Lois de conservation

Forme conservative :

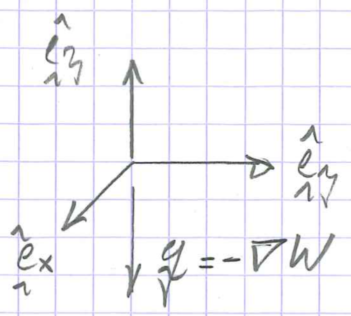
masse
$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho v_j) = 0$$

qté de mvt
$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_i v_j) = \frac{\partial}{\partial x_j}\sigma_{ij} + \rho g_i$$

(tenseur des contraintes : σ_{ij})

énergie
$$\frac{\partial}{\partial t}(\rho U) + \frac{\partial}{\partial x_j}(\rho U v_j) = \sigma_{ij} d_{ij} - \frac{\partial}{\partial x_j} q_j$$

d_{ij} : tenseur des taux de déformation ; q_j : densité de flux de chaleur



$$\underset{\sim}{g} = -\nabla W$$

$$W = g z$$

Aussi :
$$\frac{\partial}{\partial t}(\rho \varphi) + \frac{\partial}{\partial x_j}(\rho \varphi v_j) = \rho \frac{\partial \varphi}{\partial t}$$
$$= \rho \frac{\partial \varphi}{\partial t} + \rho v_j \frac{\partial \varphi}{\partial x_j} + \frac{\partial \rho}{\partial t}\varphi + \frac{\partial}{\partial x_j}(\rho v_j)\varphi$$
$$= \rho\left(\frac{\partial \varphi}{\partial t} + v_j \frac{\partial \varphi}{\partial x_j}\right) + \underbrace{\left(\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho v_j)\right)}_{= 0}\varphi$$

Forme advective :

$$\left(\frac{\partial \rho}{\partial t} + v_j \frac{\partial \rho}{\partial x_j}\right) + \rho \frac{\partial v_j}{\partial x_j} = 0$$

Note : $\frac{\partial v_j}{\partial x_j} = d_{ii} = \nabla \cdot \underset{\sim}{v}$

$$\rho\left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}\right) = \frac{\partial}{\partial x_j}\sigma_{ij} + \rho g_i$$

$$\rho\left(\frac{\partial U}{\partial t} + v_j \frac{\partial U}{\partial x_j}\right) = \sigma_{ij} d_{ij} - \frac{\partial}{\partial x_j} q_j$$

Aussi :
$$\sigma_{ij} = -p\,\delta_{ij} + \tau_{ij} \Rightarrow \frac{\partial}{\partial x_j}\sigma_{ij} = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\tau_{ij}$$

(p : pression ; τ_{ij} : tenseur des contraintes visqueuses)

et $\sigma_{ij} d_{ij} = -p\, d_{ii} + \tau_{ij} d_{ij}$

Fluide visqueux newtonien :

$$\tau_{ij} = 2\mu\underbrace{\left(d_{ij} - \frac{1}{3} d_{mm}\delta_{ij}\right)}_{\text{partie déviatoire de } d_{ij} \Rightarrow \text{ sa trace est nulle}} + \mu_v\, d_{mm}\,\delta_{ij}$$

$$d_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$ tenseur des taux de déformation

μ est la viscosité dynamique de cisaillement

μ_v est la viscosité dynamique de volume

$\tau_{ij}\, d_{ij} \geq 0 \Rightarrow \mu$ et μ_v sont ≥ 0

dissipation visqueuse

Loi de Fourier:

$$q_j = -k \frac{\partial T}{\partial x_j} \Rightarrow -\frac{\partial q_j}{\partial x_j} = \frac{\partial}{\partial x_j}\left(k \frac{\partial T}{\partial x_j}\right)$$

$$\underset{\sim}{q} = -k \nabla T \Rightarrow -\nabla\cdot \underset{\sim}{q} = \nabla\cdot(k \nabla T)$$

(k : conductivité thermique)

En général: ρ et U dépendent de T et p

μ, μ_v et k dépendent de T et p

Ecoulements incompressibles: ρ constant, $dU = c\, dT$

- $\frac{\partial v_j}{\partial x_j} = 0$ $\quad \boxed{\nabla\cdot \underset{\sim}{v} = 0} \Rightarrow \tau_{ij} = 2\mu d_{ij}$

- $$\rho\left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\tau_{ij} + \rho g_i$$

  ou $$\frac{\partial}{\partial x_j}(v_i v_j) = -\frac{\partial}{\partial x_i}(p + \rho W) + \frac{\partial}{\partial x_j}(2\mu d_{ij})$$

- $$\rho c\left(\frac{\partial T}{\partial t} + v_j \frac{\partial T}{\partial x_j}\right) = \tau_{ij} d_{ij} - \frac{\partial q_j}{\partial x_j}$$

  ou $$\frac{\partial}{\partial x_j}(T v_j) \qquad = 2\mu\, d_{ij} d_{ij} + \frac{\partial}{\partial x_j}\left(k \frac{\partial T}{\partial x_j}\right)$$

Aussi: $$\frac{\partial}{\partial x_j}(2\mu d_{ij}) = \frac{\partial}{\partial x_j}\left(\mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\right)$$

Cas μ constant: $$= \mu\left(\frac{\partial}{\partial x_j}\frac{\partial v_i}{\partial x_j} + \frac{\partial}{\partial x_i}\underbrace{\frac{\partial v_j}{\partial x_j}}_{0}\right) = \mu \frac{\partial}{\partial x_j}\frac{\partial v_i}{\partial x_j}$$

Cas k constant: $$-\frac{\partial}{\partial x_j} q_j = k \frac{\partial}{\partial x_j}\frac{\partial T}{\partial x_j}$$

Dérivée matérielle:

$$\frac{D\varphi}{Dt} \triangleq \frac{\partial \varphi}{\partial t} + v_j \frac{\partial \varphi}{\partial x_j}$$

Notation vectorielle:

$$\frac{D\varphi}{Dt} = \frac{\partial \varphi}{\partial t} + (\underset{\sim}{v} \cdot \nabla)\varphi$$

En notation vectorielle:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \underset{\sim}{v}) = \frac{D\rho}{Dt} + \rho\, \nabla\cdot\underset{\sim}{v} = 0$$

$$\frac{\partial}{\partial t}(\rho \underset{\sim}{v}) + \nabla\cdot(\rho \underset{\sim}{v}\underset{\sim}{v}) = \rho \frac{D\underset{\sim}{v}}{Dt} = -\nabla p + \nabla\cdot\underset{=}{\tau} + \rho \underset{\sim}{g}$$

$$\frac{\partial}{\partial t}(\rho U) + \nabla\cdot(\rho U \underset{\sim}{v}) = \rho \frac{DU}{Dt} = -p\, \nabla\cdot\underset{\sim}{v} + \underset{=}{\tau} : \underset{=}{d} + \nabla\cdot(k \nabla T)$$

avec $\underset{=}{\tau} = 2\mu\underbrace{\left(\underset{=}{d} - \frac{1}{3}\nabla\cdot\underset{\sim}{v}\, \underset{=}{\delta}\right)}_{\underset{=}{d}^d} + \mu_v\, \nabla\cdot\underset{\sim}{v}\, \underset{=}{\delta}$

$\underset{=}{d}^d$ partie déviatoire de $\underset{=}{d}$

$$\underset{=}{d} = \frac{1}{2}\left(\nabla\underset{\sim}{v} + (\nabla\underset{\sim}{v})^T\right)$$

Ecoulements incompressibles avec μ et k constants:

$$\nabla\cdot\underset{\sim}{v} = 0$$

$$\rho \frac{D\underset{\sim}{v}}{Dt} = -\nabla p + \mu \nabla^2 \underset{\sim}{v} + \rho \underset{\sim}{g} = -\nabla(p + \rho W) + \mu \nabla^2 \underset{\sim}{v}$$

$$\rho c \frac{DT}{Dt} = 2\mu\, \underset{=}{d} : \underset{=}{d} + k \nabla^2 T$$

On peut aussi tout diviser par ρ:

Déf:
- $P = \frac{p}{\rho}$ pression cinématique $\left[\frac{m^2}{s^2}\right]$
- $\nu = \frac{\mu}{\rho}$ viscosité cinématique $\left[\frac{m^2}{s}\right]$
- $\alpha = \frac{k}{\rho c}$ diffusivité thermique $\left[\frac{m^2}{s}\right]$

↳

$$\frac{D\underset{\sim}{v}}{Dt} = -\nabla P + \nu \nabla^2 \underset{\sim}{v} + \underset{\sim}{g} = -\nabla(P + W) + \nu \nabla^2 \underset{\sim}{v}$$

$$\frac{DT}{Dt} = \frac{2}{c}\, \underset{=}{d} : \underset{=}{d} + \alpha \nabla^2 T$$