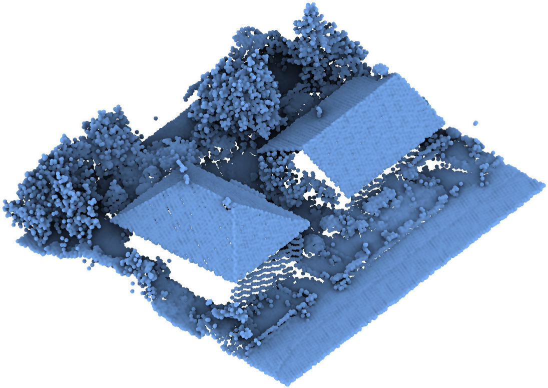
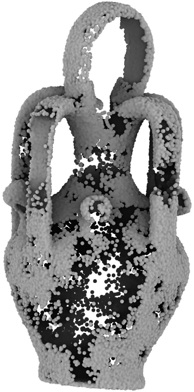
An Introduction to Point Cloud Processing

Computational Geometry (LMECA2170)

Point Clouds are the simplest kind of geometric data

$$P = \{p_1, ..., p_N\} \subset \mathbb{R}^d \qquad d = 2 \text{ or } 3$$





https://github.com/potree/potree

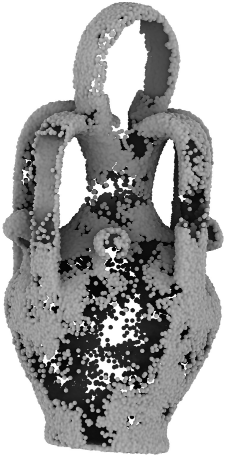
Point Cloud Acquisition

Retz (Austria)

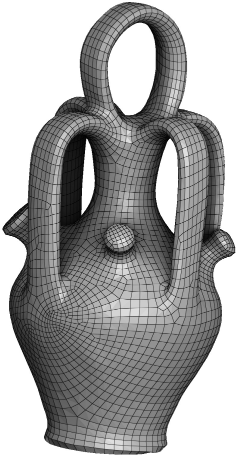
Lidar scanners

Our final goal: Surface reconstruction

From point clouds...



... to surface meshes



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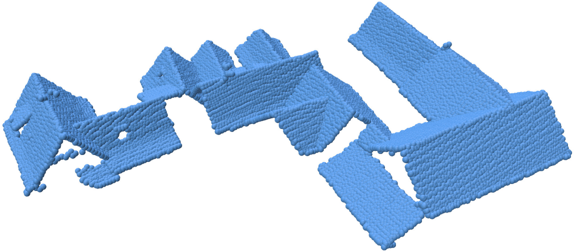

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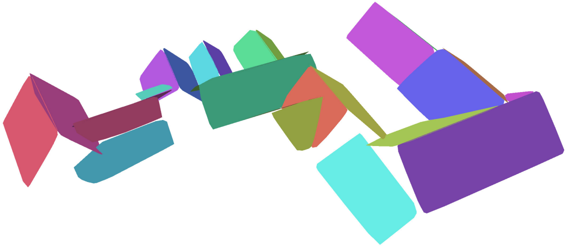
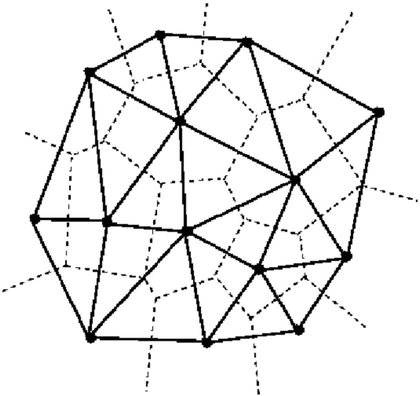
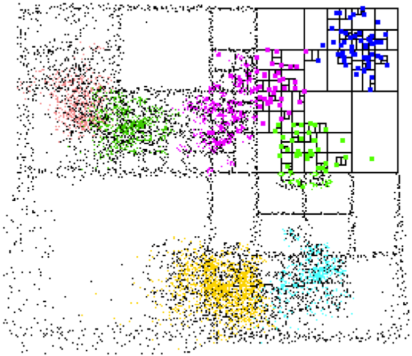


Requirements



Delaunay triangulation



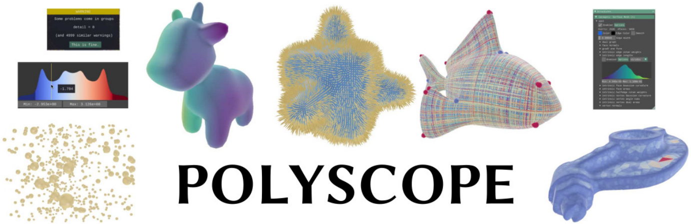
Efficient k-nearest neighbors search

... and basic linear algebra (matrices and eigenvalues)

Software Environment

Python using:

- *numpy* and *scipy*
- *mouette*[1] for data loading and geometry utilities
- *polyscope*[2] for visualization



[1] https://gcoiffier.github.io/mouette/
[2] https://polyscope.run/py/

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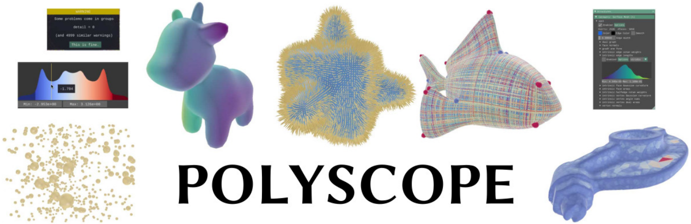

DEMO

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Part 1: The Point Cloud Processing Toolbox

1. Nearest Neighbors and the k-NN graph
2. Estimating Normals
   - Best fitting hyperplane
   - Consistent Orientation
3. Estimating Density
4. Clustering and Primitive Fitting
   - The RANSAC Algorithm

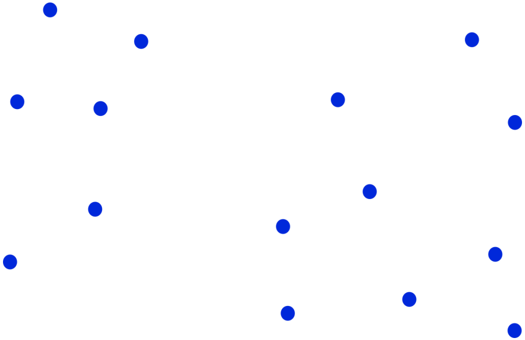
Nearest Neighbors and the k-NN graph

The k-nn graph: Finding structure through locality

Definition

A graph $G = (P, E)$ where:

- $P = \{p_1, ..., p_N\}$ are the N points of the cloud
- Oriented edge $p_i \rightarrow p_j \in E \Leftrightarrow p_j$ is among the k nearest points of p_i



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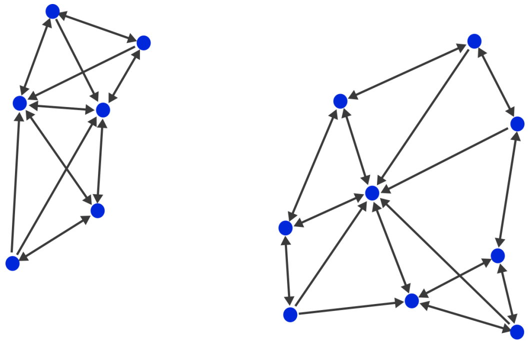

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Properties of the k-NN graph

- The graph is k-regular
- The graph is **sparse**: $|E| = kN = \Theta(N)$

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DEMO

The k-nn graph: Finding structure through locality

Representation

Adjacency list $L \in \mathbb{N}^{N \times 3}$

$L_{i,j}$ = index of the j-th nearest neighbor of point p_i.

Adjacency matrix $A \in \mathbb{R}^{N \times N}$

$$A_{i,j} = \begin{cases} 1 & \text{if } j \text{ is among the } k \\ & \text{nearest neighbors of } p_i \\ 0 & \text{otherwise} \end{cases}$$

Can be weighted (replace 1 by $d(p_i, p_j)$)

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Variants

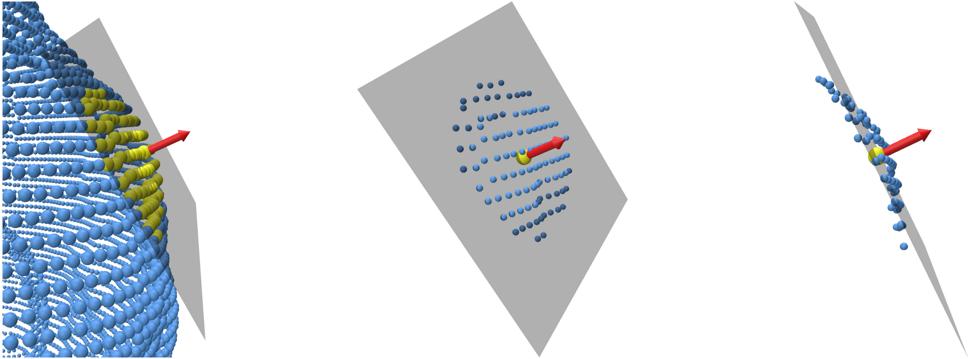
Unoriented: Consider edge (i, j) if p_j is among the k nearest points of p_i or vice-versa (Not k-regular anymore!)

Weighted: Add to edge (i, j) a weight $w_{i,j} \in \mathbb{R}$. Typically, $w_{i,j} = d(p_i, p_j)$

Estimating Normals

Estimating normals

The normal vector is the orthogonal vector of the tangent plane at point p_i
$\hookrightarrow$ Find the plane that best interpolates the points locally.



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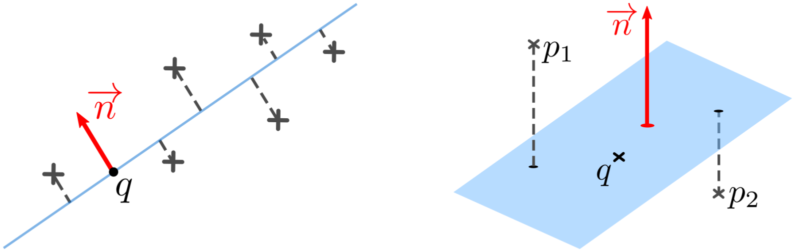
Best fitting hyperplane

A hyperplane (line in 2D, plane in 3D) Π is completely defined by a normal **unitary** vector $\overrightarrow{n} \in \mathbb{R}^d$ and any point $q \in \mathbb{R}^d$ by the following implicit equation:

$$\Pi = \left\{ p \in \mathbb{R}^d \;\middle|\; (p - q).\overrightarrow{n} = 0 \right\}$$

The distance to plane Π is given by $d(p, \Pi) = |(p - q).\overrightarrow{n}|$



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Problem formulation

Given points $p_1, ..., p_N \in \mathbb{R}^d$, find the plane $\Pi(\overrightarrow{n}, q)$ that minimizes the sum of square distances to each point:

$$\min_{\overrightarrow{n},q} E(p_i, \overrightarrow{n}, q) \quad \text{s.t.} \quad ||\overrightarrow{n}|| = 1 \quad \text{where} \quad E = \sum_{i=1}^{N} ((p_i - q).\overrightarrow{n})^2$$

Best fitting plane resolution 1: Finding the origin point

Let $c = \frac{1}{N}\sum_{i=1}^{N} p_i$ be the *centroid* of the point cloud.

> The best fitting plane goes through c. As a consequence, we can reduce the search to planes of the form $\Pi(\overrightarrow{n}, c)$.

Proof. Compute:

$$\frac{\partial E}{\partial q} = -2\sum_i ((p_i - q).n)\, n = -2\left((\sum_i p_i - q).n\right) n = -2N\,[(c-q).n]\, n$$

and since $||n|| = 1$, solving for $\frac{\partial E}{\partial q} = 0$ implies that $(c-q).n = 0$ which means that c belongs to the plane. □

Principal Axes and Best-Fit Planes, with Applications, Brown, 1976

Best fitting plane resolution 2: Finding the best normal vector

Matrix Notation

Define the **scatter matrix** M:

$$M = \begin{pmatrix} p_1 - c \\ p_2 - c \\ \vdots \\ p_N - c \end{pmatrix} \in \mathbb{R}^{N \times d}$$

and the **covariance matrix** K:

$$K = M^T M \in \mathbb{R}^{d \times d}$$

Reformulation

The problem of finding the best fitting plane can be written as:

$$\min_{n \in \mathbb{R}^d} ||Mn||^2 \quad \text{s.t.} \quad ||n|| = 1$$

or

$$\min_{n \in \mathbb{R}^d} n^T K n \quad \text{s.t.} \quad n^T n = 1$$

The function E is minimized when n is a (unitary) eigenvector associated with the smallest eigenvalue of K.

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Proof. K is symetric positive definite. Let $0 < \lambda_1 < \lambda_2 < ... < \lambda_d$ be its eigenvalues.
We can write: $K = U^T \Lambda U$ with $U^T U = I$ and $\Lambda = \text{diag}(\lambda_1, ..., \lambda_d)$.
Since U is invertible, if we write $y = Un$, we have:

$$\min_n n^T K n = \min_y y^T \Lambda y = \min_y \sum_i \lambda_i y_i^2 \geqslant \sum_i \lambda_1 y_i^2 = \lambda_1 ||y||^2.$$

Since U is orthogonal, $||y|| = ||n|| = 1$ so the minimum of $n^T K n$ is $\geqslant \lambda_1$. But this value is achieved for $y = (1, 0, ..., 0)$. In that case, $n = U^T y$ is the first column of U, i.e. an eigenvector associated to λ_1. □

Algorithm for finding the best fitting plane

Best-fitting plane

input: points $\{p_1, ..., p_N\} \subset \mathbb{R}^d$

1. Compute the scatter matrix $M = (p_i - c) \in \mathbb{R}^{N \times d}$
2. Compute the correlation matrix $K = M^T M \in \mathbb{R}^{d \times d}$
3. Compute the eigen decomposition $(\lambda_1, e_1), (\lambda_2, e_2), (\lambda_3, e_3)$ of K $(\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3)$

return e_1

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return e_1

Best-fitting plane (variant)

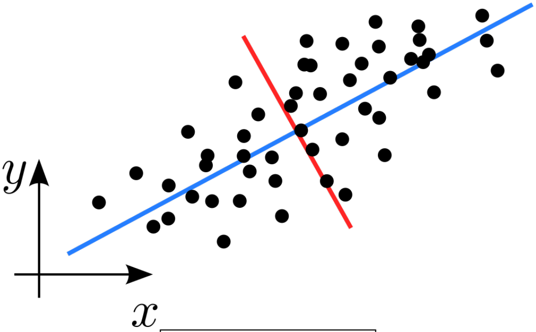
A mathematically equivalent formulation (saves the multiplication $M^T M$):

1. Compute the scatter matrix $M = (p_i - c) \in \mathbb{R}^{N \times d}$
2. Compute the SVD of M^T: $M^T = U \Sigma V^T$

return the first line of U

Digression: The Principal Component Analysis

Singular values of $M = U\Sigma V^T$ and lines of U give us the directions where the dataset varies the most/the less.



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Back to normal estimation

Normal estimation algorithm

For each point p_i in the point cloud:

- Compute $q_1, ...q_k$ the k nearest neighbors of p_i in P.
- Find the best fitting plane Π through $\{p_i, q_1, ..., q_k\}$
- The normal at p_i is the normal of the best fitting plane.

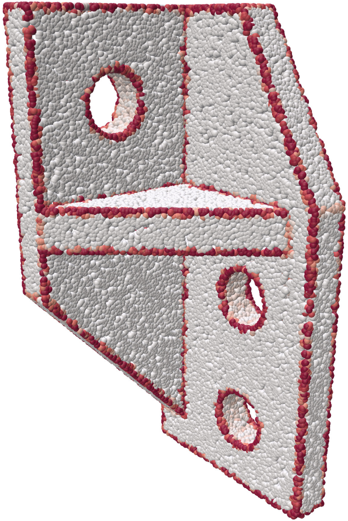
DEMO

Digression: Curvature Estimation and Edge Detection

Let $\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3$ be the eigenvalues of the correlation matrix K.

$$\eta = \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3}$$

is a good indicator of how much the considered points deviate from a perfect plane.



Efficient Simplification of Point-Sampled Surfaces, Pauly et al., 2002

Digression: Curvature Estimation and Edge Detection

Crease Detection

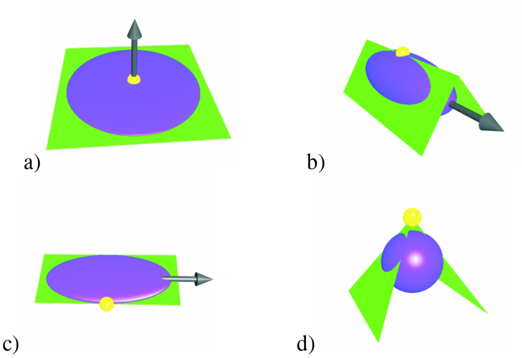
$$\eta_{\text{crease}} = \frac{\max\{\lambda_2 - \lambda_1, |\lambda_3 - \lambda_2 - \lambda_1|\}}{\lambda_3}$$

Edge Detection

$$\eta_{\text{edge}} = \frac{|\lambda_3 - 2\lambda_2|}{\lambda_3}$$

Corner detection

$$\eta_{\text{edge}} = \frac{\lambda_3 - \lambda_1}{\lambda_3}$$



Feature Extraction from Point Clouds, Gumhold et al., n.d.

Consistent Orientation

The best fitting plane only provides the normal's *direction* but not orientation.
Finding a globally consistent orientation is NP-complete.
$\hookrightarrow$ Approximation via propagation to neighbors

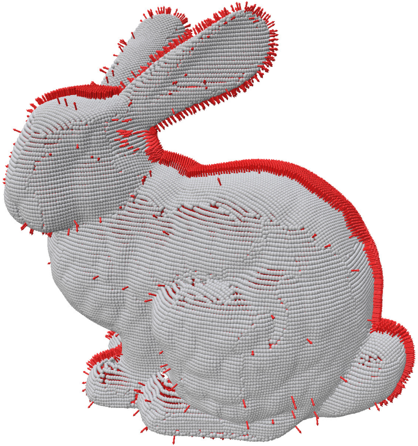
Consistent Normal Orientation

1. Compute the *unoriented* k-NN graph $G = (P, E)$.
2. Compute a minimal spanning tree of G with weights $w_{ij} = 1 - |n_i \cdot n_j|$
3. Set an arbitrary root r and traverse the spanning tree.
   For each edge (i, j) where i is the parent of j:
   if $-n_i \cdot n_j > n_i \cdot n_j$ then set $n_i = -n_i$

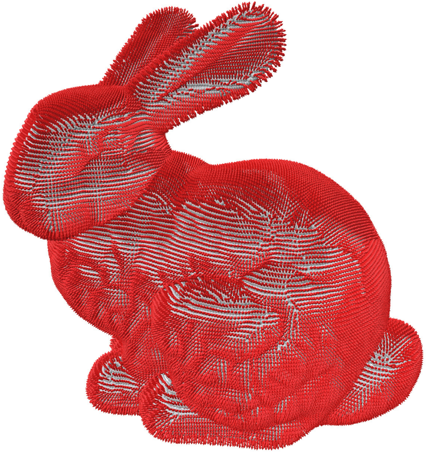
DEMO

Surface Reconstruction from Unorganized Points, Hoppe et al., 1992

Consistent Orientation



Faulty normal orientation
(raw hyperplane fitting)

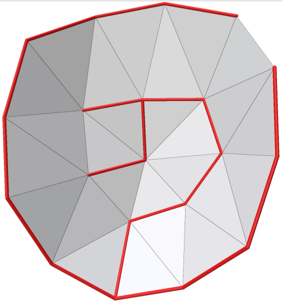


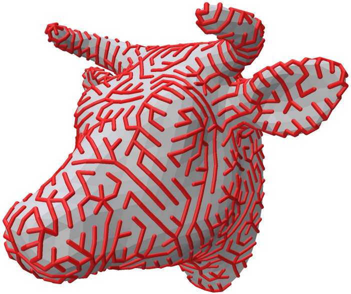
Consistent (outward) normal orientation

Minimal Spanning Tree

Let $G = (V, E)$ be a connected graph with weights $w_{ij} \in \mathbb{R}$ on edge (i, j).
A **Minimal Spanning Tree** $T = (V, E')$ is a subgraph of G such that:

- T is connected (there exists a path of edges between every vertex)
- $E' \subset E$ such that $\sum_{(i,j)\in E'} w_{ij}$ is minimal





Minimal Spanning Tree on surface meshes with Euclidean distance as edge weights

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Kruskal's algorithm

Input: A graph $G = (V, E)$
Output: A minimal spanning tree T of G

- $T = \varnothing$ set of sedges
- For each edge $(a, b) \in E$ in increasing weight order:
- If a and b were not already connecting by the tree T[a]:
- Add edge (a, b) to T

[a]Fast with a Union-Find data structure (https://fr.wikipedia.org/wiki/Union-find)

Estimating Density

Local lengths

- Distance to the closest neighbor
- Distance to the n-th neighbor
- Mean of distances to the k nearest neighbors
- etc.

Local areas

1. Consider the k nearest neighbors of a point p (or points at a distance $< r$ from p)
2. Project all points onto the tangent plane of p ($\perp$ to $\overrightarrow{n}_p$)
3. Compute the 2d Delaunay triangulation
4. Return the sum of areas of the Delaunay triangles

Fast Winding Numbers for Soups and Clouds, Barill et al., 2018

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DEMO

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Outlier removal

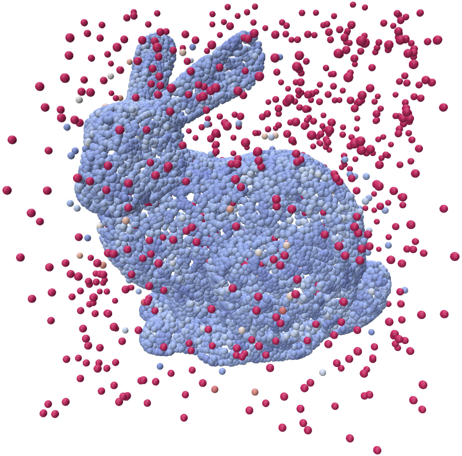
Observation

Density of outlier points is usually lower than density of inliers.

Simple outlier detection

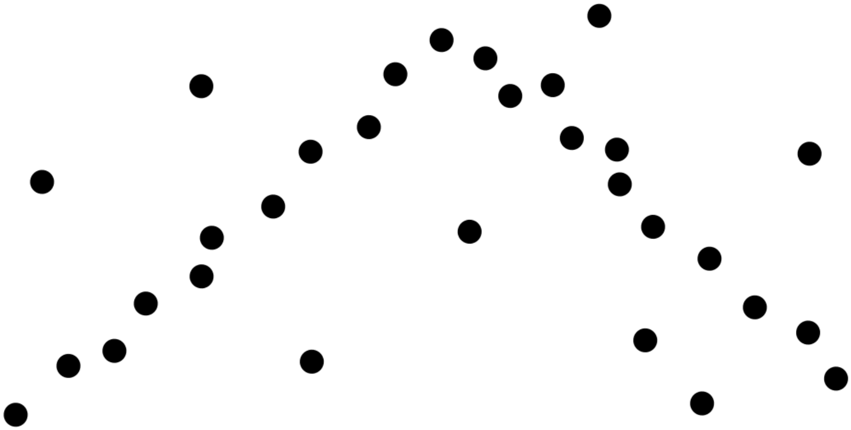
Input: points $P = \{p_1, ..., p_N\}$
Remove points whose local area is $> \eta$

DEMO



Clustering and Primitive Fitting

How to fit a plane in the presence of many outliers?



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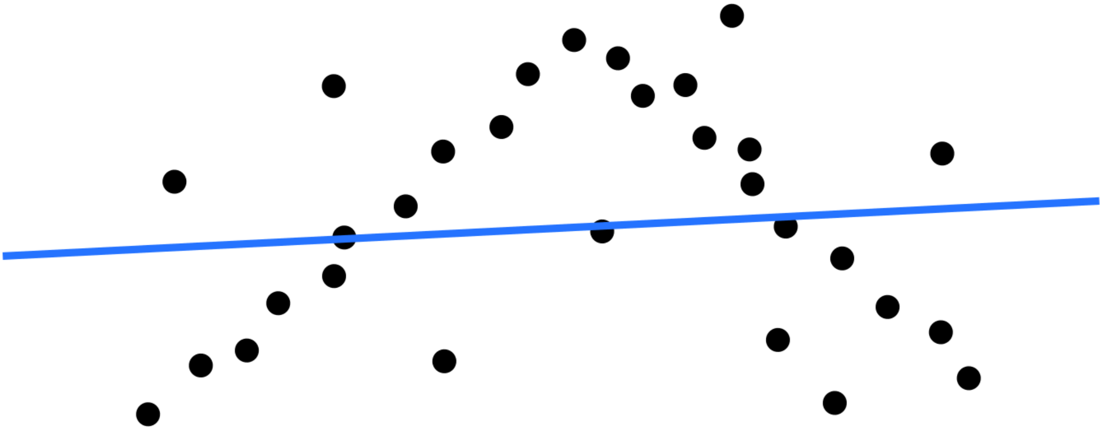

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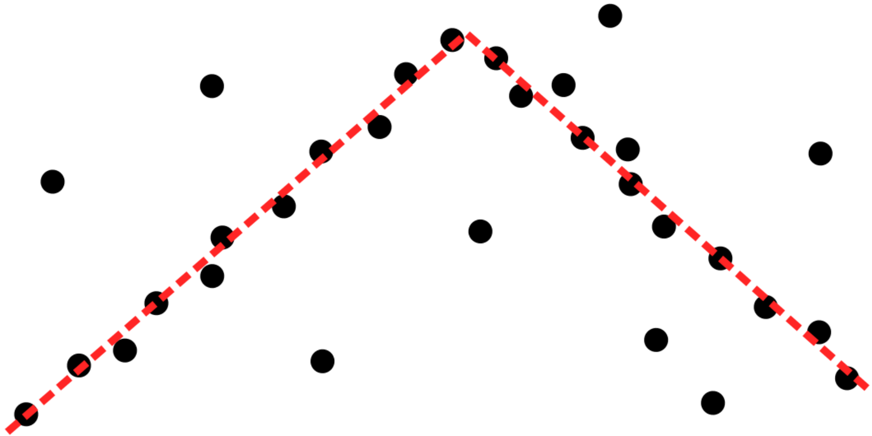

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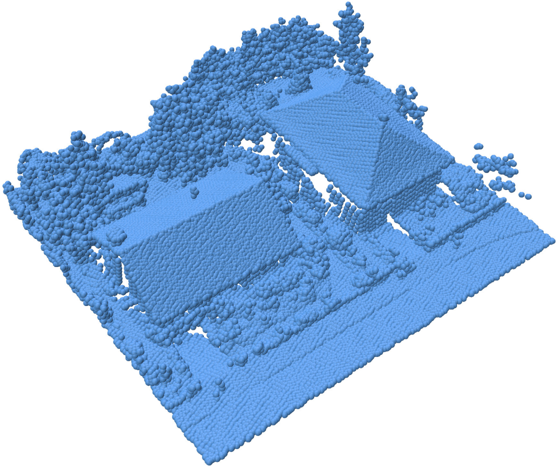
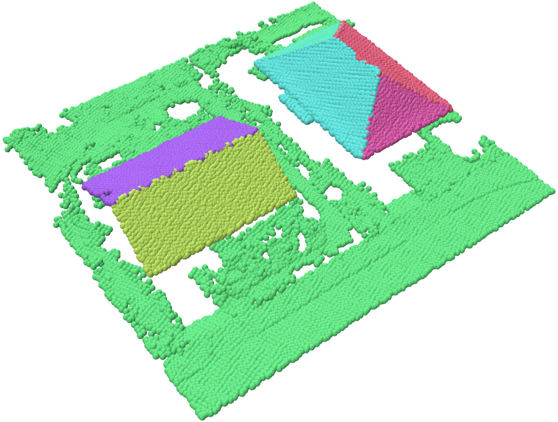




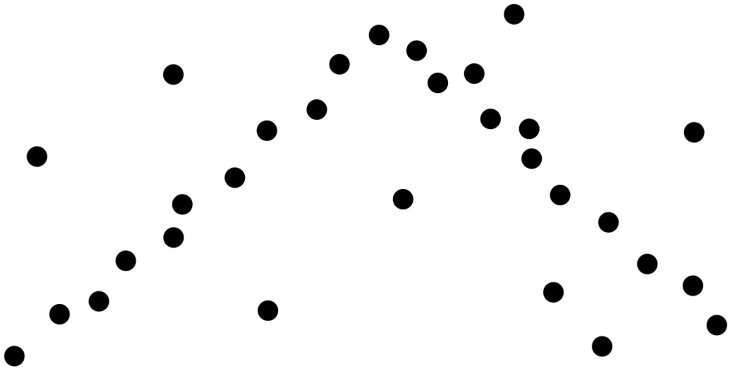
General Idea: Sample and reject random candidates

RANSAC Algorithm (RANdom SAmple Consensus)

Input: points $P = \{p_1, ..., p_N\}$
Parameters: sample size K, minimal cluster size S, inlier threshold τ
Iterate:

- Sample K planes (triplets of points in P)
- For each plane i, compute its set P_i of inliers (points in P at distance $< \tau$)
- Find the plane i_0 with the most inliers.
- If $|P_{i_0}| \geqslant S$:
  Add the plane to the found primitives
  Remove the points P_{i_0} from the samplable points ($P \leftarrow P \backslash P_i$)

Executing the RANSAC algorithm on a 2D example



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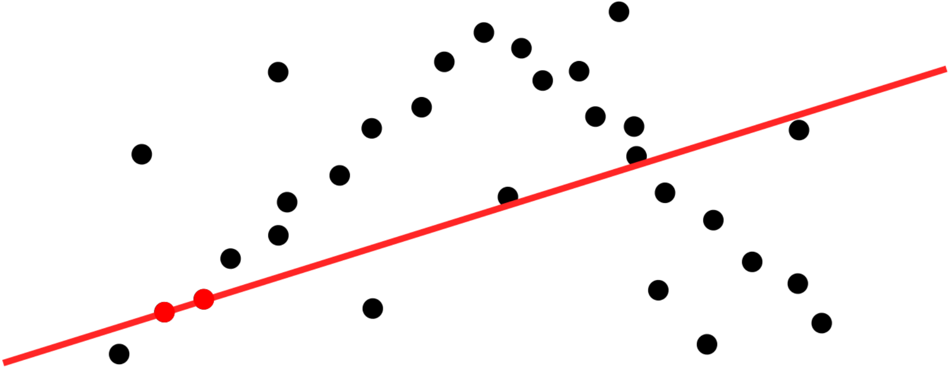

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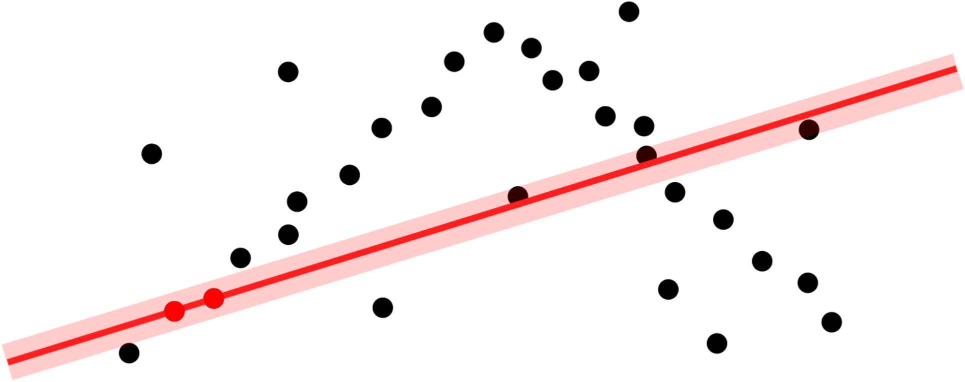

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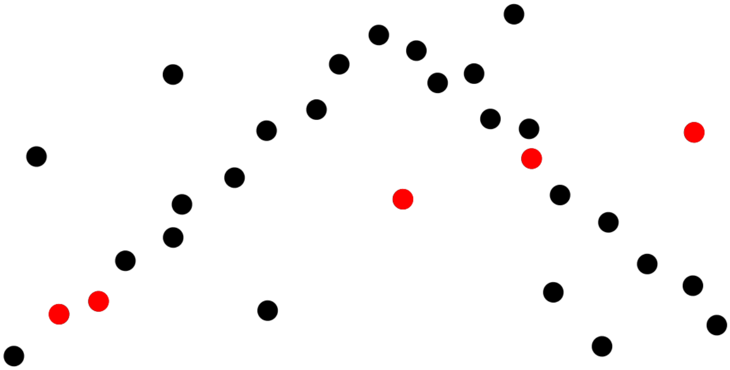

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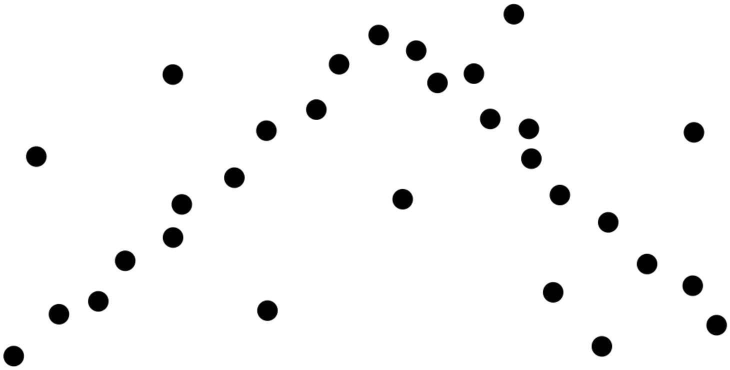

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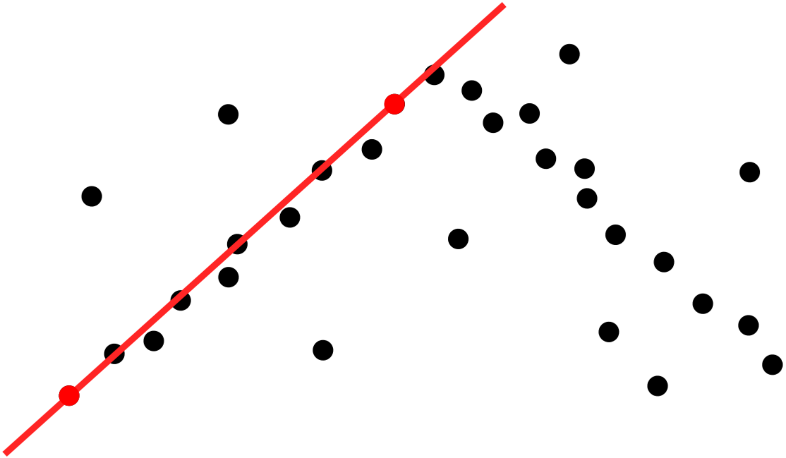

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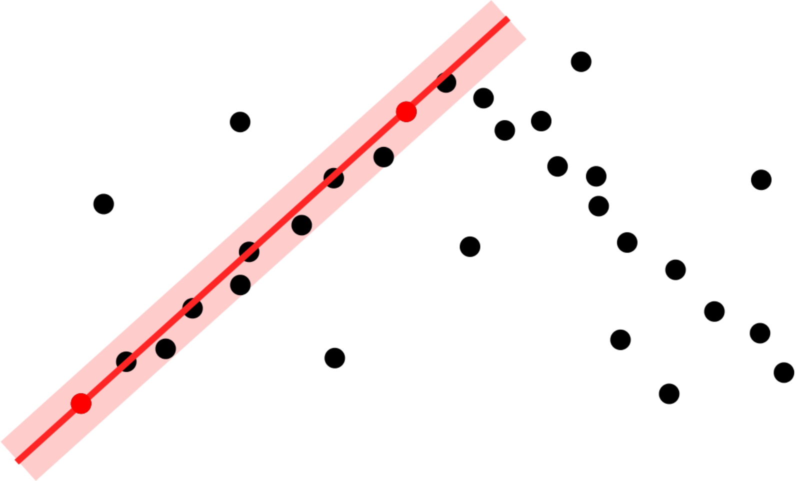
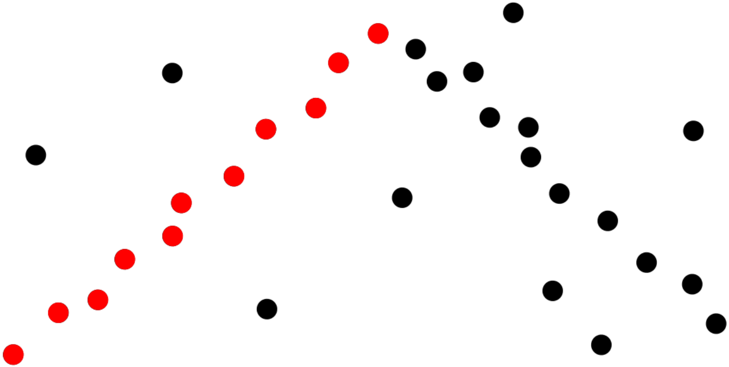


Executing the RANSAC algorithm on a 2D example



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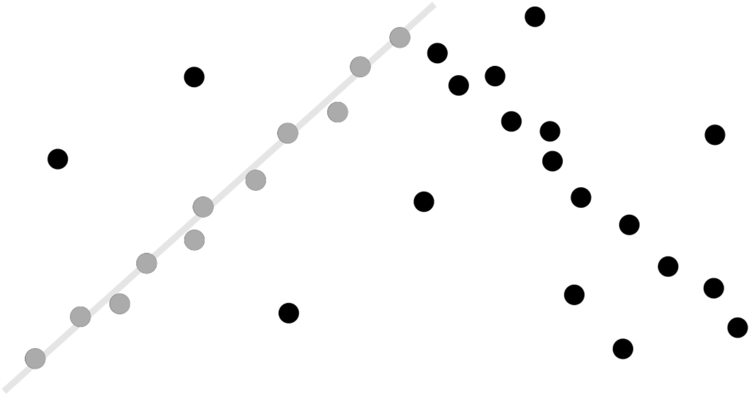

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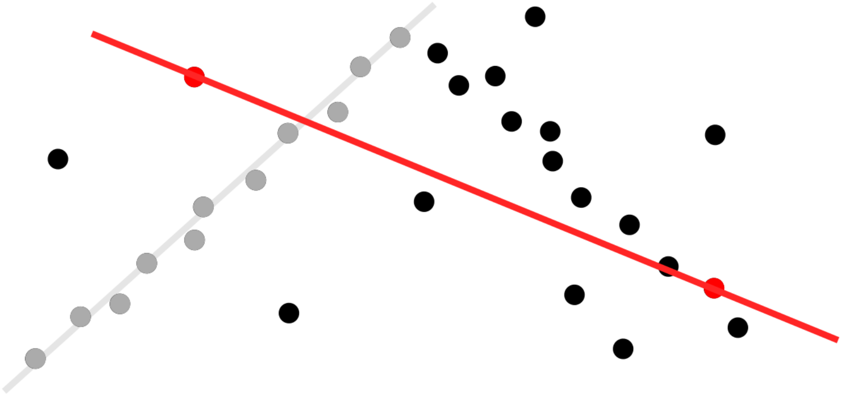

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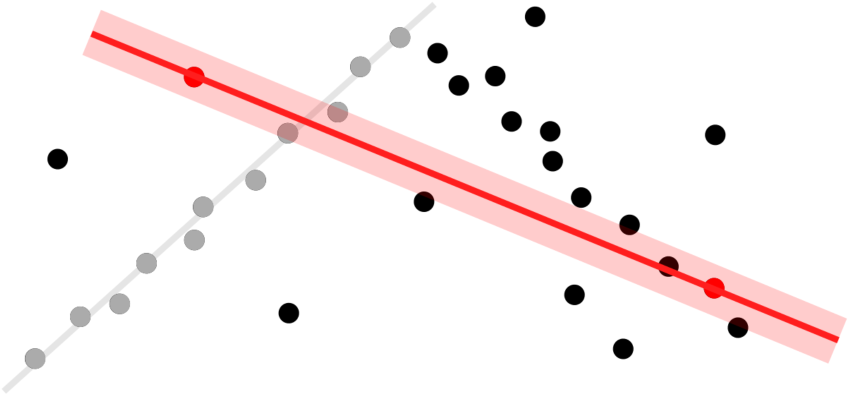

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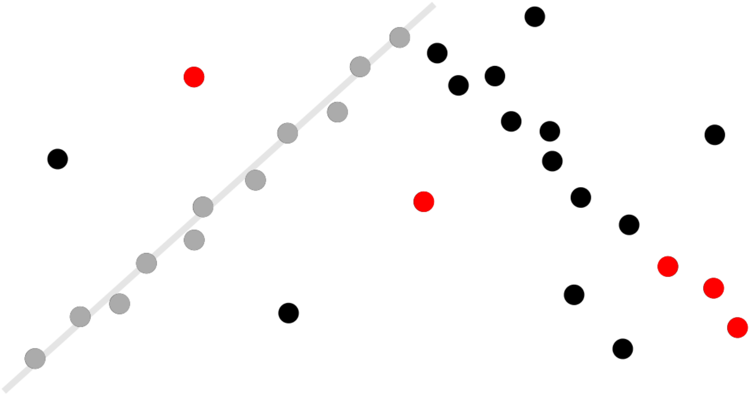

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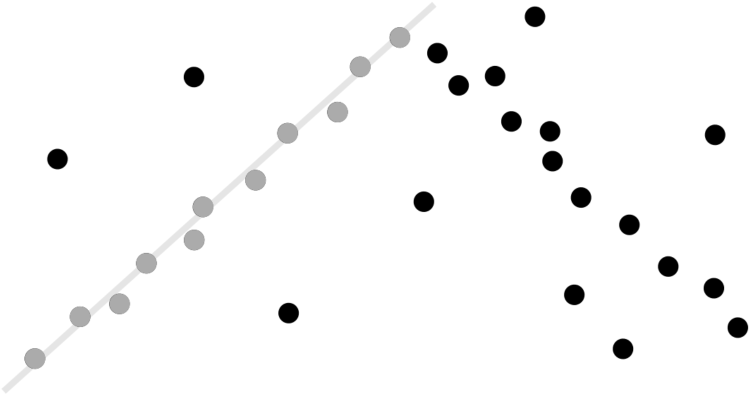

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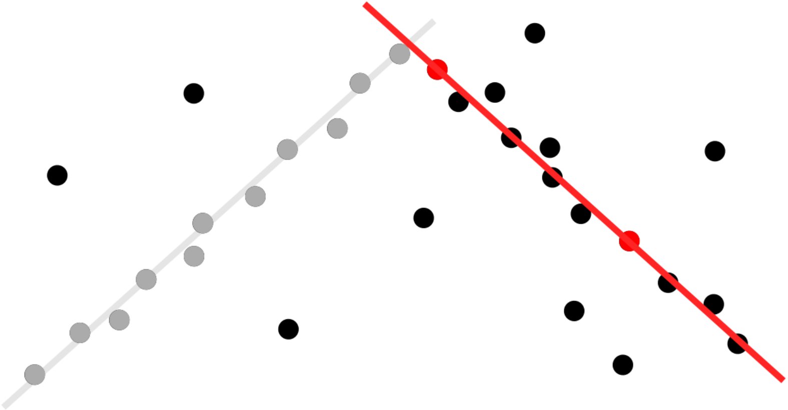

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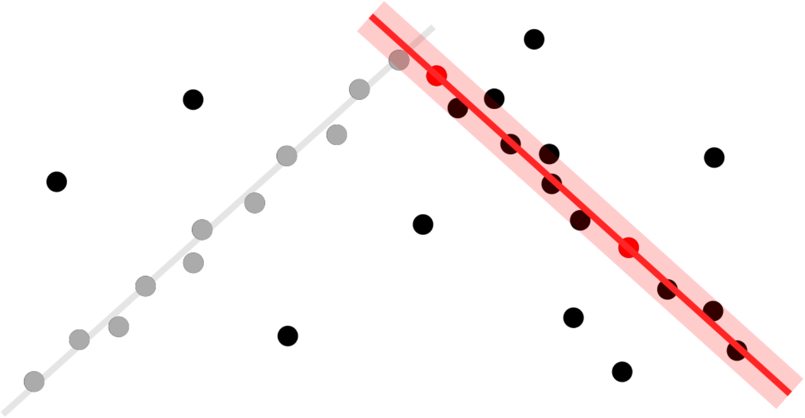

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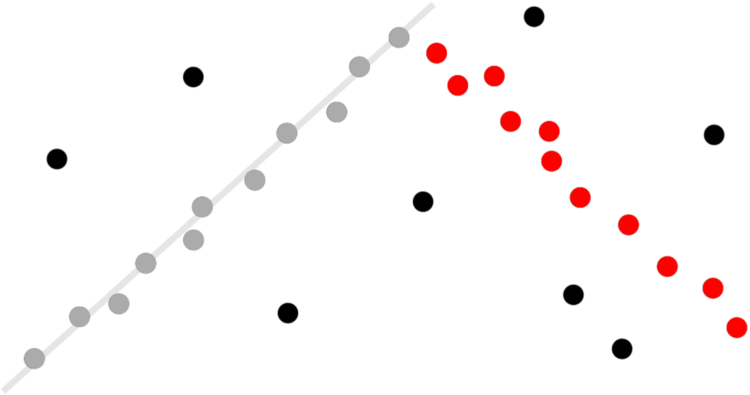

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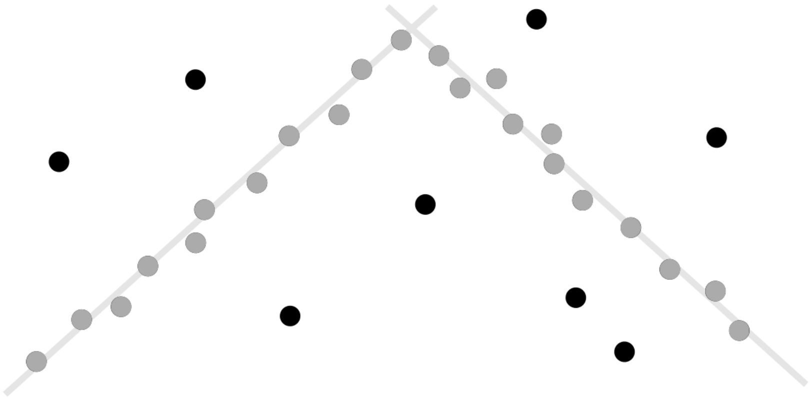

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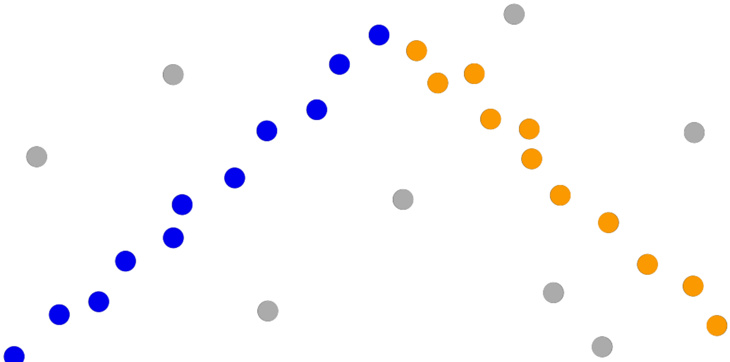


Improvements

Locality bias

Planes have more chance to be a valid primitive if all three points are "close" together. $\hookrightarrow$ Sample first point and two others among its nearest neighbors

Consider connected components

When computing inliers, only consider points in the largest connected component of the k-NN graph

Consider normals

A point is an inlier if small distance and its normal aligns (dot product with plane's normal $> \delta$)

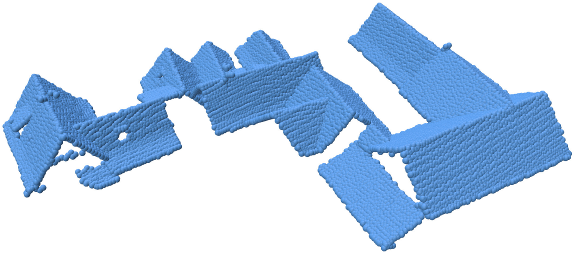
Other primitives

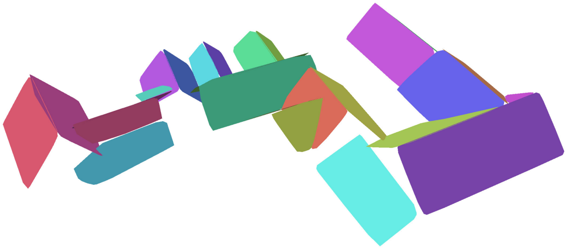
Also works when sampling spheres (4 points), cylinders, cones, etc...

Efficient RANSAC for Point-Cloud Shape Detection, Schnabel et al., 2007

Presentation of the Project

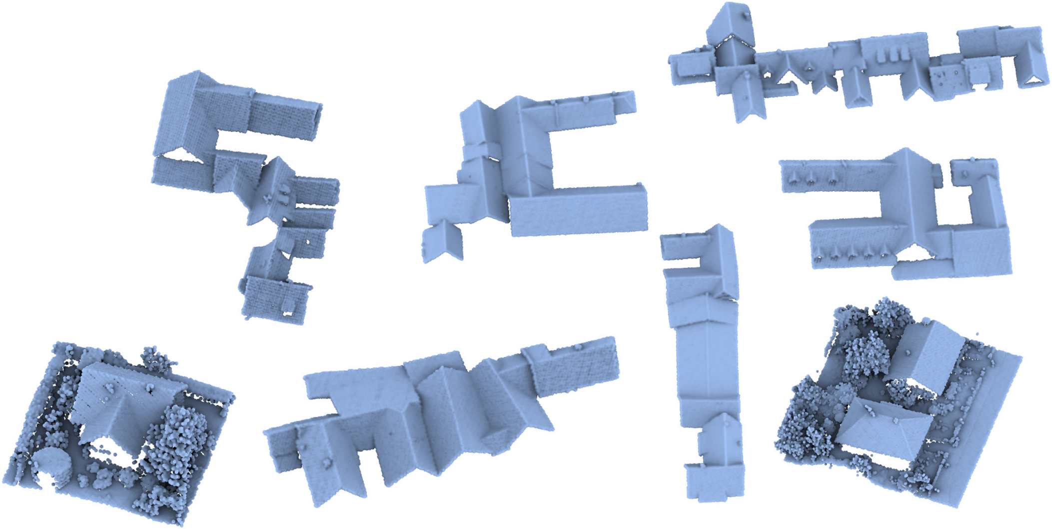
Finding plane primitives in aerial LIDAR point clouds





DEMO

Input Data: List of (x, y, z) coordinates in a text file



Features to implement in python (baseline)

- K-nearest neighbors
- Normal estimation
- RANSAC with some heuristics
- Delaunay triangulation of the primitives

... and lots of possible improvements

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... and lots of possible improvements

Evaluation

An oral evaluation with us where you show us a demo (and we ask you questions!)

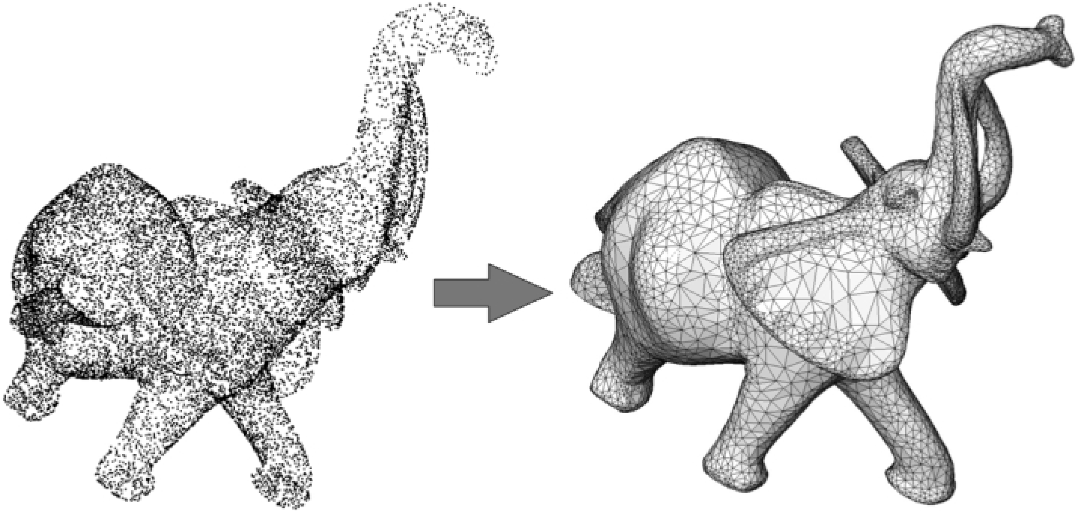
https://www.youtube.com/watch?v=X0vC3slzDmc

Part 2: Surface Reconstruction

5. Reconstruction algorithms based on Delaunay triangulations
    - α-shapes
    - Ball Pivoting
    - CRUST
    - Implicit Representations of Geometry

6. Reconstruction algorithm based on implicit representations
    - Poisson Surface Reconstruction
    - Generalized Winding Number

From point cloud to surface meshes



https://doc.cgal.org/Manual/3.5/doc_html/cgal_manual/Surface_reconstruction_points_3/Chapter_main.html

Reconstruction algorithms based on Delaunay triangulations

How to choose k in the k-NN graph?

Observation

The edges/faces that we want in the reconstruction are in the k-NN graph for some k large enough.

DEMO

How to choose k in the k-NN graph?

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The edges/faces that we want in the reconstruction are in the k-NN graph for some k large enough.

DEMO

A naive reconstruction algorithm (2D)

Keep edges of the 2-NN graph of $P = \{p_1, ..., p_N\}$

What about Delaunay?

Observation

The edges/faces that we want in the reconstruction are edges/faces of the Delaunay triangulation/tetrahedrization.

DEMO

What about Delaunay?

Observation

The edges/faces that we want in the reconstruction are edges/faces of the Delaunay triangulation/tetrahedrization.

DEMO

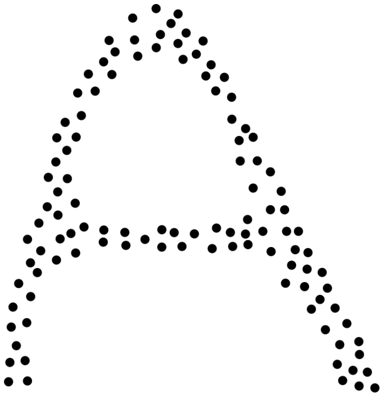
A (slightly less) naive reconstruction algorithm

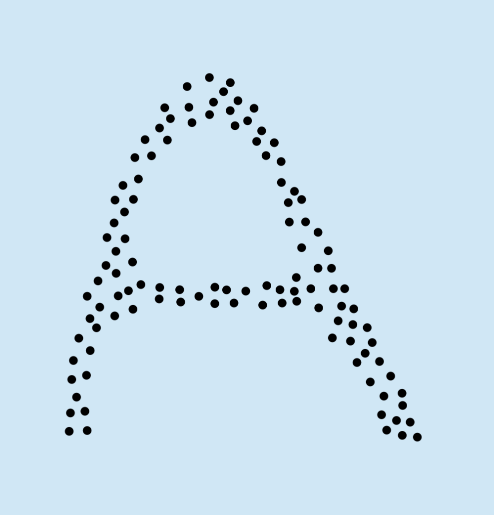
1. Compute the Delaunay Triangulation of $P = \{p_1, ..., p_N\}$
2. Keep edges with length $< L$

α-shapes

The intuition

Given points $P = \{p_1, ..., p_N\} \in \mathbb{R}^d$, remove all possible spheres of radius α that does not contain any point.





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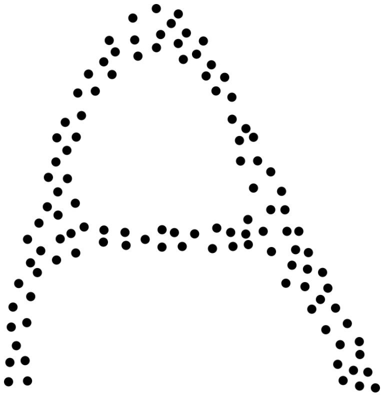

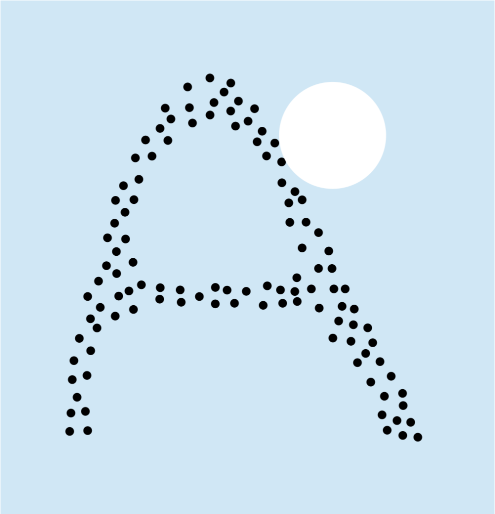

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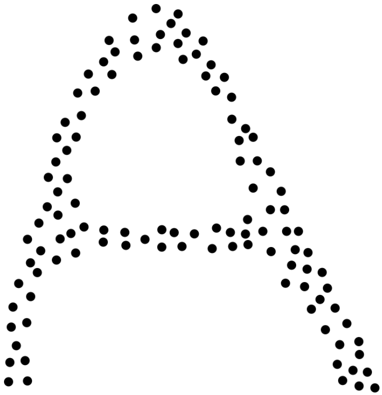

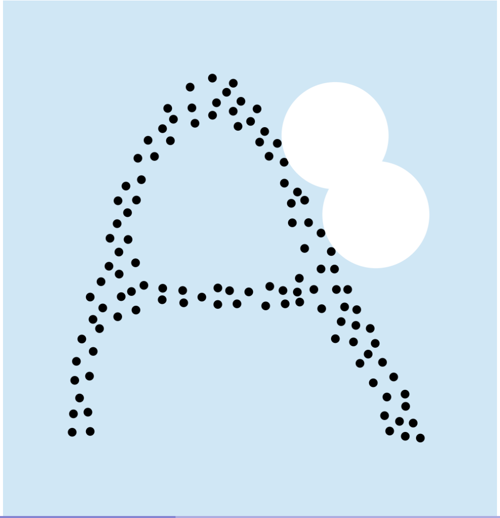

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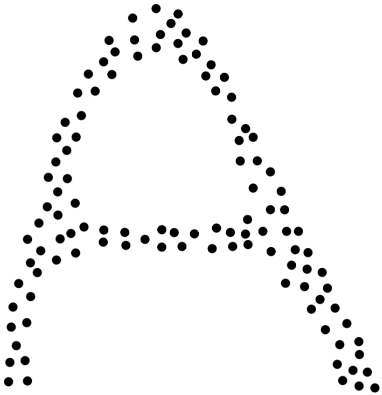

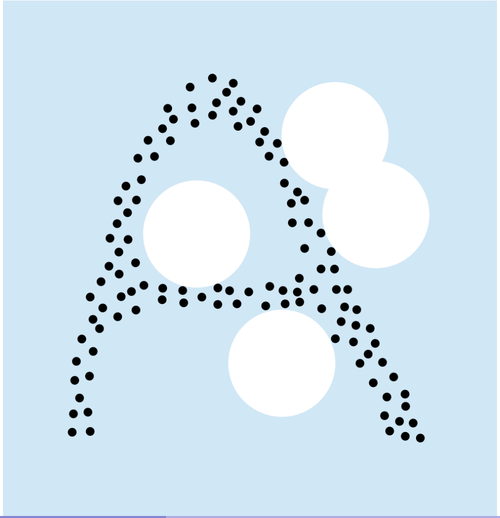

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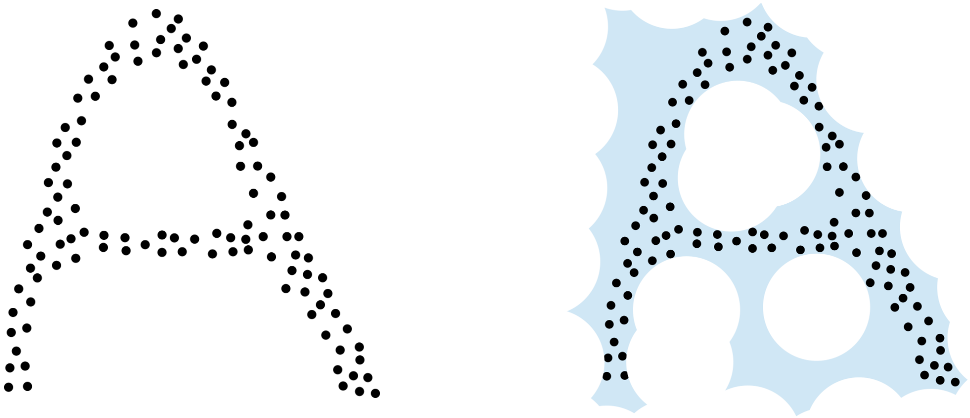

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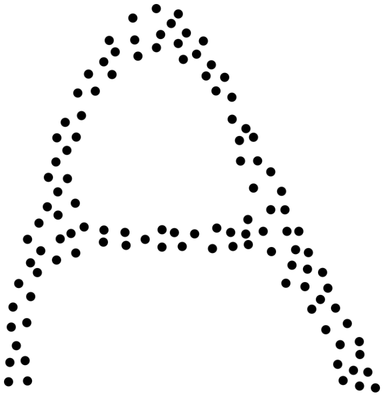
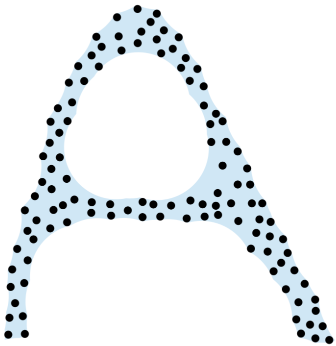
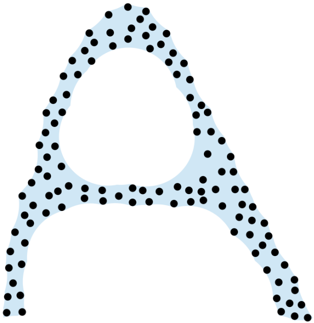




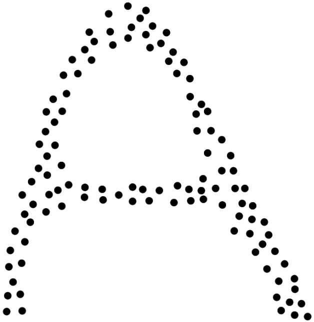
α-hull



The α-**hull** of P is what's left when we have removed all possible empty circles of radius α

On the Shape of a Set of Points in the Plane, Edelsbrunner et al., 1983

α-hull



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When $\alpha \to 0$, we get only P

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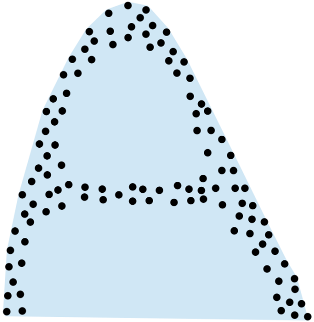
α-hull



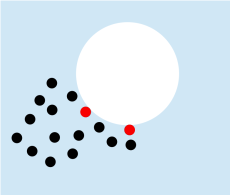
The α-**hull** of P is what's left when we have removed all possible empty circles of radius α

When $\alpha \to 0$, we get only P

When $\alpha \to \infty$, we get the convex-hull of P

On the Shape of a Set of Points in the Plane, Edelsbrunner et al., 1983

α-shape



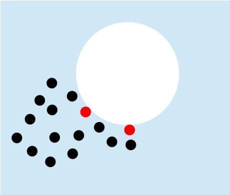
A point $p \in P$ is an α-**exposed** if there exists an empty circle of radius α such that p is on its boundary.

An edge (p_1, p_2) is α-**exposed** if there exists an empty circle of radius α such that *both* points are on its boundary.

On the Shape of a Set of Points in the Plane, Edelsbrunner et al., 1983

Three-dimensional alpha shapes, Edelsbrunner and Mücke, 1994

α-shape



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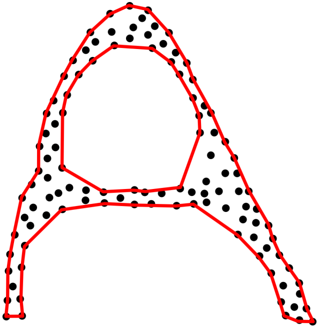
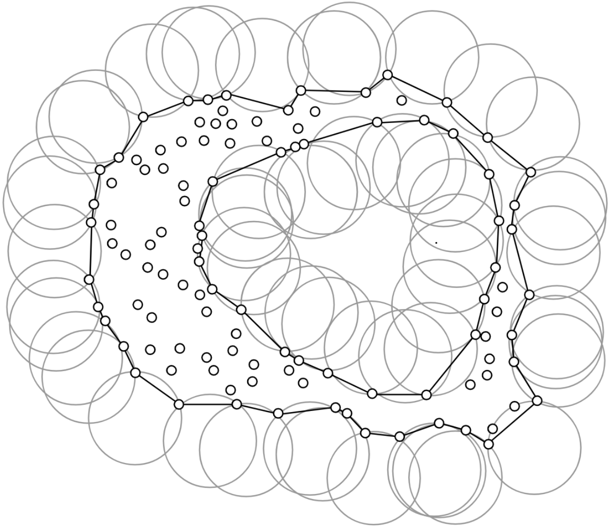


The α-**shape** is the straight line graph made of all α-exposed edges between two points of P.

On the Shape of a Set of Points in the Plane, Edelsbrunner et al., 1983
Three-dimensional alpha shapes, Edelsbrunner and Mücke, 1994

α-shape



https://graphics.stanford.edu/courses/cs268-11-spring/handouts/AlphaShapes/as_fisher.pdf

Properties of α-shapes

Theorem

For any α, the α-shape of P is a subgraph of the Delaunay triangulation of P.

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Let p and q be two α-exposed points. There exists a circle of radius α not containing any point $r \in P$ such that p and q are on its boundary. Let c be the center of this circle. Clearly, $d(p,c) = d(q,c) \leqslant d(r,c)$ for any $r \in P \backslash \{p,q\}$. This means that c is in both the Voronoi cells $Vor(p)$ and $Vor(q)$, which means that those cells are touching. In other words, p and q are neighbors in the (dual) Delaunay triangulation.

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Consequence

There exists only a finite number of different α-shapes of P when α goes from 0 to ∞.

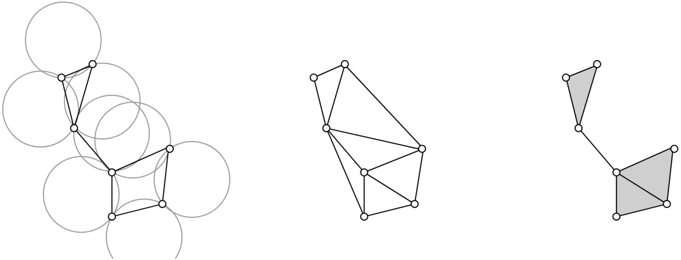
Computing α-shapes: the α-complex

https://demonstrations.wolfram.com/AlphaComplexAndUnionOfGrowingDisks/

Which Delaunay triangle belong inside the α-shape?

Let S be a simplex (segment, triangle, tetrahedra,...). Let its circumsphere be centered at μ_S with radius σ_S.
S is in the α-**complex** if:

- S is on the boundary of a simplex S' of the α-complex, or
- $\sigma_S < \alpha$ and the sphere centered in μ_S of radius σ_S is empty



Computing α-shapes: the α-complex

The α-shape is the boundary of the α-complex

Three-dimensional alpha shapes, Edelsbrunner and Mücke, 1994

Computing α-shapes: the α-complex

The α-shape is the boundary of the α-complex

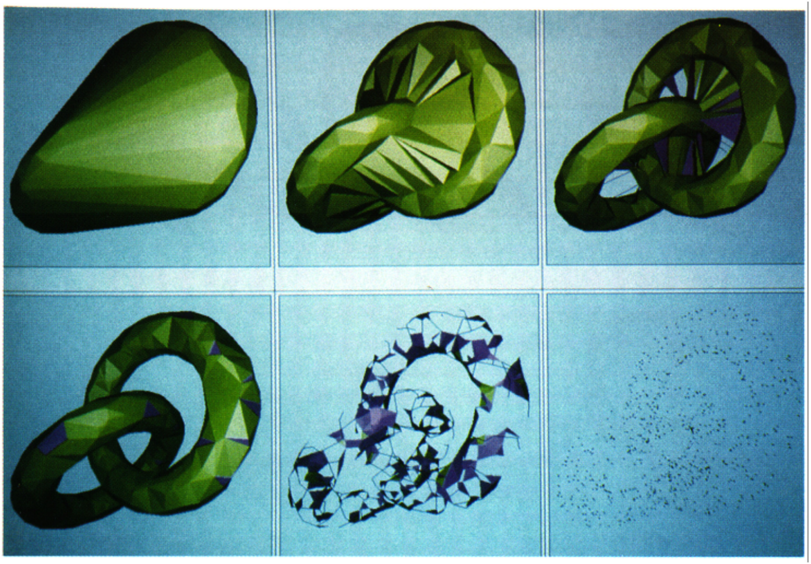
Edelsbrunner's algorithm

Input: points $P = \{p_1, ..., p_N\}$

1. Compute the Delaunay triangulation $D = (V, E, T)$ of P
2. Determine the set $\mathcal{C}_\alpha$ of triangles T inside the α-shape
3. Return the outside boundary of $\mathcal{C}_\alpha$

DEMO

Three-dimensional alpha shapes, Edelsbrunner and Mücke, 1994



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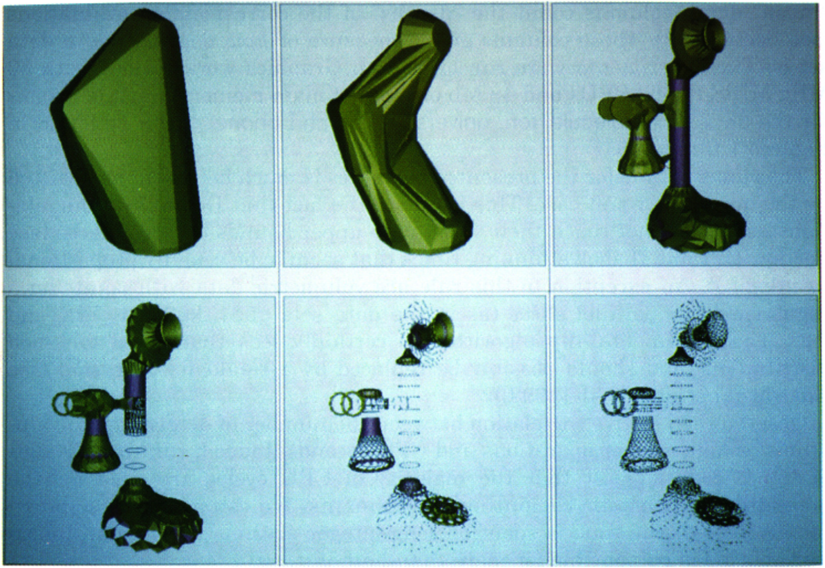

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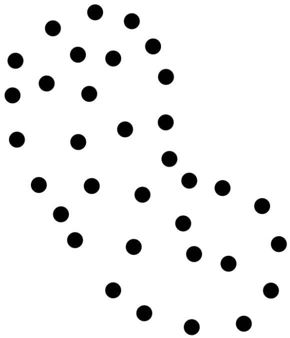
Ball Pivoting

2D example



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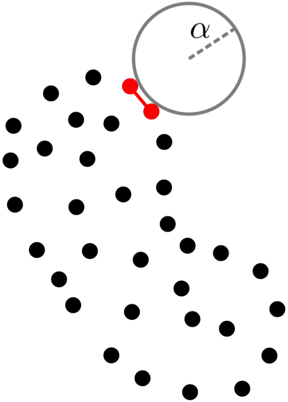

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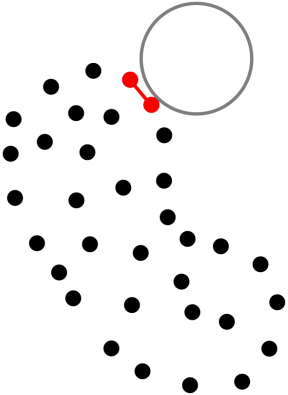

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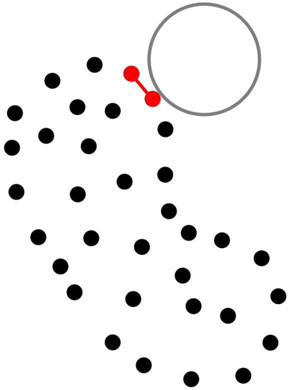

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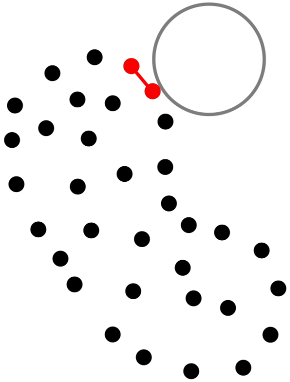

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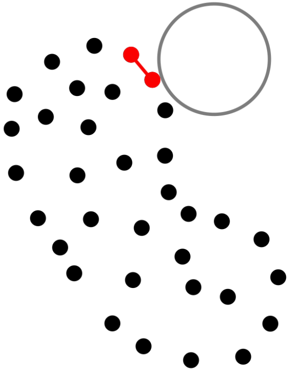

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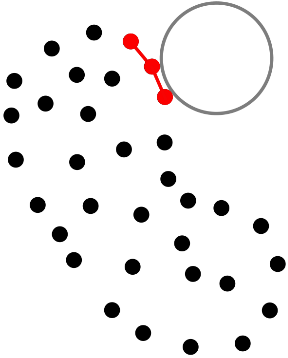

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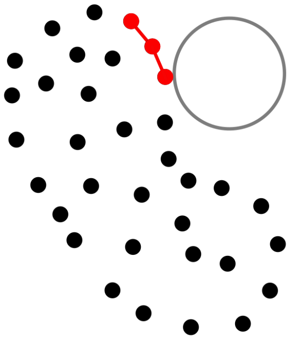

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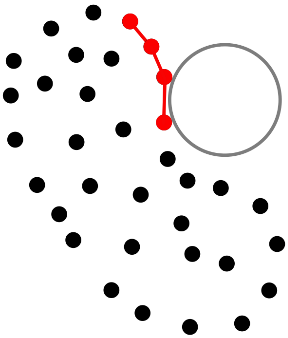

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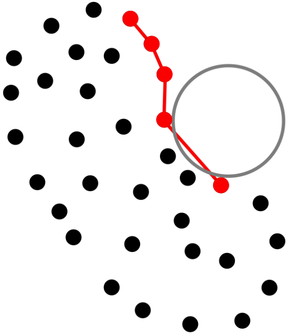

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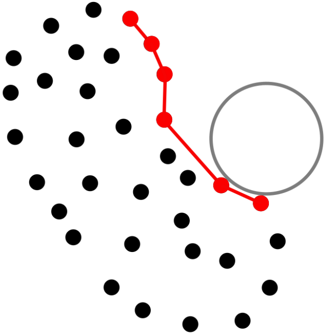

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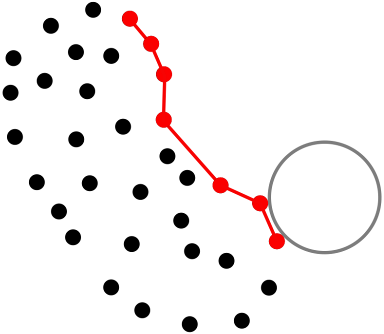

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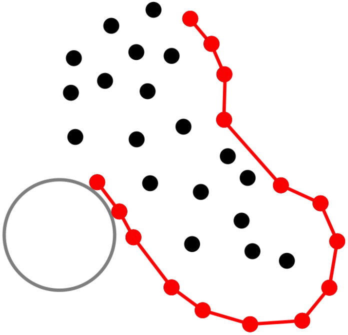

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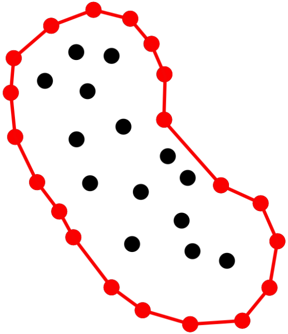

Ball Pivoting

Algorithm in 2D

Consider a ball of radius α.
Find an α-exposed edge $e = (p_0, p_1)$. Add edge e to reconstructed edges.
Set $p = p_1$ as the pivot.
Iterate:

- Pivot the ball around point p until another unvisited point q is touched
- If no point q can be reached, stop.
- Otherwise, add (p, q) as a reconstructed edge. Set p as the new pivot.

The Ball-Pivoting Algorithm for Surface Reconstruction, Bernardini et al., 1999

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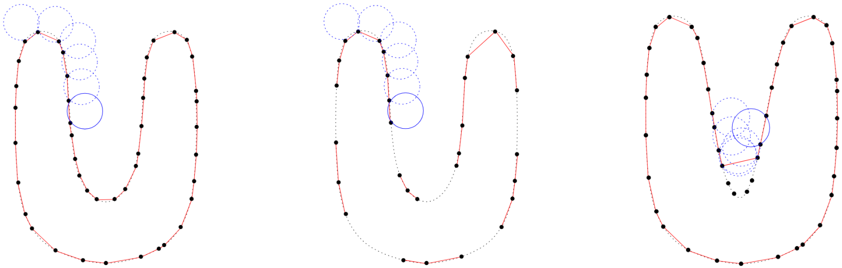
It generalizes naturally to surfaces in 3D (pivot around an edge), though normals are needed to repair some bad cases

The Ball-Pivoting Algorithm for Surface Reconstruction, Bernardini et al., 1999

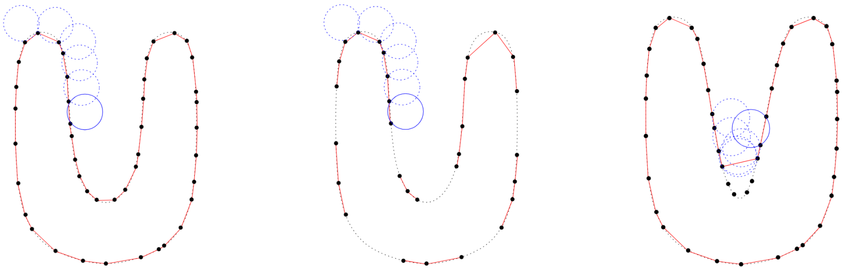
Properties



(a) (b) (c)

Every edge found by the BPA with radius α is α-exposed

Properties



(a) (b) (c)

Suppose the existence of an underlying manifold M from which the points are sampled. If:

- The intersection of any ball of radius α with M is a topological disk and
- Any ball of radius α centered on M contains at least one point

then the reconstructed surface is manifold with the correct topology.

Algorithmic perspective

Computing the next intersected point

Let p be the current pivot, r be the previous pivot and c being the center of the α-ball.

- Query the unvisited neighbors $q_1, ..., q_k$ of p at distance $< 2\alpha$
- Compute the centers $c_1, ..., c_k$ of the touching spheres for each of them
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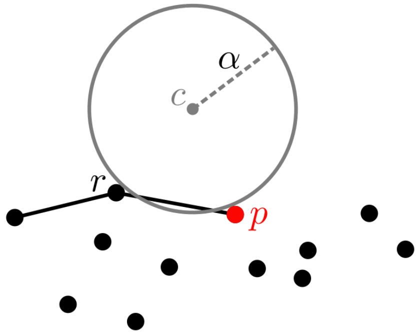

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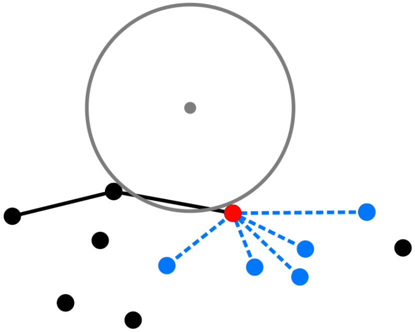

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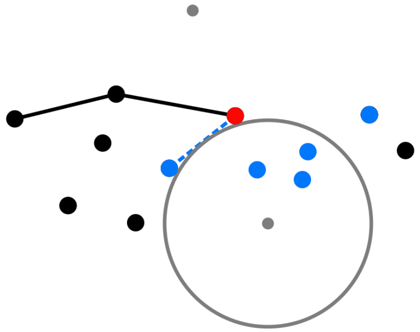

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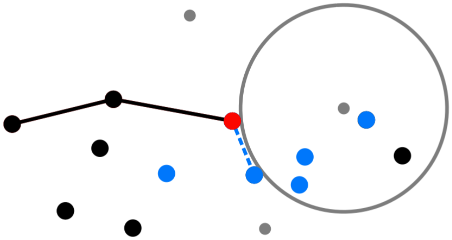

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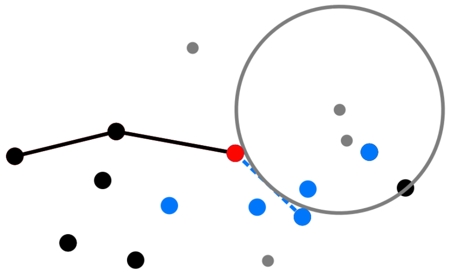

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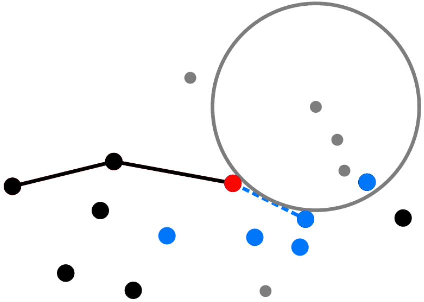

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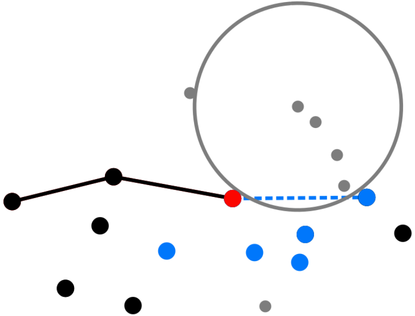

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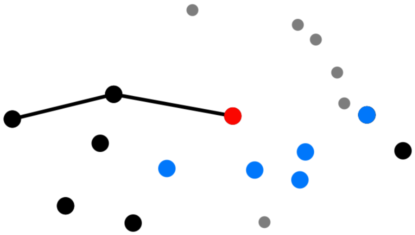

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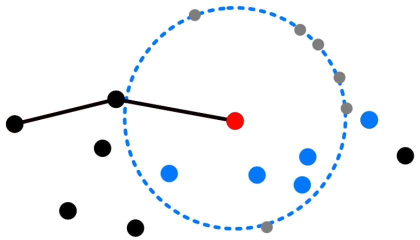

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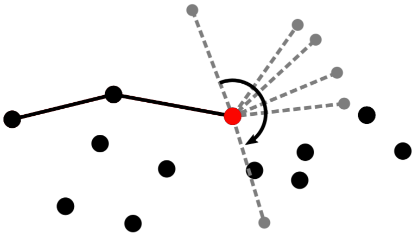

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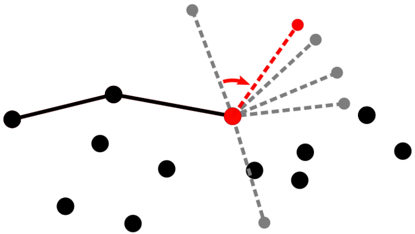

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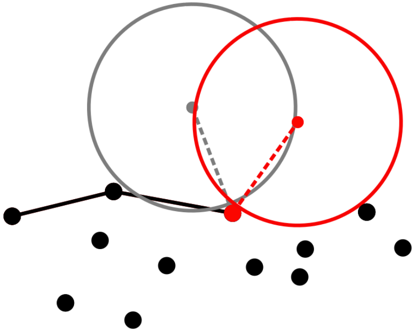

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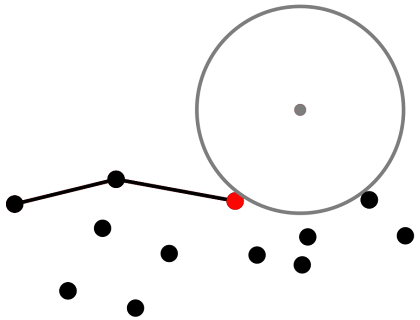

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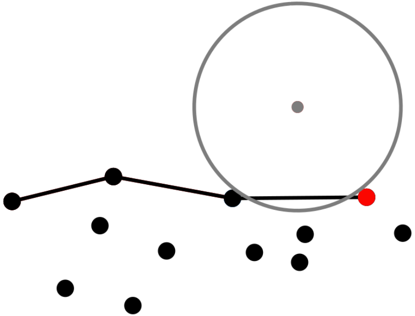

The Ball-Pivoting Algorithm for Surface Reconstruction, Bernardini et al., 1999

CRUST

CRUST

Idea

Delaunay reconstruction but prevent edges from "crossing" inside the domain

CRUST algorithm

Input: points $P = \{p_1, ..., p_N\}$

1. Compute the Voronoi diagram V of P.
2. Compute the Delaunay triangulation D of points $P \cup S$ where S are the vertices of the Voronoi diagram V
3. Return edges of D that link two points of P

A New Voronoi-based Surface Reconstruction Algorithm, Amenta et al., 1998

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DEMO

A New Voronoi-based Surface Reconstruction Algorithm, Amenta et al., 1998

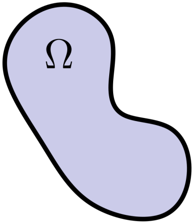
Reconstruction algorithm based on implicit representations

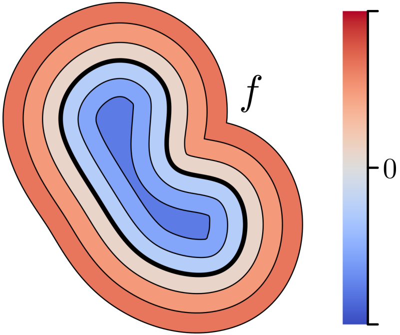
Implicit Representations of Geometry

Implicit Representation of Geometry

Represent a compact object $\Omega \subset \mathbb{R}^d$ as a level set of a continuous function:

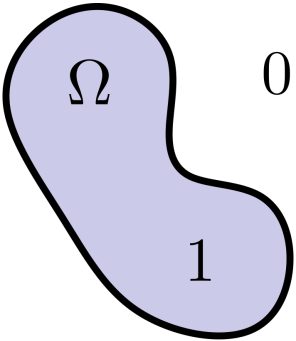
$$\Omega = \{x \in \mathbb{R}^d \mid f(x) \leqslant 0\}$$





Indicator Function

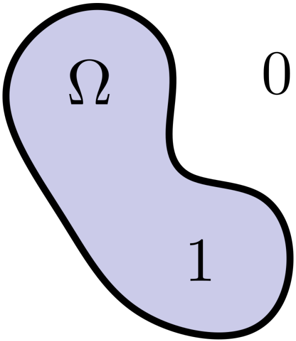
$$f(x) = \begin{cases} 1 \text{ if } x \in \Omega \\ 0 \text{ otherwise} \end{cases}$$



Indicator Function

$$f(x) = \begin{cases} 1 \text{ if } x \in \Omega \\ 0 \text{ otherwise} \end{cases}$$

- Simplest possible function
- Not differentiable...
- Not always easy to compute



Indicator Function

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- Simplest possible function
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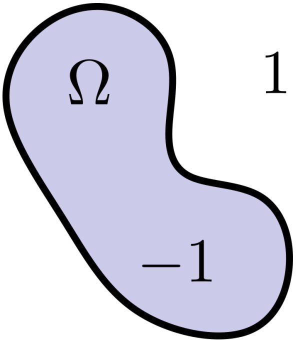
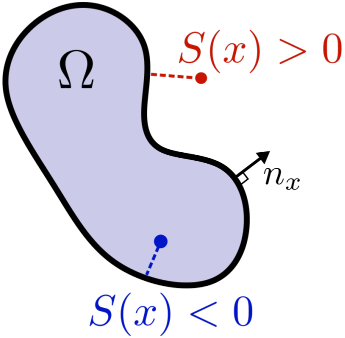
A variant: the sign function

$$y(x) = 1 - 2f(x) = \begin{cases} -1 \text{ if } x \in \Omega \\ 1 \text{ otherwise} \end{cases}$$



Signed Distance Function

$$S(x) = \begin{cases} -d(x, \partial\Omega) & \text{if } x \in \Omega \\ d(x, \partial\Omega) & \text{otherwise} \end{cases}$$

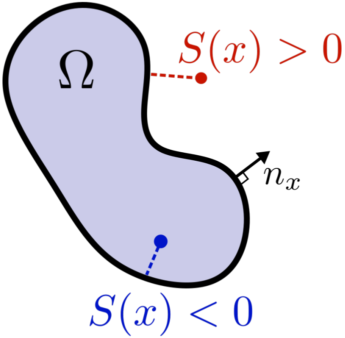


Signed Distance Function

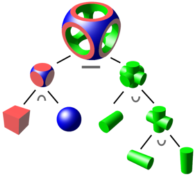
$$S(x) = \begin{cases} -d(x, \partial\Omega) & \text{if } x \in \Omega \\ d(x, \partial\Omega) & \text{otherwise} \end{cases}$$

Eikonal equation

$$\begin{cases} ||\nabla S(x)|| & = 1, \quad \forall x \in \mathbb{R}^d \\ S(x) & = 0, \quad \forall x \in \partial\Omega \\ \nabla S(x) & = n_x, \quad \forall x \in \partial\Omega \end{cases}$$

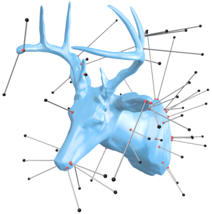


Applications



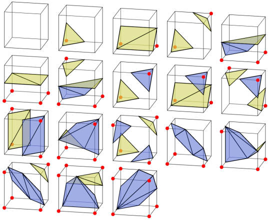
Constructive Solid Geometry

[Ricci (1973)]



Closest Point Query

[Sharp and Jacobson (2022)]

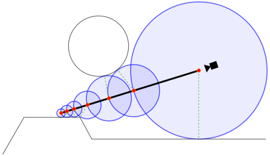


Marching Cubes

[Lorensen and Cline (1987)]

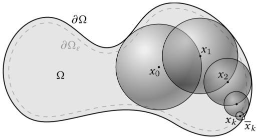
Rendering

Snail shader by Inigo Quilez



Empty Sphere Query

[Hart (1995)]



Monte-Carlo Simulation

[Sawhney and Crane (2020)]

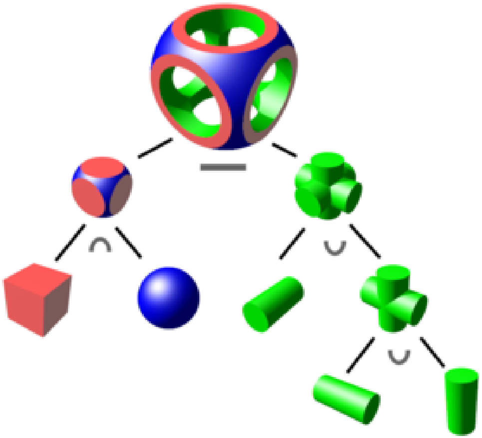
Constructive Solid Geometry

If $\Omega_a \leftrightarrow S_a$ and $\Omega_b \leftrightarrow S_b$:

- $-S_a$ represents $\overline{\Omega_a}$
- $\min(S_a, S_b)$ represents $\Omega_a \cap \Omega_b$
- $\max(S_a, S_b)$ represents $\Omega_a \cup \Omega_b$

$\min(S_a, S_b)$ is **not** a distance field[a]

[a] https://iquilezles.org/articles/interiordistance/

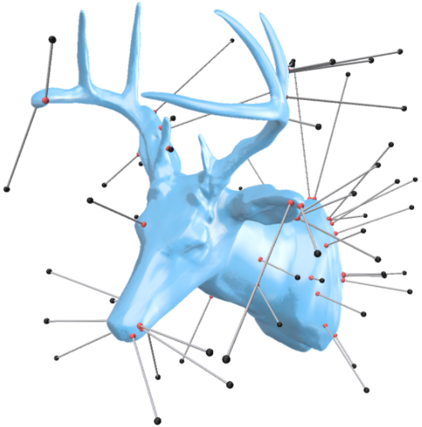


Geometrical Queries

For $x \in \mathbb{R}^d$ and $\Omega \leftrightarrow S$:

$$p = x - S(x)\nabla S(x)$$

is the closest point from x on Ω



[Sharp and Jacobson (2022)]

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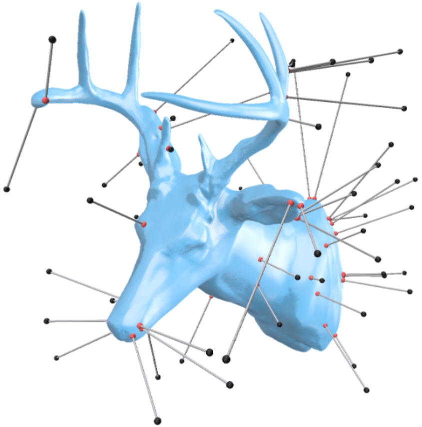
Approximated signed distance fields

This also works if $|f(x)| < S(x)$ and

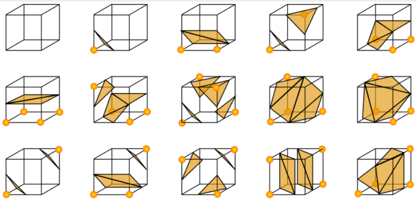
$$x_{n+1} = x_n - f(x_n)\nabla f(x_n)$$

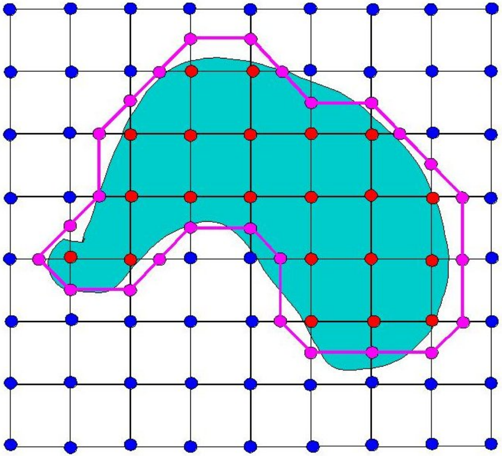


[Sharp and Jacobson (2022)]

From implicit functions to surface meshes: the Marching Cubes Algorithm

- Sample your implicit function over a grid
- For each cell, determine which points are inside/outside
- Mesh the cell according to a finite set of templates
- Possible improvements depending on the value of the function at grid vertices





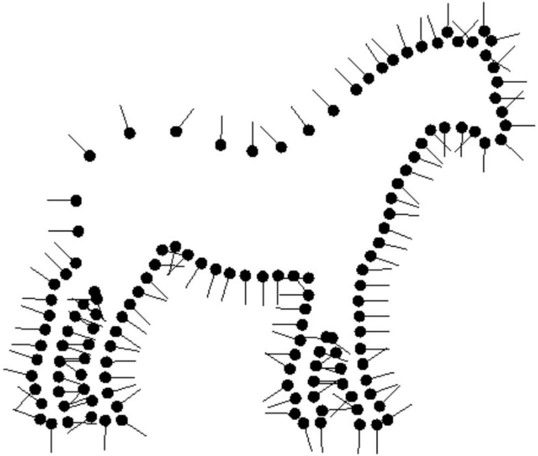
Marching Cubes: A High Resolution 3D Surface Construction Algorithm, Lorensen and Cline, 1987

Reconstruction algorithm based on implicit representations

Poisson Surface Reconstruction

Poisson Surface Reconstruction: the setting

Input: points with normals

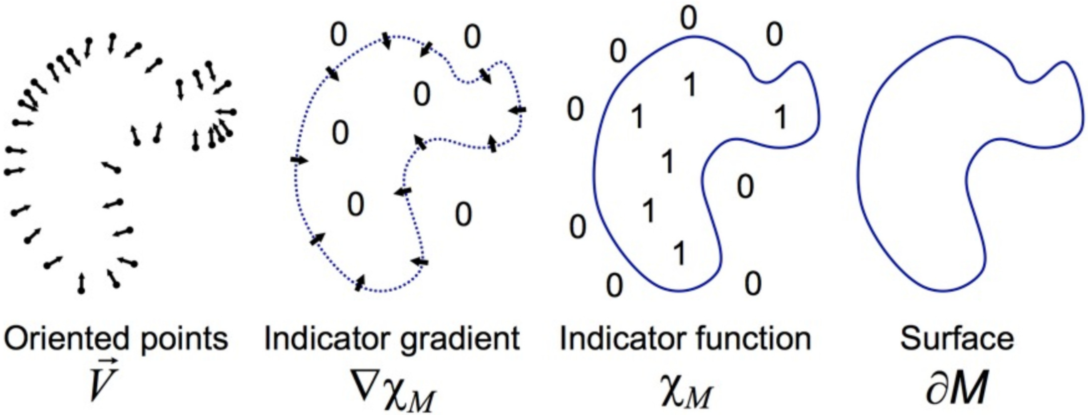


The idea

- Consider that each normal $\overrightarrow{n_p}$ at point p is the gradient ∇f of some implicit function f
- Integrate this gradient into the function f
- Recover the surface via marching cubes

Poisson surface reconstruction, Kazhdan et al., 2006

Poisson Surface Reconstruction: Principle



https://slides.cgg.unibe.ch/GP20/06_Surface_Reconstruction.html

Points $p_1, ...p_N$ with normals $\overrightarrow{n_1}, ..., \overrightarrow{n_N}$
Approximate a vector field v by $v(p_i) = \overrightarrow{n_i}$ and $v(x) = 0$ otherwise.
Solve:

$$\min_f \int_\Omega ||\nabla f(x) - \overrightarrow{n}(x)||^2 dx$$

i.e.:

$$\min_f \sum_{i=1}^{N} ||\nabla f(p_i) - \overrightarrow{n_i}||^2$$

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Poisson Problem

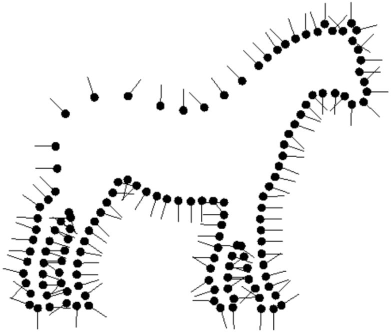
Vector field v is not integrable in general.
We apply the **divergence** operator (Euler-Lagrange equation):

$$\nabla.\nabla f = \Delta f = \nabla.\overrightarrow{n}$$

We recover a Poisson problem (of form $\Delta f = a$)

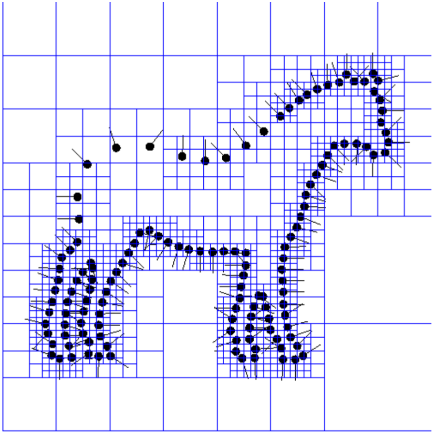
Implementation: Solving the Poisson problem on an octree via FEM

1. Consider a dataset of points p_i with normals n_i



Implementation: Solving the Poisson problem on an octree via FEM

1. Consider a dataset of points p_i with normals n_i
2. Setup an octree containing a single point per cell



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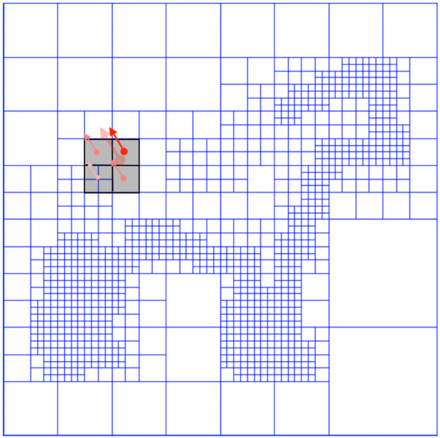
3. Splat the samples (define a FEM basis)



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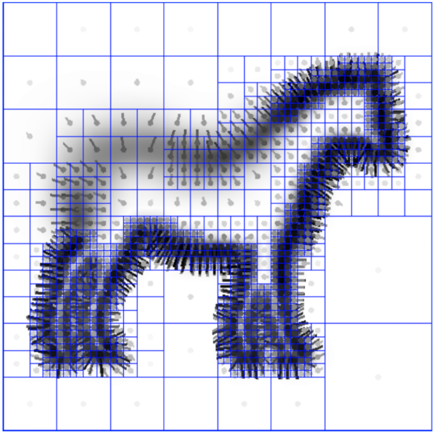
1. Consider a dataset of points p_i with normals n_i
2. Setup an octree containing a single point per cell
3. Splat the samples (define a FEM basis)



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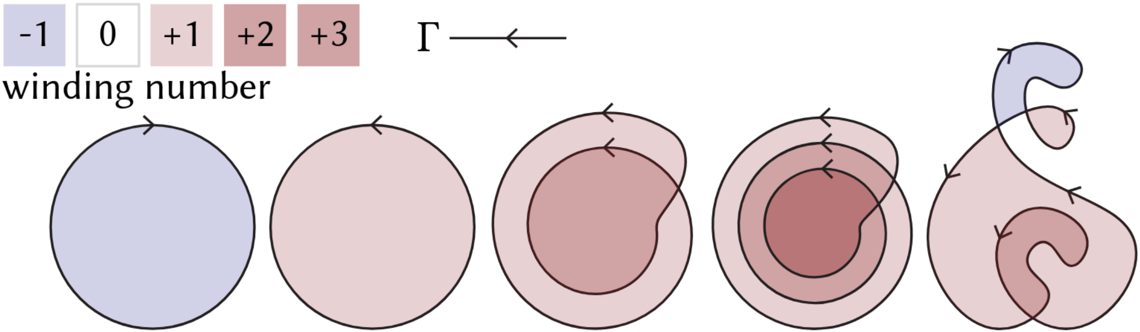
DEMO

- Result is a watertight manifold surface
- Can also be implemented on a mesh instead of an octree

- Needs consistently oriented normals
- Implementation is not trivial on an octree

Generalized Winding Number

Winding Number: another possible implicit representation



https://nzfeng.github.io/research/WNoDS/PerspectivesOnWindingNumbers.pdf

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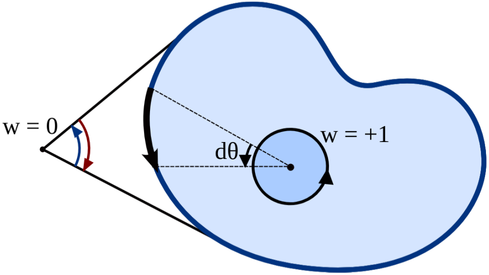
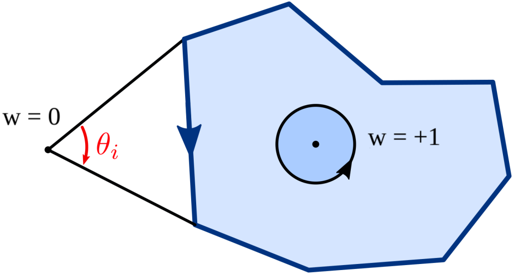


$$w(p) = \frac{1}{2\pi} \oint_{\mathcal{C}} d\theta$$

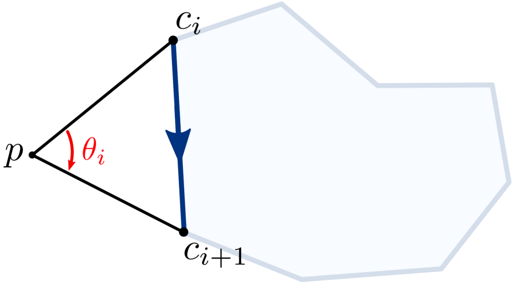


$$w(p) = \frac{1}{2\pi} \sum_i \theta_i$$

Computing Winding Number of Polylines

Let $a = c_i - p$ and $b = c_{i+1} - p$.

$$\tan(\theta_i) = \frac{\det(a,b)}{a.b} = \frac{a_x b_y - a_y b_x}{a_x b_x + a_y b_y}$$



$$w(p) = \frac{1}{2\pi} \sum_i \theta_i$$

Generalizing for imperfect geometries

Idea

Winding number is a sum of "solid angle" weighted by "area":

Robust Inside-Outside Segmentation Using Generalized Winding Numbers, Jacobson et al., 2013

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For points $p_1, ..., p_N$ with normals n_i and local areas a_i, the generalized winding number at point q is:

$$w(q) = \frac{1}{4\pi} \sum_i a_i \frac{(q - p_i).n_i}{||q - p_i||^3}$$

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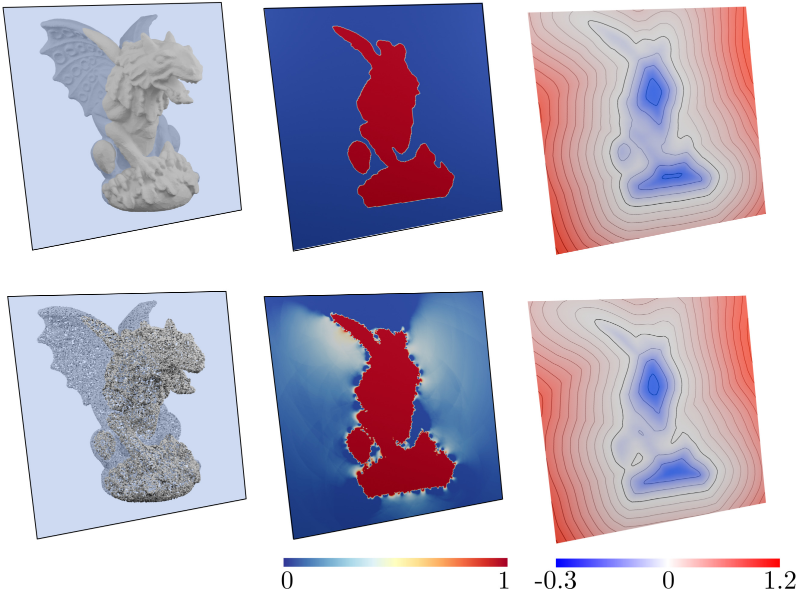
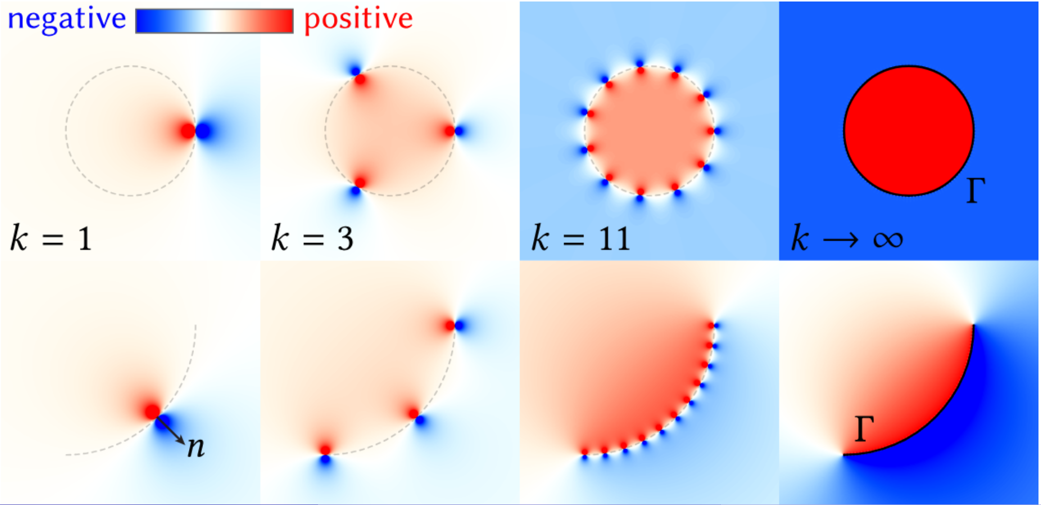
A good representation of the underlying surface of $p_1, ..., p_N$ is the isovalue $\frac{1}{2}$ of w.

https://observablehq.com/@rreusser/fast-generalized-winding-numbers-in-2d

Robust Inside-Outside Segmentation Using Generalized Winding Numbers, Jacobson et al., 2013


0
1
-0.3
0
1.2

The generalized winding number corresponds to the electric potential of infinitely many dipoles scattered on the surface ⇒ It's a harmonic function ($\Delta w = \rho$)



Relation with Poisson Surface Reconstruction

The two methods are actually equivalent in theory!

- Both solve a Laplace equation $\Delta f = a$ under constraint that f "jumps" from 0 to 1 at the interface
- Surface in Poisson is the discontinuity of the function. Surface of GWN is isovalue $\frac{1}{2}$
- Only the implementations differ. GWN does not need to splat the points $\Rightarrow$ more precise
- Both need normals, but GWN also needs an approximation of the density

Fast Winding Numbers for Soups and Clouds, Barill et al., 2018

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