cours 1/12/21

$$\frac{\delta^*}{\delta} = \frac{1}{\delta}\int_0^\delta \left(1-\frac{\bar{U}}{\bar{U}_e}\right)dy = \frac{\bar{U}_\tau}{\bar{U}_e}\,\frac{1}{\delta}\int_0^\delta \left(\frac{\bar{U}_e-\bar{U}}{\bar{U}_\tau}\right)dy$$

zone III est dominante !

$$= \underbrace{\frac{\bar{U}_\tau}{\bar{U}_e}}_{\sqrt{\frac{C_f}{2}}}\ \underbrace{\int_0^1 F(\eta)\,d\eta}_{a\,\simeq\,3.4}$$

$$\frac{(\delta^*-\theta)}{\delta} = \frac{1}{\delta}\int_0^\delta \left(1-\frac{\bar{U}}{\bar{U}_e}\right)^2 dy = \underbrace{\frac{U_\tau^2}{\bar{U}_e^2}}_{\frac{C_f}{2}}\ \underbrace{\int_0^1 (F(\eta))^2\,d\eta}_{b\,\simeq\,23}$$

Notation $\quad \zeta \triangleq \sqrt{\frac{2}{C_f}}$

$$\hookrightarrow \frac{\delta^*}{\delta} = \frac{a}{\zeta}$$

et $\displaystyle \frac{(\delta^*-\theta)}{\delta} = \frac{b}{\zeta^2} \Longrightarrow \frac{\theta}{\delta} = \frac{a}{\zeta} - \frac{b}{\zeta^2}$

Aussi $\quad H \triangleq \dfrac{\delta^*}{\theta} \equiv$ facteur de forme

$$\hookrightarrow H = \frac{\frac{a}{\zeta}}{\left(\frac{a}{\zeta}-\frac{b}{\zeta^2}\right)} = \frac{1}{\left(1-\frac{b}{a}\frac{1}{\zeta}\right)}$$

# Profils simplifiés en exposant

$$\frac{\bar{U}}{U_e} = \eta^{1/n}$$

$$\hookrightarrow \frac{\delta^*}{\delta} = \int_0^1 \left(1-\eta^{1/n}\right) d\eta = \frac{1}{(n+1)}$$

$$\text{et} \quad \frac{(\delta^*-\theta)}{\delta} = \int_0^1 \left(1-\eta^{1/n}\right)^2 d\eta = \int_0^1 \left(1-2\eta^{1/n}+\eta^{2/n}\right) d\eta = \frac{2}{(n+1)(n+2)}$$

$$\hookrightarrow \frac{\theta}{\delta} = \frac{1}{(n+1)} - \frac{2}{(n+1)(n+2)} = \frac{n}{(n+1)(n+2)}$$

$$\hookrightarrow H = \frac{(n+2)}{n}$$

## Comparaison

$$\frac{(n+2)}{n} \simeq \frac{1}{\left(1 - \frac{b}{a}\frac{1}{\zeta}\right)}$$

$$\equiv$$

$$\frac{(n+2)}{\left((n+2)-2\right)}$$

$$\equiv$$

$$\frac{1}{\left(1 - \frac{2}{(n+2)}\right)}$$

$$\hookrightarrow \frac{2}{(n+2)} \simeq \frac{b}{a}\frac{1}{\zeta} \iff \boxed{n \simeq 2\left(\frac{a}{b}\zeta - 1\right)}$$

$$\boxed{n \simeq 2\left(0.15\,\zeta - 1\right)}$$

Simple et utile !

Cas lisse

$$\frac{\bar U}{\bar U_\tau} = \left(\frac{1}{K} \log\left(\frac{y \bar U_\tau}{\nu}\right) + C\right) + G\left(\frac{y}{\delta}\right)$$

$$\hookrightarrow \frac{\bar U_e}{\bar U_\tau} = \left(\frac{1}{K} \log\left(\frac{\delta \bar U_\tau}{\nu}\right) + C\right) + G(1)$$

$$\frac{\delta \bar U_e}{\nu} \cdot \frac{\bar U_\tau}{\bar U_e}$$

$$\hookrightarrow \boxed{\zeta = \frac{1}{K} \log\left(Re_\delta \frac{1}{\zeta}\right) + C + G(1)} \equiv \text{exact (mais implicite)}$$

• Approximation de White : $\zeta \simeq 10\, Re_\delta^{1/12}$

$$\hookrightarrow \frac{d\theta}{dx} = \frac{1}{\zeta^2} \simeq \frac{1}{100} \frac{1}{\left(\frac{\delta \bar U_e}{\nu}\right)^{1/6}}$$

• Approximation : négliger la variation de $\frac{\theta}{\delta}$

et prendre la valeur pour $n = 7$

$$\hookrightarrow \frac{\theta}{\delta} \simeq \frac{7}{8 * 9} = \frac{7}{72} \simeq 0.0972$$

$$\hookrightarrow \frac{7}{72} * 100 \frac{d\delta}{dx} \simeq \frac{1}{\left(\frac{\delta \bar U_e}{\nu}\right)^{1/6}}$$

$$\delta^{1/6}\, d\delta \simeq \frac{72}{700} \frac{1}{\left(\frac{\bar U_e}{\nu}\right)^{1/6}} dx$$

$$\frac{\delta^{7/6}}{7/6} \simeq \frac{72}{700} \frac{1}{\left(\frac{\bar U_e}{\nu}\right)^{1/6}} x$$

$$\delta \simeq \left(\frac{12}{100}\right)^{6/7} \frac{x^{6/7}}{\left(\frac{\bar U_e}{\nu}\right)^{1/7}} \Rightarrow \frac{\delta}{x} \simeq \frac{0.162}{\left(\frac{\bar U_e x}{\nu}\right)^{1/7}}$$

$$\frac{\delta}{x} \simeq \frac{0.162}{Re_x^{1/7}} \qquad \text{avec } Re_x \triangleq \frac{\bar{U}_e x}{\nu}$$

$$\hookrightarrow \frac{\theta}{x} \simeq \frac{0.0158}{Re_x^{1/7}} \longrightarrow C_f = 2\frac{d\theta}{dx} \simeq \frac{0.0271}{Re_x^{1/7}}$$

$$\hookrightarrow C_{f,m} = 2\frac{\theta}{x} \simeq \frac{0.0316}{Re_x^{1/7}}$$

$$\text{et } \frac{\delta^*}{x} \simeq \frac{0.0203}{Re_x^{1/7}}$$

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<u>Analyse plus précise de White (1962)</u>

$$\hookrightarrow \ldots \rightarrow C_f \simeq \frac{0.455}{\left(\log(0.060\, Re_x)\right)^2}$$

$$C_{f,m} \simeq \frac{0.523}{\left(\log(0.060\, Re_x)\right)^2}$$

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<u>Formule de Schultz-Grunow</u>

$$C_f \simeq \frac{0.37}{\left(\log_{10}(Re_x)\right)^{2.584}}$$

Note:

$$\zeta = \frac{1}{K} \log\left(Re_\delta \frac{1}{\zeta}\right) + C + G(1)$$

$$= \frac{1}{K} \log\left(Re_\delta \frac{c}{\zeta}\right) \quad \text{avec } c = e^{K(C+G(1))}$$

$$\hookrightarrow K\zeta = \log\left(Re_\delta \frac{c}{\zeta}\right)$$

$$\hookrightarrow \boxed{Re_\delta = \frac{\zeta}{c} e^{K\zeta}} \equiv \text{exact et explicite !}$$

Aussi $\dfrac{\theta}{\delta} = \dfrac{a}{\zeta} - \dfrac{b}{\zeta^2}$ $\equiv$ exact

$$\hookrightarrow \theta = \left(\frac{a}{\zeta} - \frac{b}{\zeta^2}\right)\delta \Rightarrow \boxed{Re_\theta = \left(\frac{a}{\zeta} - \frac{b}{\zeta^2}\right) Re_\delta}$$

$$\hookrightarrow Re_\theta = \left(\frac{a}{\zeta} - \frac{b}{\zeta^2}\right)\frac{\zeta}{c} e^{K\zeta}$$

$$\boxed{Re_\theta = \frac{1}{c}\left(a - \frac{b}{\zeta}\right) e^{K\zeta}} \equiv \text{exact}$$

$$\frac{d\theta}{dx} = \frac{d(Re_\theta)}{d(Re_x)} = \frac{1}{\zeta^2} \Longrightarrow \boxed{\zeta^2\, d(Re_\theta) = d(Re_x)} \equiv \text{exact}$$

$$\hookrightarrow \boxed{\frac{\zeta^2}{c}\, d\left(\left(a - \frac{b}{\zeta}\right) e^{K\zeta}\right) = d\,Re_x} \equiv \text{exact}$$

↳ Intégrer. On obtient $Re_x$ en fonction de $\zeta$

(voor syllabus)

## Cas rugueux

$$\frac{\bar{U}}{\bar{U}_\tau} = \left(\frac{1}{K} \log\left(\frac{y}{\epsilon}\right) + B\right) + G\left(\frac{y}{\delta}\right)$$

$$\hookrightarrow \frac{\bar{U}_e}{\bar{U}_\tau} = \frac{1}{K} \log\left(\frac{\delta}{\epsilon}\right) + B + G(1)$$

$$= \frac{1}{K} \log\left(\frac{\delta}{\epsilon} c_2\right) \quad \text{avec} \quad c_2 = e^{K(B+G(1))}$$

$$\hookrightarrow \boxed{z = \frac{1}{K} \log\left(\frac{\delta}{\epsilon} c_2\right)} \equiv \text{exact et explicite}$$

$$\hookrightarrow \boxed{\frac{\delta}{\epsilon} = \frac{1}{c_2} e^{Kz}} \equiv \text{exact et explicite}$$

$$\hookrightarrow \frac{d\theta}{dx} = \frac{1}{z^2} = \frac{K^2}{\log^2\left(\frac{\delta}{\epsilon} c_2\right)}$$

On obtient $\frac{x}{\epsilon}$ en fonction de $\frac{\delta}{\epsilon}$, et donc aussi en fonction de $z$

$\hookrightarrow$ etc... voir syllabus

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## Formules empiriques de Schlichting (1979)

$$C_f \simeq \frac{1}{\left(2.87 + 1.58 \log_{10}\left(\frac{x}{\epsilon}\right)\right)^{2.5}}$$

$$C_{f,m} \simeq \frac{1}{\left(1.89 + 1.62 \log_{10}\left(\frac{x}{\epsilon}\right)\right)^{2.5}}$$

## Formule de White

$$C_f \simeq \frac{1}{\left(1.4 + 3.7 \log_{10}\left(\frac{x}{\epsilon}\right)\right)^{2}}$$

# Transfert de chaleur en couches limites

$$\rho c\left(\frac{\partial T}{\partial t} + \frac{\partial}{\partial x_j}(T v_j)\right) = \tau_{ij} d_{ij} - \frac{\partial q_j}{\partial x_j} \quad \text{avec} \quad \begin{cases} \tau_{ij} = 2\mu d_{ij} \\ q_j = -k \dfrac{\partial T}{\partial x_j} \end{cases}$$

$$= 2\mu d_{ij} d_{ij} + \frac{\partial}{\partial x_j}\left(k \frac{\partial T}{\partial x_j}\right)$$

## Moyenne de Reynolds

$$\overline{T v_j} = \bar{T} \bar{v}_j + \overline{T' v_j'}$$

$$\overline{d_{ij} d_{ij}} = \bar{d}_{ij} \bar{d}_{ij} + \overline{d_{ij}' d_{ij}'}$$

$$\hookrightarrow \rho c \frac{\partial}{\partial x_j}(\bar{T} \bar{v}_j) = 2\mu\left(\bar{d}_{ij} \bar{d}_{ij} + \overline{d_{ij}' d_{ij}'}\right) + \frac{\partial}{\partial x_j}\left(k \frac{\partial \bar{T}}{\partial x_j} - \rho c \overline{T' v_j'}\right)$$

$$\hookrightarrow \begin{cases} \bar{q}_j = -k \dfrac{\partial \bar{T}}{\partial x_j} \\ \bar{q}_j^{\,t} \triangleq \rho c \overline{T' v_j'} \end{cases}$$

$\equiv$ densité de flux de chaleur due aux fluctuations de la turbulence

**Modèle:** $\bar{q}_j^{\,t} = -k_t \dfrac{\partial \bar{T}}{\partial x_j}$

$$2\mu \overline{d_{ij} d_{ij}} \simeq 2\mu_t \bar{d}_{ij} \bar{d}_{ij}$$ (ok pour écoulements turbulents simples)

↳ Couches limites: $\bar{q}^{\,t} = \rho c \overline{T' v'} = -k_t \dfrac{\partial \bar{T}}{\partial y}$

$$\rho c\left(\bar{u} \frac{\partial \bar{T}}{\partial x} + \bar{v} \frac{\partial \bar{T}}{\partial y}\right) = \cancel{2}(\mu + \mu_t)\left(\frac{\partial \bar{u}}{\partial y}\right)^2 + \frac{\partial}{\partial y}\left((k + k_t) \frac{\partial \bar{T}}{\partial y}\right)$$

car $\dfrac{\partial}{\partial x}\left((k + k_t) \dfrac{\partial \bar{T}}{\partial x}\right)$ est négligeable

$$P_r \triangleq \frac{\mu c}{k} = \frac{\nu}{\alpha} \quad \text{car} \begin{cases} \nu \triangleq \dfrac{\mu}{\rho} \\ \alpha \triangleq \dfrac{k}{\rho c} \end{cases}$$

$$P_{r_t} \triangleq \frac{\mu_t c}{k_t} = \frac{\nu_t}{\alpha_t} \quad \text{car} \begin{cases} \nu_t = \dfrac{\mu_t}{\rho} \\ \alpha_t = \dfrac{k_t}{\rho c} \end{cases}$$

Pour la suite, on va supposer que $P_{r_t}$ est constant

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$$(0) \quad \frac{\partial \bar{U}}{\partial x} + \frac{\partial \bar{V}}{\partial y} = 0$$

on considère $\bar{U}_e$ constant

$$(1) \quad \rho\left(\bar{U}\frac{\partial \bar{U}}{\partial x} + \bar{V}\frac{\partial \bar{U}}{\partial y}\right) = \cancel{\rho \bar{U}_e \frac{d\bar{U}_e}{dx}} + \frac{\partial}{\partial y}\left((\mu+\mu_t)\frac{\partial \bar{U}}{\partial y}\right)$$

$$(2) \quad \rho c\left(\bar{U}\frac{\partial \bar{T}}{\partial x} + \bar{V}\frac{\partial \bar{T}}{\partial y}\right) = (\mu+\mu_t)\left(\frac{\partial \bar{U}}{\partial y}\right)^2 + \frac{\partial}{\partial y}\left(\left(\frac{\mu}{P_r} + \frac{\mu_t}{P_{r_t}}\right) c \frac{\partial \bar{T}}{\partial y}\right)$$

$\bar{U} * Eq.(1) + Eq.(2)$:

$$\hookrightarrow \rho\left(\bar{U}\frac{\partial}{\partial x}\left(c\bar{T} + \frac{\bar{U}^2}{2}\right) + \bar{V}\frac{\partial}{\partial y}\left(c\bar{T} + \frac{\bar{U}^2}{2}\right)\right) = \frac{\partial}{\partial y}\left(\left(\frac{\mu}{P_r} + \frac{\mu_t}{P_{r_t}}\right)\frac{\partial}{\partial y}\left(c\bar{T} + \frac{\bar{U}^2}{2}\right)\right)$$

<u>Cas $P_r = 1$ et $P_{r_t} = 1$</u>

l'EDP pour $\left(c\bar{T} + \frac{\bar{U}^2}{2}\right)$ est idem que l'EDP pour $\bar{U}$

$\hookrightarrow$ les profils sont liés linéairement

$$\boxed{c\bar{T} + \frac{\bar{U}^2}{2} = A\bar{U} + B} \equiv \text{Relation de Crocco}$$

<u>Rappel</u>: $dU = c\,dT$

en $y=0$ : $\bar{U}=0$ et $\bar{T}=\bar{T}_w$ $\Rightarrow B = c\bar{T}_w$

en $y=\delta$ : $\bar{U}=\bar{U}_e$ et $\bar{T}=\bar{T}_e$ $\Rightarrow c\bar{T}_e + \frac{\bar{U}_e^2}{2} = A\bar{U}_e + B$

$$= A\bar{U}_e + c\bar{T}_w$$

$$\hookrightarrow A = \frac{1}{\bar{U}_e}\left(c(\bar{T}_e-\bar{T}_w) + \frac{\bar{U}_e^2}{2}\right)$$

$$\hookrightarrow \boxed{c(\bar{T}-\bar{T}_w) + \frac{\bar{U}^2}{2} = \left(c(\bar{T}_e-\bar{T}_w) + \frac{\bar{U}_e^2}{2}\right)\frac{\bar{U}}{\bar{U}_e}}$$

Aussi $c\frac{\partial \bar{T}}{\partial y} + \bar{U}\frac{\partial \bar{U}}{\partial y} = A\frac{\partial \bar{U}}{\partial y}$

$\hookrightarrow$ en $y=0$ : $c\left.\frac{\partial \bar{T}}{\partial y}\right|_0 = A\left.\frac{\partial \bar{U}}{\partial y}\right|_0$

Cas lisse (zone I) $P_r = \frac{\mu c}{k} = 1 \Rightarrow c = \frac{k}{\mu}$

$$\hookrightarrow k\left.\frac{\partial \bar{T}}{\partial y}\right|_0 = A\mu\left.\frac{\partial \bar{U}}{\partial y}\right|_0$$

$$\hookrightarrow -\bar{q}_w = A\bar{\tau}_w$$

Cas rugueux (zone III) $P_{r_t} = \frac{\mu_t c}{k_t} = 1 \Rightarrow c = \frac{k_t}{\mu_t}$

$$\hookrightarrow k_t\left.\frac{\partial \bar{T}}{\partial y}\right|_0 = A\mu_t\left.\frac{\partial \bar{U}}{\partial y}\right|_0$$

$$\hookrightarrow -\bar{q}_w = A\bar{\tau}_w \quad \equiv \text{idem !}$$

Donc $\boxed{A = -\frac{\bar{q}_w}{\bar{\tau}_w}}$ dans tous les cas