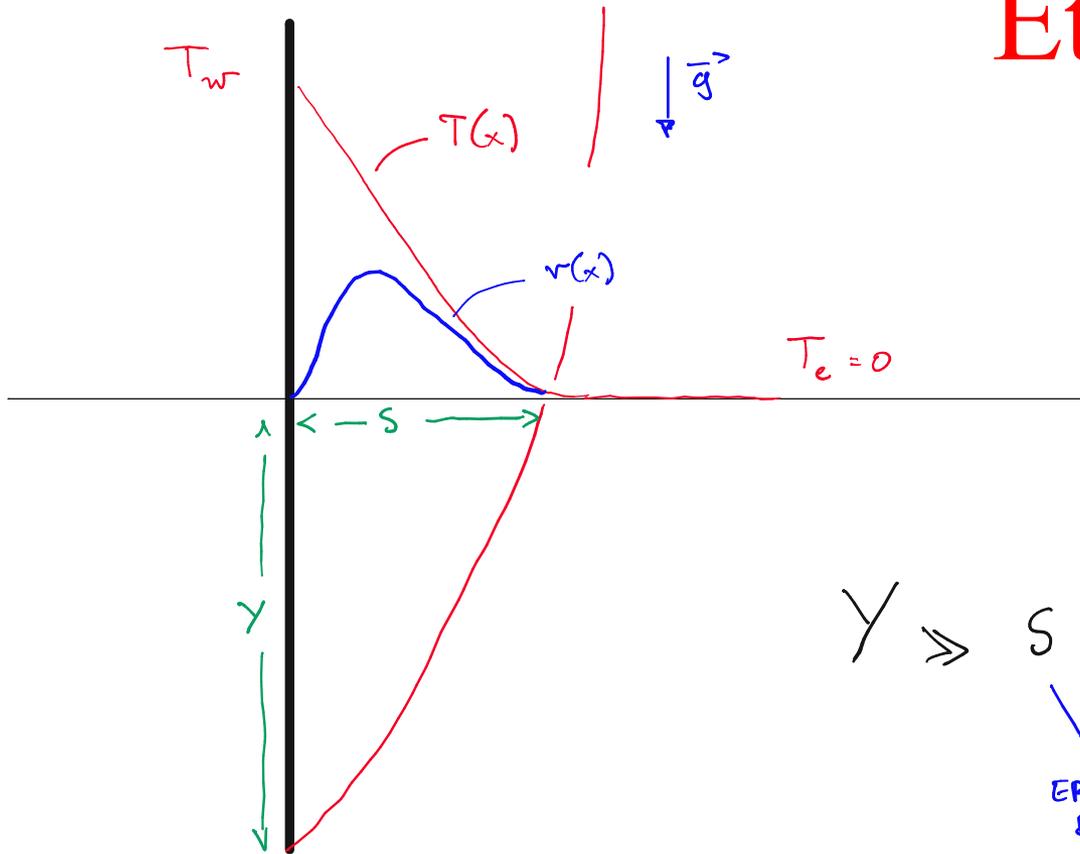


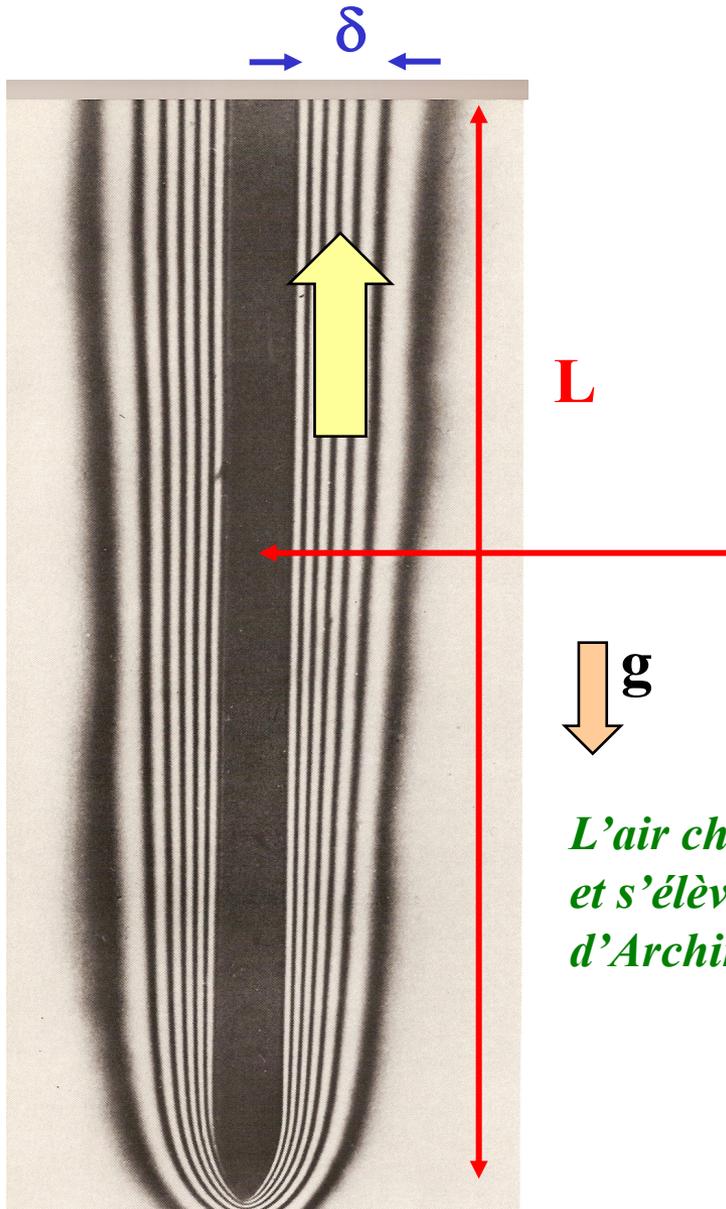
Et enfin, un peu de convection naturelle



$Y \gg S$

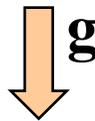


EPAISSEUR DE COUCHE LIMITE



Convection naturelle le long d'une plaque suspendue dans l'air

Plaque chaude



L'air chaud près de la plaque devient plus léger et s'élève naturellement sous l'effet de la force d'Archimède (flottabilité)(buoyancy)

*La photo a été dilatée d'un facteur six dans le long de l'axe horizontal !
(Gebhart, University of Pennsylvania)*

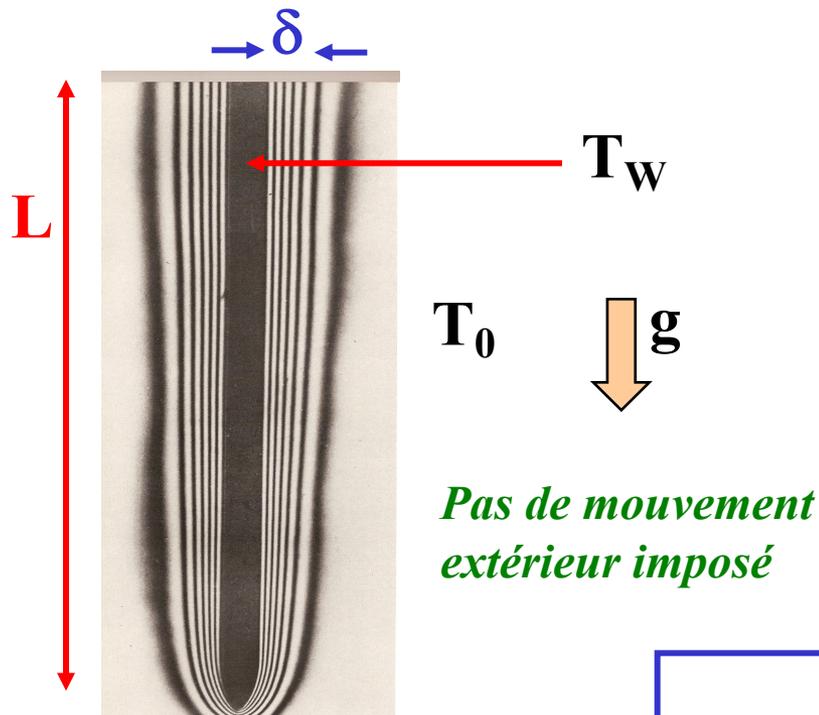
$$\rho_0(1 - \underbrace{\beta(T - T_0)}_{\ll 1}) \left(u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = \underbrace{-\frac{\partial p}{\partial y}}_{\rho_0 g} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) - \rho_0(1 - \beta(T - T_0))g$$

$$\rho_0 \left(u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + \rho_0 \beta(T - T_0)g$$

Approximation de Boussinesq

On néglige les variations de masse volumique dans tous les termes sauf dans la poussée d'Archimède...

** Au passage, on observe que cette hypothèse n'a vraiment du sens que dans le cas où la pression est hydrostatique ... En d'autres mots, on introduit soit l'approximation hydrostatique, soit l'approximation de Boussinesq, mais on doit introduire en tous cas une approximation :-)*



Reprenons le problème de convection naturelle

$$\delta \ll Y$$

$$\delta_T \ll Y$$



$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = \beta g (T - T_0) + \nu \frac{\partial^2 v}{\partial x^2}$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial x^2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{U}{S_T} = \frac{V}{Y}$$

$$\boxed{u \frac{\partial T}{\partial x}} + \boxed{v \frac{\partial T}{\partial y}} = \boxed{\alpha \frac{\partial^2 T}{\partial x^2}} + \boxed{\alpha \frac{\partial^2 T}{\partial y^2}}$$
$$\underbrace{\frac{U \Delta T}{S_T} + \frac{V \Delta T}{Y}}_{\frac{V \Delta T}{Y}}$$

$$\alpha \frac{\Delta T}{S_T^2} + \cancel{\alpha \frac{\Delta T}{Y^2}}$$

$$1 = \frac{\cancel{V \Delta T}}{Y} \frac{S_T^2}{\cancel{\alpha \Delta T}}$$

$$\frac{S_T^2}{Y} = \frac{\alpha}{VY}$$

$$\boxed{\frac{S_T}{Y} = \sqrt{\frac{\alpha}{VY}} = \sqrt{\frac{1}{Pe}}}$$

Couche limite
thermique,
on connaît :-)

Lieu où l'ordre de la convection et de la conduction sont identiques

$$\frac{\text{Convection}}{\text{Conduction}} = \frac{V\Delta T/Y}{\alpha\Delta T/\delta_T^2} = \frac{VY}{\underbrace{\alpha}_{Pe_Y}} \frac{\delta_T^2}{Y^2} = 1$$

Couche limite thermique,
on connaît :-)

$$\frac{\delta_T}{Y} = \sqrt{\frac{1}{Pe_Y}} = \sqrt{\frac{1}{Pr Re_Y}}$$

Problème : on ne connaît pas V :-)

$$\delta \ll Y$$
$$\delta_T \ll Y$$

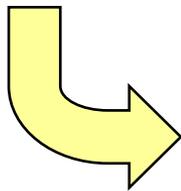
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$
$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = \beta g(T - T_0) + \nu \frac{\partial^2 v}{\partial x^2}$$
$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial x^2}$$

Longueur verticale caractéristique : Y

Longueur horizontale caractéristique : δ

Vitesse horizontale caractéristique : $U = V \delta / Y$ (incompressibilité :-)

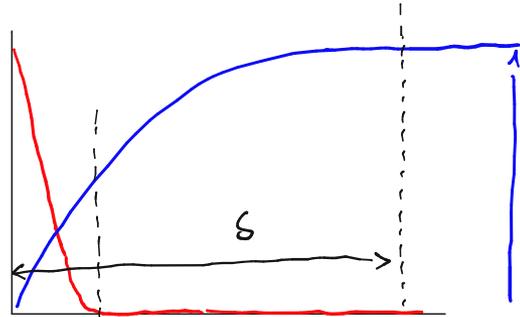
Ecart de température caractéristique : $T_w - T_0$



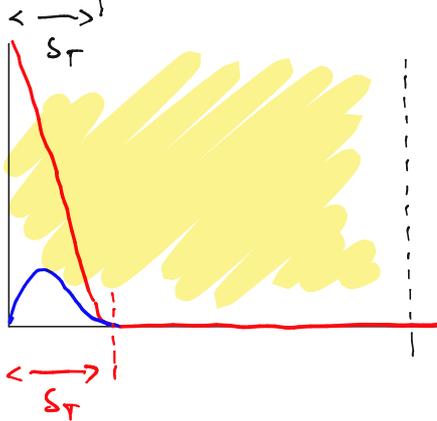
**Donc, comment choisir
la vitesse verticale
caractéristique V ?**

Buoyancy balanced by friction

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = \beta g (T - T_0) + \nu \frac{\partial^2 v}{\partial x^2}$$



CONVECTION
FORCEE



CONVECTION
NATURELLE

$$\beta g \Delta T \approx \frac{\rho V}{S_T^2}$$

$$V = \frac{\beta g \Delta T}{\rho} S_T^2$$

$$\frac{S_T}{Y} = \sqrt{\frac{\alpha}{\nu Y}} = \sqrt{\frac{\rho \alpha}{\beta g \Delta T} Y^3} \frac{Y}{S_T}$$

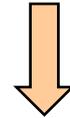
$$\frac{S_T}{Y} = \left(\frac{\alpha \rho}{\beta g \Delta T Y^3} \right)^{1/4}$$

$(Gr)^{-1/4} Pr^{-1/4}$

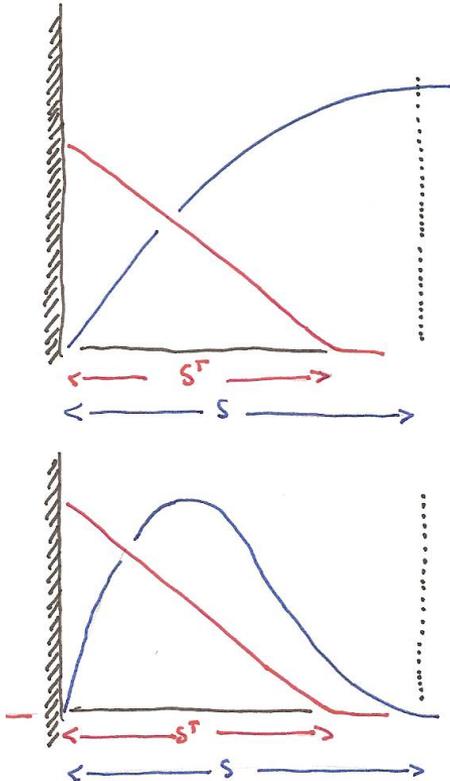
$$Gr = \frac{\beta g \Delta T Y^3}{\nu^2}$$

Buoyancy balanced by friction

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = \beta g (T - T_0) + \nu \frac{\partial^2 v}{\partial x^2}$$



$$V = \frac{\beta g \Delta T \delta_T^2}{\nu}$$



$$\frac{\delta_T}{Y} = \sqrt{\frac{\alpha}{\nu Y}} = \sqrt{\frac{\nu \alpha}{\beta g \Delta T Y \delta_T^2}} = (Gr)^{-1/4} (Pr)^{-1/4}$$

$$\frac{\delta}{Y} = (Gr)^{-1/4} (Pr)^{1/4}$$

Buoyancy balanced by inertia

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = \beta g (T - T_0) + \cancel{\nu \frac{\partial^2 v}{\partial x^2}}$$

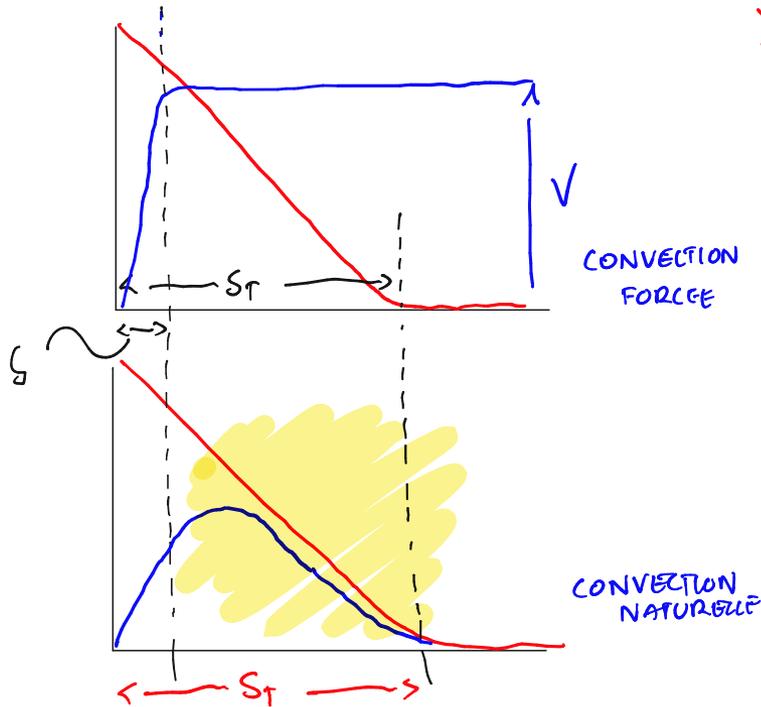
$$\frac{V^2}{Y}$$

$$\beta g \Delta T$$

$$V = \sqrt{\beta g Y \Delta T}$$

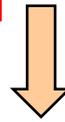
$$\frac{S_T}{Y} = \sqrt{\frac{\alpha}{\nu Y}} = \left(\frac{\alpha^2}{\beta g \Delta T Y^3} \right)^{1/4}$$

$$(Gr^{-1/4} Pr^{-1/2})$$

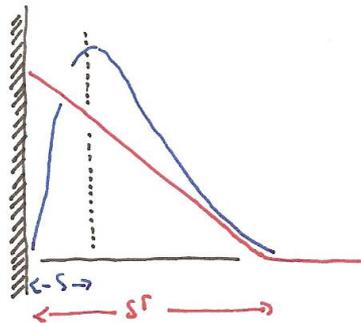
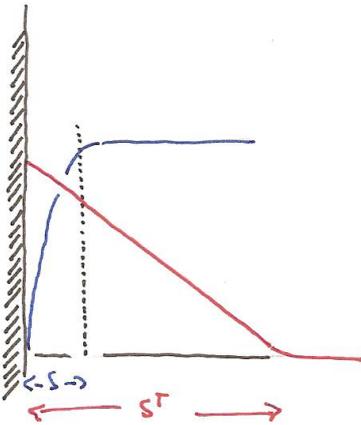


Buoyancy
balanced
by inertia

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = \beta g (T - T_0) + \cancel{\nu \frac{\partial^2 v}{\partial x^2}}$$



$$V = \sqrt{\beta g \Delta T Y}$$



$$\frac{\delta_T}{Y} = \sqrt{\frac{\alpha}{VY}} = \left(\frac{\alpha^2}{\beta g \Delta T Y^3} \right)^{1/4} = (Gr)^{-1/4} (Pr)^{-1/2}$$

$$\frac{\delta}{Y} = (Gr)^{-1/4}$$

Nombre de Grashof



$$Gr = \frac{\beta \Delta T g L^3}{\nu^2}$$

$$\frac{(ARCHIMEDE)(INERTIE)}{(VISQUEUX)^2}$$

$$\underbrace{\rho \beta g \Delta T}_{(ARCHIMEDE)}$$

$$\underbrace{\frac{L^4}{\rho^2 \nu^2 U^2}}_{\frac{1}{(VISQUEUX)^2}}$$

$$\underbrace{\frac{\rho U^2}{L}}_{(INERTIE)}$$

$$= \frac{\beta g \Delta T L^3}{\nu^2}$$

$$\rho \left(u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right)$$

$$= \rho \beta (T - T_0) g$$

$$+ \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)$$

Nombre de Grashof

$$Gr = \frac{\beta \Delta T g L^3}{\nu^2}$$



1822-1893, University of Karlsruhe, Germany
Professor of Mechanical Engineering

(Forces d'inertie) (Forces d'Archimède)

(Forces visqueuses)²

Grashof

$$\rho \left(u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = \rho \beta (T - T_0) g + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)$$

$$\frac{\text{■}}{\text{■}} = Re$$

$$\frac{\text{■}}{\text{■}} = \frac{Gr}{Re}$$

$$\frac{\text{■}}{\text{■}} = \frac{Gr}{Re^2}$$

$$Gr = \frac{(\text{Forces d'Archimède})(\text{Forces d'inertie})}{(\text{Forces visqueuses})^2} = \frac{\beta \Delta T g L^3}{\nu^2}$$

Grashof - Reynolds

*Forces
visqueuses*

$$\rho \left(u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = \rho \beta (T - T_0) g + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)$$

$\mathcal{O}(\rho U^2/L)$ $\mathcal{O}(\rho \beta \Delta T g)$ $\mathcal{O}(\mu U/L^2)$

*Forces
d'inertie*

*Forces
d'Archimède*

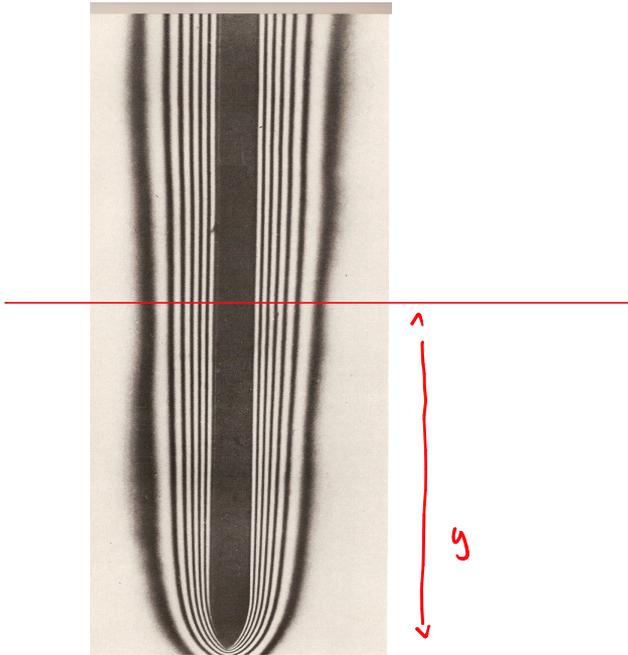
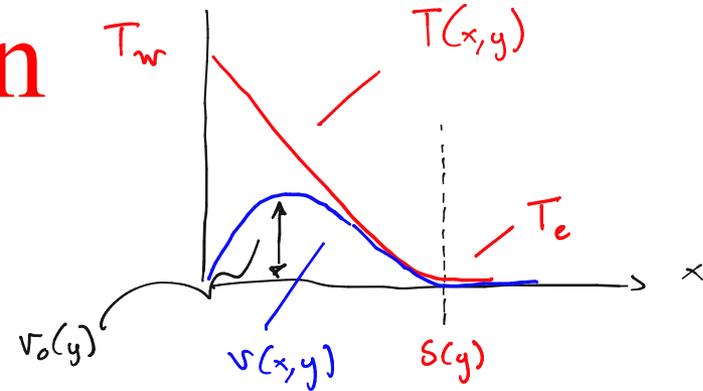
$$Re = \frac{\text{■}}{\text{■}}$$

$$\frac{Gr}{Re} = \frac{\text{■}}{\text{■}}$$

$$\frac{Gr}{Re^2} = \frac{\text{■}}{\text{■}}$$

$$Gr = \frac{(\text{Forces d'Archimède})(\text{Forces d'inertie})}{(\text{Forces visqueuses})^2} = \frac{\beta \Delta T g L^3}{\nu^2}$$

Une solution approchée pour la convection naturelle...



$$v(x,y) = v_0(y) \frac{x}{\delta(y)} \left(1 - \frac{x}{\delta(y)}\right)^2$$

$$\frac{T(x,y) - T_e}{T_w - T_e} = \left(1 - \frac{x}{\delta(y)}\right)^2$$



IL FAUT MAINTENANT
DETERMINER $\delta(y)$ ET $v_0(y)$

11/11 → CONGÉ :-)
PAS DE TP!

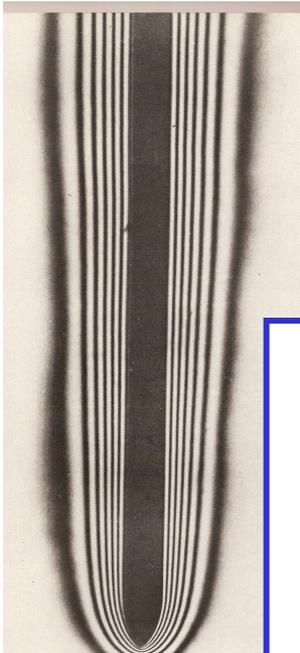
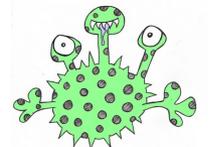


DONC
PAS DE TP
EN 10/10

Une solution approchée pour la convection naturelle...

$$v(x, y) = v_0(y) \frac{x}{\delta(y)} \left(1 - \frac{x}{\delta(y)}\right)^2$$

$$\frac{T(x, y) - T_0}{T_w - T_0} = \left(1 - \frac{x}{\delta(y)}\right)^2$$



On considère des profils semblables avec conditions de raccord très sommaires à la couche limite... mais cela va nous donner une idée des ordres de grandeurs

$$\begin{array}{llll}
 v(0, y) = 0 & v(\delta, y) = 0 & \frac{\partial v}{\partial x}(\delta, y) = 0 & \mu \frac{\partial^2 v}{\partial x^2}(0, y) = -\rho \beta g (T_w - T_0) \\
 T(0, y) = T_w & T(\delta, y) = T_0 & \frac{\partial T}{\partial x}(\delta, y) = 0 &
 \end{array}$$

$$\underbrace{\int_0^S u \frac{\partial v}{\partial x}} + \underbrace{\int_0^S v \frac{\partial v}{\partial y}} = \beta g \int_0^S (T - T_0) + \underbrace{\int_0^S \frac{\partial^2 v}{\partial x^2} dx}_{= \left. \frac{\partial v}{\partial x} \right|_0}$$

$$\cancel{[uv]_0^S} - \underbrace{\int_0^S v \frac{\partial u}{\partial x}}_{= \frac{\partial v}{\partial y}}$$

$$\frac{d}{dy} \int_0^{S(y)} v^2 = \beta g \int_0^S T - T_0 - \left. \frac{\partial v}{\partial x} \right|_0$$

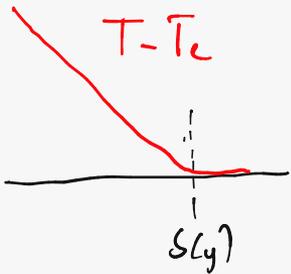
Calculons

$\delta_T(y)$ et $v_0(y)$!

$$\int_0^s u \frac{\partial T}{\partial x} + \int_0^J v \frac{\partial T}{\partial y} = \int_0^s \alpha \frac{\partial^2 T}{\partial x^2} - \alpha \frac{\partial T}{\partial x} \Big|_0$$

$$[uT] - \int_0^s T \frac{\partial u}{\partial x}$$

$$[\cancel{u(T-T_c)}] - \int_0^J (T-T_c) \frac{\partial v}{\partial x} = -\frac{\partial v}{\partial y}$$



$$\frac{d}{dy} \int_0^s (T-T_0) v \, dx = -\alpha \frac{\partial T}{\partial x} \Big|_0$$

Il faut
encore
estimer
 $\delta_T(y)$
et $v_0(y)$!

$$\int_0^\delta u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} dx = \beta g \int_0^\delta (T(x, y) - T_0) dx + \nu \int_0^\delta \frac{\partial^2 v}{\partial x^2} dx$$

$$[uv]_0^\delta - \int_0^\delta v \frac{\partial u}{\partial x} dx + \int_0^\delta v \frac{\partial v}{\partial y} dx = \beta g \int_0^\delta (T(x, y) - T_0) dx - \nu \frac{\partial v}{\partial x} \Big|_{x=0}$$

$$\frac{d}{dy} \int_0^\delta v^2(x, y) dx = \beta g \int_0^\delta (T(x, y) - T_0) dx - \nu \frac{\partial v}{\partial x} \Big|_{x=0}$$

$$\int_0^\delta u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} dx = \alpha \int_0^\delta \frac{\partial^2 T}{\partial x^2} dx$$

$$\frac{d}{dy} \int_0^\delta v(x, y) (T(x, y) - T_0) dx = -\alpha \frac{\partial T}{\partial x} \Big|_{x=0}$$

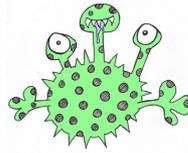
Intégrons les équations de conservation de la quantité de mouvement et de l'énergie dans la couche limite en tirant profit de l'incompressibilité...

Que deviennent ces équations intégrales ?

$$\left\{ \begin{array}{l} \frac{d}{dy} \int_0^\delta v^2 dx = \beta g \int_0^\delta (T - T_0) dx - \nu \frac{\partial v}{\partial x} \Big|_{x=0} \\ \frac{d}{dy} \int_0^\delta v(T - T_0) dx = -\alpha \frac{\partial T}{\partial x} \Big|_{x=0} \end{array} \right.$$

$T_w - T_e$
3

En intégrant les expressions approchées que nous avons introduites...



$$v = v_0 \eta (1-\eta)^2$$

$$\Theta = (1-\eta)^2$$

$$\int_0^1 (1-\eta)^4 \eta^2$$

$$\left[\frac{1}{3} - \frac{4}{4} + \frac{6}{5} - \frac{4}{6} + \frac{1}{7} \right]$$

$$= \frac{35 - 105 + 126 - 70 + 15}{105}$$

$$= \frac{1}{105}$$

$$v_0 \int_0^1 (1-\eta)^4 \eta$$

$$(1-2\eta+\eta^2)(1-2\eta+\eta^2)$$

$$\left(\begin{array}{l} 1-2\eta+\eta^2 \\ -2\eta+4\eta^2-2\eta^3 \\ \eta^2-2\eta^3+\eta^4 \end{array} \right) \eta$$

$$(1-4\eta+6\eta^2-4\eta^3+\eta^4) \eta$$

$$\left[\frac{\eta^2}{2} - \frac{4\eta^3}{3} + \frac{6\eta^4}{4} - \frac{4\eta^5}{5} + \frac{\eta^6}{6} \right]_0^1 = \frac{15 - 40 + 45 - 24 + 5}{30} = \frac{1}{30}$$

$$\int_0^1 (1-\eta)^2$$

$$= \left[\eta - \eta^2 + \frac{\eta^3}{3} \right]_0^1$$

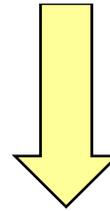
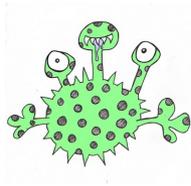
$$= 1 - 1 + \frac{1}{3} = \frac{1}{3}$$

Que deviennent ces équations

intégrales

$$\left\{ \begin{array}{l} \frac{d}{dy} \int_0^\delta v^2 dx = \beta g \int_0^\delta (T - T_0) dx - \nu \left. \frac{\partial v}{\partial x} \right|_{x=0} \\ \frac{d}{dy} \int_0^\delta v(T - T_0) dx = -\alpha \left. \frac{\partial T}{\partial x} \right|_{x=0} \end{array} \right.$$

En intégrant les expressions
approchées que nous avons
introduites...



$$\left\{ \begin{array}{l} \frac{1}{105} \frac{d}{dy} \left(v_0^2(y) \delta(y) \right) = \frac{\beta g \Delta T \delta(y)}{3} - \frac{\nu v_0(y)}{\delta(y)} \\ \frac{1}{30} \frac{d}{dy} \left(v_0(y) \delta(y) \right) = \frac{2\alpha}{\delta(y)} \end{array} \right.$$

On obtient finalement de deux équations
différentielles ordinaires avec deux
fonctions inconnues...

Résolution des équations intégrales

$$\begin{cases} \frac{1}{105} \frac{d}{dy} \left(v_0^2(y) \delta(y) \right) = \frac{\beta g \Delta T \delta(y)}{3} - \frac{\nu v_0(y)}{\delta(y)} \\ \frac{1}{30} \frac{d}{dy} \left(v_0(y) \delta(y) \right) = \frac{2\alpha}{\delta(y)} \end{cases}$$

IDÉE $v_0(y) = V y^m$
 $\delta(y) = D y^n$

$$\begin{cases} \frac{1}{105} V^2 D (2m+n) y^{2m+n-1} = \beta g \frac{\Delta T}{3} D y^n - \nu \frac{V}{D} y^{m-n} \\ \frac{1}{30} V D (m+n) y^{m+n-1} = \frac{2\alpha}{D} y^{-n} \end{cases}$$

$2m$

$2m+n-1 = n = m-n$

$m+n-1 = -n$
 $2m$

$m = 2n$

~~$4m+n-1 = n$~~

$$\begin{aligned} n &= 1/4 \\ m &= 1/2 \end{aligned}$$

$2m+n-1 = -n$

$4n = 1$

Résolution des équations intégrales

$$\begin{cases} \frac{1}{105} \frac{d}{dy} \left(v_0^2(y) \delta(y) \right) = \frac{\beta g \Delta T \delta(y)}{3} - \frac{\nu v_0(y)}{\delta(y)} \\ \frac{1}{30} \frac{d}{dy} \left(v_0(y) \delta(y) \right) = \frac{2\alpha}{\delta(y)} \end{cases}$$

Essayons une solution de
la forme...

$$\begin{aligned} v_0(y) &= V y^m \\ \delta(y) &= D y^n \end{aligned}$$

$$\begin{cases} \frac{1}{105} V^2 D (2m + n) y^{2m+n-1} = \frac{\beta g \Delta T D}{3} y^n - \frac{\nu V}{D} y^{m-n} \\ \frac{1}{30} V D (m + n) y^{m+n-1} = \frac{2\alpha}{D} y^{-n} \end{cases}$$

Et cela marche avec $2m = 1$ et $4n = 1$...

C'est presque
fini !

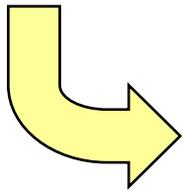
$$\begin{cases} \frac{V^2 D}{84} = \frac{\beta g \Delta T D}{3} - \frac{\nu V}{D} \\ \frac{VD}{40} = \frac{2\alpha}{D} \end{cases}$$

$V = \frac{80\alpha}{D^2}$

$$\frac{80^2 \alpha^4}{D^4} \frac{\cancel{D}}{84} = \beta g \frac{\Delta T \cancel{D}}{3} - \nu \frac{80\alpha}{D^2} \frac{1}{D} \frac{\cancel{D}}{D}$$

$$D^4 = \frac{3}{\beta g \Delta T} \left[\frac{80^2}{84} \alpha^2 - 80\alpha \nu \right]$$

$$\begin{cases} \frac{V^2 D}{84} = \frac{\beta g \Delta T D}{3} - \frac{\nu V}{D} \\ \frac{VD}{40} = \frac{2\alpha}{D} \end{cases}$$

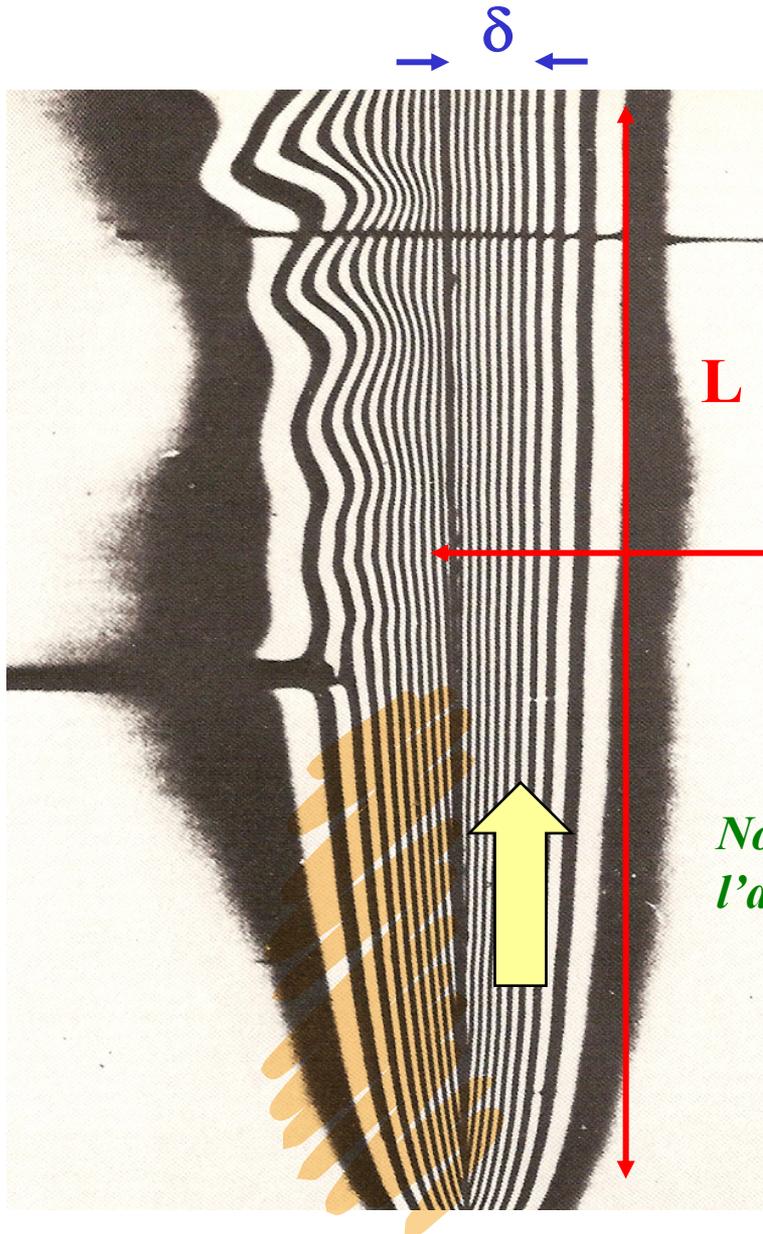


$$D^4 = 240L^3 \left(\frac{20}{21} + \frac{\nu}{\alpha} \right) \frac{\nu^2}{\beta g \Delta T L^3} \frac{\alpha^2}{\nu^2}$$

Solution ...

$$\frac{\delta_T(y)}{y} = 3,936 \left(Pr \right)^{-1/2} \left(Gr(y) \right)^{-1/4} \left(\frac{20}{21} + Pr \right)^{1/4}$$

On peut ensuite calculer le profil de vitesse et de température, le transfert de chaleur et des nombres de Nusselt locaux et moyens.



Est-ce que cette
solution est
stable ?

*Plaque
chaude*

*Non, après une certaine distance, on constate
l'apparition d'instabilités : c'est la turbulence !*

Cela, c'est très très très très compliqué.

Grégoire est de
retour

