Triangulations / Quadrangulations

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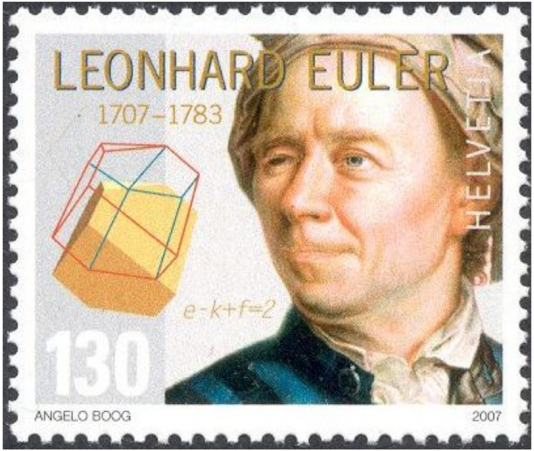
Algebraic topology for meshes

Delaunay triangulations in the plane

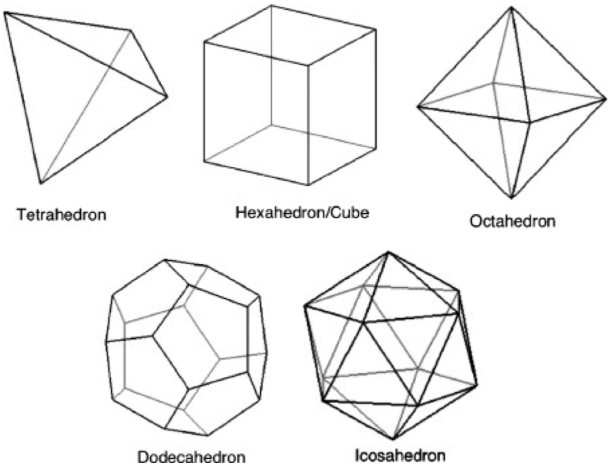
N-symmetry direction fields

Euler's second most famous result



Platonic solids

There exist exactly 5 "ideal" polyedra:



There a are exactly 5 platonic solids

- Consider a polyhedron with n vertices, n_e edges, n_f planar facets. Euler Formula is written

$$n - n_e + n_f = 2. \tag{1}$$

- Let m denote the number of edges and vertices of each facet and k the degree of each vertex i.e. the number of facets adjacent to the vertex.
- Each vertex has k adjacent faces and each face has m vertices. This implies that

$$mn_f = kn \quad \rightarrow \quad n_f = \frac{kn}{m}. \tag{2}$$

- Each edge has 2 adjacent faces and each face has m edges. This implies

$$mn_f = 2n_e \quad \rightarrow \quad n_e = \frac{mn_f}{2} = \frac{kn}{2}. \tag{3}$$

- Putting (1), (2) and (3) together gives

$$n\left(1 + \left(\frac{k}{m} - \frac{k}{2}\right)\right) = 2.$$

There a are exactly 5 platonic solids

- We can expand

$$n\left(1+\left(\frac{k}{m}-\frac{k}{2}\right)\right)=2.$$

  into

$$(2m+2k-mk)n=4.$$

- Since $n>0$ and $m>0$, we must have

$$2m+2k-mk>0.$$

- Since

$$2m+2k-mk=-(k-2)(m-2)+4>0$$

  the condition is transformed into

$$(k-2)(m-2)<4.$$

- Since $k\geq 3$ and $m\geq 3$, the only possible values for (m,k) are $(3,3)$, $(4,3)$, $(5,3)$, $(3,4)$ and $(3,5)$.

There a are exactly 5 platonic solids

$$n\left(1+\left(\frac{k}{m}-\frac{k}{2}\right)\right)=2 \quad , \quad n_f=\frac{kn}{m}.$$

- Tetrahedron : $(m,k)=(3,3) \quad \rightarrow \quad n=4 \quad , \quad n_f=4.$
- Hexahedron : $(m,k)=(4,3) \quad \rightarrow \quad n=8 \quad , \quad n_f=6.$
- Octahedron : $(m,k)=(3,4) \quad \rightarrow \quad n=6 \quad , \quad n_f=8.$
- Dodecahedron : $(m,k)=(5,3) \quad \rightarrow \quad n=20 \quad , \quad n_f=12.$
- Icosahedron : $(m,k)=(3,5) \quad \rightarrow \quad n=12 \quad , \quad n_f=20.$

Euler-Poincare Characteristic

- The topology of a 3D surface S can be described by a topological invariant that is its Euler-Poincare characteristic χ.
  - Two surfaces S_1 and S_2 with the same χ are topologically equivalent: it is possible to deform S_1 onto S_2 smoothly.
  - Assume that our surface is a sphere with n_b holes and n_h handles. We have :

$$\chi = 2 - n_h - 2n_h$$

  - A disk can be seen (topologically) as a sphere with one hole in it so $\chi = 2 - 1 = 1$.
  - The surface of a cylinder can be seen (topologically) as a sphere with two holes in it so $\chi = 2 - 2 = 0$.
- The Euler-Poincare formula is a generalization of Euler's formula for general 3D surfaces that may have a topology that is not the one of a sphere. Assume a polyhedron (n vertices, n_e edges and n_f facets) that covers a surface of topology χ, we have

$$n - n_e + n_f = \chi.$$

Euler-Poincare – Triangular meshes

Assume a triangular mesh mesh with n vertices, n_e edges and n_f triangular facets that covers a domain that has the topology of a sphere ($\chi = 2$):

$$n - n_e + n_f = \chi.$$

- Each edge has exactly two neighboring triangles and each triangle has three edges:

$$3n_f = 2n_e$$

- With Euler's formula:

$$n_f = 2(n-2) \quad , \quad n_e = 3(n-2).$$

Euler-Poincare – Triangular meshes

Assume a triangular mesh mesh with n vertices, n_e edges and n_f triangular facets that covers a domain with topology χ:

$$n - n_e + n_f = \chi.$$

Assume that n_h edges and vertices are located on the boundaries of the surface.

- Each triangle has 3 edges. Each internal edge has two triangles and each edge on the boundary is asjacent to on triangle:

$$3n_f = 2(n_e - n_h) + n_h$$

- With Euler's formula:

$$n_f = 2(n - \chi) - n_h \quad , \quad n_e = 3(n - \chi) - n_h.$$

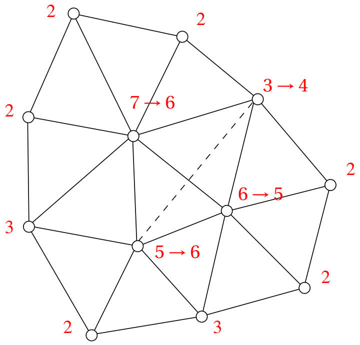
- There are asymptotically 3 times more edges than nodes and 2 times more triangles than nodes in a triangular mesh.

Euler-Poincare – Triangular meshes

- A triangle has three vertices and each vertex is adjacent in average to n_{vf} triangles. This leads to

$$n_{vf} n = 3n_f = 3\left(2(n-\chi) - n_h\right) \quad \rightarrow \quad n_{vf} = 6 - \frac{3n_h + 3\chi}{n}.$$

- This means that, for large meshes, there is in average 6 triangles surrounding every vertex.
- There is, in average, exactly 6 triangles surrounding each vertex on a triangular mesh of a torus ($n_h = \chi = 0$).



A triangulation T with $n = 12$ and $n_h = 9$. The average number of triangles adjacent to a vertex is $n_{vf} = 6 - \frac{3\times9+6}{12} = 3,25$. This average can also be computed explicitely: $n_{vf} = \frac{39}{12} = 3,25$.

Regular triangulations

$$n_f = 2(n - \chi) - n_h.$$

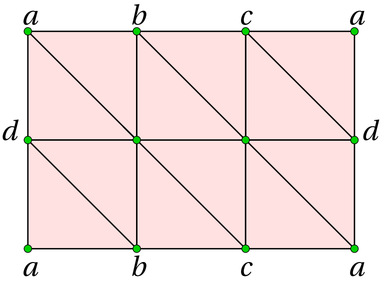
- Closed surface, no boundaries, $n_h = 0$.
- Regular topology: exactly 6 triangles adjacent to a vertex:

$$3n_f = 6n \quad \rightarrow \quad n_f = 2n.$$

- Restriction:

$$2n = 2(n - \chi) \quad \rightarrow \quad \chi = 0.$$

- Regular triangulations of closed surfaces are only possible for torus topologies ($\chi = 0$).



Regular triangulations with boundaries

$$n_f = 2(n - \chi) - n_h.$$

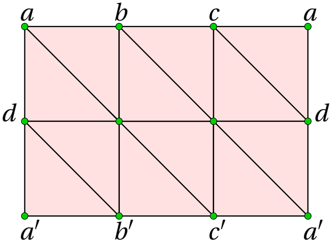
- We have n_h edges/vertices on the boundaries of the surface.
- Regular topology: exactly 6 triangles adjacent to an internal vertex and 3 triangles adjacent to a boundary vertex.

$$3n_f = 6(n - n_h) + 3n_h \quad \rightarrow \quad n_f = 2n - n_h.$$

- Same restriction:

$$2n - n_h = 2(n - \chi) - n_h \quad \rightarrow \quad \chi = 0.$$

- Regular triangulations of general surfaces are only available for $\chi = 0$ i.e. surface of a cylinder or torus.



Quasi-regular triangulations

- Introduction of n_k , $k = -2, -1, 1, 2$ non-regular internal vertices of degree $6 - k$.
- Introduction of m_l , $l = -2, -1, 1, 2$ non-regular boundary vertices of degree $3 - k$.
- This leads to

$$3n_f = \sum_k [(6-k)n_k + (3-k)m_k + 6(n - n_k - n_h) + 3(n_h - m_k)]$$

Finally , using $n_f = 2(n - \chi) - n_h$, we get

$$6n - 6\chi - 3n_h = \sum_k [(6-k)n_k + (3-k)m_k + 6(n - n_k - m_k) + 3(n_h - m_k)]$$

that simplifies into

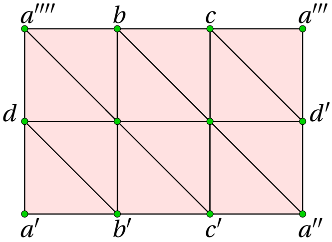
$$\chi = -\sum_k \frac{k}{6}(n_k + m_k).$$

Quasi-regular triangulations

$$\chi = -\sum_k \frac{k}{6}(n_k + m_k).$$

This formula has quite interresting implications

- It is possible to compute χ only by counting singularities
- Each singularity of index k count as $-k/6$ in the Poincare characteristic.
- A vertex with 5 neighboring triangles counts for $1/6$
- A vertex with 7 neighboring triangles counts for $-1/6$
- In the example, $\chi = 1$ and vertices a, a', a'' and a''' are irregular: a and a''' have indices $k = -1$ and a' and a'' have indices $k = -2$, which leads to $1/6 + 1/6 + 2/6 + 2/6 = 1$.



Euler-Poincare – Quadrangular meshes

Assume a quad-mesh with n vertices, n_e edges and n_f quad facets that covers a domain with topology χ:

$$n - n_e + n_f = \chi.$$

Assume that n_h edges and vertices are located on the boundaries of the surface.

- Each quad has 4 edges. Each internal edge has two adjacent quads and each edge on the boundary is adjacent to on quad:

$$4n_f = 2(n_e - n_h) + n_h$$

- With Euler's formula:

$$n_f = n - \chi - \frac{n_h}{2}$$

- Quad meshes are only possible if n_f is even!

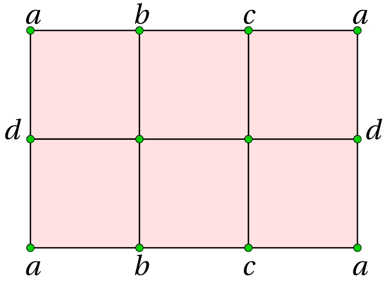
Regular quadrangulations

$$n_f = n - \chi - \frac{n_h}{2}$$

- Closed surface, no boundaries, $n_h = 0$.
- Regular topology: exactly 6 triangles adjacent to a vertex:

$$4n_f = 4n \quad \rightarrow \quad n_f = n.$$

- Regular quadrangulations of closed surfaces are only possible for torus topologies ($\chi = 0$).



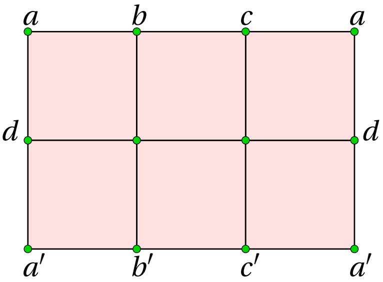
Regular quadrangulations with boundaries

$$n_f = 2(n - \chi) - n_h.$$

- We have n_h edges/vertices on the boundaries of the surface.
- Regular topology: exactly 4 quads adjacent to an internal vertex and 2 quads adjacent to a boundary vertex.

$$4n_f = 4(n - n_h) + 2n_h \quad \rightarrow \quad n_f = n - \frac{n_h}{2}.$$

- Regular quadrangulations of general surfaces are only available for $\chi = 0$ i.e. surface of a cylinder or torus.



Quasi-regular quadrangulations

- Introduction of n_k , $k = -2, -1, 1, 2$ non-regular internal vertices of degree $4 - k$.
- Introduction of m_l , $l = -2, -1, 1, 2$ non-regular boundary vertices of degree $2 - k$.
- This leads to

$$4n_f = \sum_k [(4-k)n_k + (2-k)m_k + 4(n - n_k - n_h) + 2(n_h - m_k)]$$

Finally , using $n_f = 2(n - \chi) - n_h$, we get

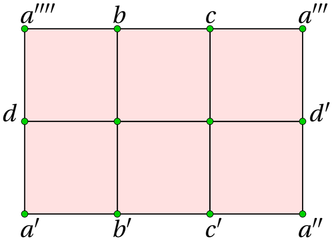
$$\chi = -\sum_k \frac{k}{4}(n_k + m_k).$$

Quasi-regular quadrangulations

$$\chi = -\sum_k \frac{k}{4}(n_k + m_k).$$

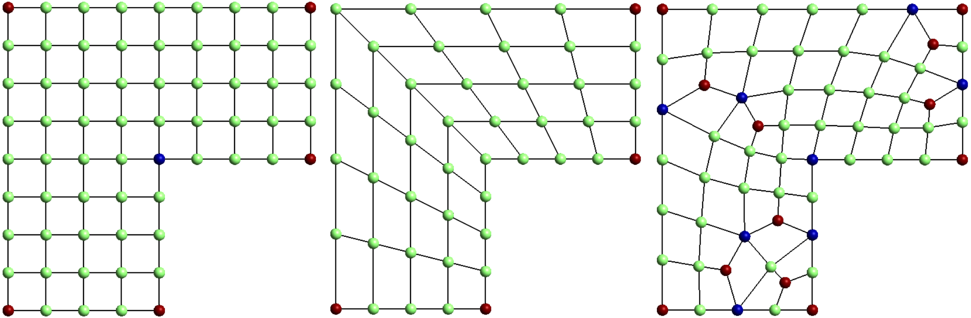
This formula has quite interresting implications

- It is possible to compute χ only by counting singularities
- Each singularity of index k count as $-k/4$ in the Poincare characteristic.
- A vertex with 3 neighboring triangles counts for $1/4$
- A vertex with 5 neighboring triangles counts for $-1/4$
- In the example, $\chi = 1$ and vertices a, a', a'' and a''' are irregular and of index $k = -1$, which leads to $1/4 + 1/4 + 1/4 + 1/4 = 1$.



Quasi-regular quadrangulations

- Quadrilateral meshes of a non smooth domain. Five singularities of index $1/4$ (in red) and one singularity of index $-1/4$ (in blue) are required to have the sum of the indices to be one (left).
- It is also possible to use 4 irregular nodes only (right), leading to a different result.
- Quadrilateral mesh with 8 vertices of index $-1/4$, and 12 of index $1/4$, leading to $\chi = 12/4 - 8/4 = 1$.

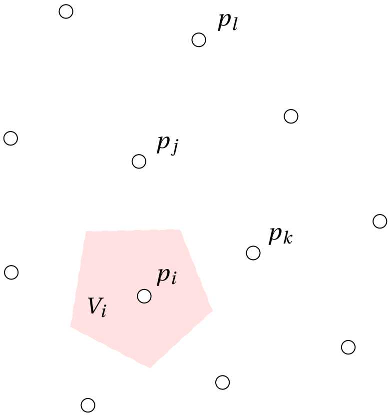


The Voronoï Diagram

Definition: Consider a finite set $S = \{p_1, \dots, p_n\} \subseteq \mathbb{R}^2$ of n distinct points in the plane. The *Voronoi cell* V_i of $p_i \in S$ is the set of points x that are closer to p_i than to any other points of the set:

$$V_i = \left\{x \in \mathbb{R}^2 \mid \|x - p_i\| < \|x - p_j\|, \ \forall 1 \leq i \leq n, i \neq j\right\}$$

where $\|x - y\|$ is the euclidian distance between x and y.

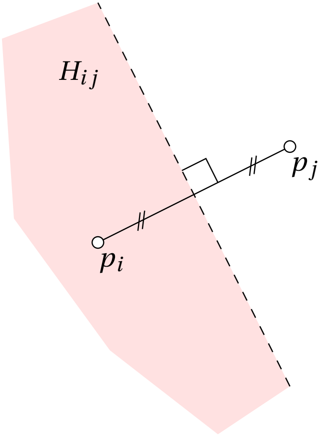


The Voronoï Diagram of 2 points p_i and p_j

The perpendicular bissector of p_ip_j divides $\mathbb{R}^2$ into two halfplanes H_{ij} and H_{ji}:

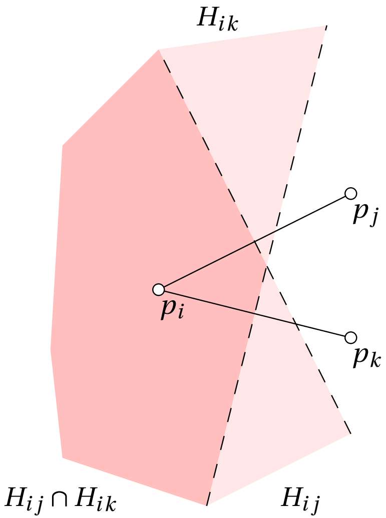
$$H_{ij} = \left\{x \in \mathbb{R}^2 \mid \|x - p_i\| < \|x - p_j\|\right\}.$$

We have $V_i = H_{ij}$.



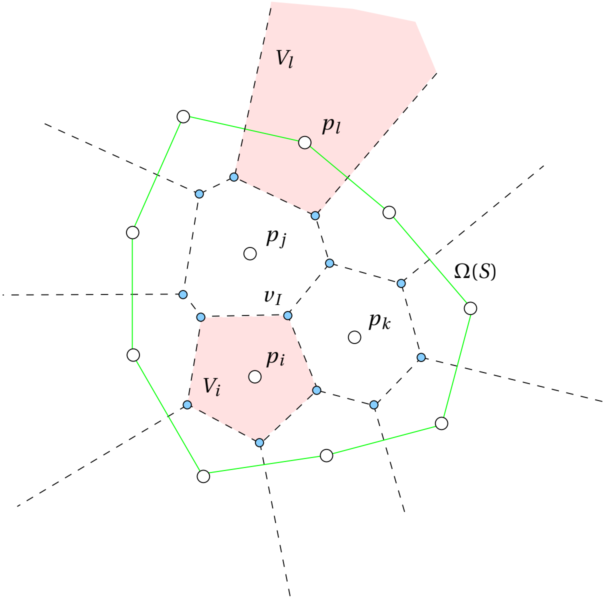
The Voronoï Diagram of 3 points

Let's make the problem a little more complicated and consider a set $S = \{p_i, p_j, p_k\}$ of 3 points. The Voronoi cell associated to p_i is the intersection of half planes H_{ij} and H_{ik}: $V_i = H_{ij} \cap H_{ik}$.



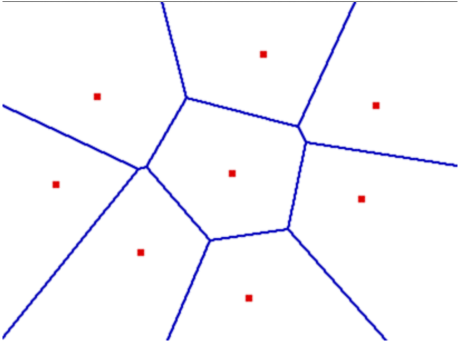
The Voronoï Diagram

The Voronoi diagram $V(S)$ is the unique subdivision of the plane into n cells. Its is the union of all Voronoi cells V_p:

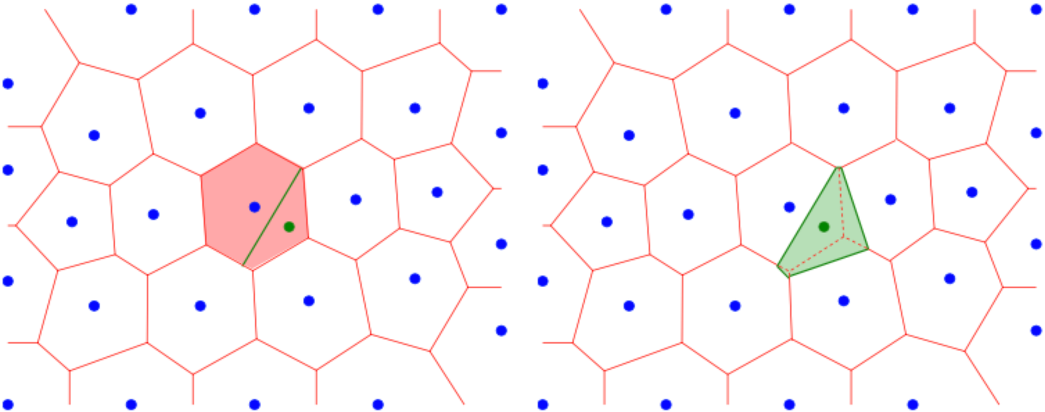


Green and Sibson's algorithm ($\mathcal{O}(n^2)$)

- Incremental: adding a point only modifies the diagram locally
- Let $S_n = \{p_1, p_2, \ldots, p_n\}$ and $V(S_n)$. Add p_{n+1} to form $V(S_{n+1})$ with $S_{n+1} = \{p_1, p_2, \ldots, p_{n+1}\}$.
  1. Find voronoi cell V_i such that $p_{n+1} \in V_i$.
  2. Draw orthogonal bissector of $p_{n+1}p_i$ and compute x_1 and x_2 its intersections with V_i (only 2 intersections because V_i is convex.
  3. x_1x_2 is the Voronoï edge that separates V_{n+1} and V_i. Start with x_2 that sits on a Voronoï edge of $V(S)$ that separates V_i with V_j.
  4. Replace i by j and goto 2 until x_2 goes back to x_1.
  5. The Voronoï cell V_{n+1} relative to p_{n+1} has been created. Remove the parts of all V_i's that have been "eaten" by V_{n+1}.



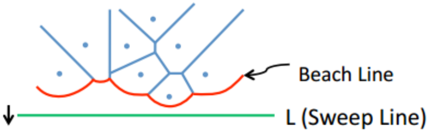
Green and Sibson's algorithm



(a) (b)

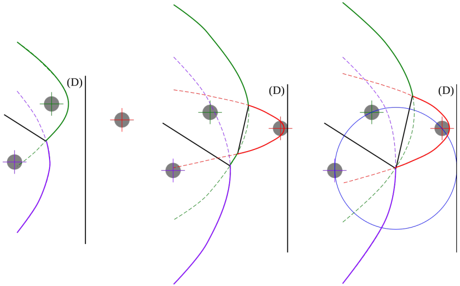
Fortune's algorithm ($\mathcal{O}(n \log(n))$)

- Line sweep (like intersection of lines) e.g. from left to right. Main issue, a part of the diagram on the left of the line depends on points on the right of the line.
- Fortune solves the issue by introducing a "beach line" that is (i) made of parabolas and that is (ii) delayed with respect to the sweep line.
- For each point left of the sweep line, one can define a parabola of points equidistant from that point and from the sweep line; the beach line is the boundary of the union of these parabolas.
- As the sweep line progresses, the vertices of the beach line, at which two parabolas cross, trace out the edges of the Voronoi diagram.
- The algorithm maintains as data structures a binary search tree describing the combinatorial structure of the beach line, and a priority queue listing potential future events that could change the beach line structure.



Fortune's algorithm ($\mathcal{O}(n\log(n))$)

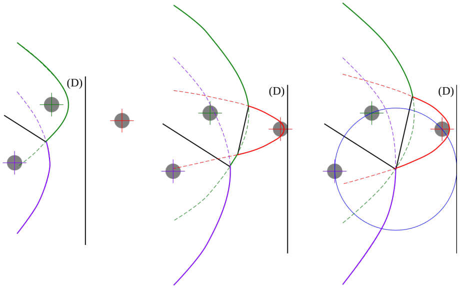
- Sweep line L passes through a first point p_1 and initiates a parabola P_1 s.t. $d(L, P_1) = d(p_1, P_1)$.
- Sweep line L passes through a second point p_2 and initiates a parabola P_2. Intersection point I between P_1 and P_2 verifies $d(I, p_1) = d(I, p_2)$ so I belongs to the Voronoï edge between p_1 and p_2.
- Sweep line L passes through a third point p_3 and initiates a parabola P_3. If points are in general position, there exist a circle C containing the 3 points. When L is tangent to C, its center is a Voronoï vertex. At that point, a part of P_1 must be removed from the beachline.



Fortune's algorithm ($\mathcal{O}(n\log(n))$)

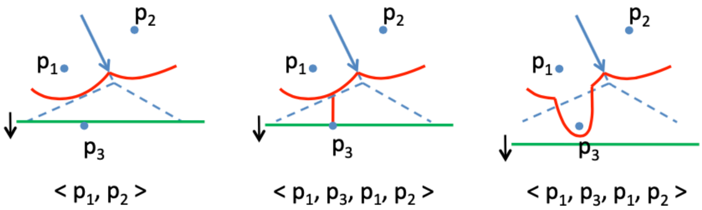
Two types of events:

- **Point event** A new parabola P_i is created whenever the sweep lines encounters seed p_i.
- **Circle/Vertex event** Disparition of a piece of parabola when the sweep line encounters a vertex i.e. the circumcircle of three "seeds".
- Both the point event and the vertex event can be handled in $\mathcal{O}(\log(n))$ time.
- Fortune's algorithm computes the Voronoï diagram in $\mathcal{O}(n\log(n))$ time. The storage space requirement is $\mathcal{O}(n)$.



Point Event

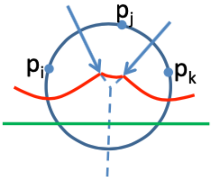
Point Event:



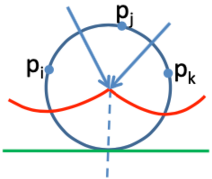
- To process a point event:
  - Determine the arc of the beach line directly above the new point
  - Split the arc into two by inserting a new infinitesimally small arc at this point
  - As the sweep proceeds this arc will start to widen

Circle/Vertex Event

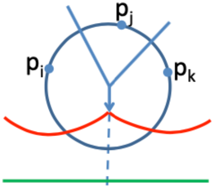
1. P_i, P_j, and P_k whose arcs appear consecutively on the beach line. The circumcircle lies partially below the sweep line



2. Circumcircle is empty and the center is equidistant to p_i, p_j, p_k, and L. The center is a Voronoi vertex.

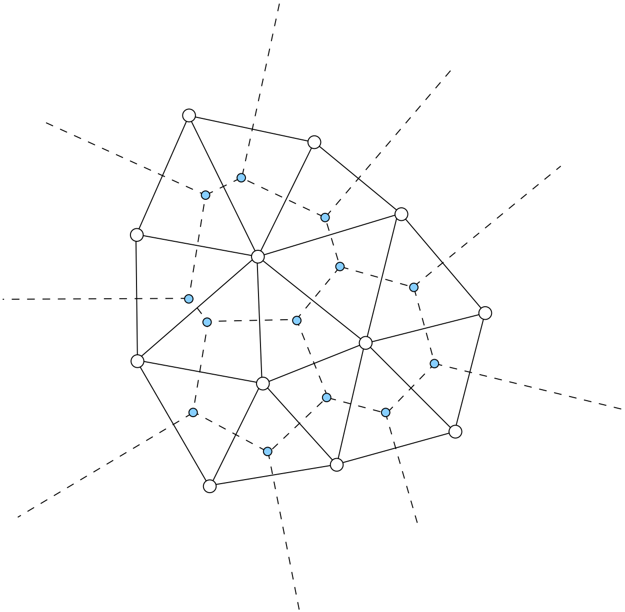


3. The arc of p_j disappears from the beach line



The Delaunay triangulation

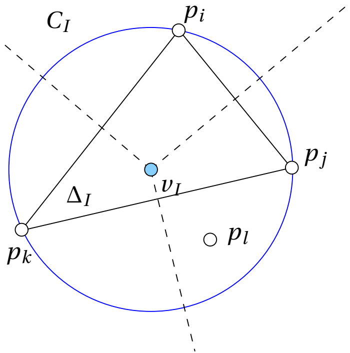
The Delaunay triangulation $DT(S)$ is the geometric dual of the Voronoï diagram



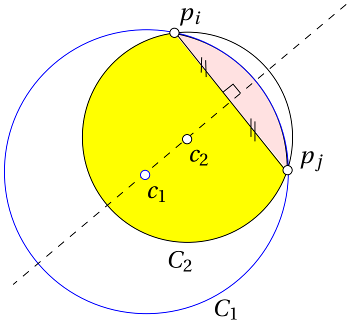
The empty circle property

The circumcircle of any triangle in the Delaunay triangulation is empty i.e. it contains no point of S.

- Consider the Delaunay triangle $\Delta_I = p_i p_j p_k$. Assume now that point $p_l \in C_I$ where C_I is the circumcircle of Δ_I.
- By definition, the triple point v_I is at equal distance to p_i, p_j and p_k and no other points of S are closer to v_I than those three points.
- Then, if a point like p_l exist in S, v_I is not a triple point and triangle Δ_I cannot be a Delaunay triangle.



Delaunay Edges

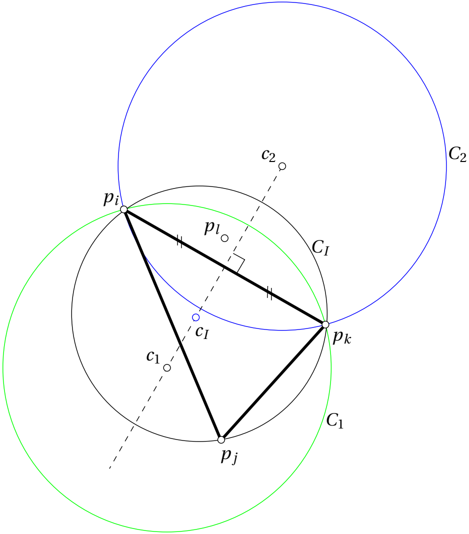


- Two circles C_1 and C_2 sharing an edge $p_i p_j$. The centers of the circles c_1 and c_2 lie on the perpendicular bissector of segment $p_i p_j$ (in dashed lines).
- Edge $p_i p_j$ divides disk C_1 into two disk sectors and one of the two sectors completely lies inside C_2. On the Figure, the pink sector of C_1 is inside C_2 and the yellow sector of C_2 lies inside C_1.

Delaunay Edges

An edge $p_i p_j$ of a triangulation is a *Delaunay edge* if there exist a circle that contains p_i and p_j and that is empty i.e. that contain no point of S.

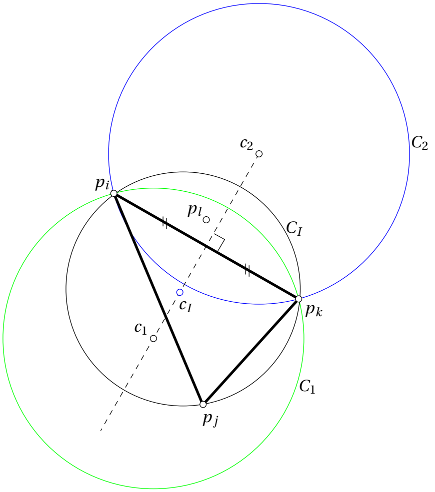
A mesh is a Delaunay Triangulation if and only if all its edges are Delaunay edges.



Delaunay Edges

Let us first show that a Delaunay triangulation has only Delaunay edges.

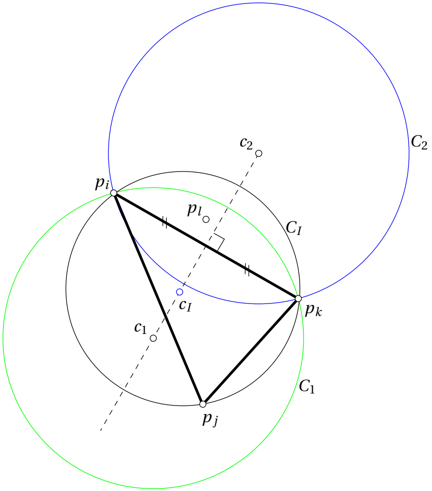
- Assume a Delaunay triangulation $T(S)$ and an edge $p_i p_j$ that is not Delaunay.
- This means that there exist no circle passing through p_i and p_j that is empty.
- Consider Delaunay triangle $\Delta_I = p_i p_j p_k$ that contains edge $p_i p_j$.
- Its circumcircle is empty by definition because T is a Delaunay triangulation.
- This is in contradiction with the hypothesis that there exist no circle passing through p_i and p_j and that is not empty.



Delaunay Edges

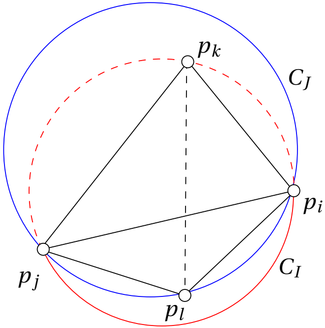
Now let's proof that if every edge of a triangulation is Delaunay, then every triangle is Delaunay as well.

- Assume that triangle $\Delta_I = p_i p_j p_k$ is not Delaunay (p_l is inside its circle), but all its 3 edges $p_i p_j$, $p_i p_k$ and $p_j p_k$ are Delaunay.
- Point p_l cannot be inside triangle Δ_I. It is then situated inside one of the three circular sectors delimited by p_i, p_j and p_k.
- Assume that p_l and p_j are on opposite sides of $p_i p_k$. By hypothesis, there exist a circle passing through p_i and p_k and that is empty. The center of such a circle lies on the orthogonal bissector of $p_i p_k$. Any circle like C_1 with its center c_1 that is below c_I contains p_j any circle C_2 that is above c_I contains p_l, which is in contradiction with the hypothesis that there exist a circle passing through $p_i p_k$ and that is empty.



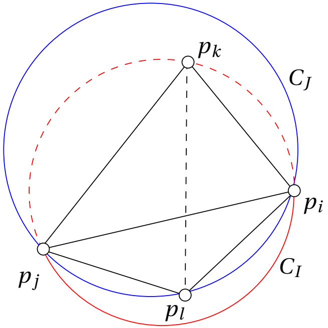
Local Delaunayhood

- Given a triangulation $T(S)$ and an edge p_ip_j in the triangulation that is adjacent to two triangles $\Delta_I = p_ip_jp_k$ and $\Delta_J = p_ip_lp_j$. We call edge p_ip_j *locally Delaunay* if p_l lies outside the circumcircle of Δ_I.
- Edge p_ip_j is not locally Delaunay on the Figure.
- It is easy to see that this condition is symmetric: if point p_l lies inside circle C_I, then point p_k lies inside circle C_J. We'll prove that below.



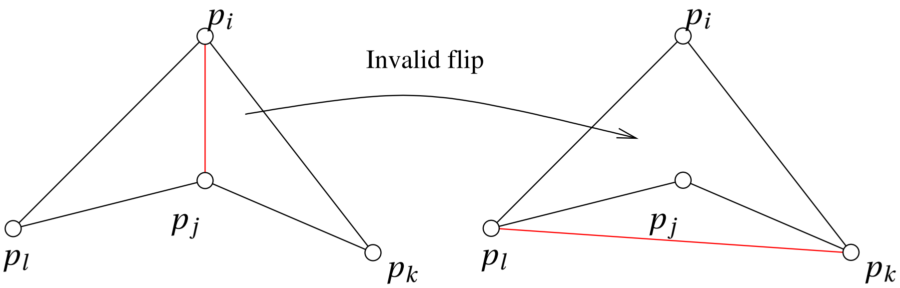
Edge Flip

- Consider again the situation of two triangles adjacent to edge $p_i p_j$ as depicted in the Figure.
- Flipping edge $p_i p_j$ consist in replacing triangles $p_i p_j p_k$ and $p_j p_i p_l$ by triangles $p_l p_k p_i$ and $p_k p_l p_j$.
- Edge $p_i p_j$ has been flipped and replaced by edge $p_k p_l$.



Edge Flip

The edge flip operator can only be applied to a pair of triangles that form a convex quadrilateral. If it is concave, then flipping the edge leads to an invalid configuration with two overlapping triangles



- An edge that is not locally Delaunay is flippable and the new edge resulting of the flip operation is locally Delaunay.
- If all edges of triangulation $T(S)$ are locally Delaunay, then T is the Delaunay triangulation $DT(S)$.

The Flip Algorithm

Flip until you drop:

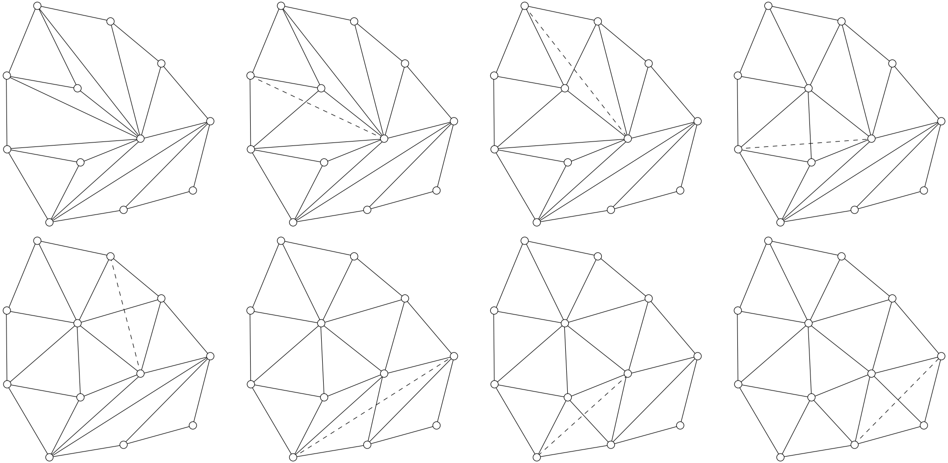
- Insert all the internal edges of $T(S)$ in a `stack`.
- Do while the `stack` is not empty
  - Take edge p_ip_j at the top of the `stack`. This edge is adjacent to triangles $p_ip_jp_k$ and $p_jp_ip_l$. If p_ip_j is not locally Delaunay, then flip it and add edges p_ip_k,p_kp_j, p_jp_l and p_lp_i in the stack. If one of those edges was already present in the stack, update its neighbors.
  - Remove p_ip_j from the `stack`.

Two questions should be asked at that point: (i) does this algorithm produce the Delaunay triangulation of S and (ii) if it achieves to create $DT(S)$, what is its complexity (does it simply terminate)?

The edge flip algorithm converges to $DT(S)$ in at most $\mathcal{O}(n^2)$ flips

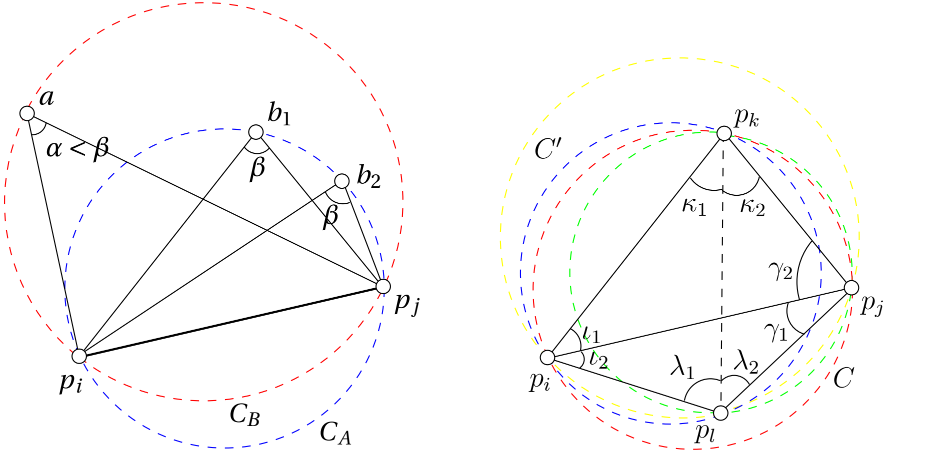
This result is outmost importance. It means that every triangulation $T(S)$ is "connected" to the Delaunay triangulation $DT(S)$ by at most $\mathcal{O}(n^2)$ flips. It also means that any two triangulations T and T' are flip connected.

The Flip Algorithm



The MaxMin property

The Delaunay triangulation $DT(S)$ is angle-optimal: it maximizes the minimum angle among all possible triangulations.

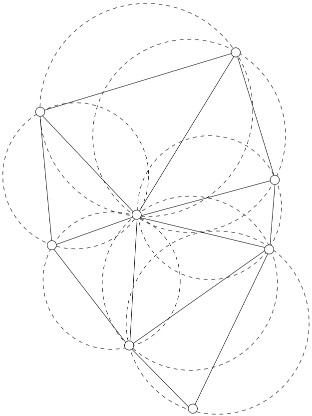


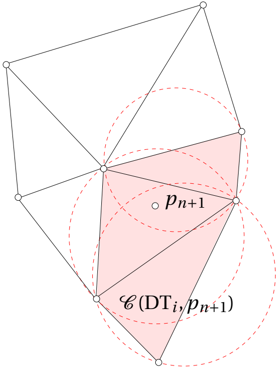
Thales theorem (left) and MaxMin property illustrated (right)

Bowyer-Watson Algorithm

Let DT_n be the Delaunay triangulation of a point set $S_n = \{p_1, \dots, p_n\} \subset \mathbb{R}^2$ that are in general position. We describe an incremental process allowing the insertion of a given point $p_{n+1} \in \Omega(S_n)$ into DT_n and to build the Delaunay triangulation DT_{n+1} of $S_{n+1} = \{p_1, \dots, p_n, p_{n+1}\}$.

$$DT_{n+1} = DT_n - \mathcal{C}(DT_n, p_{n+1}) + \mathcal{B}(DT_n, p_{n+1}). \quad (4)$$



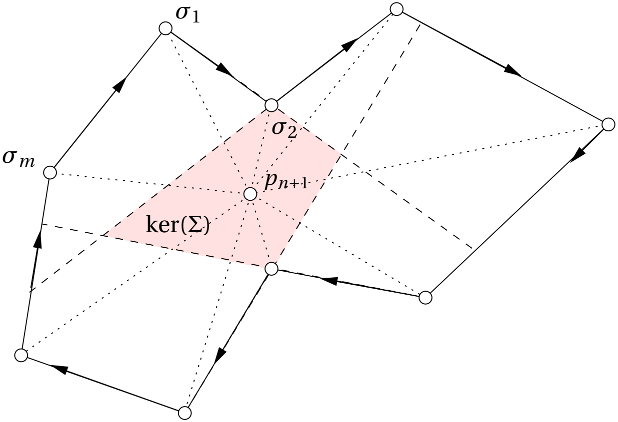


Bowyer-Watson Algorithm

Consider a polygon Σ with m corners $\sigma_1, \ldots, \sigma_m$ that is bounded by m edges $\sigma_i, \sigma_{(i+1)\%m}$, $1 \leq i \leq m$.

The kernel $\ker(\Sigma)$ is the set of point $x \in \mathbb{R}^2$ that are visible to every σ_j i.e. the line segment $x\sigma_j$ them do not intersect any edges of the polygon.

The kernel $\ker(\Sigma)$ can be computed by intersection of the halfplanes that correspond to all oriented edges of the polygon (see Figure).



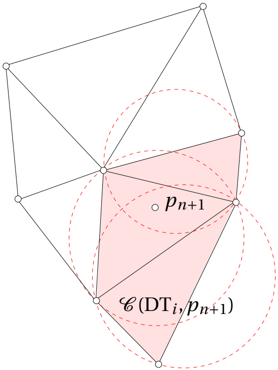
Bowyer-Watson Algorithm

The Delaunay cavity $\mathcal{C}(T_n, p_{n+1})$ is the set of m triangles $\Delta_1, \ldots, \Delta_m \in DT_n$ for which their circumcircle contains p_{n+1}.

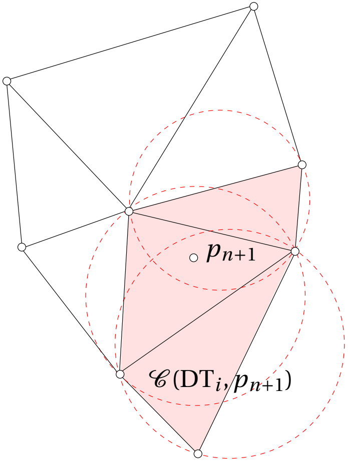
The Delaunay cavity contains the set of triangles that cannot belong to T_{n+1}. The region covered by those invalid triangles should be emptied and re-triangulated in a Delaunay fashion.

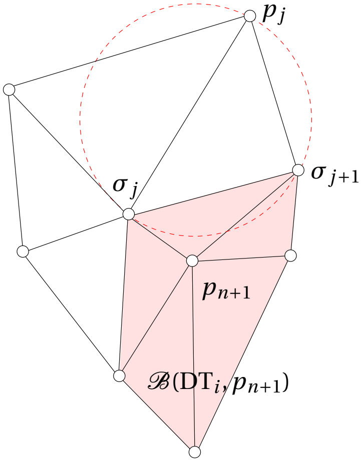
The Delaunay cavity has some interresting properties.

Theorem: The Delaunay cavity $\mathcal{C}(T_n, p_{n+1})$ is a non empty connected set of triangles which the union form a star shaped polygon with p_{n+1} in its kernel.



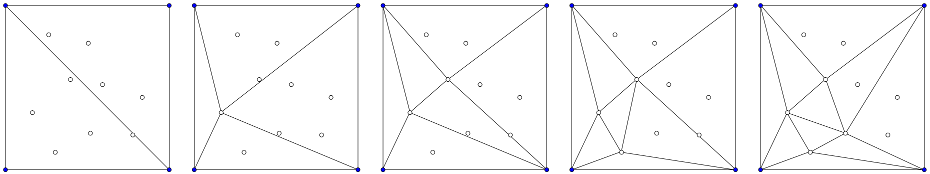
Bowyer-Watson Algorithm

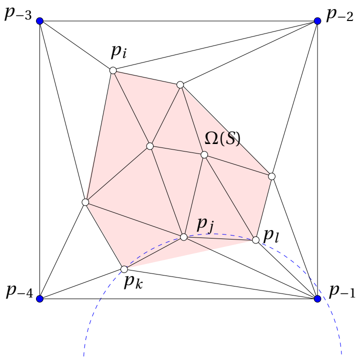




Bowyer-Watson Algorithm

Super triangles :





N-symmetry direction fields