Incompressible turbulent flows: equations for the Reynolds-averaged velocity and temperature fields, and simple cases: channel and boundary layer

Grégoire Winckelmans

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NS equations (incompressible)

$$\frac{\partial u_j}{\partial x_j} = 0$$

$$d_{ij} \overset{\text{def}}{=} \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$\sigma_{ij} \overset{\text{def}}{=} -\, p\,\delta_{ij} + \tau_{ij} \qquad \tau_{ij} = 2\,\mu\, d_{ij}$$

$$\begin{aligned} \rho\left(\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j}(u_i\,u_j)\right) &= \frac{\partial \sigma_{ij}}{\partial x_j} \\ &= -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} \\ &= -\frac{\partial p}{\partial x_i} + \mu\,\frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial x_j} \end{aligned}$$

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j\,\frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + \mu\,\frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial x_j}$$

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Reynolds averaging

$$\overline{\phi}(t) \stackrel{\text{def}}{=} \frac{1}{T} \int_{t-T/2}^{t+T/2} \phi(\tau)\, d\tau$$

time window used for the averaging

$$T_f \ll T \ll T_v$$

fast time scale of the turbulent fluctuations

slow time scale of interest

$$\phi = \overline{\phi} + \phi'$$

$$\psi = \overline{\psi} + \psi'$$

$$\overline{\phi\,\psi} = \overline{\phi}\,\overline{\psi} + \overline{\phi'\,\psi'}$$

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Reynolds averaged NS (RANS) equations

$$u_i = \overline{u}_i + u_i' \qquad p = \overline{p} + p'$$

$$u_j = \overline{u}_j + u_j' \qquad d_{ij} = \overline{d}_{ij} + d_{ij}'$$

$$\tau_{ij} = \overline{\tau}_{ij} + \tau_{ij}' \qquad \overline{\tau}_{ij} = 2\,\mu\,\overline{d}_{ij}$$

$$\overline{u_i\,u_j} = \overline{u}_i\,\overline{u}_j + \overline{u_i'\,u_j'} \qquad \overline{\sigma}_{ij} = -\overline{p}\,\delta_{ij} + \overline{\tau}_{ij}$$

$$\frac{\partial \overline{u}_j}{\partial x_j} = 0$$

$$\rho\left(\frac{\partial \overline{u}_i}{\partial t} + \frac{\partial}{\partial x_j}(\overline{u}_i\,\overline{u}_j)\right) = -\frac{\partial \overline{p}}{\partial x_i} + \frac{\partial \overline{\tau}_{ij}}{\partial x_j} - \rho\,\frac{\partial}{\partial x_j}\left(\overline{u_i'\,u_j'}\right)$$

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Reynolds averaged NS (RANS) equations

Reynolds stress : $\overline{\sigma}^t_{ij} \stackrel{\text{def}}{=} -\rho\,\overline{u'_i\,u'_j}$

turbulent kinetic energy (TKE) : $\overline{k} \stackrel{\text{def}}{=} \dfrac{\overline{u'_k\,u'_k}}{2}$

$$\overline{\sigma}^t_{kk} = -\rho\,\overline{u'_k\,u'_k} = -2\,\rho\,\overline{k}$$

turbulent shear stress : $\overline{\tau}^t_{ij} \stackrel{\text{def}}{=} \overline{\sigma}^t_{ij} - \dfrac{1}{3}\,\overline{\sigma}^t_{kk}\,\delta_{ij} = \overline{\sigma}^t_{ij} + \dfrac{2}{3}\,\rho\,\overline{k}\,\delta_{ij}$

$$\rho\left(\frac{\partial \overline{u}_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\overline{u}_i\,\overline{u}_j\right)\right) = \frac{\partial}{\partial x_j}\left(\overline{\sigma}_{ij} + \overline{\sigma}^t_{ij}\right)$$

$$\rho\left(\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j\,\frac{\partial \overline{u}_i}{\partial x_j}\right) = -\frac{\partial}{\partial x_i}\left(\overline{p} + \frac{2}{3}\,\rho\,\overline{k}\right) + \frac{\partial}{\partial x_j}\left(\overline{\tau}_{ij} + \overline{\tau}^t_{ij}\right)$$

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Effective « turbulent viscosity » model

$$\rho \left(\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} \right) = -\frac{\partial}{\partial x_i} \left(\overline{p} + \frac{2}{3} \rho \overline{k} \right) + \frac{\partial}{\partial x_j} \left(\overline{\tau}_{ij} + \overline{\tau}^t_{ij} \right)$$

model the turbulent shear stress using an effective « turbulent viscosity » :

$$\overline{\tau}^t_{ij} \stackrel{\text{model}}{=} 2\mu_t \, \overline{d}_{ij}$$

effective pressure: $\overline{p}_{\text{eff}} \stackrel{\text{def}}{=} \overline{p} + \frac{2}{3} \rho \overline{k}$

$$\rho \left(\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} \right) = -\frac{\partial \overline{p}_{\text{eff}}}{\partial x_i} + \frac{\partial}{\partial x_j} \left(2 \left(\mu + \mu_t \right) \overline{d}_{ij} \right)$$

We still need to model μ_t **!**

Momentum equation divided by the density

kinematic viscosity : $\nu \stackrel{\text{def}}{=} \dfrac{\mu}{\rho}$

turbulent kinematic viscosity : $\nu_t \stackrel{\text{def}}{=} \dfrac{\mu_t}{\rho}$

kinematic pressure : $\overline{P} \stackrel{\text{def}}{=} \dfrac{\overline{p}}{\rho}$

effective kinematic pressure : $\overline{P}_{\text{eff}} \stackrel{\text{def}}{=} \overline{P} + \dfrac{2}{3}\,\overline{k}$

$$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} = -\frac{\partial \overline{P}_{\text{eff}}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(2\,(\nu + \nu_t)\overline{d}_{ij}\right)$$

We still need to model ν_t !

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Temperature equation (incompressible)

$$q_j = -k \, \frac{\partial T}{\partial x_j}$$

$$\begin{aligned} \rho \, c \left(\frac{\partial T}{\partial t} + \frac{\partial}{\partial x_j} (T \, u_j) \right) &= \tau_{ij} \, d_{ij} - \frac{\partial q_j}{\partial x_j} \\ &= 2\mu \, d_{ij} \, d_{ij} + k \, \frac{\partial}{\partial x_j} \frac{\partial T}{\partial x_j} \end{aligned}$$

$$\rho \, c \left(\frac{\partial T}{\partial t} + u_j \, \frac{\partial T}{\partial x_j} \right) = 2\mu \, d_{ij} \, d_{ij} + k \, \frac{\partial}{\partial x_j} \frac{\partial T}{\partial x_j}$$

Reynolds averaged temperature equation

$$T = \overline{T} + T'$$

$$u_j = \overline{u}_j + u'_j \qquad\qquad d_{ij} = \overline{d}_{ij} + d'_{ij}$$

$$\overline{T\,u_j} = \overline{T}\,\overline{u}_j + \overline{T'\,u'_j} \qquad\qquad \overline{d_{ij}\,d_{ij}} = \overline{d}_{ij}\,\overline{d}_{ij} + \overline{d'_{ij}\,d'_{ij}}$$

$$\rho\,c \left(\frac{\partial \overline{T}}{\partial t} + \frac{\partial}{\partial x_j} (\overline{T}\,\overline{u}_j) \right) = 2\mu\, \left(\overline{d}_{ij}\,\overline{d}_{ij} + \overline{d'_{ij}\,d'_{ij}} \right) - \frac{\partial \overline{q}_j}{\partial x_j} - \rho\,c\,\frac{\partial}{\partial x_j} \left(\overline{T'\,u'_j} \right)$$

turbulent heat flux density : $\overline{q}^t_j \stackrel{\text{def}}{=} \rho\,c\,\overline{T'\,u'_j}$

rate of dissipation of the TKE : $\rho\,\overline{\epsilon} \stackrel{\text{def}}{=} 2\mu\,\overline{d'_{ij}\,d'_{ij}}$

$$\rho\,c \left(\frac{\partial \overline{T}}{\partial t} + \overline{u}_j\,\frac{\partial \overline{T}}{\partial x_j} \right) = 2\mu\,\overline{d}_{ij}\,\overline{d}_{ij} + \rho\,\overline{\epsilon} - \frac{\partial}{\partial x_j} \left(\overline{q}_j + \overline{q}^t_j \right)$$

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Effective « turbulent thermal conductivity » model

$$\rho\, c \left(\frac{\partial \overline{T}}{\partial t} + \overline{u}_j \frac{\partial \overline{T}}{\partial x_j} \right) = 2\mu\, \overline{d}_{ij}\, \overline{d}_{ij} + \rho\, \overline{\epsilon} - \frac{\partial}{\partial x_j} \left(\overline{q}_j + \overline{q}_j^t \right)$$

model the turbulent heat flux density using an effective « turbulent thermal conductivity » :

$$\overline{q}_j^t \stackrel{\text{model}}{=} -\, k_t \frac{\partial \overline{T}}{\partial x_j}$$

turbulence « at equilibrium » : rate of dissipation of TKE = rate of production of TKE (see the TKE equation; not developed in this course) :

$$\rho\, \overline{\epsilon} = \overline{\tau}_{ij}^t\, \overline{d}_{ij} = 2\mu_t\, \overline{d}_{ij}\, \overline{d}_{ij}$$

$$\rho\, c \left(\frac{\partial \overline{T}}{\partial t} + \overline{u}_j \frac{\partial \overline{T}}{\partial x_j} \right) = 2\left(\mu + \mu_t\right) \overline{d}_{ij}\, \overline{d}_{ij} + \frac{\partial}{\partial x_j} \left((k + k_t) \frac{\partial \overline{T}}{\partial x_j} \right)$$

turbulent Prandtl number : $Pr_t \stackrel{\text{def}}{=} \dfrac{\mu_t\, c}{k_t}$ **assumed constant !**

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Temperature equation divided by the density and the specific heat

thermal diffusivity : $\alpha \stackrel{\text{def}}{=} \dfrac{k}{\rho\, c}$

turbulent thermal diffusivity : $\alpha_t \stackrel{\text{def}}{=} \dfrac{k_t}{\rho\, c}$

$$\frac{\partial \overline{T}}{\partial t} + \overline{u}_j \frac{\partial \overline{T}}{\partial x_j} = \frac{2}{c}\,(\nu + \nu_t)\,\overline{d}_{ij}\,\overline{d}_{ij} + \frac{\partial}{\partial x_j}\left((\alpha + \alpha_t)\frac{\partial \overline{T}}{\partial x_j}\right)$$

Prandtl number : $Pr \stackrel{\text{def}}{=} \dfrac{\mu\, c}{k} = \dfrac{\nu}{\alpha}$

turbulent Prandtl number : $Pr_t \stackrel{\text{def}}{=} \dfrac{\mu_t\, c}{k_t} = \dfrac{\nu_t}{\alpha_t}$ **assumed constant !**

$$\frac{\partial \overline{T}}{\partial t} + \overline{u}_j \frac{\partial \overline{T}}{\partial x_j} = \frac{2}{c}\,(\nu + \nu_t)\,\overline{d}_{ij}\,\overline{d}_{ij} + \frac{\partial}{\partial x_j}\left(\left(\frac{\nu}{Pr} + \frac{\nu_t}{Pr_t}\right)\frac{\partial \overline{T}}{\partial x_j}\right)$$

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Steady and 2-D mean flows

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} = 0$$

$$\frac{\partial}{\partial x}\left(\overline{u}\,\overline{u} + \overline{u'u'}\right) + \frac{\partial}{\partial y}\left(\overline{u}\,\overline{v} + \overline{u'v'}\right) = -\frac{\partial \overline{P}}{\partial x} + \nu\left(\frac{\partial^2 \overline{u}}{\partial x^2} + \frac{\partial^2 \overline{u}}{\partial y^2}\right)$$

$$\frac{\partial}{\partial x}\left(\overline{v}\,\overline{u} + \overline{v'u'}\right) + \frac{\partial}{\partial y}\left(\overline{v}\,\overline{v} + \overline{v'v'}\right) = -\frac{\partial \overline{P}}{\partial y} + \nu\left(\frac{\partial^2 \overline{v}}{\partial x^2} + \frac{\partial^2 \overline{v}}{\partial y^2}\right)$$

$$\overline{u}\,\frac{\partial \overline{u}}{\partial x} + \overline{v}\,\frac{\partial \overline{u}}{\partial y} = -\frac{\partial \overline{P}}{\partial x} + \frac{\partial}{\partial x}\left(\nu\,\frac{\partial \overline{u}}{\partial x} - \overline{u'u'}\right) + \frac{\partial}{\partial y}\left(\nu\,\frac{\partial \overline{u}}{\partial y} - \overline{u'v'}\right)$$

$$\overline{u}\,\frac{\partial \overline{v}}{\partial x} + \overline{v}\,\frac{\partial \overline{v}}{\partial y} = -\frac{\partial \overline{P}}{\partial y} + \frac{\partial}{\partial x}\left(\nu\,\frac{\partial \overline{v}}{\partial x} - \overline{u'v'}\right) + \frac{\partial}{\partial y}\left(\nu\,\frac{\partial \overline{v}}{\partial y} - \overline{v'v'}\right)$$

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Steady and 2-D mean flows: model

$$\frac{\overline{\tau}^t_{xx}}{\rho} = \frac{\overline{\sigma}^t_{xx}}{\rho} + \frac{2}{3}\,\overline{k} = -\overline{u'u'} + \frac{2}{3}\,\overline{k} \overset{\text{model}}{=} 2\nu_t\,\frac{\partial \overline{u}}{\partial x}$$

$$\frac{\overline{\tau}^t_{yy}}{\rho} = \frac{\overline{\sigma}^t_{yy}}{\rho} + \frac{2}{3}\,\overline{k} = -\overline{v'v'} + \frac{2}{3}\,\overline{k} \overset{\text{model}}{=} 2\nu_t\,\frac{\partial \overline{v}}{\partial y}$$

$$\frac{\overline{\tau}^t_{xy}}{\rho} = \frac{\overline{\sigma}^t_{xy}}{\rho} = -\overline{u'v'} \overset{\text{model}}{=} \nu_t\left(\frac{\partial \overline{u}}{\partial y} + \frac{\partial \overline{v}}{\partial x}\right)$$

$$\overline{u}\,\frac{\partial \overline{u}}{\partial x} + \overline{v}\,\frac{\partial \overline{u}}{\partial y} = -\frac{\partial \overline{P}_{\text{eff}}}{\partial x} + \frac{\partial}{\partial x}\left(2\left(\nu+\nu_t\right)\frac{\partial \overline{u}}{\partial x}\right) + \frac{\partial}{\partial y}\left(\left(\nu+\nu_t\right)\left(\frac{\partial \overline{u}}{\partial y} + \frac{\partial \overline{v}}{\partial x}\right)\right)$$

$$\overline{u}\,\frac{\partial \overline{v}}{\partial x} + \overline{v}\,\frac{\partial \overline{v}}{\partial y} = -\frac{\partial \overline{P}_{\text{eff}}}{\partial y} + \frac{\partial}{\partial x}\left(\left(\nu+\nu_t\right)\left(\frac{\partial \overline{v}}{\partial x} + \frac{\partial \overline{u}}{\partial y}\right)\right) + \frac{\partial}{\partial y}\left(2\left(\nu+\nu_t\right)\frac{\partial \overline{v}}{\partial y}\right)$$

Steady and 2-D mean flows: model

$$\frac{\partial}{\partial x}\left(\overline{T}\,\overline{u}+\overline{T'u'}\right)+\frac{\partial}{\partial y}\left(\overline{T}\,\overline{v}+\overline{T'v'}\right)=\frac{2}{c}\,\nu\left(\overline{d}_{ij}\,\overline{d}_{ij}+\overline{d'_{ij}d'_{ij}}\right)+\alpha\left(\frac{\partial^2\overline{T}}{\partial x^2}+\frac{\partial^2\overline{T}}{\partial y^2}\right)$$

$$\overline{u}\,\frac{\partial\overline{T}}{\partial x}+\overline{v}\,\frac{\partial\overline{T}}{\partial y}=\frac{2}{c}\,\nu\left(\overline{d}_{ij}\,\overline{d}_{ij}+\overline{d'_{ij}d'_{ij}}\right)+\frac{\partial}{\partial x}\left(\alpha\,\frac{\partial\overline{T}}{\partial x}-\overline{T'u'}\right)+\frac{\partial}{\partial y}\left(\alpha\,\frac{\partial\overline{T}}{\partial y}-\overline{T'v'}\right)$$

model

$$\overline{u}\,\frac{\partial\overline{T}}{\partial x}+\overline{v}\,\frac{\partial\overline{T}}{\partial y}=\frac{2}{c}\,(\nu+\nu_t)\,\overline{d}_{ij}\,\overline{d}_{ij}+\frac{\partial}{\partial x}\left((\alpha+\alpha_t)\,\frac{\partial\overline{T}}{\partial x}\right)+\frac{\partial}{\partial y}\left((\alpha+\alpha_t)\,\frac{\partial\overline{T}}{\partial y}\right)$$

with $\overline{d}_{ij}\,\overline{d}_{ij}=\left(\frac{\partial\overline{u}}{\partial x}\right)^2+2\left(\frac{1}{2}\left(\frac{\partial\overline{u}}{\partial y}+\frac{\partial\overline{v}}{\partial x}\right)\right)^2+\left(\frac{\partial\overline{v}}{\partial y}\right)^2$

$$\alpha=\frac{\nu}{Pr}\qquad\qquad \alpha_t=\frac{\nu_t}{Pr_t}$$

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Fully developed turbulent channel flow

$$\overline{v} = 0$$

$\overline{u}\,,\overline{u'u'}\,,\overline{v'v'}\,,\overline{w'w'}\,,\overline{k}\,,\overline{u'v'}$ **are solely functions of** y

$$0 = -\frac{\partial \overline{P}}{\partial x} + \frac{d}{dy}\left(\nu\,\frac{d\overline{u}}{dy} - \overline{u'v'}\right)$$

also : $\dfrac{\partial \overline{P}}{\partial x} = \dfrac{\partial \overline{P}_{\text{eff}}}{\partial x}$

$$0 = -\frac{\partial \overline{P}}{\partial y} - \frac{d}{dy}\left(\overline{v'v'}\right)$$

$$\overline{P} + \overline{v'v'} = \overline{P}_w(x)$$

$$\frac{\partial \overline{P}}{\partial x} = \frac{d\overline{P}_w}{dx}$$

$$\frac{d}{dy}\left(\nu\,\frac{d\overline{u}}{dy} - \overline{u'v'}\right) = \frac{d\overline{P}_w}{dx} = \text{Constant}$$

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Fully developed turbulent channel flow

$$\frac{d}{dy}\left(\nu\,\frac{d\overline{u}}{dy} - \overline{u'v'}\right) = \frac{d\overline{P}_w}{dx} = \text{Constant}$$

Note: y **is here measured from the bottom wall**

$$\nu\,\frac{d\overline{u}}{dy} - \overline{u'v'} = -\frac{d\overline{P}_w}{dx}\,(h-y) = -\frac{d\overline{P}_w}{dx}\,h\left(1-\frac{y}{h}\right)$$

$$\mu\,\frac{d\overline{u}}{dy} - \rho\,\overline{u'v'} = -\frac{d\overline{p}_w}{dx}\,h\left(1-\frac{y}{h}\right)$$

$$\overline{\tau}_{xy} + \overline{\tau}^t_{xy} = -\frac{d\overline{p}_w}{dx}\,h\left(1-\frac{y}{h}\right)$$

The profile of the total shear stress (molecular + turbulent) is linear !

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Fully developed turbulent channel flow

$$\overline{\tau}_{xy}(y) + \overline{\tau}^t_{xy}(y) = -\frac{d\overline{p}_w}{dx}\, h \left(1 - \frac{y}{h}\right)$$

short notation : $\overline{\tau}(y) = \overline{\tau}_{xy}(y)$

$$\overline{\tau}^t(y) = \overline{\tau}^t_{xy}(y)$$

$$\overline{\tau}_w = -\frac{d\overline{p}_w}{dx}\, h$$

$$\overline{\tau}(y) + \overline{\tau}^t(y) = \overline{\tau}_w \left(1 - \frac{y}{h}\right)$$

friction velocity : $\overline{u}_\tau \stackrel{\text{def}}{=} \sqrt{\frac{\overline{\tau}_w}{\rho}}$

$$\nu \frac{d\overline{u}}{dy}(y) - \overline{u'v'}(y) = \overline{u}^2_\tau \left(1 - \frac{y}{h}\right)$$

$$\nu \frac{d\overline{u}}{dy}(y) + \nu_t(y) \frac{d\overline{u}}{dy}(y) = \overline{u}^2_\tau \left(1 - \frac{y}{h}\right)$$

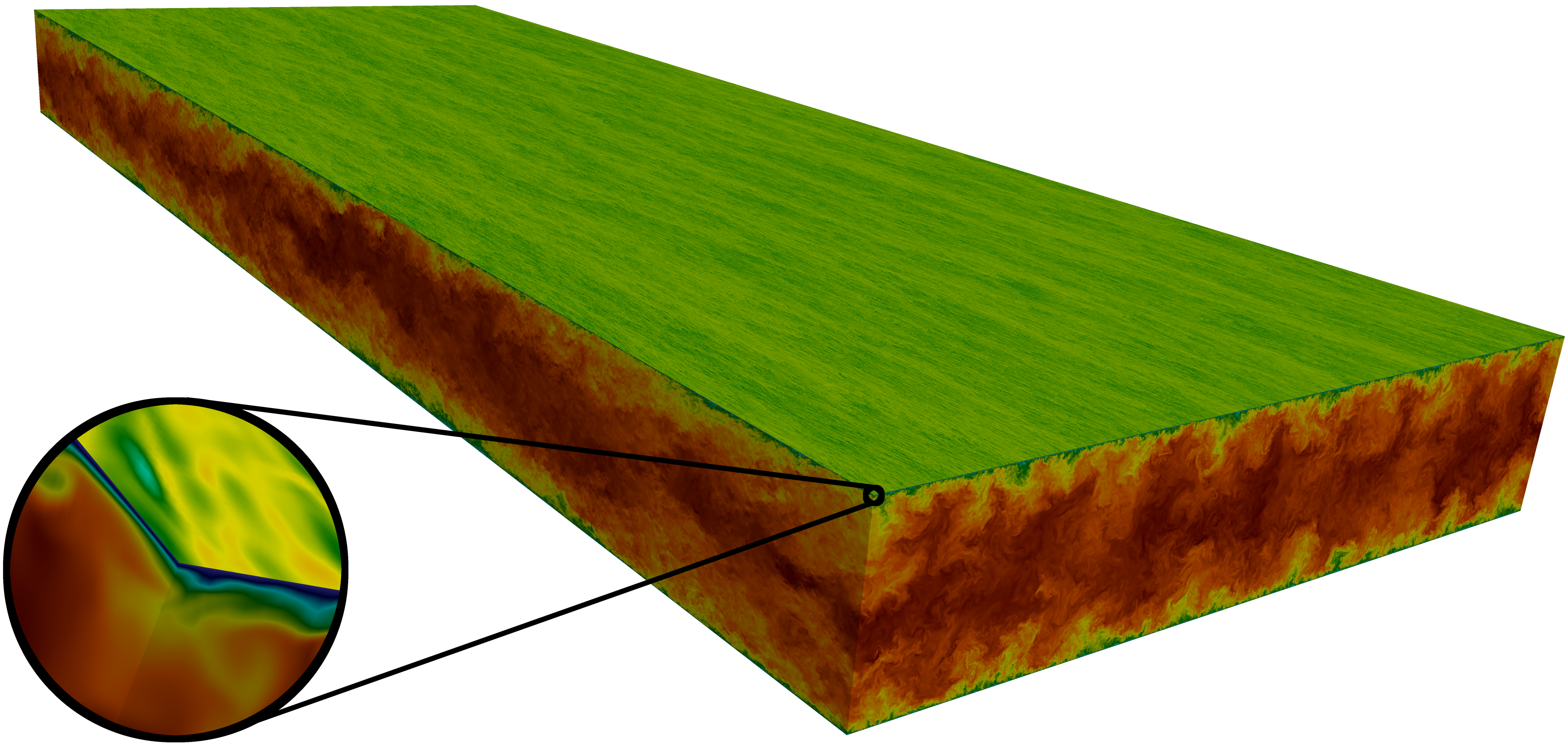
High Re channel flow simulation

$$Re_\tau \stackrel{\text{def}}{=} \frac{h\,\overline{u}_\tau}{\nu} = 5200$$

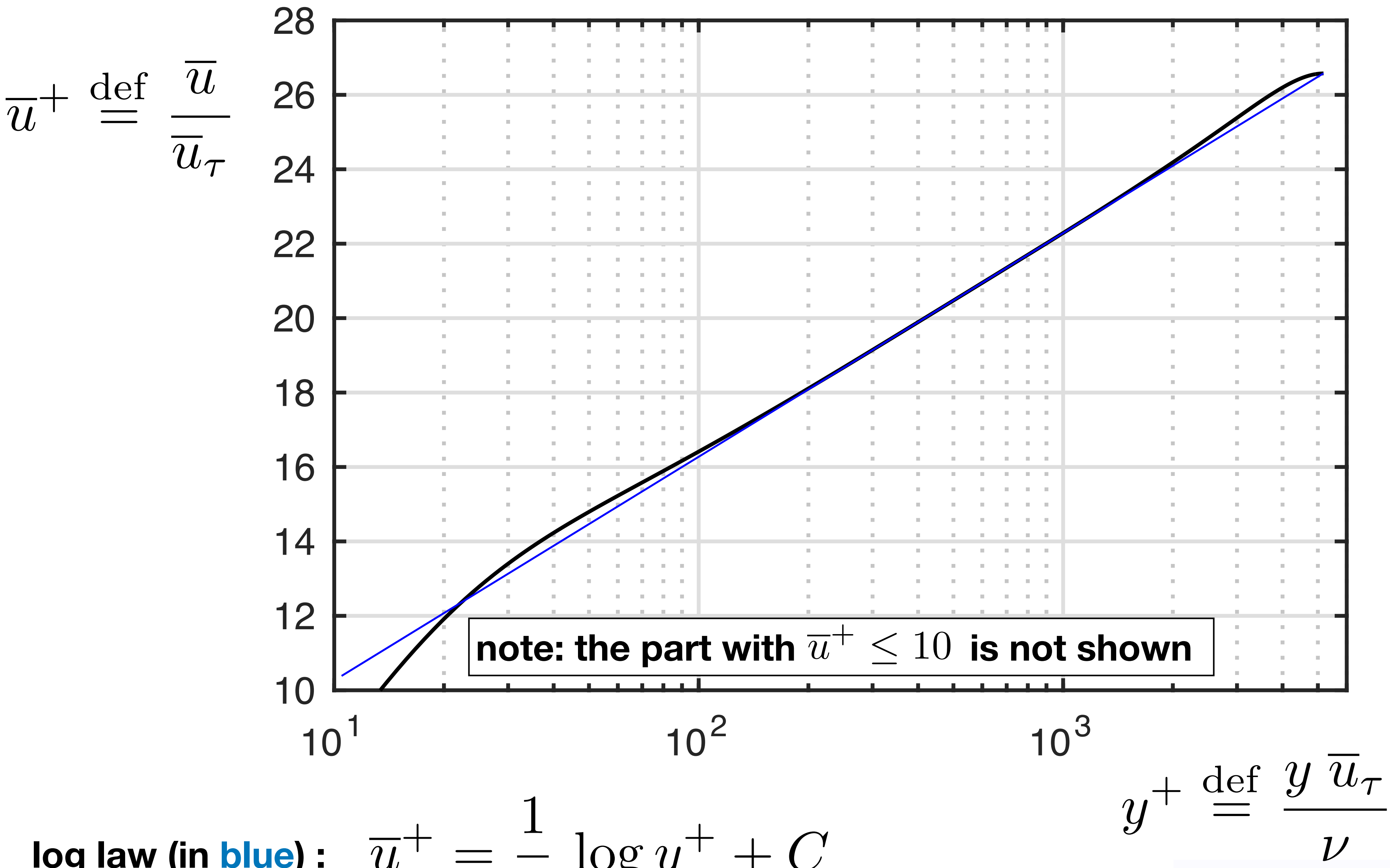
Lee and Moser 2015

DNS = Direct Numerical Simulation

all the scales are resolved !

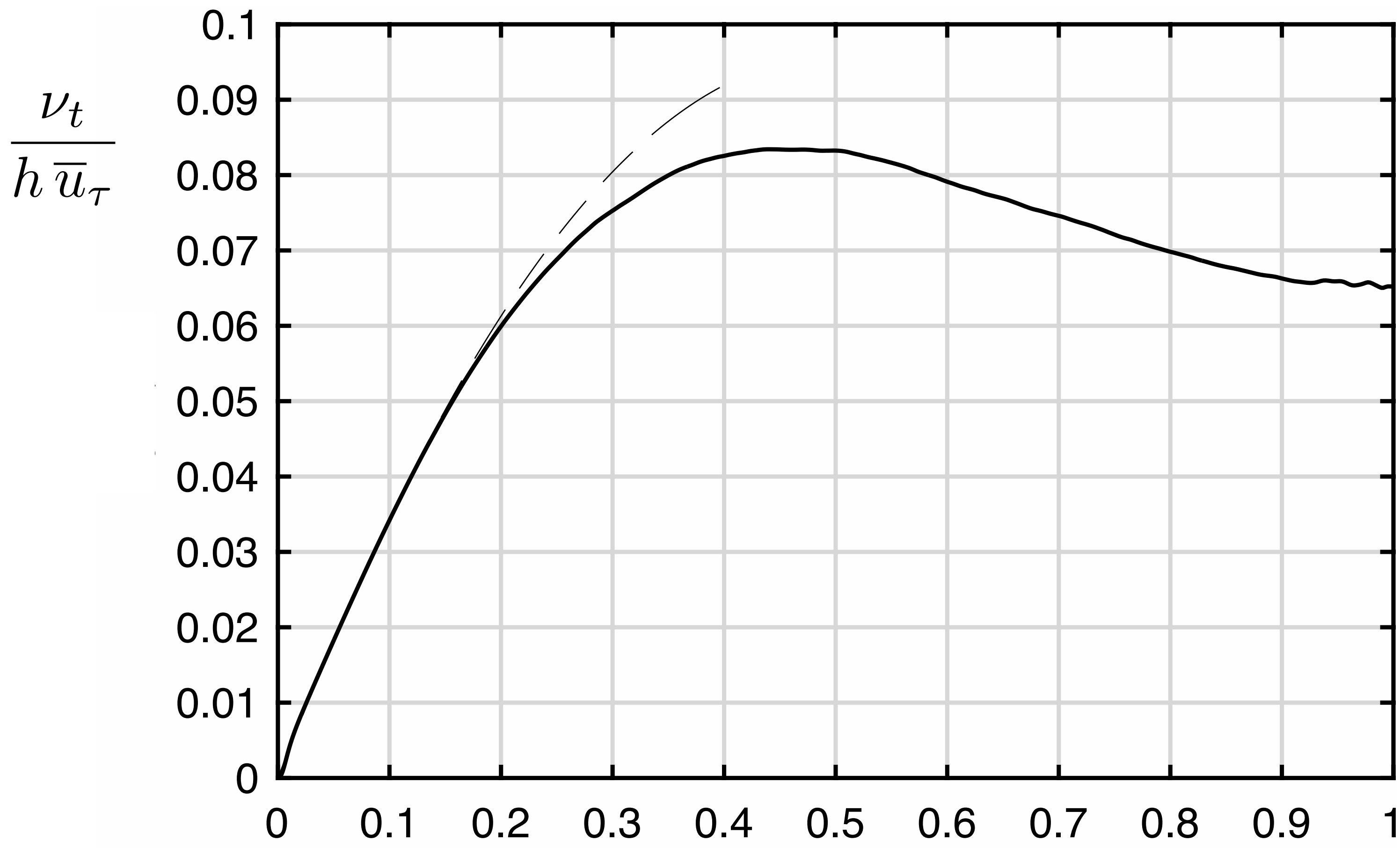


velocity profile obtained from the DNS



log law (in blue) : $\overline{u}^+ = \dfrac{1}{\kappa} \log y^+ + C$

turbulent viscosity obtained from the DNS



$$\eta = \frac{y}{h}$$

behaviour in the near wall log layer (in dash):

$$\frac{\nu_t}{h\,\overline{u}_\tau} = \kappa\,\frac{y}{h}\,\left(1 - \frac{y}{h}\right)$$

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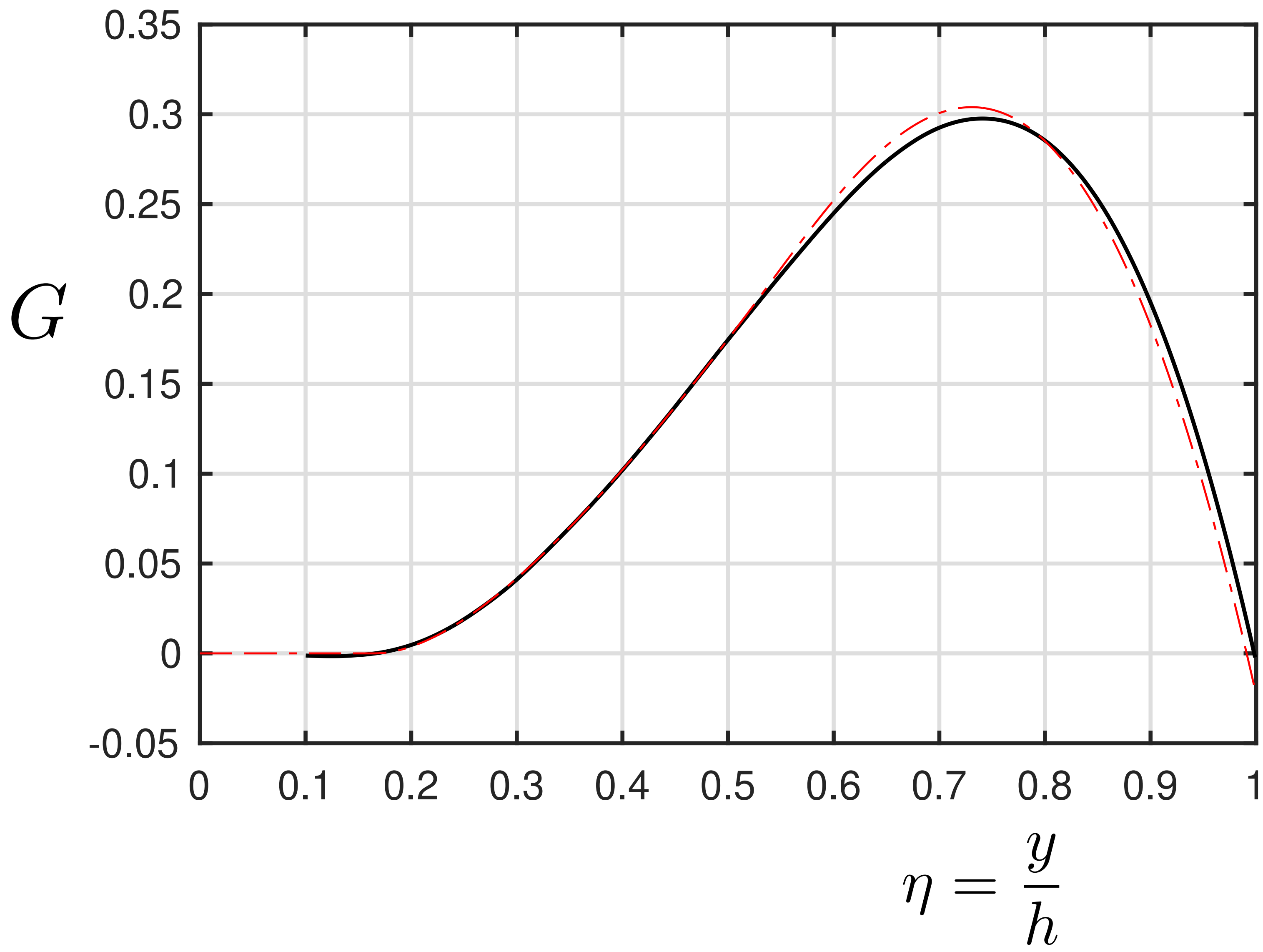
wake function obtained from the DNS



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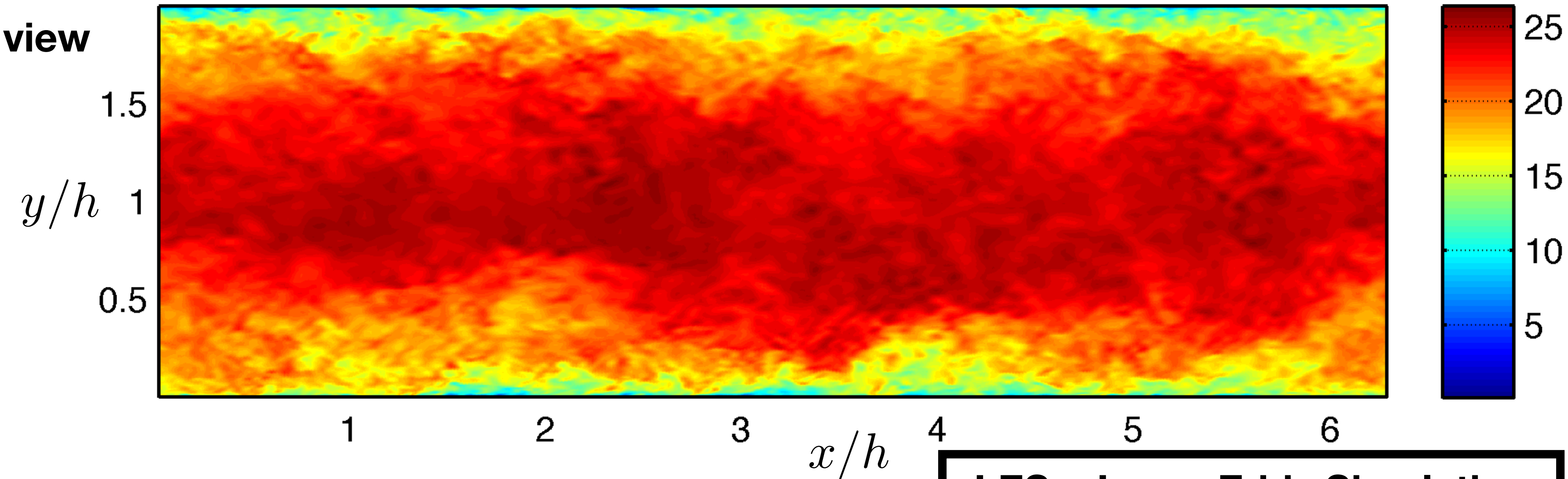
red : model by Winckelmans and Duponcheel, PhysRev Fluids (2021)

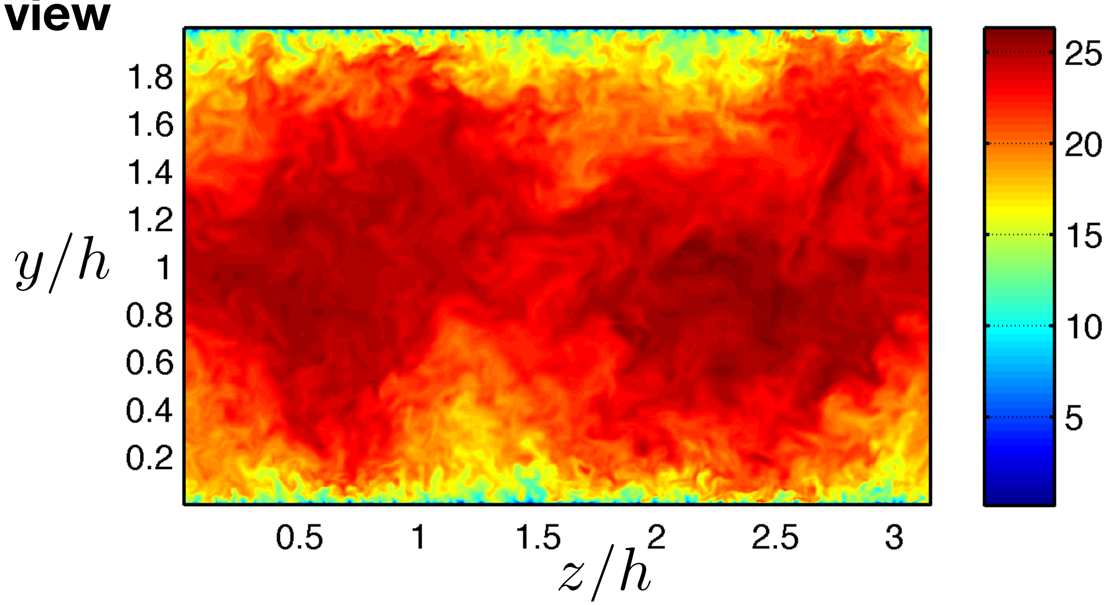
wall-resolved LES of a channel flow

snapshot of instantaneous u field $Re_\tau \stackrel{\text{def}}{=} \frac{h\,\overline{u}_\tau}{\nu} = 2000 \quad Re_d \stackrel{\text{def}}{=} \frac{(2\,h)\,\overline{u}_m}{\nu} \simeq 87200$

side view



cross view



LES = Large Eddy Simulation

not all the scales are resolved, yet many scales are !

hence the need to model the effect of the sub-grid scales (SGS) on the resolved scales

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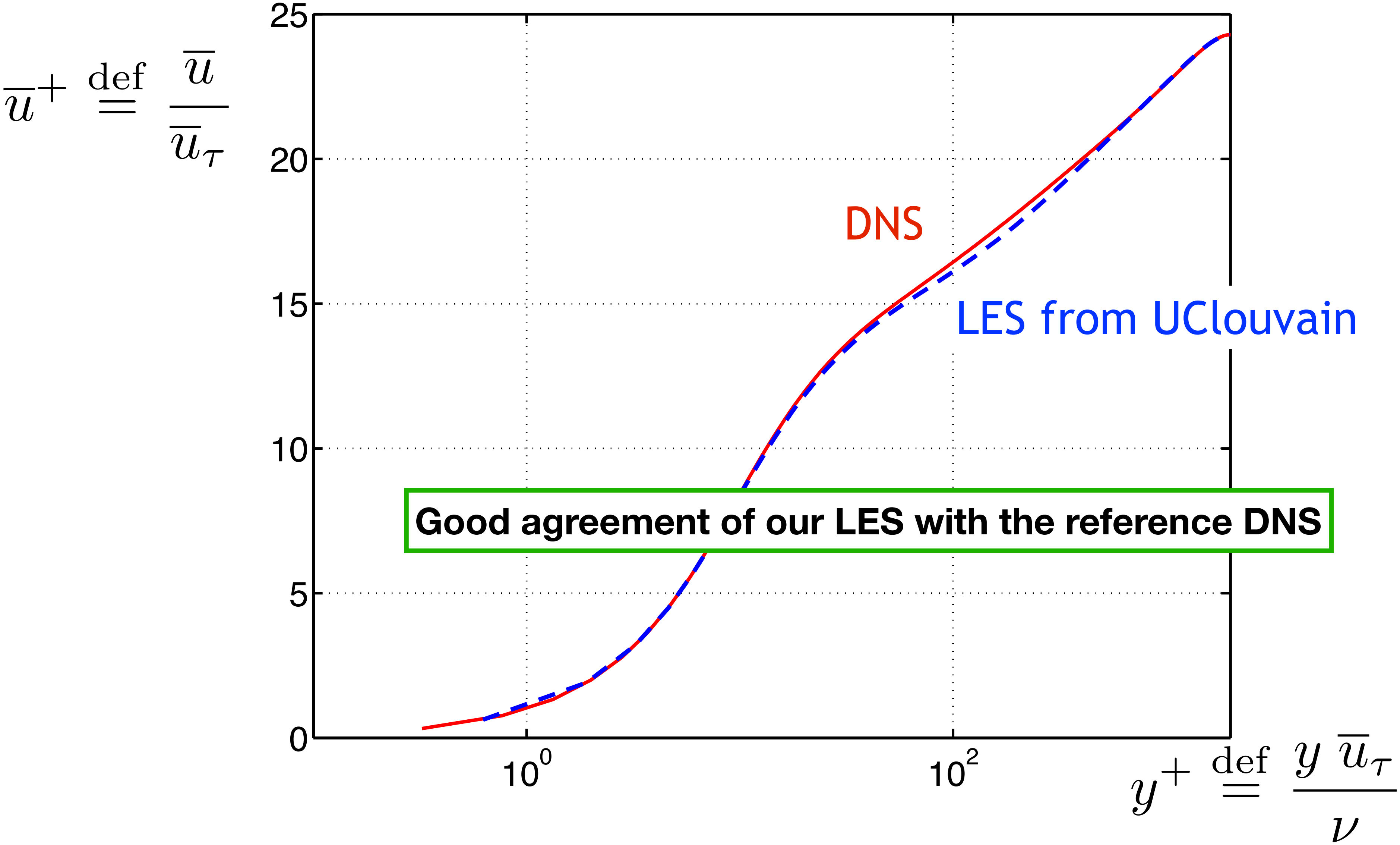
LES of a channel flow: velocity profile



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Thermally developed turbulent channel flow

mean flow velocity : $$\overline{u}_m = \frac{1}{h}\int_0^h \overline{u}(y)\,dy$$

mean flow temperature (cup mixing temperature) : $$\overline{T}_m = \frac{1}{\overline{u}_m\,h}\int_0^h \overline{T}(y)\,\overline{u}(y)\,dy$$

dimensionless temperature profile : $$\frac{(\overline{T}-\overline{T}_w)}{(\overline{T}_m-\overline{T}_w)}$$

thermally developed flow = this dimensionless profile is solely function of y:

$$\frac{\partial}{\partial x}\left(\frac{(\overline{T}-\overline{T}_w)}{(\overline{T}_m-\overline{T}_w)}\right) = 0$$

$$\Downarrow$$

$$\left(\frac{\partial\overline{T}}{\partial x} - \frac{d\overline{T}_w}{dx}\right) = \frac{(\overline{T}-\overline{T}_w)}{(\overline{T}_m-\overline{T}_w)}\left(\frac{d\overline{T}_m}{dx} - \frac{d\overline{T}_w}{dx}\right)$$

Thermally developed turbulent channel flow

$$\left(\frac{\partial \overline{T}}{\partial x} - \frac{d\overline{T}_w}{dx}\right) = \frac{(\overline{T} - \overline{T}_w)}{(\overline{T}_m - \overline{T}_w)} \left(\frac{d\overline{T}_m}{dx} - \frac{d\overline{T}_w}{dx}\right)$$

Case with constant heat flux density imposed at the wall :

$$\frac{\partial \overline{T}}{\partial x} = \frac{d\overline{T}_m}{dx} = \frac{d\overline{T}_w}{dx} = \text{Constant}$$

Case with constant temperature imposed at the wall :

$$\frac{\partial \overline{T}}{\partial x} = \frac{(\overline{T} - \overline{T}_w)}{(\overline{T}_m - \overline{T}_w)} \frac{d\overline{T}_m}{dx}$$

to put into the equation for the temperature profile :

$$\overline{u}\,\frac{\partial \overline{T}}{\partial x} = \frac{1}{c}\,(\nu + \nu_t)\,\left(\frac{\partial \overline{u}}{\partial y}\right)^2 + \frac{\partial}{\partial y}\left((\alpha + \alpha_t)\,\frac{\partial \overline{T}}{\partial y}\right)$$

Fully developed turbulent boundary layer

$\overline{v} \neq 0$ **but is much smaller than the external forcing velocity** $\overline{u}_e(x)$

$\overline{u}\;, \overline{u'u'}\;, \overline{v'v'}\;, \overline{w'w'}\;, \overline{k}\;, \overline{u'v'}$ **also vary in** x **but slowly**

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} = 0$$

$$\overline{u}\,\frac{\partial \overline{u}}{\partial x} + \overline{v}\,\frac{\partial \overline{u}}{\partial y} = -\frac{\partial \overline{P}}{\partial x} - \frac{\partial}{\partial x}\left(\overline{u'u'}\right) + \frac{\partial}{\partial y}\left(\nu\,\frac{\partial \overline{u}}{\partial y} - \overline{u'v'}\right)$$

$$0 = -\frac{\partial \overline{P}}{\partial y} - \frac{\partial}{\partial y}\left(\overline{v'v'}\right) \quad \Rightarrow \quad \overline{P} + \overline{v'v'} = \overline{P}_e(x)$$

$$\Rightarrow \quad \frac{\partial \overline{P}}{\partial x} = \frac{d\overline{P}_e}{dx} - \frac{\partial}{\partial x}\left(\overline{v'v'}\right)$$

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Fully developed turbulent boundary layer

$$\overline{u}\,\frac{\partial \overline{u}}{\partial x} + \overline{v}\,\frac{\partial \overline{u}}{\partial y} + \frac{\partial}{\partial x}\left(\overline{u'u'} - \overline{v'v'}\right) = -\frac{d\overline{P}_e}{dx} + \frac{\partial}{\partial y}\left(\nu\,\frac{\partial \overline{u}}{\partial y} - \overline{u'v'}\right)$$

Bernoulli equation : $\overline{P}_e(x) + \dfrac{\overline{u}_e^2(x)}{2} = \text{Constant}$

$$-\frac{d\overline{P}_e}{dx} = \overline{u}_e\,\frac{d\overline{u}_e}{dx}$$

$$\overline{u}\,\frac{\partial \overline{u}}{\partial x} + \overline{v}\,\frac{\partial \overline{u}}{\partial y} + \frac{\partial}{\partial x}\left(\overline{u'u'} - \overline{v'v'}\right) = \overline{u}_e\,\frac{d\overline{u}_e}{dx} + \frac{\partial}{\partial y}\left(\nu\,\frac{\partial \overline{u}}{\partial y} - \overline{u'v'}\right)$$

this is the « Townsend term »

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Fully developed turbulent boundary layer

The boundary layer equations for the velocity field are finally taken as :

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} = 0$$

$$\overline{u}\,\frac{\partial \overline{u}}{\partial x} + \overline{v}\,\frac{\partial \overline{u}}{\partial y} = \overline{u}_e\,\frac{d\overline{u}_e}{dx} + \frac{\partial}{\partial y}\left((\nu + \nu_t)\,\frac{\partial \overline{u}}{\partial y} \right)$$

(thus neglecting the variation in x of the Townsend term !)

As to the boundary layer equation for the temperature field, it is obtained as :

$$\overline{u}\,\frac{\partial \overline{T}}{\partial x} + \overline{v}\,\frac{\partial \overline{T}}{\partial y} = \frac{1}{c}\,(\nu + \nu_t)\,\left(\frac{\partial \overline{u}}{\partial y} \right)^2 + \frac{\partial}{\partial y}\left((\alpha + \alpha_t)\,\frac{\partial \overline{T}}{\partial y} \right)$$

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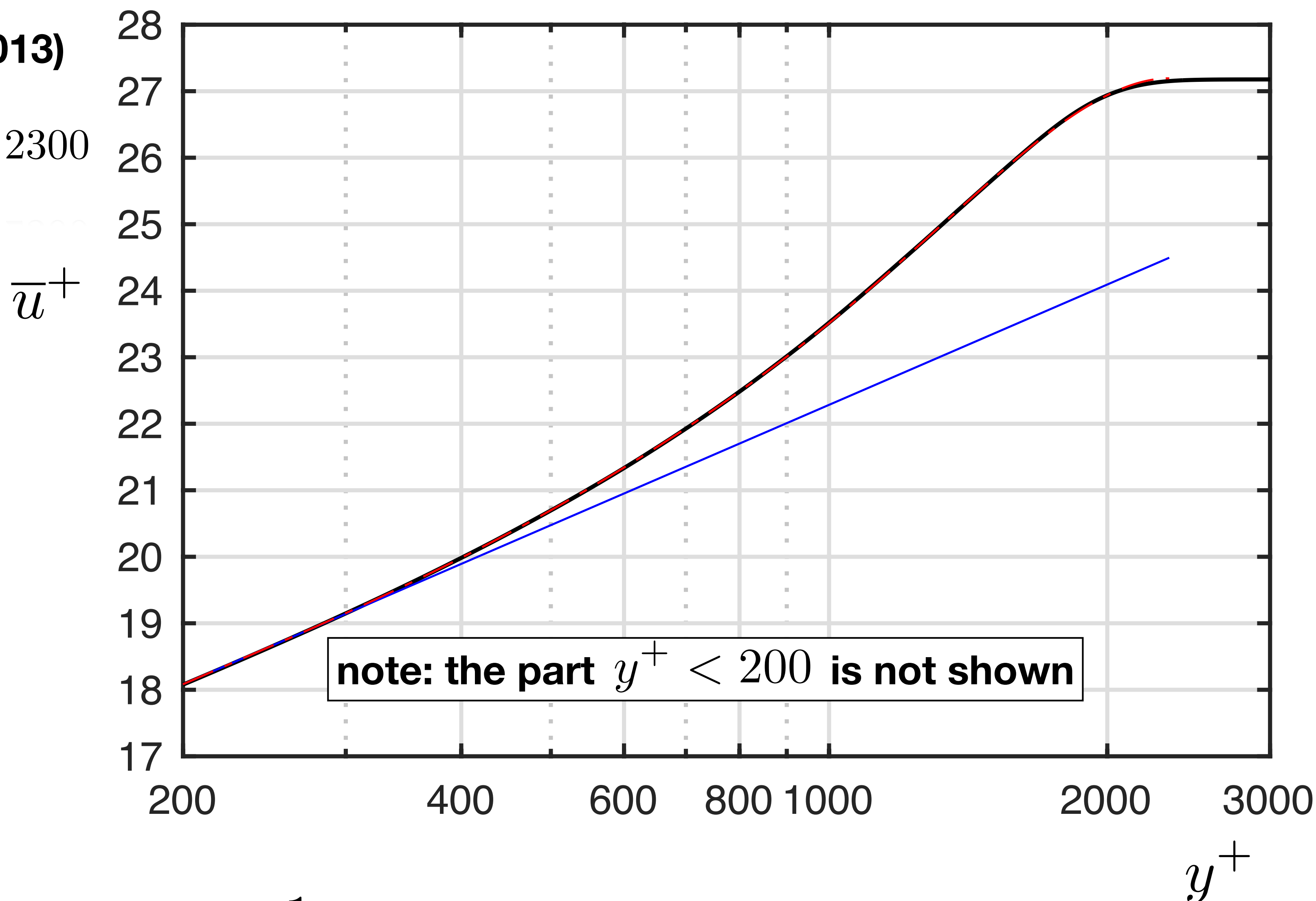
velocity profile obtained from a DNS

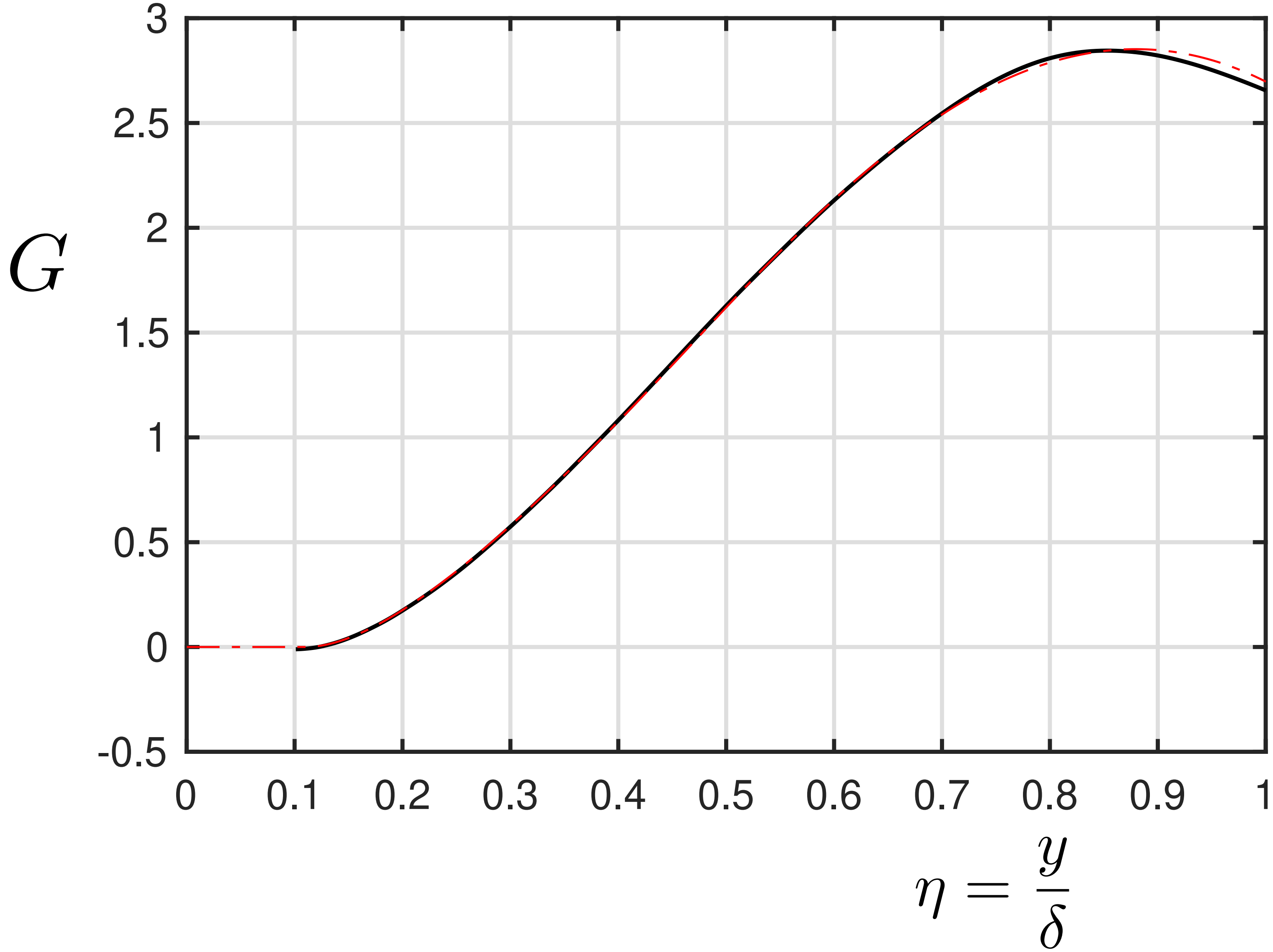
Sillero et al. (2013)

$$Re_\tau \stackrel{\text{def}}{=} \frac{\delta\, \overline{u}_\tau}{\nu} \simeq 2300$$



log law (in blue) : $$\overline{u}^+ = \frac{1}{\kappa} \log y^+ + C$$

wake function obtained from a DNS



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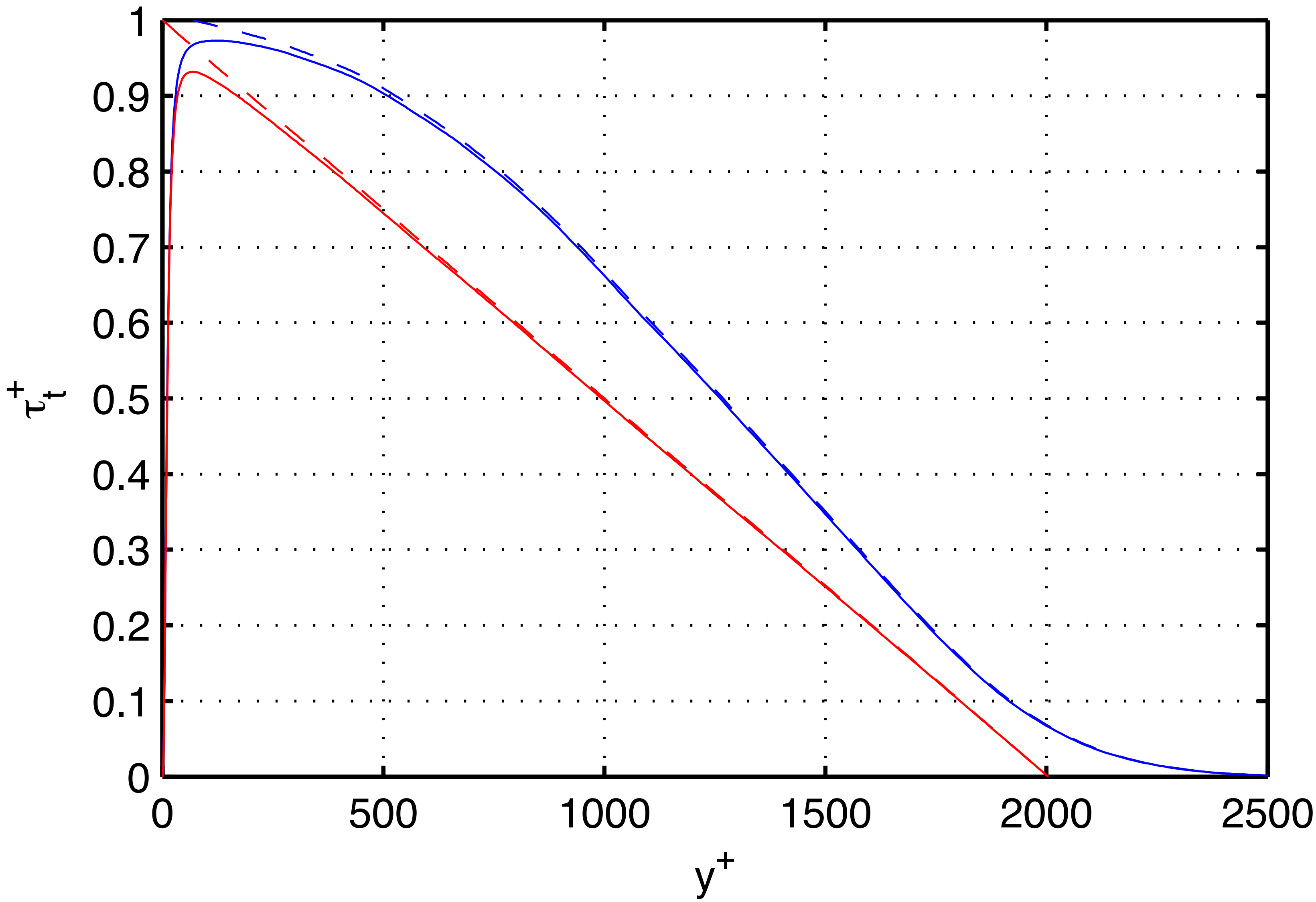
red : model by Winckelmans and Duponcheel, PhysRev Fluids (2021)

Turbulent shear stress profile: channel versus boundary layer



Adding gravity

- **Case with constant density**:
  - it suffices to include the hydrostatic pressure into the definition of the kinematic pressure:

$$P \stackrel{\text{def}}{=} \frac{(p + \rho\, g\, y)}{\rho} = \frac{p}{\rho} + g\, y$$

  - then nothing changes: we obtain the same Navier-Stokes equations for incompressible flows as before, and thus also the same RANS equations, and the same physics.

Adding gravity

- **Case with small variations of density** (yet still assuming that the flow remains incompressible = Boussinesq approximation):

$$\rho = \rho_0 \, (1 - \beta \, (T - T_0))$$

- we then define the kinematic pressure as:

$$P \stackrel{\text{def}}{=} \frac{(p + \rho_0 \, g \, y)}{\rho_0} = \frac{p}{\rho_0} + g \, y$$

- we obtain an added « buoyancy term » on the RHS of the momentum equation for v : $\beta \, (T - T_0) \, g$

- and thus also in the RANS equation for $\overline{v}$: $\beta \, (\overline{T} - T_0) \, g$

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