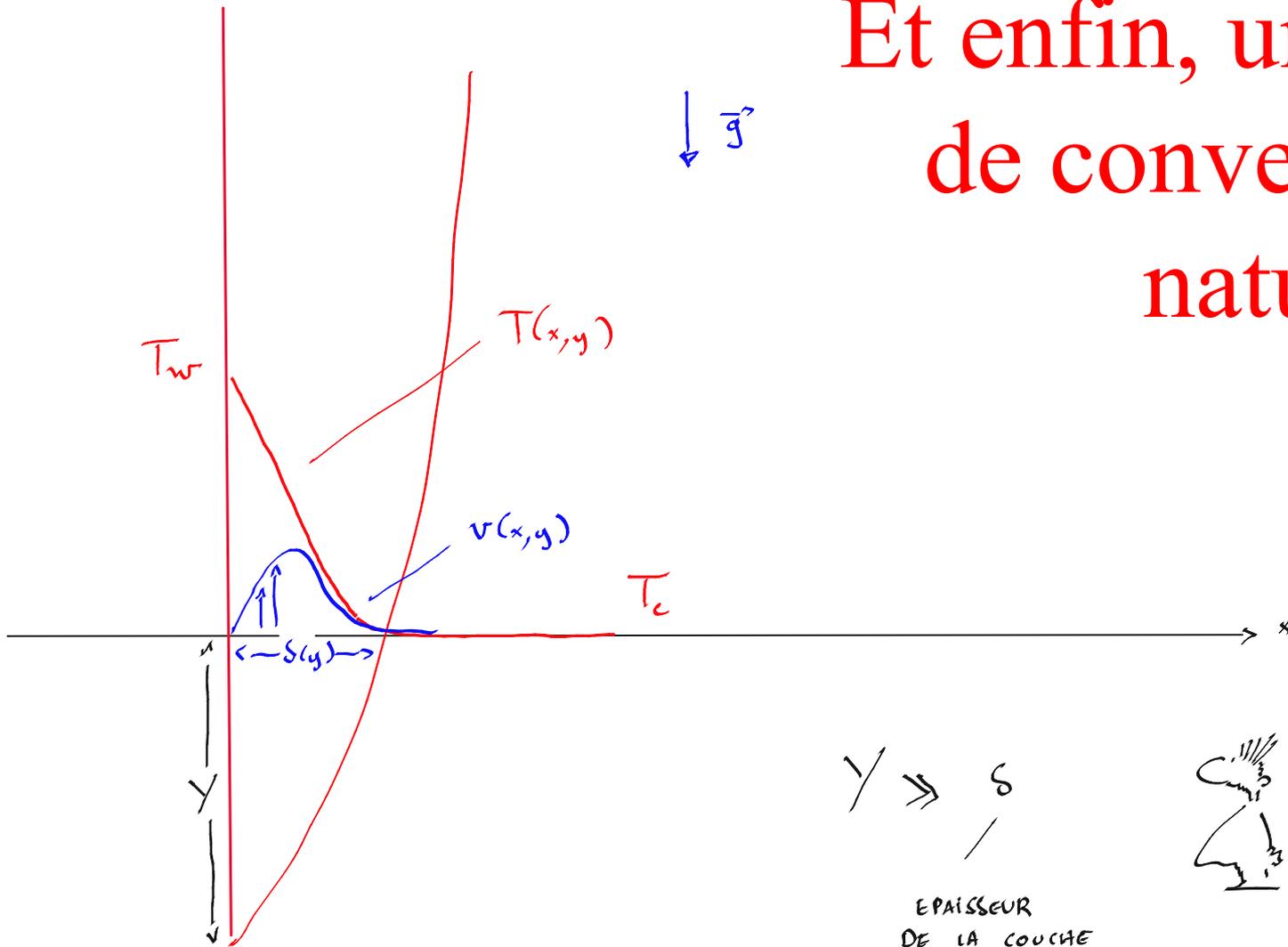


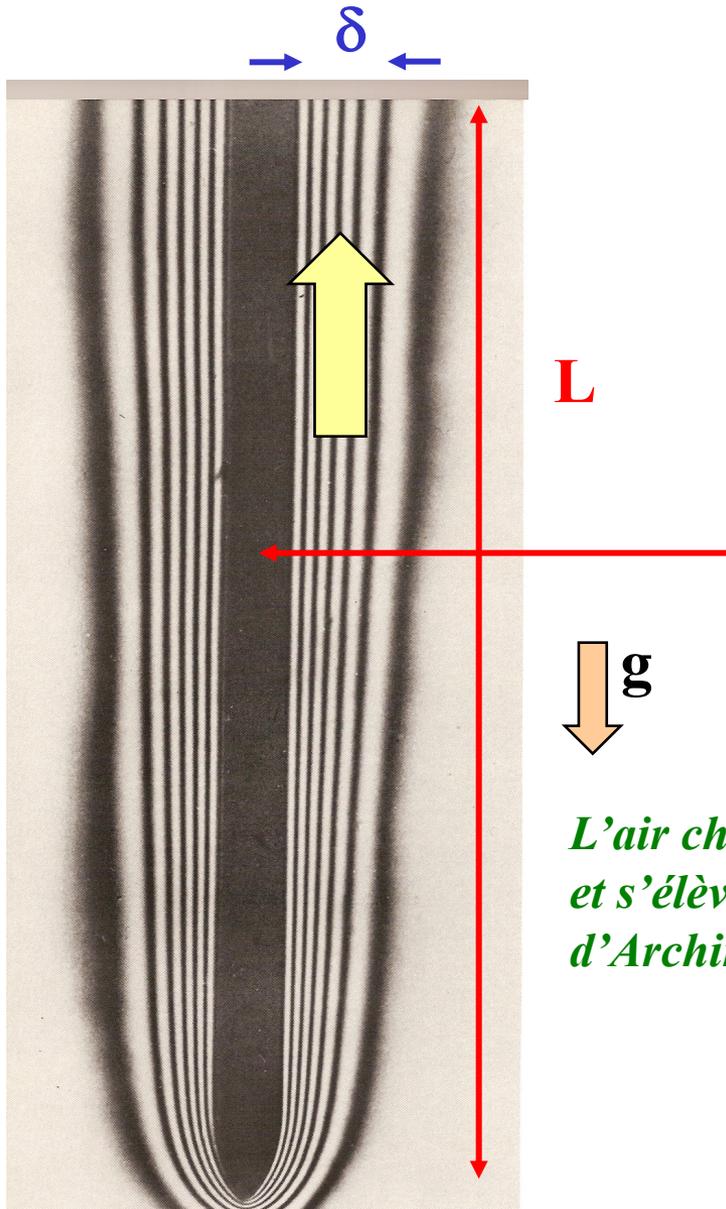
# Et enfin, un peu de convection naturelle



$y \Rightarrow \delta$

EPAISSEUR  
DE LA COUCHE  
LIMITES





# Convection naturelle le long d'une plaque suspendue dans l'air

*Plaque chaude*

*L'air chaud près de la plaque devient plus léger et s'élève naturellement sous l'effet de la force d'Archimède (flottabilité)(buoyancy)*

*La photo a été dilatée d'un facteur six dans le long de l'axe horizontal !  
(Gebhart, University of Pennsylvania)*

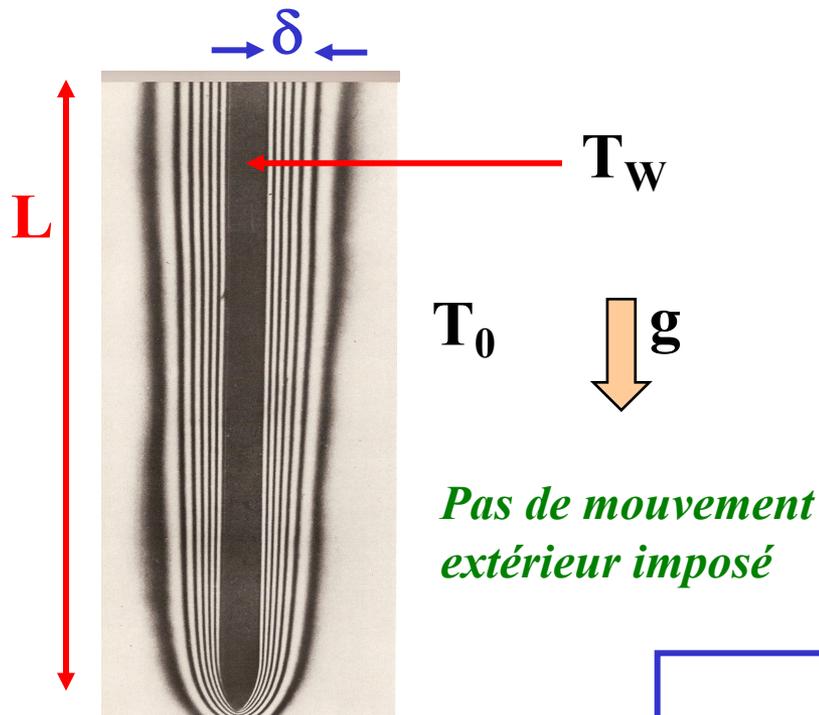
$$\rho_0 \underbrace{(1 - \beta(T - T_0))}_{\ll 1} \left( u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = \underbrace{-\frac{\partial p}{\partial y}}_{\rho_0 g} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) - \rho_0 (1 - \beta(T - T_0)) g$$

$$\rho_0 \left( u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + \rho_0 \beta (T - T_0) g$$

# Approximation de Boussinesq

*On néglige les variations de masse volumique dans tous les termes sauf dans la poussée d'Archimède...*

*\* Au passage, on observe que cette hypothèse n'a vraiment du sens que dans le cas où la pression est hydrostatique ... En d'autres mots, on introduit soit l'approximation hydrostatique, soit l'approximation de Boussinesq, mais on doit introduire en tous cas une approximation :-)*



# Reprenons le problème de convection naturelle

$$\delta \ll Y$$

$$\delta_T \ll Y$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = \beta g (T - T_0) + \nu \frac{\partial^2 v}{\partial x^2}$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial x^2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{u}{S_T} + \frac{v}{Y} = \alpha \frac{\partial^2 T}{\partial x^2} + \alpha \frac{\partial^2 T}{\partial y^2}$$

$$\frac{u}{S_T} + \frac{v}{Y} = \alpha \frac{\Delta T}{S_T^2} + \alpha \frac{\Delta T}{Y^2}$$

$$1 = \frac{v}{Y} \frac{S_T^2}{\alpha}$$

$$\frac{S_T^2}{Y^2} = \frac{\alpha}{VY}$$

Couche limite thermique, on connaît :-)

$$\frac{S_T}{Y} = \sqrt{\frac{\alpha}{VY}} = \sqrt{\frac{1}{Pr_e}}$$

*Lieu où l'ordre de la convection et de la conduction sont identiques*

$$\frac{\text{Convection}}{\text{Conduction}} = \frac{V\Delta T/Y}{\alpha\Delta T/\delta_T^2} = \frac{VY}{\underbrace{\alpha}_{Pe_Y}} \frac{\delta_T^2}{Y^2} = 1$$

Couche limite thermique,  
on connaît :-)

$$\frac{\delta_T}{Y} = \sqrt{\frac{1}{Pe_Y}} = \sqrt{\frac{1}{Pr Re_Y}}$$

Problème : on ne connaît pas  $V$  :-)

$$\delta \ll Y$$
$$\delta_T \ll Y$$

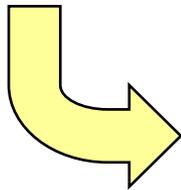
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$
$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = \beta g(T - T_0) + \nu \frac{\partial^2 v}{\partial x^2}$$
$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial x^2}$$

**Longueur verticale caractéristique :  $Y$**

**Longueur horizontale caractéristique :  $\delta$**

**Vitesse horizontale caractéristique :  $U = V \delta / Y$  (incompressibilité :-)**

**Ecart de température caractéristique :  $T_w - T_0$**



**Donc, comment choisir  
la vitesse verticale  
caractéristique  $V$  ?**

# Buoyancy balanced by friction

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = \beta g (T - T_0) + \nu \frac{\partial^2 v}{\partial x^2}$$

$$\beta g \Delta T \approx \frac{\rho V}{s_T^2}$$

$$V = \frac{\beta g \Delta T}{\rho} s_T^2$$

$$\frac{s_T}{y} = \sqrt{\frac{\alpha}{\nu y}} = \sqrt{\frac{\alpha \rho}{\beta g \Delta T} y^3} \frac{y}{s_T}$$

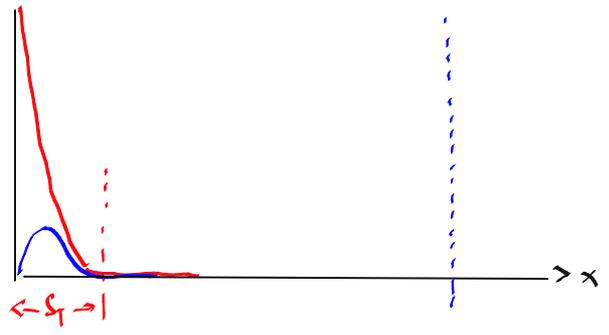
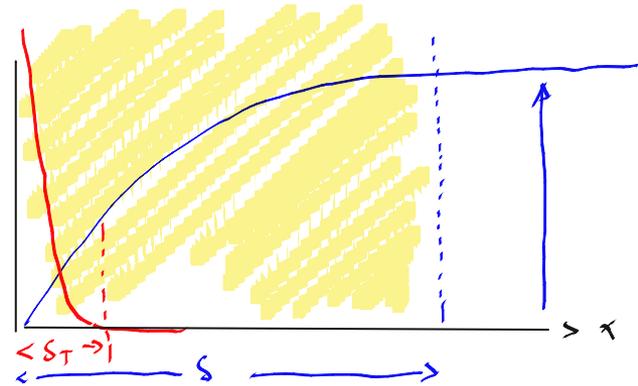
$$\frac{s_T}{y} = \left( \frac{\alpha \rho}{\beta g \Delta T} y^3 \right)^{1/4}$$

CONVECTION  
FORCE

CONVECTION  
NATURELLE

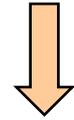
$$Gr = \frac{\beta g \Delta T y^3}{\nu^2}$$

$$Gr^{-1/4} Pr^{-1/4}$$

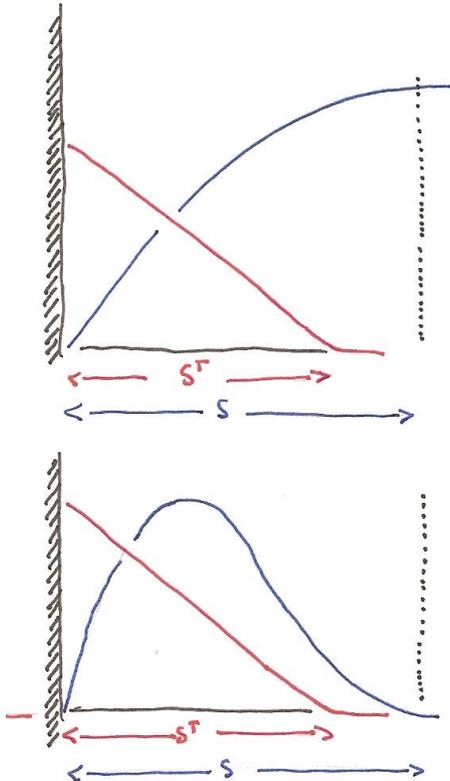


# Buoyancy balanced by friction

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = \beta g (T - T_0) + \nu \frac{\partial^2 v}{\partial x^2}$$



$$V = \frac{\beta g \Delta T \delta_T^2}{\nu}$$



$$\frac{\delta_T}{Y} = \sqrt{\frac{\alpha}{VY}} = \sqrt{\frac{\nu \alpha}{\beta g \Delta T Y \delta_T^2}} = (Gr)^{-1/4} (Pr)^{-1/4}$$

$$\frac{\delta}{Y} = (Gr)^{-1/4} (Pr)^{1/4}$$

# Buoyancy balanced by inertia

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = \beta g (T - T_0) + \cancel{\nu \frac{\partial^2 v}{\partial x^2}}$$

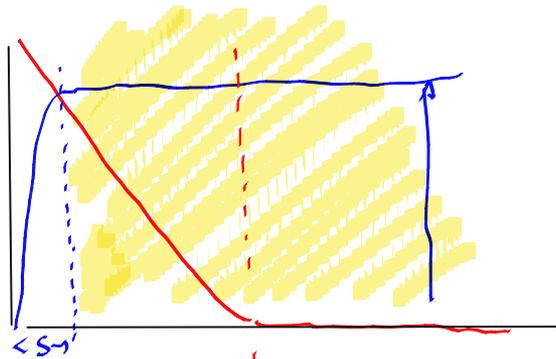
$$\frac{V^2}{Y}$$

$$\text{is } \beta g \Delta T$$

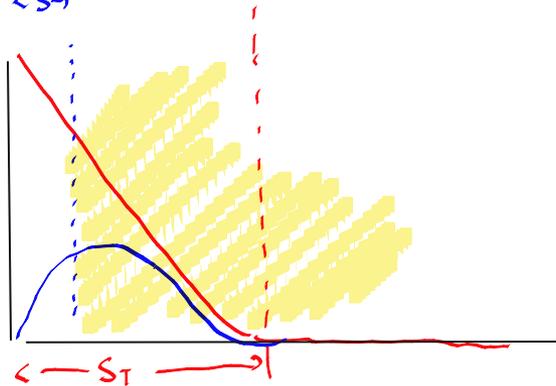
$$V = \sqrt{\beta g \Delta T Y}$$

$$\frac{S_T}{Y} = \sqrt{\frac{\alpha}{V Y}} = \left( \frac{\alpha^2}{\beta g \Delta T Y^3} \right)^{1/4}$$

$$Gr^{-1/4} Pr^{-1/2}$$



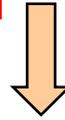
CONVECTION  
FORCEE



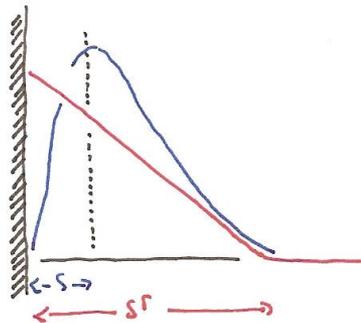
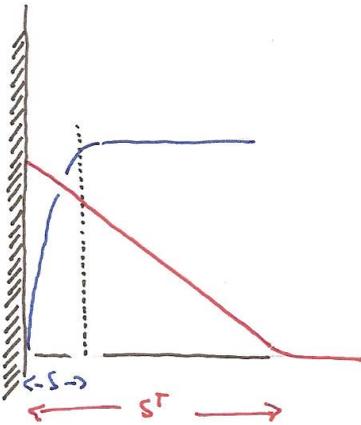
CONVECTION  
NATURELLE

# Buoyancy balanced by inertia

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = \beta g (T - T_0) + \cancel{\nu \frac{\partial^2 v}{\partial x^2}}$$



$$V = \sqrt{\beta g \Delta T Y}$$



$$\frac{\delta_T}{Y} = \sqrt{\frac{\alpha}{VY}} = \left( \frac{\alpha^2}{\beta g \Delta T Y^3} \right)^{1/4} = (Gr)^{-1/4} (Pr)^{-1/2}$$

$$\frac{\delta}{Y} = (Gr)^{-1/4}$$

# Nombre de Grashof



$$Gr = \frac{\beta \Delta T g L^3}{\nu^2}$$

$$\frac{(\text{ARCHIMÈDE}) (\text{INERTIE})}{(\text{VISQUEUX})^2}$$

$$\underbrace{\rho \beta g \Delta T}_{\text{ARCHIMÈDE}} \underbrace{\frac{L^4}{\rho^2 \nu^2 U^2}}_{\frac{1}{(\text{VISQUEUX})^2}} \underbrace{\frac{\rho U^2}{L}}_{\text{INERTIE}} = \frac{\beta g \Delta T L^3}{\nu^2}$$

$$\rho \left( u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = \rho \beta (T - T_0) g + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)$$

# Nombre de Grashof

$$Gr = \frac{\beta \Delta T g L^3}{\nu^2}$$



1822-1893, University of Karlsruhe, Germany  
Professor of Mechanical Engineering

**(Forces d'inertie) (Forces d'Archimède)**

---

**(Forces visqueuses)<sup>2</sup>**

# Grashof

$$Gr = \frac{\text{blue} \text{ red}}{\text{green} \text{ blue}}$$

$$\rho \left( u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right)$$

INERTIE

$$= \rho \beta (T - T_0) g$$

ARCHIMEDE

$$+ \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)$$

VISQUEUX

$$\frac{\text{red}}{\text{green}} = Re$$

$$\frac{\text{blue}}{\text{green}} = \frac{Gr}{Re}$$

$$\frac{\text{blue}}{\text{red}} = \frac{Gr}{Re^2}$$

$$Gr = \frac{(\text{Forces d'Archimède})(\text{Forces d'inertie})}{(\text{Forces visqueuses})^2} = \frac{\beta \Delta T g L^3}{\nu^2}$$

# Grashof - Reynolds

*Forces  
visqueuses*

$$\rho \left( u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = \rho \beta (T - T_0) g + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)$$

$\mathcal{O}(\rho U^2/L)$                        $\mathcal{O}(\rho \beta \Delta T g)$                        $\mathcal{O}(\mu U/L^2)$

*Forces  
d'inertie*

*Forces  
d'Archimède*

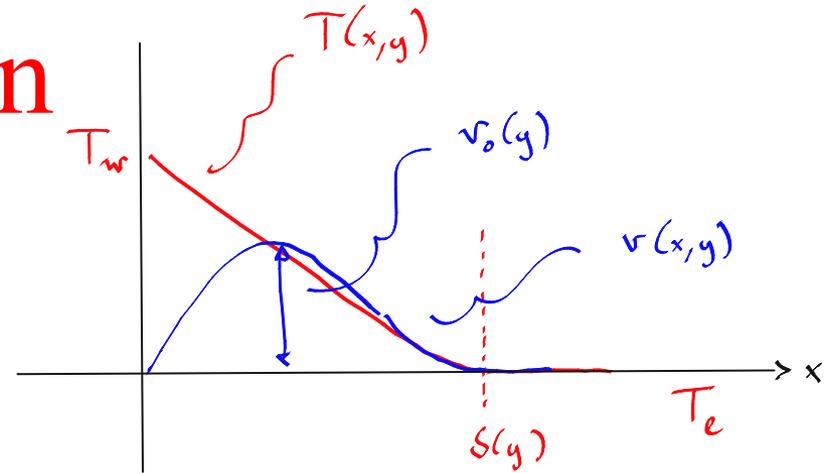
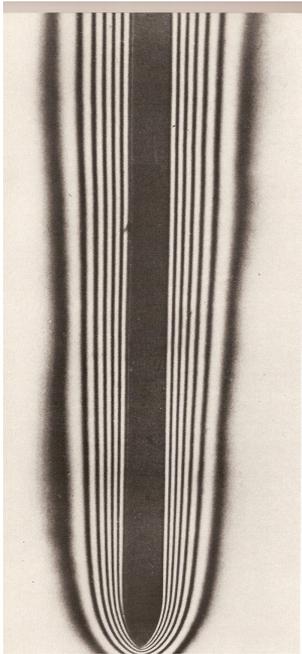
$$Re = \frac{\text{■}}{\text{■}}$$

$$\frac{Gr}{Re} = \frac{\text{■}}{\text{■}}$$

$$\frac{Gr}{Re^2} = \frac{\text{■}}{\text{■}}$$

$$Gr = \frac{(\text{Forces d'Archimède})(\text{Forces d'inertie})}{(\text{Forces visqueuses})^2} = \frac{\beta \Delta T g L^3}{\nu^2}$$

# Une solution approchée pour la convection naturelle...



$$v(x,y) = v_0(y) \frac{x}{S(y)} \left(1 - \frac{x}{S(y)}\right)^2$$
$$\frac{T(x,y) - T_e}{T_w - T_e} = \left(1 - \frac{x}{S(y)}\right)^2$$

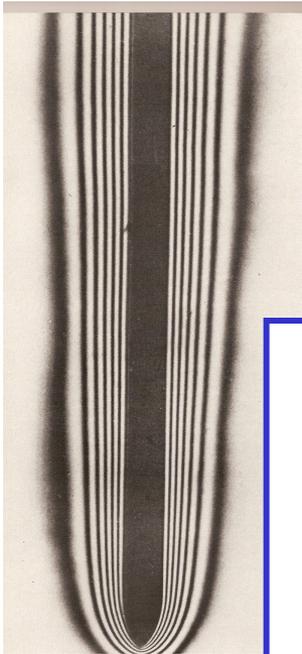
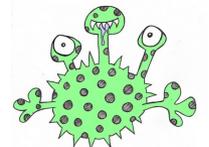


IL FAUT DETERMINER  
 $S(y)$   $v_0(y)$

# Une solution approchée pour la convection naturelle...

$$v(x, y) = v_0(y) \frac{x}{\delta(y)} \left(1 - \frac{x}{\delta(y)}\right)^2$$

$$\frac{T(x, y) - T_0}{T_w - T_0} = \left(1 - \frac{x}{\delta(y)}\right)^2$$



On considère des profils semblables avec conditions de raccord très sommaires à la couche limite... mais cela va nous donner une idée des ordres de grandeurs

$$\begin{array}{llll}
 v(0, y) = 0 & v(\delta, y) = 0 & \frac{\partial v}{\partial x}(\delta, y) = 0 & \mu \frac{\partial^2 v}{\partial x^2}(0, y) = -\rho \beta g (T_w - T_0) \\
 T(0, y) = T_w & T(\delta, y) = T_0 & \frac{\partial T}{\partial x}(\delta, y) = 0 & 
 \end{array}$$

$$\underbrace{\int_0^S u \frac{\partial v}{\partial x}} + \int_0^S v \frac{\partial v}{\partial y} = \beta g \int_0^S (T - T_c) + \underbrace{\rho \int_0^S \frac{\partial^2 v}{\partial x^2} dx}_{-\rho \frac{\partial v}{\partial x} \Big|_0}$$

$$\underbrace{\int_0^S v \frac{\partial u}{\partial x}}_{\leftarrow -\frac{\partial v}{\partial x}}$$



$$2 \int_0^S v \frac{\partial v}{\partial y} = \int_0^S \frac{\partial}{\partial y} (v^2)$$



$$\frac{d}{dy} \int_0^S v^2 = \beta g \int_0^S (T - T_c) - \rho \frac{\partial v}{\partial x} \Big|_0$$

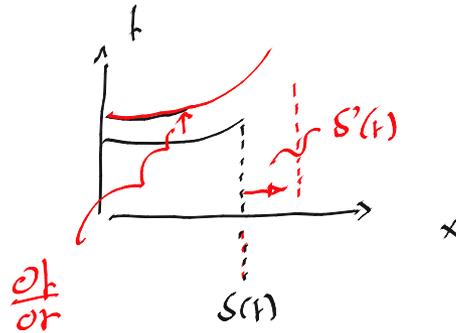
Calculons

$\delta_T(y)$  et  $v_0(y)$  !

$$\frac{d}{dy} \int_0^{S(y)} v^2(x, y) dx \stackrel{?}{=} \int_0^{S(y)} \frac{\partial}{\partial y} (v^2(x, y)) dx$$

Oui  
CAR  $v = 0$   
EN  $S(y)$

$$\frac{d}{dt} \int_0^{S(t)} f(x, t) dx = \int_0^{S(t)} \frac{\partial f}{\partial t}(x, t) dx + \underbrace{f(S(t), t) S'(t)}_{= 0}$$



= 0  
si  $f(S(t), t) = 0$

# Calculons $\delta_T(y)$ et $v_0(y)$ !

$T - T_c = 0$



$$\int_0^S v \frac{\partial T}{\partial x} + \int_0^S v \frac{\partial T}{\partial y} = \alpha \int_0^S \frac{\partial^2 T}{\partial x^2} - \alpha \frac{\partial T}{\partial x} \Big|_0$$

$$\int_0^S v \frac{\partial T}{\partial x} - \int_0^S (T - T_c) \frac{\partial v}{\partial x} - \frac{\partial v}{\partial y}$$

$$\int_0^S v \frac{\partial T}{\partial y} + (T - T_c) \frac{\partial v}{\partial y} =$$

$$\frac{\partial}{\partial y} (v(T - T_c))$$

Calculons

$\delta_T(y)$  et  $v_0(y)$  !

$$\frac{d}{dy} \int_0^S v (T - T_c) = -\alpha \frac{\partial T}{\partial x} \Big|_0$$

Il faut  
encore  
estimer  
 $\delta_T(y)$   
et  $v_0(y)$  !

$$\int_0^\delta u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} dx = \beta g \int_0^\delta (T(x, y) - T_0) dx + \nu \int_0^\delta \frac{\partial^2 v}{\partial x^2} dx$$

$$[uv]_0^\delta - \int_0^\delta v \frac{\partial u}{\partial x} dx + \int_0^\delta v \frac{\partial v}{\partial y} dx = \beta g \int_0^\delta (T(x, y) - T_0) dx - \nu \frac{\partial v}{\partial x} \Big|_{x=0}$$

$$\frac{d}{dy} \int_0^\delta v^2(x, y) dx = \beta g \int_0^\delta (T(x, y) - T_0) dx - \nu \frac{\partial v}{\partial x} \Big|_{x=0}$$

$$\int_0^\delta u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} dx = \alpha \int_0^\delta \frac{\partial^2 T}{\partial x^2} dx$$

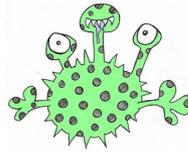
$$\frac{d}{dy} \int_0^\delta v(x, y) (T(x, y) - T_0) dx = -\alpha \frac{\partial T}{\partial x} \Big|_{x=0}$$

**Intégrons les équations de conservation de la quantité de mouvement et de l'énergie dans la couche limite en tirant profit de l'incompressibilité...**

# Que deviennent ces équations intégrales ?

$$\left\{ \begin{array}{l} \frac{d}{dy} \int_0^\delta v^2 dx = \beta g \int_0^\delta (T - T_0) dx - \nu \left. \frac{\partial v}{\partial x} \right|_{x=0} \\ \frac{d}{dy} \int_0^\delta v(T - T_0) dx = -\alpha \left. \frac{\partial T}{\partial x} \right|_{x=0} \end{array} \right.$$

En intégrant les expressions approchées que nous avons introduites...



$$v = v_0 \eta (1-\eta)^2 \quad \eta = \frac{x}{\delta}$$

$$\theta = (1-\eta)^2$$

$$\int_0^1 (1-\eta)^4 \eta^2$$

$$= \left[ \frac{1}{3} - \frac{4}{4} + \frac{6}{5} - \frac{4}{6} + \frac{1}{7} \right]$$

$$= \frac{35 - 105 + 126 - 70 + 15}{105} = \frac{1}{105}$$

15 x 7

$$v_0 \int_0^1 (1-\eta)^4 \eta$$

$$(1-2\eta+\eta^2)(1-2\eta+\eta^2)$$

$$1-2\eta+\eta^2$$

$$-2\eta+4\eta^2-2\eta^3$$

$$\eta^2-2\eta^3+\eta^4$$

$$(1-4\eta+6\eta^2-4\eta^3+\eta^4)\eta$$

$$\int_0^1 (1-\eta)^2$$

$$= \left[ \eta - \eta^2 + \frac{\eta^3}{3} \right]_0^1$$

$$= 1 - 1 + \frac{1}{3} = \frac{1}{3}$$

$$\frac{1}{3} = 35 \text{ (-)}$$

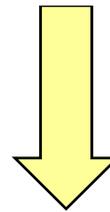
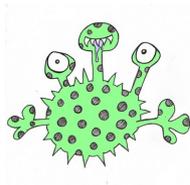
$$\frac{1}{5} = 21 \text{ (-)}$$

$$\left[ \frac{\eta^2}{2} - \frac{4\eta^3}{3} + \frac{6\eta^4}{4} - \frac{4\eta^5}{5} + \frac{\eta^6}{6} \right]_0^1 = \frac{15 - 40 + 45 - 24 + 5}{30} = \frac{1}{30}$$

# Que deviennent ces équations intégrales

$$\left\{ \begin{array}{l} \frac{d}{dy} \int_0^\delta v^2 dx = \beta g \int_0^\delta (T - T_0) dx - \nu \left. \frac{\partial v}{\partial x} \right|_{x=0} \\ \frac{d}{dy} \int_0^\delta v(T - T_0) dx = -\alpha \left. \frac{\partial T}{\partial x} \right|_{x=0} \end{array} \right.$$

En intégrant les expressions approchées que nous avons introduites...



$$v' = v_0 \left[ (1-\eta)^2 - 2\eta(1-\eta) \right]_0^1$$

$$\theta' = \left[ -2(1-\eta) \right]_0^1$$

$$\left\{ \begin{array}{l} \frac{1}{105} \frac{d}{dy} \left( v_0^2(y) \delta(y) \right) = \frac{\beta g \Delta T \delta(y)}{3} - \frac{\nu v_0(y)}{\delta(y)} \\ \frac{1}{30} \frac{d}{dy} \left( v_0(y) \delta(y) \right) = \frac{2\alpha}{\delta(y)} \end{array} \right.$$

On obtient finalement de deux équations différentielles ordinaires avec deux fonctions inconnues...

# Résolution des équations intégrales

$$\begin{cases} \frac{1}{105} \frac{d}{dy} \left( v_0^2(y) \delta(y) \right) = \frac{\beta g \Delta T \delta(y)}{3} - \frac{\nu v_0(y)}{\delta(y)} \\ \frac{1}{30} \frac{d}{dy} \left( v_0(y) \delta(y) \right) = \frac{2\alpha}{\delta(y)} \end{cases}$$

Essayons une solution de  
la forme...

$$\begin{aligned} v_0(y) &= V y^m \\ \delta(y) &= D y^n \end{aligned}$$

$$\begin{cases} \frac{1}{105} V^2 D (2m+n) y^{2m+n-1} = \frac{\beta g \Delta T D}{3} y^n - \frac{\nu V}{D} y^{m-n} \\ \frac{1}{30} V D (m+n) y^{m+n-1} = \frac{2\alpha}{D} y^{-n} \end{cases}$$

Et cela marche avec  $2m = 1$  et  $4n = 1...$

$$m = 2n$$



$$2m+n-1 = n = m-n$$

$$m+n-1 = -n$$

$$\begin{cases} 4m+n-1 = n \\ 2m+n-1 = -n \end{cases}$$

$$4m = 1$$

$$4m = 1$$

$$m = 1/4$$

$$m = 1/2$$

# Résolution des équations intégrales

$$\begin{cases} \frac{1}{105} \frac{d}{dy} \left( v_0^2(y) \delta(y) \right) = \frac{\beta g \Delta T \delta(y)}{3} - \frac{\nu v_0(y)}{\delta(y)} \\ \frac{1}{30} \frac{d}{dy} \left( v_0(y) \delta(y) \right) = \frac{2\alpha}{\delta(y)} \end{cases}$$

$$\frac{1}{105} V^2 D \overbrace{(2m+n)}^{1+1/4=5/4} y^{2m+n-1} = \frac{\beta g \Delta T}{3} D y^n - \nu \frac{V}{D} y^{m-n}$$

$$\frac{1}{30} V D \overbrace{(m+n)}^{1/4+1/2=3/4} y^{m+n-1} = \frac{2\alpha}{D} y^{-n}$$

$$\left. \begin{array}{l} n = 1/4 \\ n = 1/2 \end{array} \right\}$$

C'est presque fini !

$$1 + \frac{1}{4} = \frac{5}{4}$$

$$\begin{cases} \frac{V^2 D}{84} = \frac{\beta g \Delta T D}{3} - \frac{\nu V}{D} \\ \frac{V D}{40} = \frac{2\alpha}{D} \end{cases}$$

$$V = \frac{80\alpha}{D^2}$$

$$\begin{cases} \frac{1}{105} V^2 D (2m+n) y^{2m+n-1} = \frac{\beta g \Delta T D}{3} y^n - \nu \frac{V}{D} y^{m-n} \\ \frac{1}{30} V D (m+n) y^{m+n-1} = \frac{2\alpha}{D} y^{-n} \end{cases}$$

$$\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

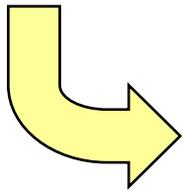
$$\frac{5}{4} \frac{1}{21 \times 5}$$

$$\frac{80^2}{84} \frac{\alpha^2}{D^3} = \frac{\beta g \Delta T D}{3} - \nu \frac{80\alpha}{D^3}$$

$$\alpha^2 \frac{80^2}{84} + \nu \frac{80\alpha}{D^3} = \frac{\beta g \Delta T D}{3}$$

$$\begin{cases} \frac{V^2 D}{84} = \frac{\beta g \Delta T D}{3} - \frac{\nu V}{D} \\ \frac{VD}{40} = \frac{2\alpha}{D} \end{cases}$$

$$\frac{D^4}{y^3} = 240 \left( \frac{20}{21} + \frac{\nu}{\alpha} \right) \frac{1}{Gr(y)} \underbrace{\frac{\alpha^2}{\nu^2}}_{Pr = 2}$$



$$D^4 = 240L^3 \left( \frac{20}{21} + \frac{\nu}{\alpha} \right) \frac{\nu^2}{\beta g \Delta T L^3} \frac{\alpha^2}{\nu^2}$$

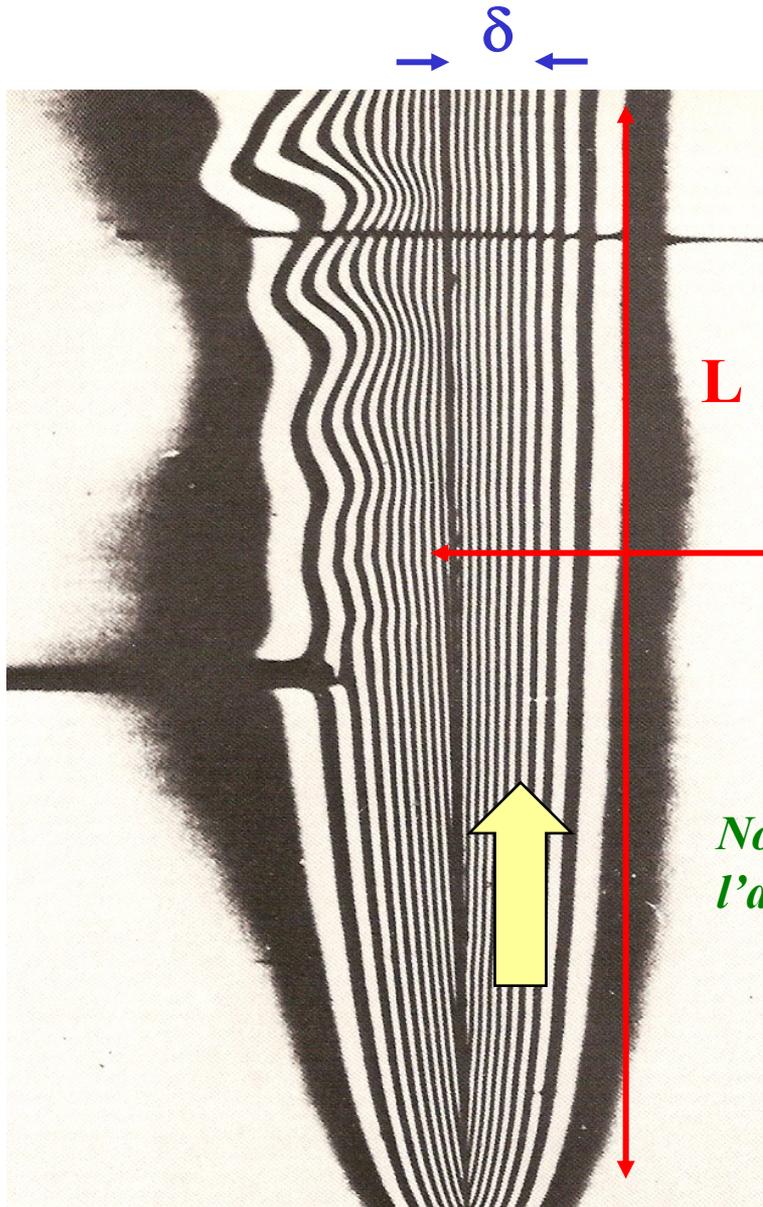
$$\frac{S}{y} = \frac{Dg^{1/4}}{y}$$

**Solution ...**

$$\frac{S}{y} = \left( \frac{D^4}{y^3} \right)^{1/4} \leftarrow \frac{S^4}{y^4} = \frac{D^4 y}{y^4}$$

$$\frac{\delta_T(y)}{y} = 3,936 \left( Pr \right)^{-1/2} \left( Gr(y) \right)^{-1/4} \left( \frac{20}{21} + Pr \right)^{1/4}$$

*On peut ensuite calculer le profil de vitesse et de température, le transfert de chaleur et des nombres de Nusselt locaux et moyens.*



Est-ce que cette  
solution est  
stable ?

*Plaque  
chaude*

*Non, après une certaine distance, on constate  
l'apparition d'instabilités : c'est la turbulence !*

*Cela, c'est très très très très compliqué.*

Grégoire est de  
retour en  
Belgique !

