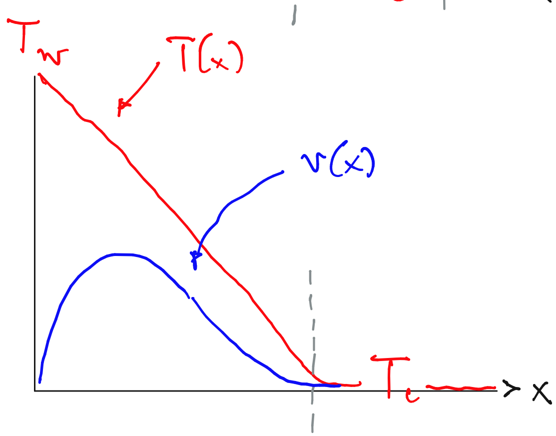
$$\frac{\delta_T}{Y} = \sqrt{\frac{\alpha}{V Y}} = \sqrt{\frac{\nu \alpha}{\beta g \Delta T Y^3}} \qquad \frac{Y}{\delta_T}$$

$$\frac{\delta_T}{Y} = \left(\frac{\alpha \nu}{\beta g \Delta T Y^3} \right)^{1/4}$$

$$Gr^{-1/4} \quad Pr^{-1/4}$$



CONVECTION FORCEE



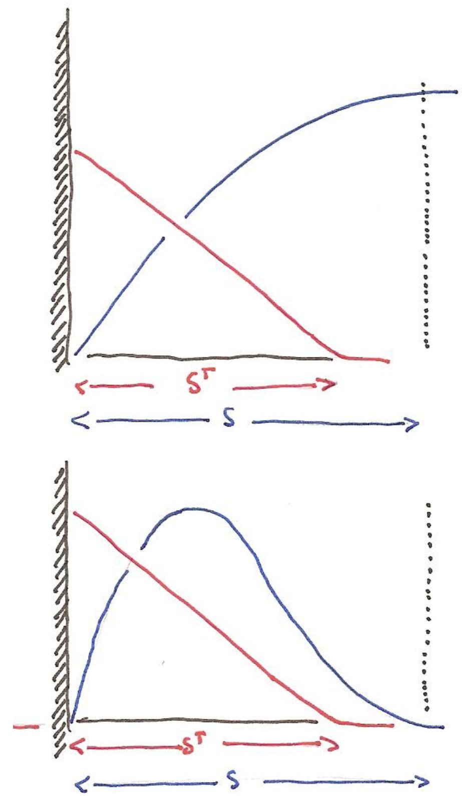
CONVECTION NATURELLE

Buoyancy balanced by friction

$$\cancel{u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}} = \boxed{\beta g(T - T_0)} + \boxed{\nu\frac{\partial^2 v}{\partial x^2}}$$

$$\Downarrow$$

$$V = \frac{\beta g \Delta T \delta_T^2}{\nu}$$



$$\boxed{\begin{aligned} \frac{\delta_T}{Y} &= \sqrt{\frac{\alpha}{VY}} = \sqrt{\frac{\nu\ \alpha}{\beta g \Delta T\ Y \delta_T^2}} = (Gr)^{-1/4}(Pr)^{-1/4} \\ \frac{\delta}{Y} &= (Gr)^{-1/4}(Pr)^{1/4} \end{aligned}}$$

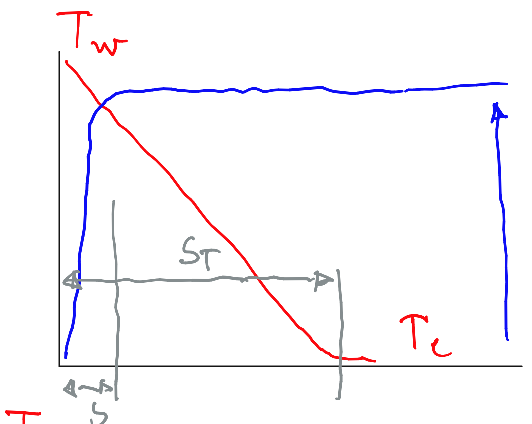
Buoyancy balanced by inertia

$$\boxed{u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}} = \boxed{\beta g(T-T_0)} + \cancel{\nu\frac{\partial^2 v}{\partial x^2}}$$

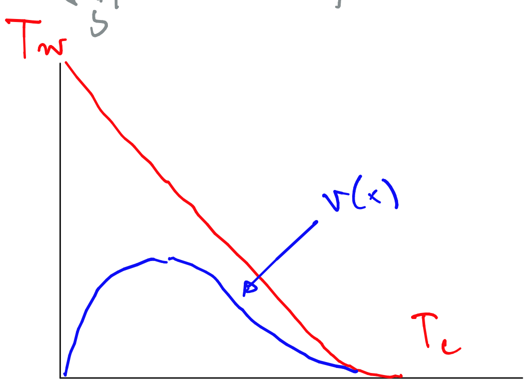
$$V^2/Y \simeq \beta \Delta T g$$

$$V = \sqrt{\beta g \Delta T Y}$$

$$\frac{\delta_T}{Y} = \sqrt{\frac{\alpha}{VY}} = \underbrace{\left(\frac{\alpha^2}{\beta g \Delta T\, Y^3}\right)^{1/4}}_{\left(Gr^{-1/4}\ Pr^{-1/2}\right)}$$



CONVECTION FORCEE

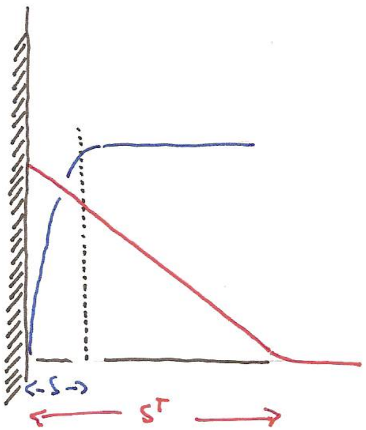


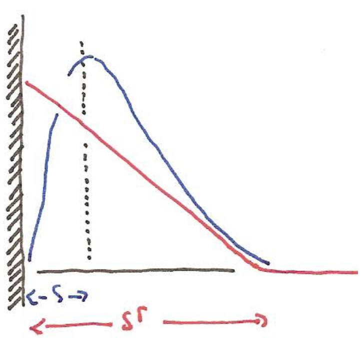
CONVECTION NATURELLE

Buoyancy balanced by inertia

$$\boxed{u\,\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial y}} = \boxed{\beta g (T - T_0)} + \nu\,\cancel{\frac{\partial^2 v}{\partial x^2}}$$

$$V \;=\; \sqrt{\beta g \Delta T\, Y}$$





$$\frac{\delta_T}{Y} \;=\; \sqrt{\frac{\alpha}{VY}} \;=\; \left(\frac{\alpha^2}{\beta g \Delta T\, Y^3}\right)^{1/4} \;=\; (Gr)^{-1/4}\,(Pr)^{-1/2}$$

$$\frac{\delta}{Y} \;=\; (Gr)^{-1/4}$$

Nombre de Grashof

$$\boxed{Gr = \frac{\beta \Delta T g L^3}{\nu^2}}$$

$$\frac{(\text{ARCHIMEDE})(\text{INERTIE})}{(\text{VISQUEUX})^2}$$

$$\underbrace{\cancel{\rho}\beta\Delta T g}_{\text{ARCHIMEDE}} \quad \underbrace{\frac{L^{\cancel{4}\,3}}{\nu^2\cancel{\rho^2}\cancel{U^2}}}_{\frac{1}{(\text{VISQUEUX})^2}} \quad \underbrace{\frac{\cancel{\rho}\,\cancel{U^2}}{\cancel{L}}}_{(\text{INERTIE})}$$

$$\overset{\rho U^2/L}{\boxed{\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right)}} = \overset{\rho\beta\Delta T g}{\boxed{\rho\beta(T - T_0)g}} + \overset{\mu U/L^2}{\boxed{\mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)}}$$

Nombre de Grashof

$$Gr = \frac{\beta \Delta T g L^3}{\nu^2}$$

1822-1893, University of Karlsruhe, Germany
Professor of Mechanical Engineering

$$\frac{\text{(Forces d'inertie) (Forces d'Archimède)}}{\text{(Forces visqueuses)}^2}$$

$$Gr = \frac{{\color{blue}\blacksquare}\,{\color{red}\blacksquare}}{{\color{green}\blacksquare}\,{\color{green}\blacksquare}}$$

Grashof

$$\boxed{\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)} = \boxed{\rho\beta(T-T_0)g} + \boxed{\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)}$$

$$\frac{{\color{red}\blacksquare}}{{\color{green}\blacksquare}} = Re \qquad \frac{{\color{blue}\blacksquare}}{{\color{green}\blacksquare}} = \frac{Gr}{Re} \qquad \frac{{\color{blue}\blacksquare}}{{\color{red}\blacksquare}} = \frac{Gr}{Re^2}$$

$$Gr = \frac{(\text{Forces d'Archimède})(\text{Forces d'inertie})}{(\text{Forces visqueuses})^2} = \frac{\beta\Delta T g L^3}{\nu^2}$$

Grashof - Reynolds

Forces visqueuses

$$\underbrace{\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)}_{\mathcal{O}(\rho U^2/L)} = \underbrace{\rho\beta(T-T_0)g}_{\mathcal{O}(\rho\beta\Delta T g)} + \underbrace{\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)}_{\mathcal{O}(\mu U/L^2)}$$

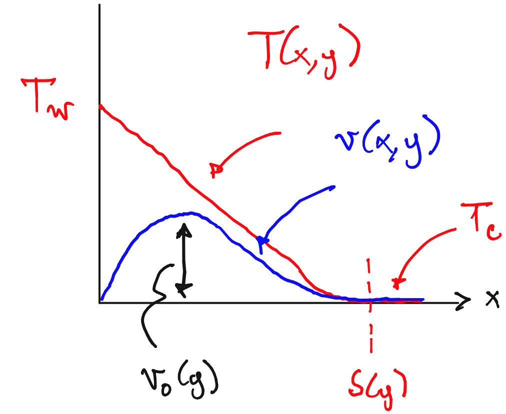
Forces d'inertie

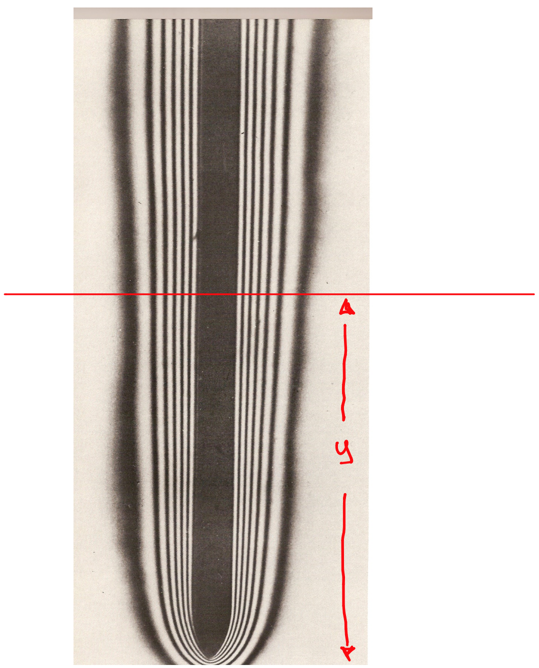
Forces d'Archimède

$$Re = \frac{\color{red}\blacksquare}{\color{green}\blacksquare} \qquad \frac{Gr}{Re} = \frac{\color{blue}\blacksquare}{\color{green}\blacksquare} \qquad \frac{Gr}{Re^2} = \frac{\color{blue}\blacksquare}{\color{red}\blacksquare}$$

$$Gr = \frac{(\text{Forces d'Archimède})(\text{Forces d'inertie})}{(\text{Forces visqueuses})^2} = \frac{\beta\Delta T g L^3}{\nu^2}$$

Une solution approchée pour la convection naturelle…





$$v(x,y) = v_0(y)\ \frac{x}{\delta(y)}\left(1 - \frac{x}{\delta(y)}\right)^2$$

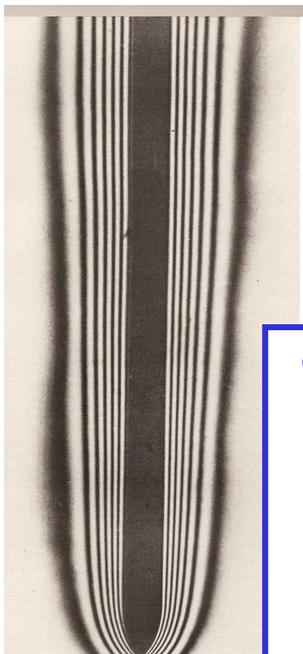
$$\frac{T(x,y) - T_e}{T_w - T_e} = \left(1 - \frac{x}{\delta(y)}\right)^2$$

IL FAUT MAINTENANT DETERMINER $v_0(y)$

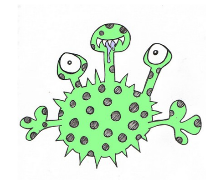
$\delta(y)$

Une solution approchée pour la convection naturelle…

$$v(x,y) = v_0(y)\,\frac{x}{\delta(y)}\left(1-\frac{x}{\delta(y)}\right)^2$$



$$\frac{T(x,y)-T_0}{T_w-T_0} = \left(1-\frac{x}{\delta(y)}\right)^2$$



On considère des profils semblables avec conditions de raccord très sommaires à la couche limite… mais cela va nous donner une idée des ordres de grandeurs

$$\begin{aligned} v(0,y) &= 0 & v(\delta,y) &= 0 & \frac{\partial v}{\partial x}(\delta,y) &= 0 & \mu\frac{\partial^2 v}{\partial x^2}(0,y) &= -\rho\beta g(T_w-T_0) \\ T(0,y) &= T_w & T(\delta,y) &= T_0 & \frac{\partial T}{\partial x}(\delta,y) &= 0 \end{aligned}$$

$$\int_0^\delta u \frac{\partial v}{\partial x} + \int_0^\delta v \frac{\partial v}{\partial y} = \beta g \int_0^\delta (T - T_0) + \nu \int_0^\delta \frac{\partial^2 v}{\partial x^2}$$

$$- \nu \left[\frac{\partial v}{\partial x} \right]_0$$

$$[\cancel{uv}] - \int v \underbrace{\frac{\partial u}{\partial x}}_{-\frac{\partial v}{\partial y}}$$

$$\frac{d}{dy} \int_0^\delta v^2 = \beta g \int_0^\delta T - T_0 - \nu \left[\frac{\partial v}{\partial x} \right]_0$$

$$\frac{d}{dy} \int_0^\delta (T - T_0)\, v = - \alpha \left[\frac{\partial T}{\partial x} \right]_0$$

EQUATIONS INTEGRALES DE VON KARMAN

Calculons $\delta_T(y)$ et $v_0(y)$!

Il faut encore estimer $\delta_T(y)$ et $v_0(y)$!

$$\int_0^\delta u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}dx = \beta g \int_0^\delta \left(T(x,y) - T_0\right)dx + \nu \int_0^\delta \frac{\partial^2 v}{\partial x^2}dx$$

$$\downarrow$$

$$[uv]_0^\delta - \int_0^\delta v\frac{\partial u}{\partial x}dx + \int_0^\delta v\frac{\partial v}{\partial y}dx = \beta g \int_0^\delta \left(T(x,y) - T_0\right)dx - \nu \left.\frac{\partial v}{\partial x}\right|_{x=0}$$

$$\downarrow$$

$$\boxed{\frac{d}{dy}\int_0^\delta v^2(x,y)dx = \beta g \int_0^\delta \left(T(x,y) - T_0\right)dx - \nu \left.\frac{\partial v}{\partial x}\right|_{x=0}}$$

$$\int_0^\delta u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}dx = \alpha \int_0^\delta \frac{\partial^2 T}{\partial x^2}dx$$

$$\downarrow$$

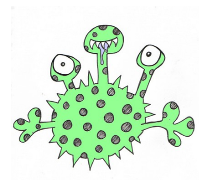
$$\boxed{\frac{d}{dy}\int_0^\delta v(x,y)\Big(T(x,y) - T_0\Big)dx = -\alpha \left.\frac{\partial T}{\partial x}\right|_{x=0}}$$

Intégrons les équations de conservation de la quantité de mouvement et de l 'énergie dans la couche limite en tirant profit de l'incompressibilité…

Que deviennent ces équations intégrales ?

$$\left\{\begin{array}{rcl} \dfrac{d}{dy}\boxed{\displaystyle\int_0^\delta v^2 dx} &=& \beta g \boxed{\displaystyle\int_0^\delta (T-T_0)\,dx} - \nu \left.\dfrac{\partial v}{\partial x}\right|_{x=0} \\ \dfrac{d}{dy}\boxed{\displaystyle\int_0^\delta v(T-T_0)dx} &=& -\alpha \left.\dfrac{\partial T}{\partial x}\right|_{x=0} \end{array}\right.$$

En intégrant les expressions approchées que nous avons introduites…



$v = v_0\, \eta\, (1-\eta)^2$

$\theta = (1-\eta)^2$

$v_0^2 \int_0^1 \underbrace{(1-\eta)^4 \eta^2}$

$\left[\frac{1}{3} - \frac{4}{4} - \frac{6}{5} - \frac{4}{6} + \frac{1}{7}\right]$

$= \dfrac{35 - 105 - 126 - 70 + 15}{105} = \dfrac{1}{105}$

15×7 :-)

$v_0 \int_0^1 \underbrace{(1-\eta)^4}\, \eta$

$\underbrace{(1-2\eta+\eta^2)(1-2\eta+\eta^2)}\,\eta$

$(\;1 - 2\eta + \eta^2$
$-2\eta + 4\eta^2 - 2\eta^3$
$\eta^2 - 2\eta^3 + 4\eta^4\;)\,\eta$

$(1 - 4\eta + 6\eta^2 - 4\eta^3 + 1\eta^4)\,\eta$

$\left[\frac{\eta^2}{2} - \frac{4\eta^3}{3} + \frac{6\eta^4}{4} - \frac{4\eta^5}{5} + \frac{\eta^6}{6}\right]_0^1 = \dfrac{15 - 40 + 45 - 24 + 5}{30} = \dfrac{1}{30}$

$\int_0^1 (1-\eta)^2$

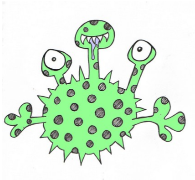
$= \left[\eta - \eta^2 + \frac{\eta^3}{3}\right]_0^1$

$= 1 - 1 + \frac{1}{3} = \frac{1}{3}$

Que deviennent ces équations intégrales

$$\begin{cases} \dfrac{d}{dy}\displaystyle\int_0^\delta v^2 dx &= \beta g \displaystyle\int_0^\delta (T - T_0)\, dx - \nu \left.\dfrac{\partial v}{\partial x}\right|_{x=0} \\ \dfrac{d}{dy}\displaystyle\int_0^\delta v(T-T_0) dx &= -\alpha \left.\dfrac{\partial T}{\partial x}\right|_{x=0} \end{cases}$$

En intégrant les expressions approchées que nous avons introduites…



$$\begin{cases} \dfrac{1}{105}\dfrac{d}{dy}\left(v_0^2(y)\delta(y) \right) &= \dfrac{\beta g\, \Delta T \delta(y)}{3} - \dfrac{\nu\, v_0(y)}{\delta(y)} \\ \dfrac{1}{30}\dfrac{d}{dy}\left(v_0(y)\delta(y) \right) &= \dfrac{2\alpha}{\delta(y)} \end{cases}$$

On obtient finalement de deux équations différentielles ordinaires avec deux fonctions inconnues…

Résolution des équations intégrales

$$
\begin{cases}
\dfrac{1}{105}\,\dfrac{d}{dy}\left(v_0^2(y)\delta(y)\right) = \dfrac{\beta g\,\Delta T\delta(y)}{3} - \dfrac{\nu\, v_0(y)}{\delta(y)} \\[2ex]
\dfrac{1}{30}\,\dfrac{d}{dy}\left(v_0(y)\delta(y)\right) = \dfrac{2\alpha}{\delta(y)}
\end{cases}
$$

IDEE $\quad v_0(y) = V y^m$

$\qquad\quad \delta(y) = D y^n$

$$
\begin{cases}
\dfrac{1}{105} V^2 D\,(2m+n)\, y^{2m+n-1} = \dfrac{\beta g \Delta T}{3} D y^n - \dfrac{\nu V}{D} y^{m-n} \\[2ex]
\dfrac{1}{30} V D\,(m+n)\, y^{m+n-1} = \dfrac{2\alpha}{D} y^{-n}
\end{cases}
$$

$$\overbrace{2m+n-1}^{1/4\ :-)} = \underbrace{n}_{1/4} = \overbrace{m-n}^{\cdots\ 1/4}$$

$$\underbrace{m+n}_{3/4}-1 = \underbrace{-n}_{-1/4}$$

$m = 1/2$

$n = 1/4$

Résolution des équations intégrales

$$\begin{cases} \dfrac{1}{105}\,\dfrac{d}{dy}\Big(v_0^2(y)\delta(y)\Big) & = & \dfrac{\beta g\,\Delta T\delta(y)}{3} - \dfrac{\nu\, v_0(y)}{\delta(y)} \\[2ex] \dfrac{1}{30}\,\dfrac{d}{dy}\Big(v_0(y)\delta(y)\Big) & = & \dfrac{2\alpha}{\delta(y)} \end{cases}$$

Essayons une solution de la forme…

$$\begin{aligned} v_0(y) &= V\; y^m \\ \delta(y) &= D\; y^n \end{aligned}$$

$$\begin{cases} \dfrac{1}{105}\,V^2D(2m+n)\;y^{2m+n-1} & = & \dfrac{\beta g\,\Delta T\;D}{3}\;y^n - \dfrac{\nu\,V}{D}\;y^{m-n} \\[2ex] \dfrac{1}{30}\,VD(m+n)\;y^{m+n-1} & = & \dfrac{2\alpha}{D}\;y^{-n} \end{cases}$$

Et cela marche avec 2m = 1 et 4n = 1…

C'est presque fini !

$$\begin{cases} \dfrac{V^2 D}{84} = \dfrac{\beta g \, \Delta T \; D}{3} - \dfrac{\nu \, V}{D} \\[2ex] \dfrac{V D}{40} = \dfrac{2\alpha}{D} \end{cases}$$

$$V = \frac{80\alpha}{D^2}$$

$$\frac{80^2 \alpha^2}{D^4} \frac{D}{84} = \frac{1}{3} \beta g \, \Delta T \, D - \frac{\nu}{D} \frac{80\alpha}{D^2}$$

$$D^4 = \frac{3}{\beta g \Delta T} \left[\frac{80^2}{84} \alpha^2 - 80 \, \alpha \, \nu \right]$$

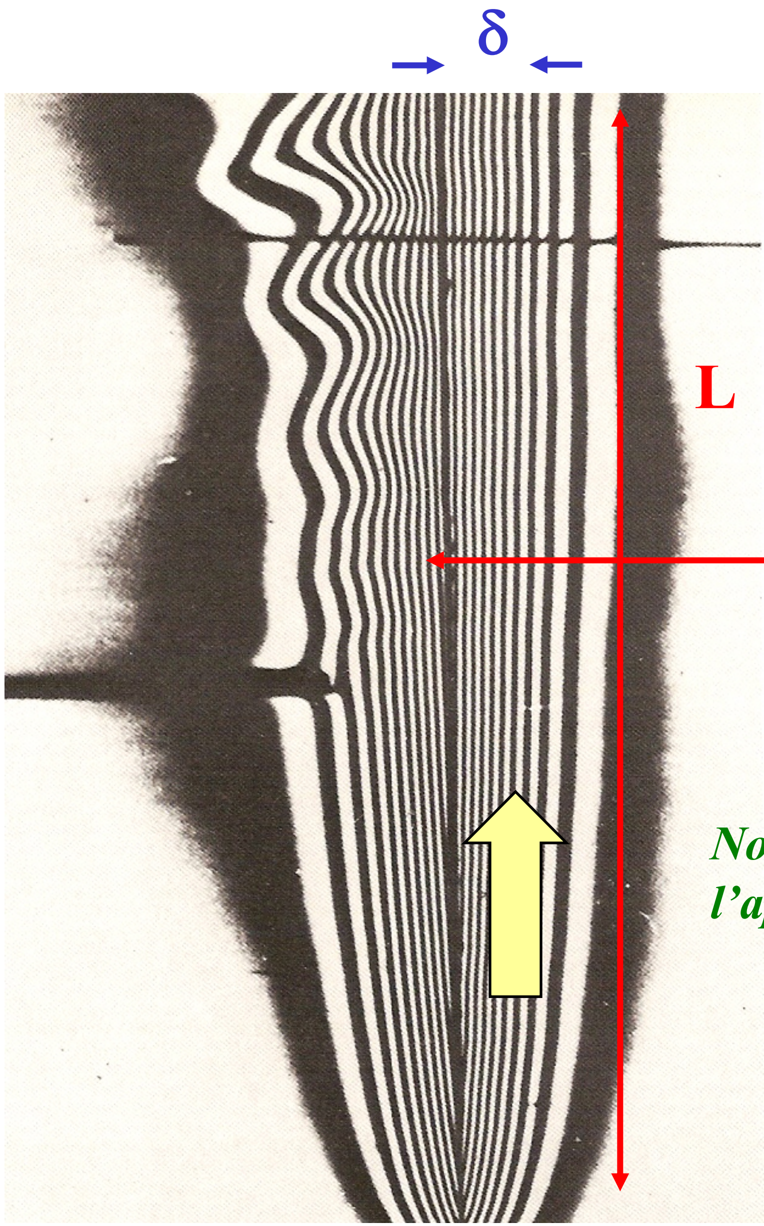
$$\left\{ \begin{array}{rcl} \dfrac{V^2 D}{84} & = & \dfrac{\beta g\ \Delta T\ D}{3} - \dfrac{\nu\ V}{D} \\ \dfrac{V D}{40} & = & \dfrac{2\alpha}{D} \end{array} \right.$$

$$D^4 = 240 L^3 \left(\frac{20}{21} + \frac{\nu}{\alpha} \right) \frac{\nu^2}{\beta g \Delta T L^3} \frac{\alpha^2}{\nu^2}$$

Solution …

$$\frac{\delta_T(y)}{y} = 3,936 \left(Pr \right)^{-1/2} \left(Gr(y) \right)^{-1/4} \left(\frac{20}{21} + Pr \right)^{1/4}$$

On peut ensuite calculer le profil de vitesse et de température, le transfert de chaleur et des nombres de Nusselt locaux et moyens.



Est-ce que cette solution est stable ?

Plaque chaude

Non, après une certaine distance, on constate l'apparition d'instabilités : c'est la <u>turbulence</u> !

Cela, c'est très très très très compliqué.

Grégoire est de retour en Belgique !