LMECA2170 – Computational Geometry
Triangles and Quadrilaterals

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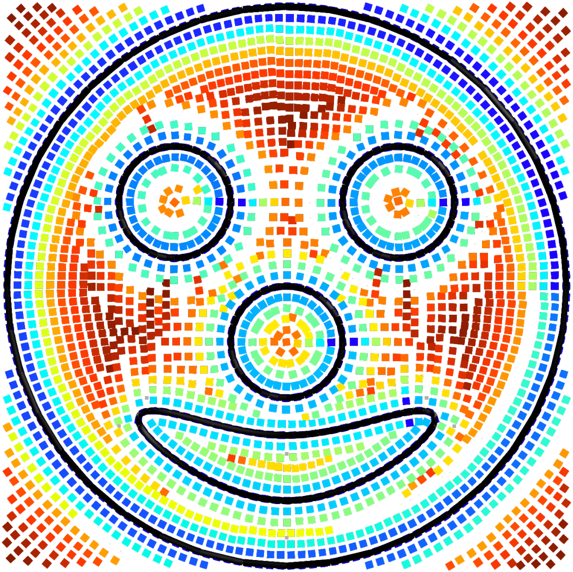


Euler's second most famous result

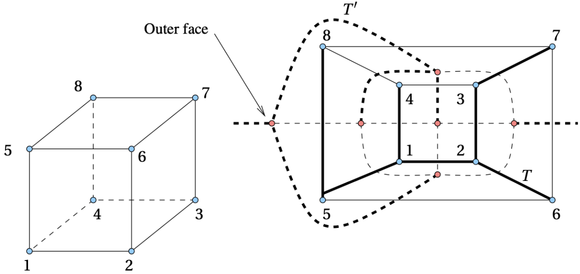
A polyhedron is a 3-dimensional solid made by joining together polygons.

$n_f - n_e + n_v = 2$ – Proof

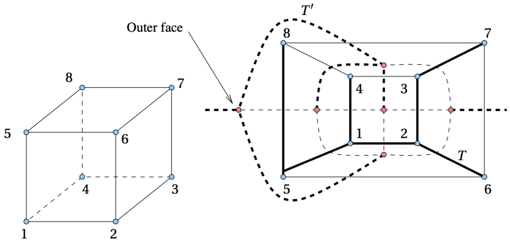
A Spanning tree of a graph is a set of edges of the graph that contains ne cycles but that "touches" all vertices.

Start with node n, add all edges (n, n_j) adjacent to n fow which n_j has not been "touched" yet. Do the same for all n_j. At the end, $n_v - 1$ edges will be added in the spanning tree.

Spanning tree T of the vertex/edge graph and T' the spanning tree of the dual face/edge graph.



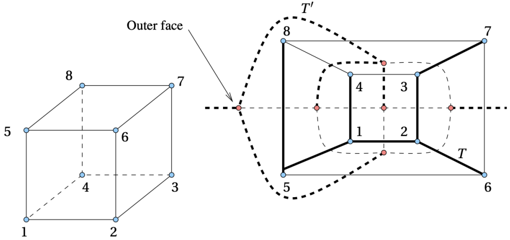
$n_f - n_e + n_v = 2$ – Proof



Spanning trees T and T' contain "edges" of the polyhedron.

The co-tree T^* of T is a spanning tree of Γ'. At first, T^* "touches" all faces of the polyhedron. If it was not the case there would be a cycle of edges in T that encloses a face. Then T^* does not contain cycles. If it was the case, it would separate the dual graph in two parts and T would not be a tree.

$n_f - n_e + n_v = 2$ – Proof



Any spanning tree T of Γ is of size $n_v - 1$.

Any spanning tree T' of Γ' is of size $n_f - 1$.

The co-tree T^* of T is of size $n_e - (n_v - 1)$.

T' and T^* have the same size so

$$n_e - (n_v - 1) = n_f - 1 \quad \rightarrow \quad \boxed{n_f - n_e + n_v = 2.}$$

A first “Gmsh” python script

Open `euler.py`.

```
import gmsh
import numpy as np
gmsh.initialize(sys.argv)
# Let’s create a simple model and mesh it:
gmsh.model.occ.addBox(0, 0, 0, 1, 1, 1)
gmsh.model.occ.synchronize()
gmsh.option.setNumber("Mesh.MeshSizeMin", 2.)
gmsh.model.mesh.generate(2)
...
```

Create a simple geometry

Generate a surface mesh

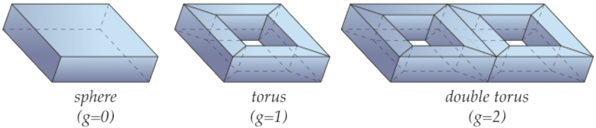
Create all mesh edges including internal edges that are not in the data model of Gmsh

Compute $n_v - n_e + n_f$.

Genus

The genus g of a connected, orientable surface is an integer representing the maximum number of cuttings along non-intersecting closed simple curves without rendering the resultant manifold disconnected.

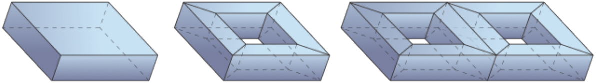
It is equal to the number of handles on the surface.



Euler Characteristic

For every combinatorial cell complex, one defines the Euler characteristic as the number of 0-cells, minus the number of 1-cells, plus the number of 2-cells, etc.,

$$\chi = n_v - n_e + n_f.$$



- For the sphere ($g = 0$): $n_f = 6$, $n_e = 12$, $n_v = 8$ so $\chi = 2$.
- For the torus ($g = 1$): $n_f = 16$, $n_e = 32$, $n_v = 16$ so $\chi = 0$.
- For the double torus ($g = 2$): $n_f = 30$, $n_e = 60$, $n_v = 28$ so $\chi = -2$.

Thus

$$\boxed{\chi = 2 - 2g.}$$

Boundaries

- Start from the sphere ($\chi = 2$)
- Remove 2 to χ for every handle
- Remove 1 to χ for every hole. Every hole is a boundary
- The annulus has $\chi = 0$ but has not the topology of a torus
- Two surfaces with the same topology have the same χ but two surfaces with the same χ may not have the same topology.

$$\boxed{\chi = 2 - 2g - b = n_f - n_e + n_v}$$

Go back to `euler.py`.

Platonic Solids

A Platonic solid is a convex, regular polyhedron in three-dimensional Euclidean space.

Being a regular polyhedron means that the faces are congruent (identical in shape and size) regular polygons (all angles congruent and all edges congruent), and the same number of faces meet at each vertex.

There are exacly 5 platonic solids.

Platonic Solids

Let $x \geq 3$ denote the number of edges/vertices of the faces of the solid and $y \geq 3$ the valence of each vertex (nr. of adjacent faces/edges).

Classic reasoning: dual graphs contain the same amount of information i.e. $\#E \times \#F$. Each edge has two adjacent triangles and each face has m edges:

$$2n_e = xn_f.$$

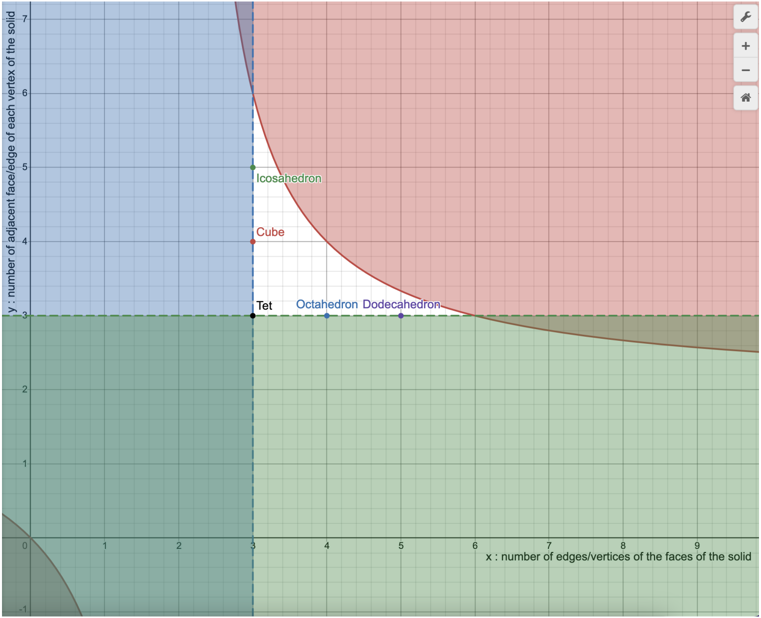
Each vertex has y adjacent edges and each edge has two vertices:

$$yn_v = 2n_e \quad \rightarrow \quad yn_v = xn_f = 2n_e \quad \overbrace{\rightarrow}^{n_v - n_e + n_f = 2} \quad n_v\left(1 - \frac{y}{2} + \frac{y}{x}\right) = 2.$$

$$n_v(2x + 2y - xy) = 4x.$$

Condition $(2x + 2y - xy) > 0$ can be visualized using desmos : `https://www.desmos.com/calculator/g5nepfdrsl?lang=fr`.

Platonic solids



Euler characteristic in dimension 3

In dimension 2

$$n_p - n_e + n_f = \chi.$$

The disk : $\chi = 1$.

In dimension 3

$$n_p - n_e + n_f - n_v = \chi.$$

The 3-ball : $\chi = 1$ as well.

Euler-Poincare – Triangular meshes

Assume a triangular mesh mesh with n vertices, n_e edges and n_f triangular facets that covers a domain with topology χ:

$$n - n_e + n_f = \chi.$$

Assume that n_h edges and vertices are located on the boundaries of the surface.

- Each triangle has 3 edges. Each internal edge has two triangles and each edge on the boundary is adjacent to on triangle:

$$3n_f = 2(n_e - n_h) + n_h$$

- With Euler-Poincare formula:

$$\boxed{n_f = 2(n_v - \chi) - n_h} \quad , \quad \boxed{n_e = 3(n_v - \chi) - n_h.}$$

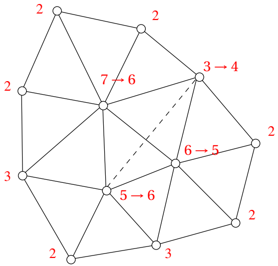
- There are asymptotically 3 times more edges than nodes and 2 times more triangles than nodes in a triangular mesh.

Euler-Poincare – Triangular meshes

- A triangle has three vertices and each vertex is adjacent in average to n_{vf} triangles. This leads to

$$n_{vf} n_v = 3n_f = 3\left(2(n_v - \chi) - n_h\right) \quad \rightarrow \quad n_{vf} = 6 - \frac{3n_h + 3\chi}{n_v}.$$

- This means that, for large meshes, there is in average 6 triangles surrounding every vertex.
- There is, in average, exactly 6 triangles surrounding each vertex on a triangular mesh of a torus ($n_h = \chi = 0$).



A triangulation T with $n_v = 12$ and $n_h = 9$. The average number of triangles adjacent to a vertex is $n_{vf} = 6 - \frac{3\times 9+6}{12} = 3.25$. This average can also be computed explicitely: $n_{vf} = \frac{39}{12} = 3.25$.

Regular triangulations

$$n_f = 2(n - \chi) - n_h.$$

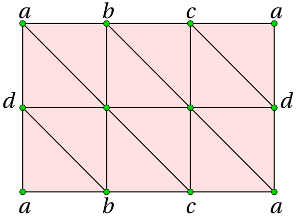
- Closed surface, no boundaries, $n_h = 0$.
- Regular topology: exactly 6 triangles adjacent to a vertex:

$$3n_f = 6n \quad \rightarrow \quad n_f = 2n.$$

- Restriction (we knew there was no such a platonic solid):

$$2n = 2(n - \chi) \quad \rightarrow \quad \chi = 0.$$

- Regular triangulations of closed surfaces are only possible for torus topologies ($\chi = 0$).



Regular triangulations with boundaries

$$n_f = 2(n - \chi) - n_h.$$

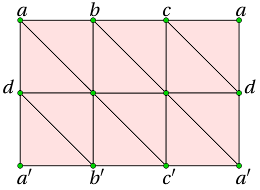
- We have n_h edges/vertices on the boundaries of the surface.
- Regular topology: exactly 6 triangles adjacent to an internal vertex and 3 triangles adjacent to a boundary vertex.

$$3n_f = 6(n - n_h) + 3n_h \quad \rightarrow \quad n_f = 2n - n_h.$$

- Same restriction:

$$2n - n_h = 2(n - \chi) - n_h \quad \rightarrow \quad \chi = 0.$$

- Regular triangulations of general surfaces are only available for $\chi = 0$ i.e. surface of a cylinder or torus.



Quasi-regular triangulations

- Introduction of n_k , $k = -2, -1, 1, 2$ non-regular internal vertices of degree $6 - k$.
- Introduction of m_l , $l = -2, -1, 1, 2$ non-regular boundary vertices of degree $3 - k$.
- This leads to

$$3n_f = \sum_k \left[(6-k)n_k + (3-k)m_k + 6(n - n_k - n_h) + 3(n_h - m_k)\right]$$

Finally , using $n_f = 2(n - \chi) - n_h$, we get

$$6n - 6\chi - 3n_h = \sum_k \left[(6-k)n_k + (3-k)m_k + 6(n - n_k - m_k) + 3(n_h - m_k)\right]$$

that simplifies into

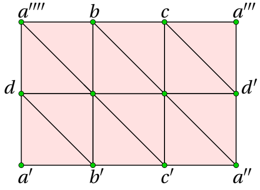
$$\chi = -\sum_k \frac{k}{6}(n_k + m_k).$$

Quasi-regular triangulations

$$\chi = -\sum_k \frac{k}{6}(n_k + m_k).$$

This formula has quite interresting implications

- It is possible to compute χ only by counting singularities (let's code that in `euler.py`).
- Each singularity of index k count as $-k/6$ in the Poincare characteristic.
- A vertex with 5 neighboring triangles counts for $1/6$
- A vertex with 7 neighboring triangles counts for $-1/6$
- In the example, $\chi = 1$ and vertices a, a', a'' and a''' are irregular: a and a''' have indices $k = -1$ and a' and a'' have indices $k = -2$, which leads to $1/6 + 1/6 + 2/6 + 2/6 = 1$.



Tetrahedra

Consider the 3-sphere ($\chi = 0$). We have

$$n_p - n_e + n_f - n_t = 0.$$

Each terahedra has 4 faces and each face has 2 tetrahedra so

$$4n_t = 2n_f.$$

There are thus twice more faces in a tet mesh than tetrahedra. **Now, it is not possible to elaborate an "exact formula" like for triangles.**

A regular tetrahedron is a tetrahedron in which all four faces are equilateral triangles. Disappointingly, *it is not a space-filling polyhedron*, see *Filling space with tetrahedra* (Naylor, 1999) or *Space-filling Tetrahedra in Euclidean Space* (Sommerville, 1923)!

Tetrahedra

Dihedral angle of a regular tetrahedron is $\arccos(1/3) \approx 70^o32'$. We thus can put a little more than 5 regular tetrahedra around an edge to approximatively fill the space $(5 \times 70^o = 350^o)$. We thus can write

$$6n_t \approx \frac{360}{70} n_e \quad \rightarrow \quad 7n_t \approx 6n_e.$$

Thus, in a regular mesh,

$$6n_p - 6n_e + 6n_f - 6n_t = 0 \;\; \rightarrow \;\; 6n_p - 7n_t + 12n_t - 6n_t \approx 0 \;\; \rightarrow \;\; n_t \approx 6n_p$$

that is observed in practice in good tet meshes.

Euler-Poincare – Quadrangular meshes

Assume a quad-mesh with n vertices, n_e edges and n_f quad facets that covers a domain with topology χ:

$$n - n_e + n_f = \chi.$$

Assume that n_h edges and vertices are located on the boundaries of the surface.

- Each quad has 4 edges. Each internal edge has two adjacent quads and each edge on the boundary is adjacent to on quad:

$$4n_f = 2(n_e - n_h) + n_h$$

- With Euler's formula:

$$n_f = n - \chi - \frac{n_h}{2}$$

- Quad meshes are only possible if n_f is even!

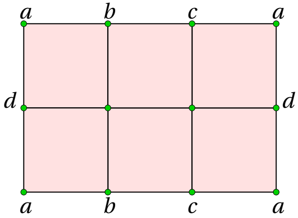
Regular quadrangulations

$$n_f = n - \chi - \frac{n_h}{2}$$

- Closed surface, no boundaries, $n_h = 0$.
- Regular topology: exactly 6 triangles adjacent to a vertex:

$$4n_f = 4n \quad \rightarrow \quad n_f = n.$$

- Regular quadrangulations of closed surfaces are only possible for torus topologies ($\chi = 0$).



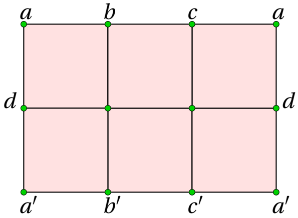
Regular quadrangulations with boundaries

$$n_f = 2(n - \chi) - n_h.$$

- We have n_h edges/vertices on the boundaries of the surface.
- Regular topology: exactly 4 quads adjacent to an internal vertex and 2 quads adjacent to a boundary vertex.

$$4n_f = 4(n - n_h) + 2n_h \quad \rightarrow \quad n_f = n - \frac{n_h}{2}.$$

- Regular quadrangulations of general surfaces are only available for $\chi = 0$ i.e. surface of a cylinder or torus.



Quasi-regular quadrangulations

- Introduction of n_k , $k = -2, -1, 1, 2$ non-regular internal vertices of degree $4 - k$.
- Introduction of m_l , $l = -2, -1, 1, 2$ non-regular boundary vertices of degree $2 - k$.
- This leads to

$$4n_f = \sum_k [(4-k)n_k + (2-k)m_k + 4(n - n_k - n_h) + 2(n_h - m_k)]$$

Finally , using $n_f = 2(n - \chi) - n_h$, we get

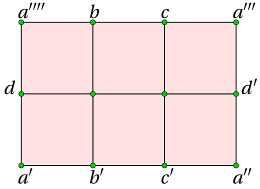
$$\chi = -\sum_k \frac{k}{4}(n_k + m_k).$$

Quasi-regular quadrangulations

$$\chi = -\sum_k \frac{k}{4}(n_k + m_k).$$

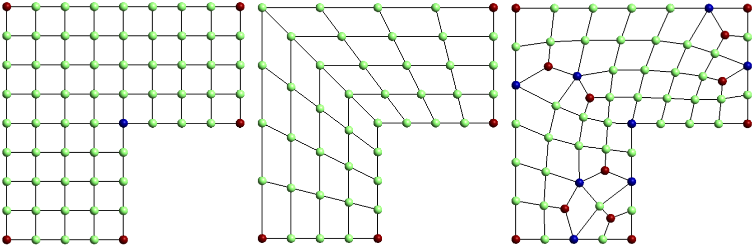
This formula has quite interresting implications

- It is possible to compute χ only by counting singularities
- Each singularity of index k count as $-k/4$ in the Poincare characteristic.
- A vertex with 3 neighboring triangles counts for $1/4$
- A vertex with 5 neighboring triangles counts for $-1/4$
- In the example, $\chi = 1$ and vertices a, a', a'' and a''' are irregular and of index $k = -1$, which leads to $1/4 + 1/4 + 1/4 + 1/4 = 1$.



Quasi-regular quadrangulations

- Quadrilateral meshes of a non smooth domain. Five singularities of index $1/4$ (in red) and one singularity of index $-1/4$ (in blue) are required to have the sum of the indices to be one (left).
- It is also possible to use 4 irregular nodes only (right), leading to a different result.
- Quadrilateral mesh with 8 vertices of index $-1/4$, and 12 of index $1/4$, leading to $\chi = 12/4 - 8/4 = 1$.



Representation of Polygonal Meshes
Half Edges

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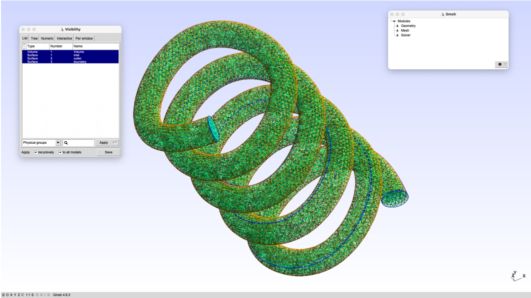
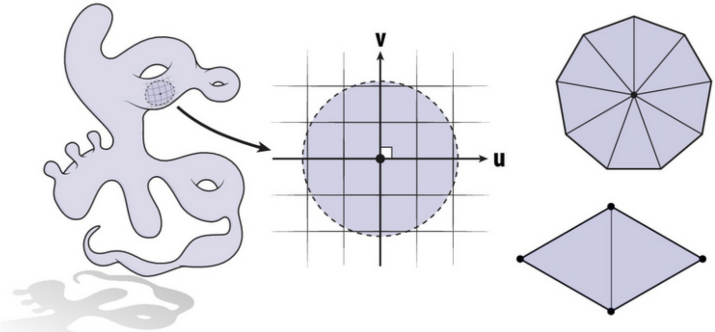
Geometry

- Many types of geometries in nature
- Two major categories
  - IMPLICIT – test if a point is inside/outside the shape
  - EXPLICIT – list of points, edges...
- Lots of representations for both
- Today – EXPLICIT representation for manifold surfaces.



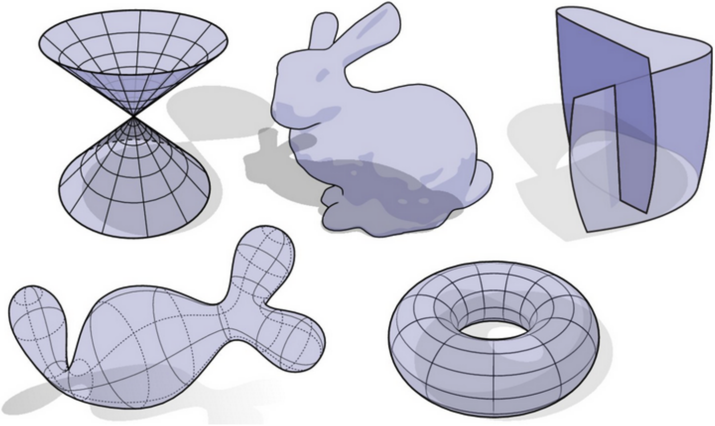
Manifold Surface

- Any point is locally 2d
- Polygonal surface is manifold if every edge is exactly adjacent to one or two faces.



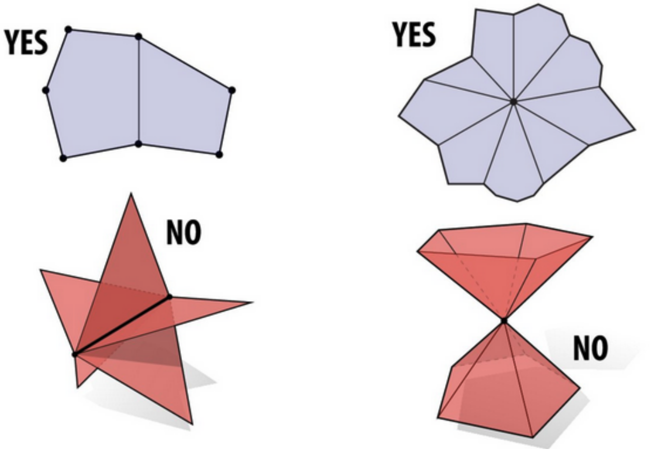
Manifold Surface

- Which surface is manifold?



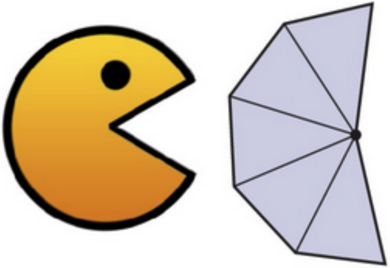
Manifold Polygonal Surface

- Which surface is manifold?



Manifold Polygonal Surface

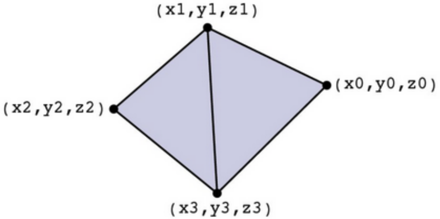
- What about boundaries?
- Boundary vertex's adjacent triangles look like PACMAN...



Encoding Manifold Polygonal Surfaces

Polygonal soup

- For each triangle encode coordinates
- No information about adjacencies
- Really simple :-)
- Redundant :-(
- Only for drawing :-(



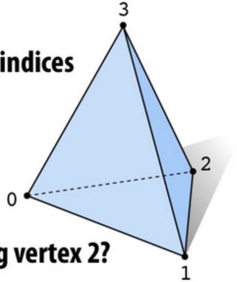
x0,y0,z0	x1,y1,z1	x3,y3,z3
x1,y1,z1	x2,y2,z2	x3,y3,z3

Encoding Manifold Polygonal Surfaces

Adjacency List (Array-like)

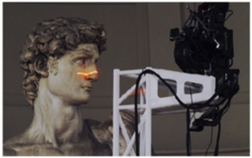
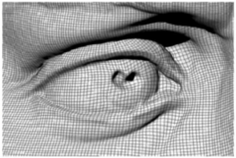
- Store triples of coordinates (x,y,z), tuples of indices
- E.g., tetrahedron:

VERTICES	x	y	z	POLYGONS	i	j	k
0:	−1	−1	−1		0	2	1
1:	1	−1	1		0	3	2
2:	1	1	−1		3	0	1
3:	−1	1	1		3	1	2



- Q: How do we find all the polygons touching vertex 2?
- Ok, now consider a more complicated mesh:

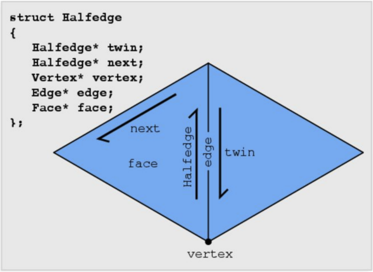
~1 *billion* polygons

Very expensive to find the neighboring triangles! (What's the cost?)

Half Edge Datastructure

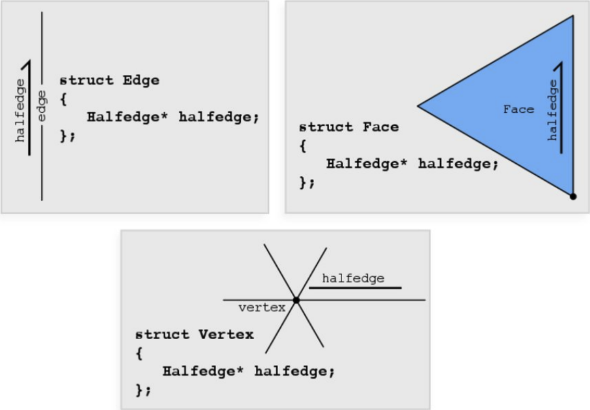
- Only for manifold surfaces
- Allows to deal with boundaries
- Allows to access adjacencies in constant time
- Allows simple algorithmics to modify locally a representation
- Key idea two half edges act as a glue between two neighboring polygons.



Half Edge Datastructure

The *magic* of half edge datsatructure

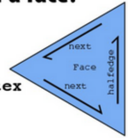
- Each vertex, edge of face points to one of its half edges.



Halfedge makes mesh traversal easy

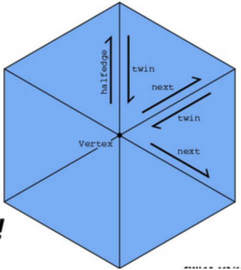
- Use "twin" and "next" pointers to move around mesh
- Use "vertex", "edge", and "face" pointers to grab element
- Example: visit all vertices of a face:

```
Halfedge* h = f->halfedge;
do {
    h = h->next;
    // do something w/ h->vertex
}
while( h != f->halfedge );
```



- Example: visit all neighbors of a vertex:

```
Halfedge* h = v->halfedge;
do {
    h = h->twin->next;
}
while( h != v->halfedge );
```



- Note: only makes sense if mesh is *manifold!*

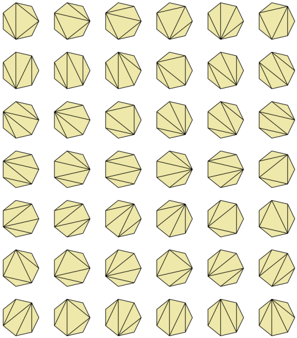
Polygon Triangulation

Polygon triangulation is the partition of a polygonal area P into a *set of triangles*.

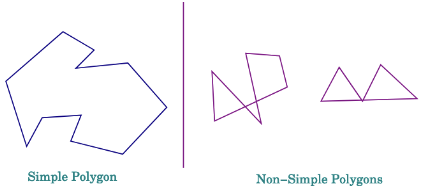
The total number of ways to triangulate a convex n-gon by is the $(n-2)$nd Catalan number, which equals

$$\frac{n(n+1)...(2n-4)}{(n-2)!}.$$

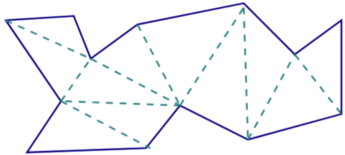
For $n = 7$, we have 42 triangulations.



Polygon Triangulation

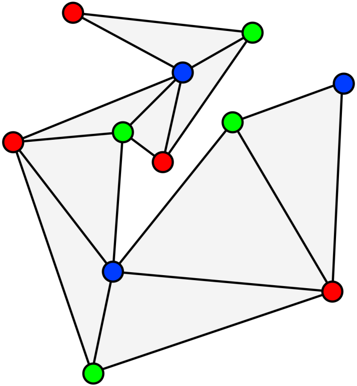


- Is this always possible for any *simple polygon*
- Partition a simple polygon P into non-overlapping triangles using diagonals only.
- Every triangulation of a n-gon, has exactly $n-2$ triangles (Euler-Poincaré).



Art Gallery Theorem

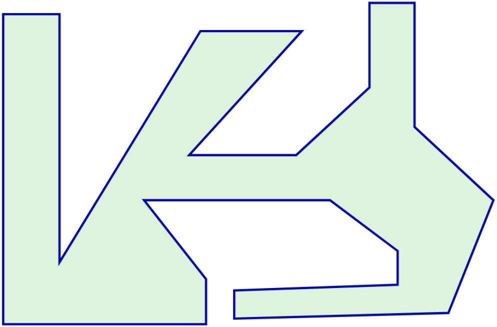
- Every simple polygon has at least two ears.
- What is an ear? If one takes two consecutive edges uv and vw of a polygon, and the line segment that joins vertices u and w is an internal diagonal, then vertex v is called an ear of the polygon.
- The two ears theorem is equivalent to the existence of polygon triangulations.



Art Gallery Theorem

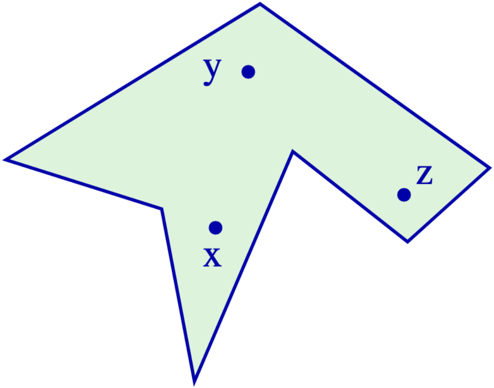
How many guards needed to see the whole room?

Story: Problem posed to Vasek Chvatal by Victor Klee at a math conference in 1973. Chvatal solved it quickly with a complicated proof, which has since been simplified significantly using triangulation.



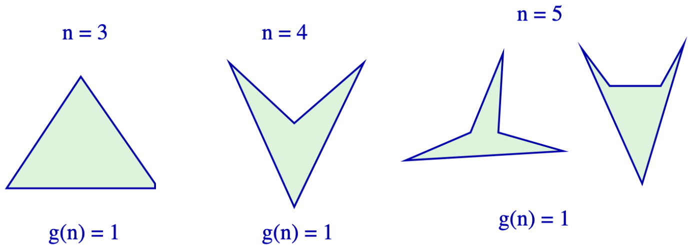
Art Gallery Theorem

- x and y visible if $xy \in P$.
- $g(P)$ minimum of guards to see P.
- $g(n)$ maximum $G(P)$, $|P| = n$.
- Art Gallery Theorem asks for bounds on function $g(n)$: what is the smallest $g(n)$ that always works for any n-gon?



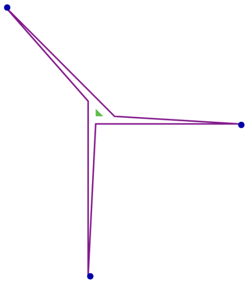
Art Gallery Theorem

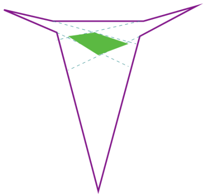
- For $n = 3, 4, 5$, $g(n) = 1$.
- For $n = 3, 4, 5$, P is star shaped.
- Is there a general formula for $g(n)$?



Art Gallery Theorem

- Fig. on left shows that seeing the boundary $\neq$ seeing the whole interior!
- Putting a guard at every "blue" vertex is not sufficient (Left Figure)
- Putting guards on vertices alone might not give the best solution (Right Figure)

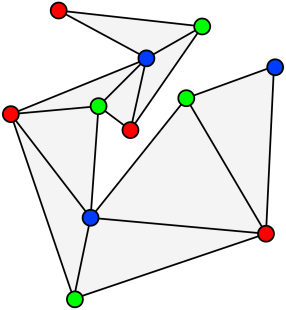




Art Gallery Theorem

Theorem: $g(n) = \left\lfloor \frac{n}{3} \right\rfloor$

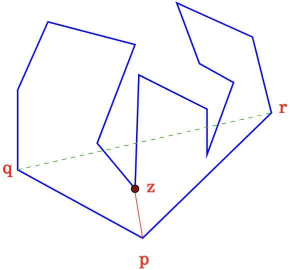
- Triangulation graph can be 3-colored
- 3-coloring means vertices can be labeled 1,2, or 3 so that no edge or diagonal has both endpoints with same label
- Remove an ear.
- Inductively 3-color the rest
- Put ear back, coloring new vertex with the label not used by the boundary diagonal.
- For g, choose vertices with the color that is the less present thus $g(n) = \left\lfloor \frac{n}{3} \right\rfloor$.



Triangulation theory

Theorem: Every polygon has a triangulation.

- **Proof by Induction. Base case $n = 3$.**



- **Pick a convex corner p. Let q and r be pred and succ vertices.**
- **If qr a diagonal, add it. By induction, the smaller polygon has a triangulation.**
- **If qr not a diagonal, let z be the reflex vertex farthest to qr inside $\triangle pqr$.**
- **Add diagonal pz; subpolygons on both sides have triangulations.**