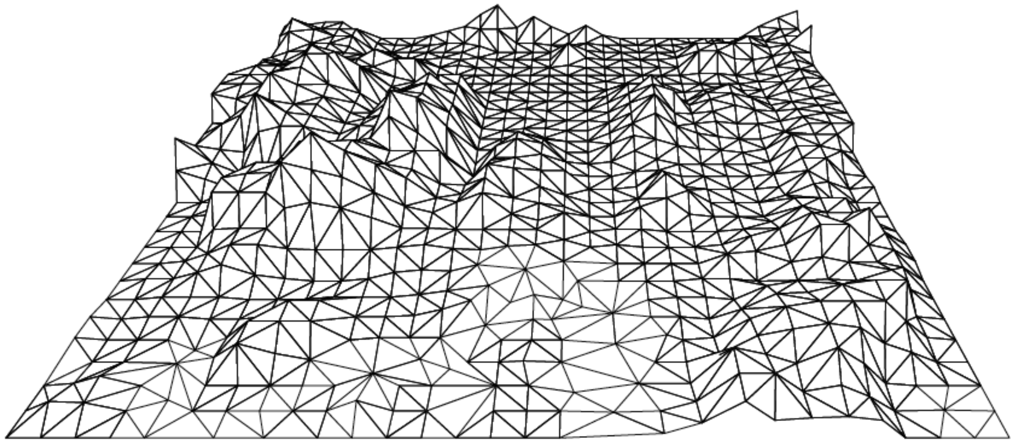
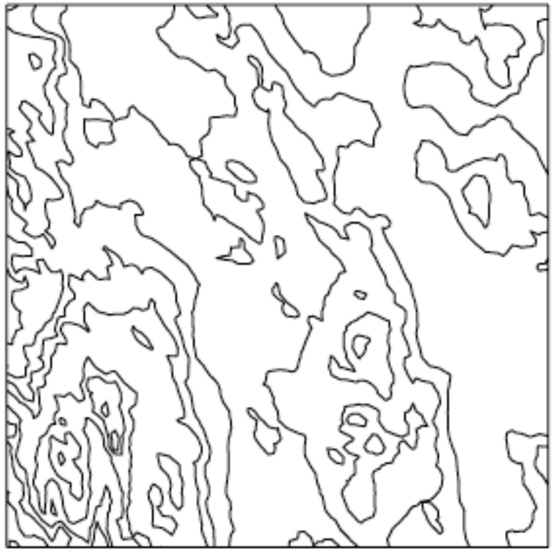
# Height Interpolation

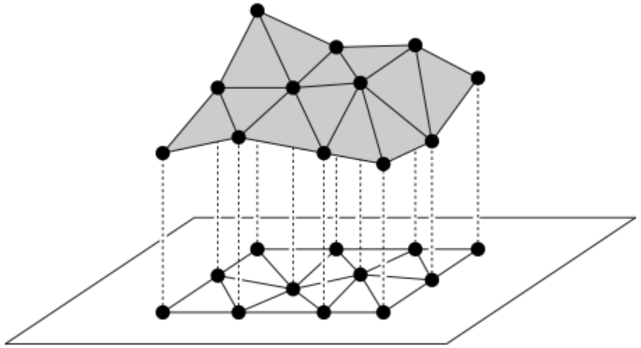


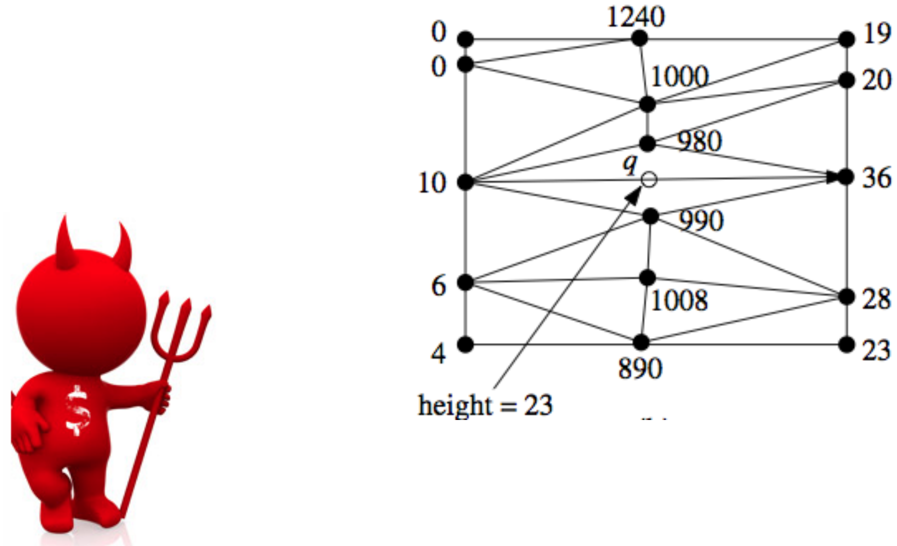


First, we build a triangulation of P : a planar subdivision whose bounded faces are triangles and whose vertices are the points of P.

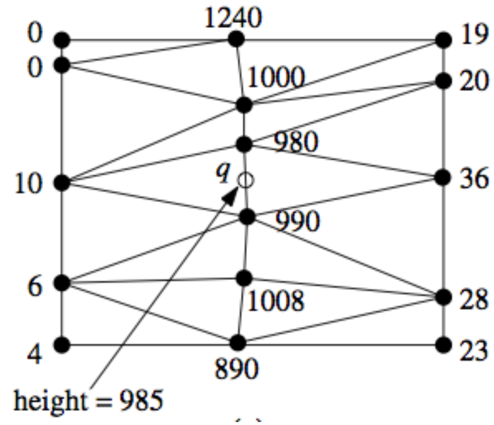
Then, we get is a polyhedral terrain, the graph of a continuous function that is piecewise linear. The polyhedral terrain can be used as an approximation of the original terrain.

Computational Geometry
9 : Delaunay Triangulation
pages 191-214



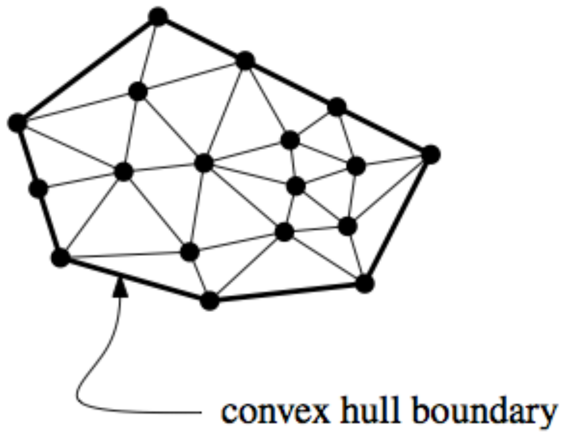



# What is the most appropriate triangulation ?




***It must be an optimal triangulation***

# Triangulation of Planar Point Sets




A triangulation of P is defined as a maximal planar subdivision of a planar point set.

No edge connecting two vertices can be added to such a subdivision without destroying its planarity.

**Theorem 9.1** *Let $P$ be a set of $n$ points in the plane, not all collinear, and let $k$ denote the number of points in $P$ that lie on the boundary of the convex hull of $P$. Then any triangulation of $P$ has $2n-2-k$ triangles and $3n-3-k$ edges.*

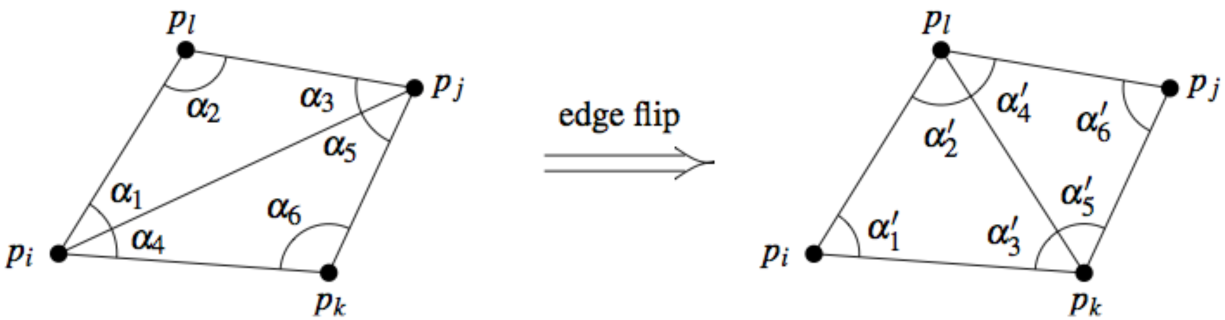
# Optimal Triangulation

$A(\mathcal{T}) \geqslant A(\mathcal{T}')$ for all triangulations $\mathcal{T}'$ of $P$.

$$A(\mathcal{T}') := (\alpha'_1, \alpha'_2, \ldots, \alpha'_{3m})$$

*The define*
*as the angle vector of a triangulation*
*the list of all angles of a triangulation*
*sorted by increasing value.*

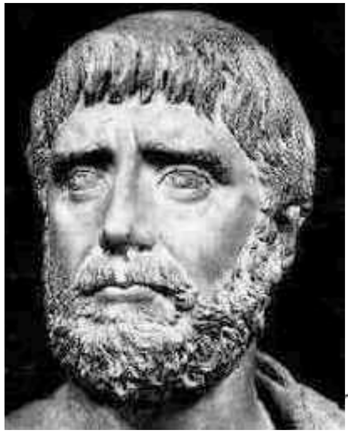
*Then, we lexicographically*
*compare those vectors.*

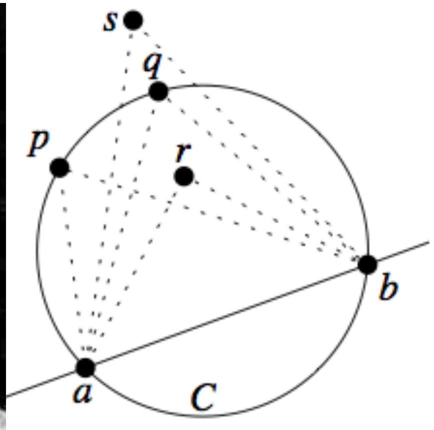




We call the edge $e = \overline{p_i p_j}$ an *illegal edge* if

$$\min_{1 \leqslant i \leqslant 6} \alpha_i < \min_{1 \leqslant i \leqslant 6} \alpha'_i.$$

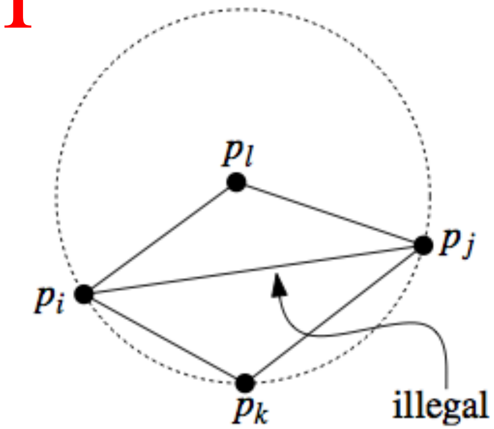





$$\measuredangle arb > \measuredangle apb = \measuredangle aqb > \measuredangle asb.$$

**Thales's Theorem**

# Empty circle criterion

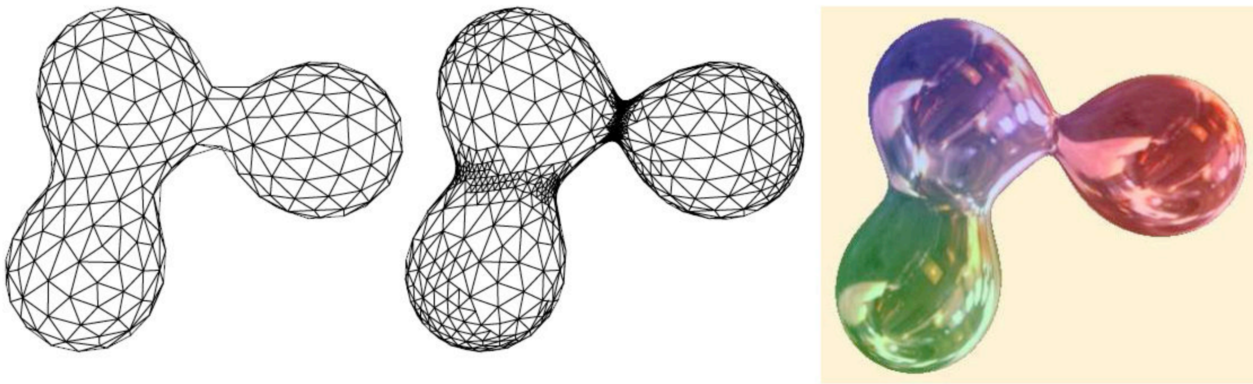



**Lemma 9.4** *Let edge $\overline{p_i p_j}$ be incident to triangles $p_i p_j p_k$ and $p_i p_j p_l$, and let $C$ be the circle through $p_i$, $p_j$, and $p_k$. The edge $\overline{p_i p_j}$ is illegal if and only if the point $p_l$ lies in the interior of $C$. Furthermore, if the points $p_i$, $p_j$, $p_k$, $p_l$ form a convex quadrilateral and do not lie on a common circle, then exactly one of $\overline{p_i p_j}$ and $\overline{p_k p_l}$ is an illegal edge.*

# Computing a legal triangulation

**Algorithm** LEGALTRIANGULATION($\mathcal{T}$)
*Input.* Some triangulation $\mathcal{T}$ of a point set $P$.
*Output.* A legal triangulation of $P$.

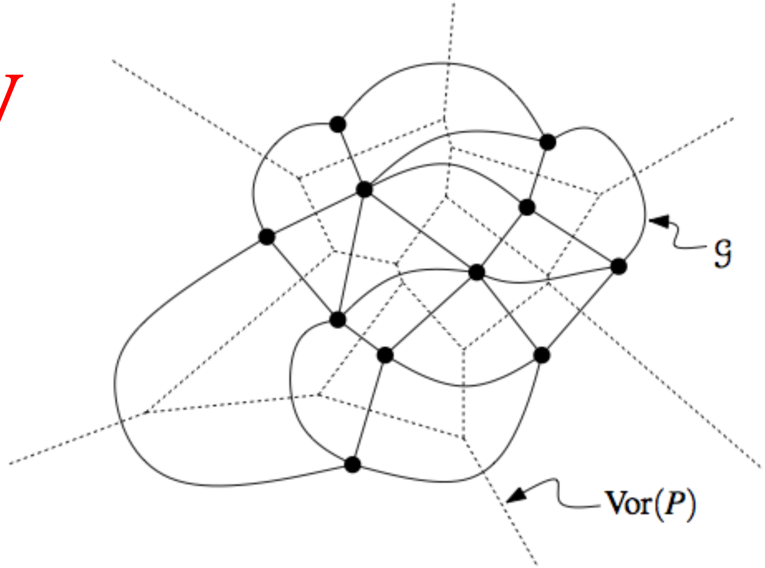
**while** $\mathcal{T}$ contains an illegal edge $\overline{p_i p_j}$
    **do** (* Flip $\overline{p_i p_j}$ *)
        Let $p_i p_j p_k$ and $p_i p_j p_l$ be the two triangles adjacent to $\overline{p_i p_j}$.
        Remove $\overline{p_i p_j}$ from $\mathcal{T}$, and add $\overline{p_k p_l}$ instead.
**return** $\mathcal{T}$



**Why does this algorithm terminate ?**
**The angle-vector increases in every iteration of the loop.**
**Since there is only a finite number of different triangulations,**
**this proves termination of the algorithm.**

**Once it terminates, the result is a legal triangulation.**
**Although the algorithm is guaranteed to terminate, it is too slow to be interesting.**

# The Delaunay Triangulation




## Gueorgui Voronoï

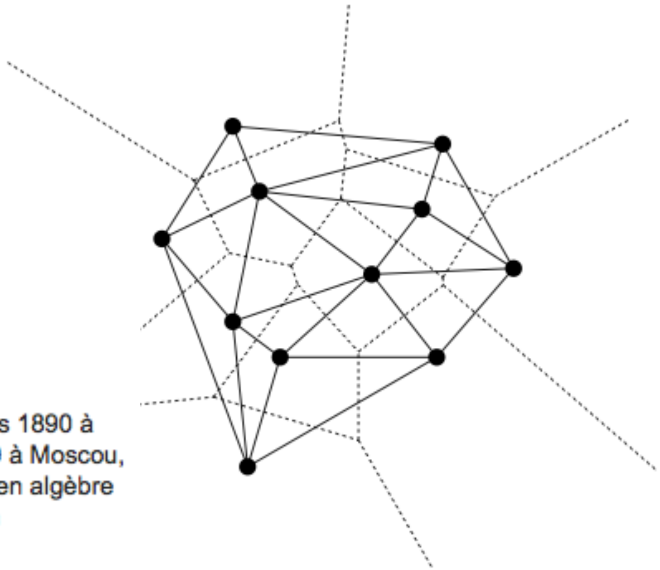
Mathématicien

Gueorgui Feodossievitch Voronoï né 28 avril 1868 à Jouravka, un village de l'oblast de Poltava en Russie, mort 20 novembre 1908 à Varsovie, est un mathématicien connu pour son diagramme de Voronoï qui ... Wikipédia

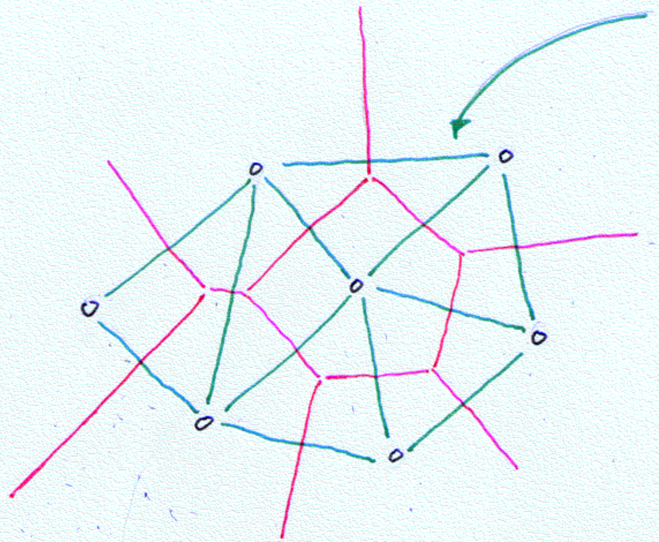
## Boris Delaunay

Mathématicien

Boris Nikolaïevitch Delaunay, né le 15 mars 1890 à Saint-Pétersbourg et mort le 17 juillet 1980 à Moscou, était un mathématicien russe. Il a travaillé en algèbre moderne, en géométrie des nombres et en cristallographie mathématique. Wikipédia

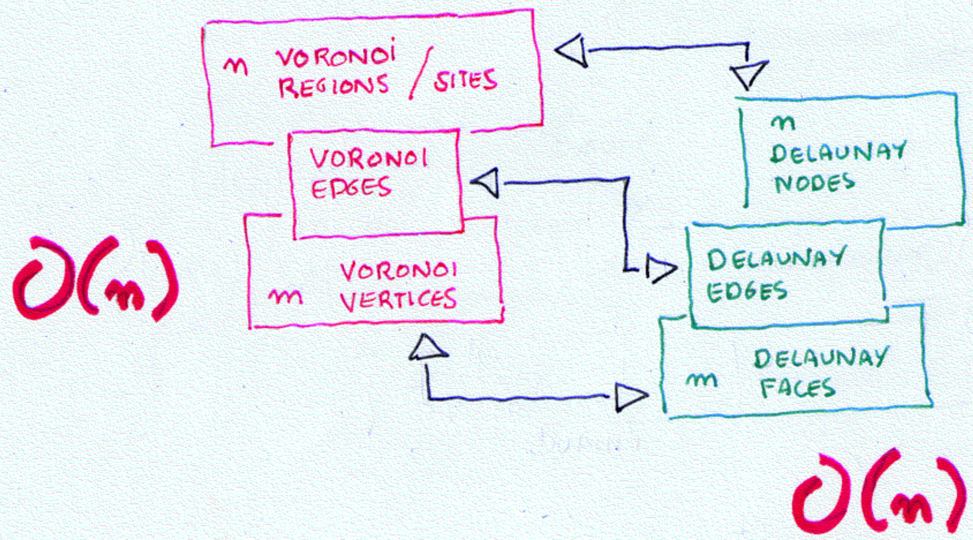


# DUAL GRAPH OF VORONOI = DELAUNAY TRIANGULATION*



EULER'S FORMULA
$3f = 2e$

3n - 6 DELAUNAY EDGES
2n - 4 DELAUNAY FACES
INCLUDING THE "OUTSIDE" FACE

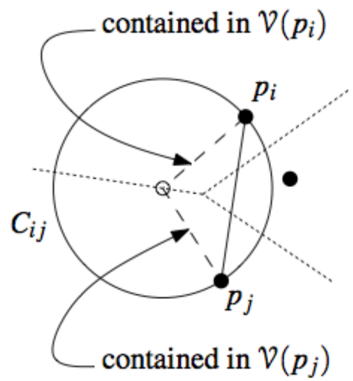



* IN FACT,

IT IS A TRIANGULATION, ONLY IF WE ASSUME, FOR SIMPLICITY, NO 4 COCIRCULAR POINTS

FROM A PRACTICAL POINT OF VIEW, JUST DIVIDE QUADRILATERALS INTO 2 TRIANGLES

# All Delaunay triangulations are legal !




**Theorem 9.5** *The Delaunay graph of a planar point set is a plane graph.*

**Theorem 9.6** *Let $P$ be a set of points in the plane.*

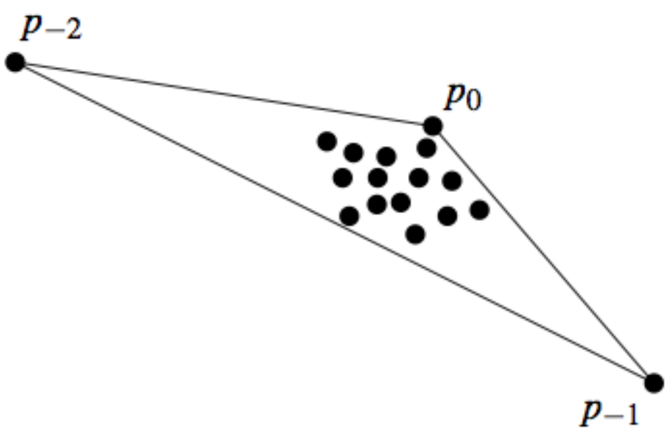
*(i) Three points $p_i, p_j, p_k \in P$ are vertices of the same face of the Delaunay graph of $P$ if and only if the circle through $p_i$, $p_j$, $p_k$ contains no point of $P$ in its interior.*

*(ii) Two points $p_i, p_j \in P$ form an edge of the Delaunay graph of $P$ if and only if there is a closed disc $C$ that contains $p_i$ and $p_j$ on its boundary and does not contain any other point of $P$.*

**Theorem 9.7** *Let $P$ be a set of points in the plane, and let $\mathcal{T}$ be a triangulation of $P$. Then $\mathcal{T}$ is a Delaunay triangulation of $P$ if and only if the circumcircle of any triangle of $\mathcal{T}$ does not contain a point of $P$ in its interior.*

**Theorem 9.8** *Let $P$ be a set of points in the plane. A triangulation $\mathcal{T}$ of $P$ is legal if and only if $\mathcal{T}$ is a Delaunay triangulation of $P$.*

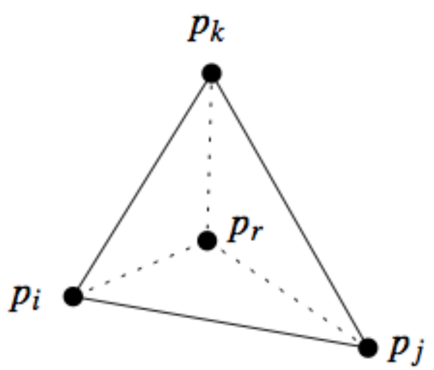
# Computing the Delaunay triangulation



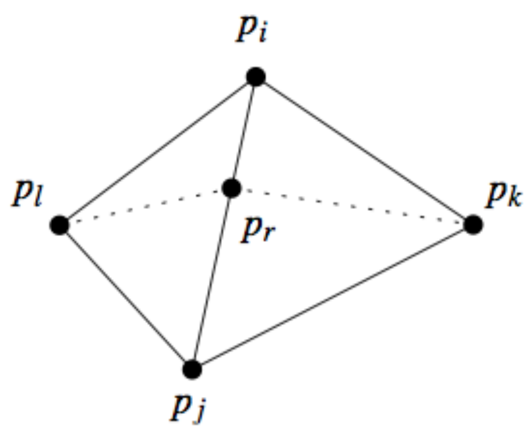

**First, we start with a large triangle that contains the point set**

**The algorithm is randomized incremental,**
**so it adds the points in random order**
**and it maintains a Delaunay triangulation of the current point set.**

$p_r$ lies in the interior of a triangle




$p_r$ falls on an edge

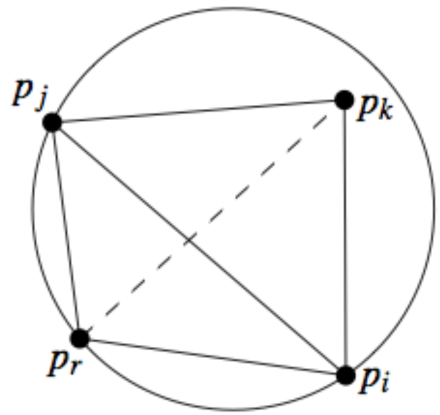


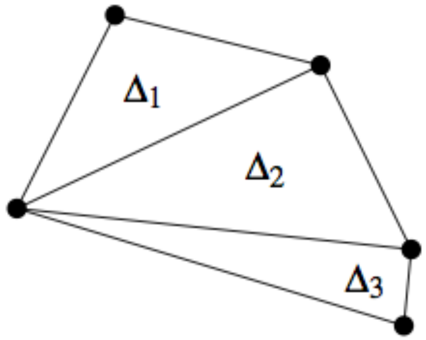
# How to maintain a legal triangulation ?

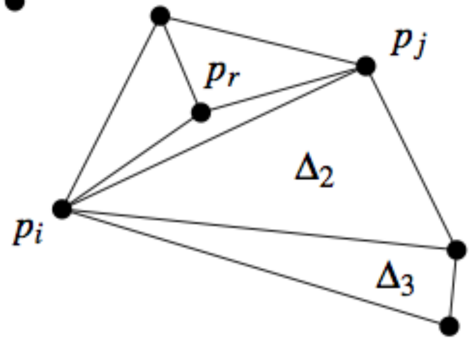
LEGALIZEEDGE($p_r, \overline{p_i p_j}, \mathcal{T}$)

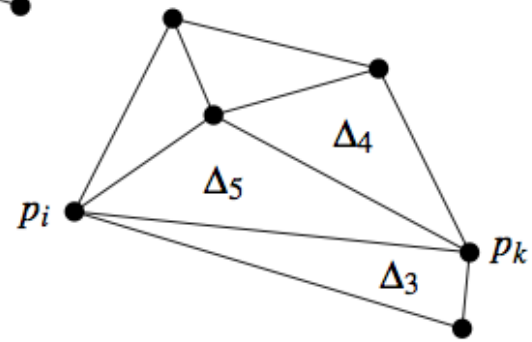
1. (* The point being inserted is $p_r$, and $\overline{p_i p_j}$ is the edge of $\mathcal{T}$ that may need to be flipped. *)
2. **if** $\overline{p_i p_j}$ is illegal
3. **then** Let $p_i p_j p_k$ be the triangle adjacent to $p_r p_i p_j$ along $\overline{p_i p_j}$.
4. (* Flip $\overline{p_i p_j}$: *) Replace $\overline{p_i p_j}$ with $\overline{p_r p_k}$.
5. LEGALIZEEDGE($p_r, \overline{p_i p_k}, \mathcal{T}$)
6. LEGALIZEEDGE($p_r, \overline{p_k p_j}, \mathcal{T}$)

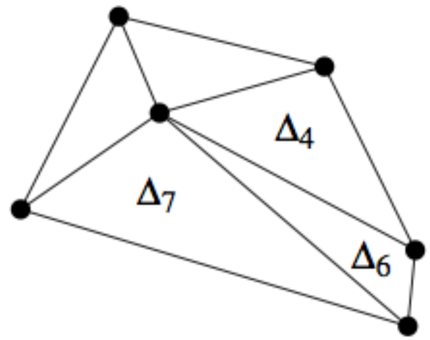
*Test of empty circle*
*to deduce if the flip is required*



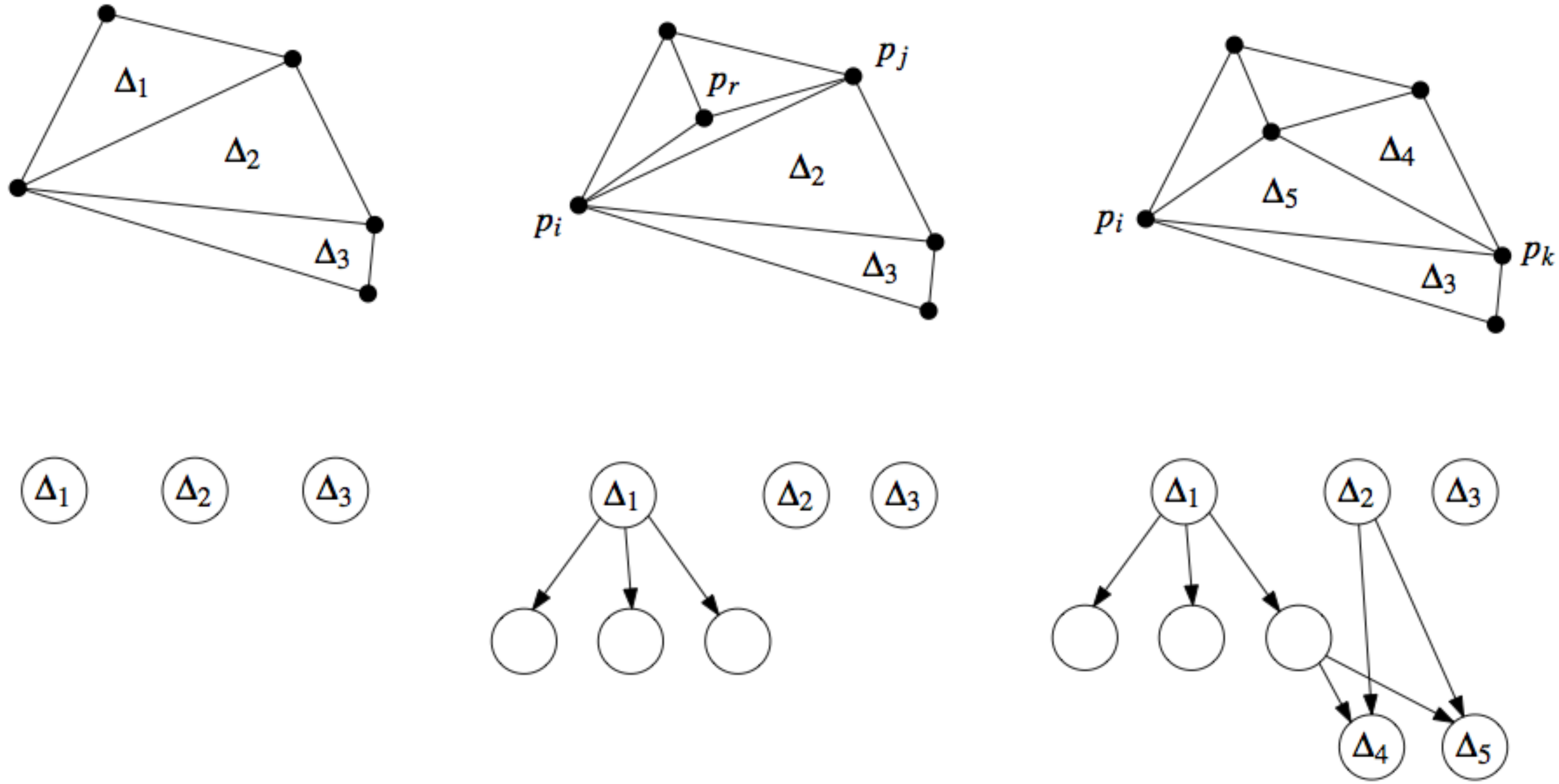


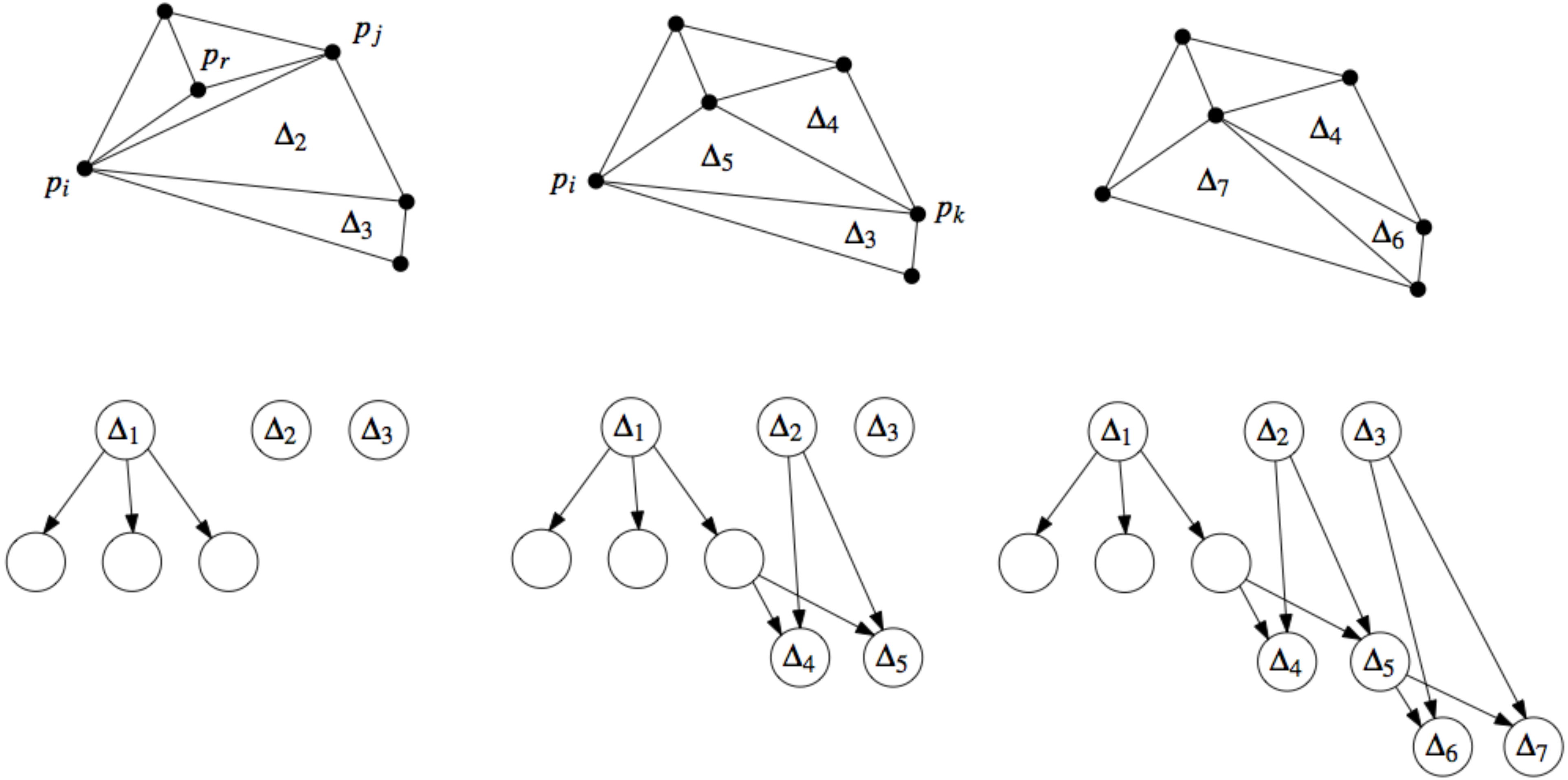




# How to find the triangle containing a point ?




A directed acyclic graph !




A directed acyclic graph !

# Complexity analysis

**Lemma 9.11** *The expected number of triangles created by algorithm* DELAUNAYTRIANGULATION *is at most* $9n+1$.

$$\begin{aligned} \mathrm{E}\left[\text{number of triangles created in step } r\right] &\leqslant \mathrm{E}\left[2\deg(p_r, \mathcal{DG}_r) - 3\right] \\ &= 2\mathrm{E}\left[\deg(p_r, \mathcal{DG}_r)\right] - 3 \\ &\leqslant 2\cdot 6 - 3 \;=\; 9 \end{aligned}$$

**Theorem 9.12** *The Delaunay triangulation of a set* $P$ *of* $n$ *points in the plane can be computed in* $O(n\log n)$ *expected time, using* $O(n)$ *expected storage.*

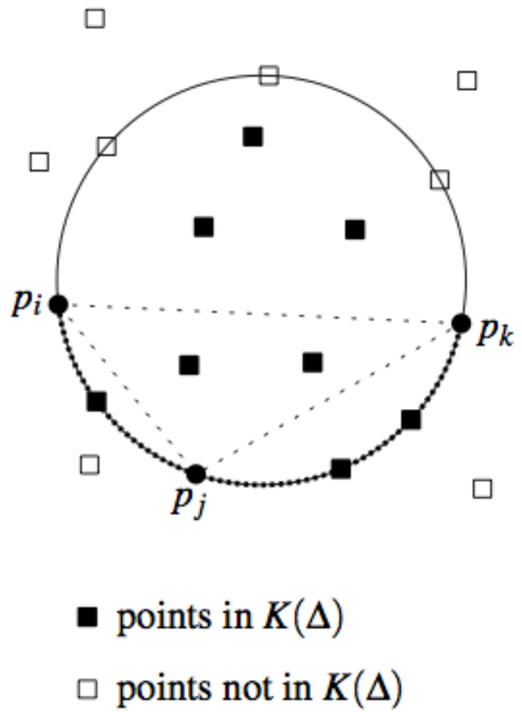
# It remains to account for the point location steps

*The visit to a triangle during the location of for a point is charged when the point belongs to K(Δ).*

*It is easy to see that a triangle Δ can be charged at most once for every one of the points in K(Δ).*

*Therefore the total time for the point location steps is*

$$O(n + \sum_{\Delta} \text{card}(K(\Delta))).$$



# It remains to bound the expected size of the sets K

**Lemma 9.13** *If $P$ is a point set in general position, then*

$$\sum_{\Delta} \mathrm{card}(K(\Delta)) = O(n \log n),$$

*where the summation is over all Delaunay triangles $\Delta$ created by the algorithm.*

$$\mathrm{E}\Big[\sum_{\Delta \in \mathcal{T}_r \setminus \mathcal{T}_{r-1}} \mathrm{card}(K(\Delta))\Big] \leqslant 12\left(\frac{n-r}{r}\right)$$

# Exercice 9

9.11 A *Euclidean minimum spanning tree* (EMST) of a set $P$ of points in the plane is a tree of minimum total edge length connecting all the points. EMST's are interesting in applications where we want to connect sites in a planar environment by communication lines (local area networks), roads, railroads, or the like.

a. Prove that the set of edges of a Delaunay triangulation of $P$ contains an EMST for $P$.
b. Use this result to give an $O(n \log n)$ algorithm to compute an EMST for $P$.