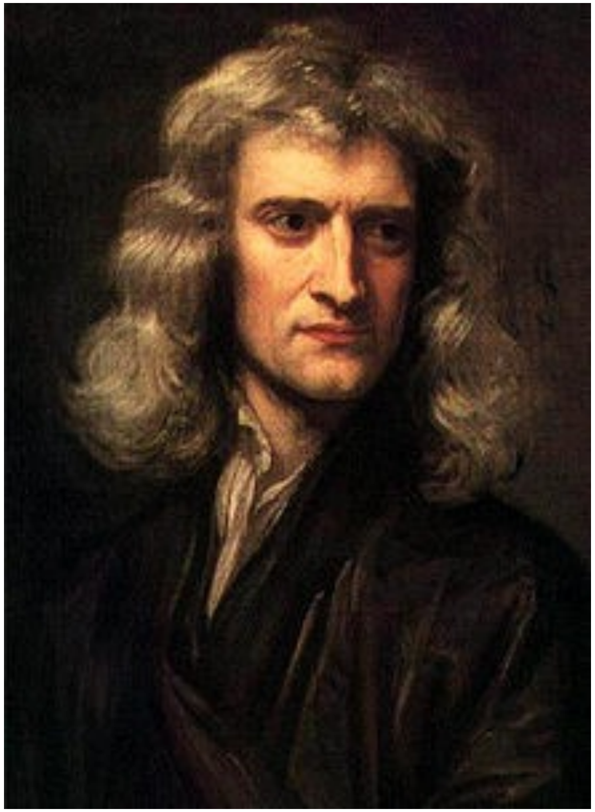
Quelques mots sur les écoulements compressibles...



ou l'erreur de notre ami Newton :-)

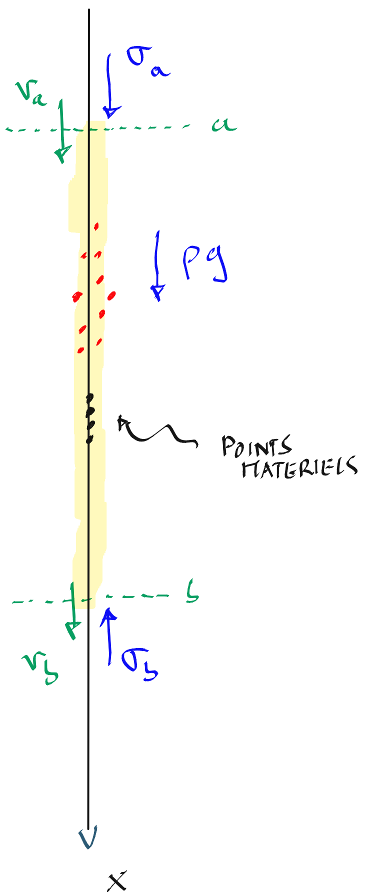
Bilan de quantité de mouvement

$$\underbrace{\underline{p}(\underline{x},t)}_{\text{DENSITE DE QUANTITE DE MVT}} = \rho(\underline{x},t)\,\underline{v}(\underline{x},t)$$

$$\boxed{\begin{aligned}\rho(\underline{x},t) &= \frac{M}{V}\\ \underline{p}(x,t) &= \frac{\underline{N}}{V}\end{aligned}}$$

$$\boxed{\underline{v}(\underline{x},t) \triangleq \frac{\underline{p}(\underline{x},t)}{\rho(\underline{x},t)}}$$

CECI EST LA DEFINITION FORMELLE DE LA VITESSE MACROSCOPIQUE



$\forall a,b$

$$\frac{d}{dt}\int_a^b \rho v\,dx = \underbrace{\rho_a v_a^2 - \rho_b v_b^2}_{-[\rho v^2]_a^b} + \int_a^b \rho g\,dx + \overbrace{(-\sigma_a) - (-\sigma_b)}^{[\sigma]_a^b}$$

CONTRAINTE

COMPRESSION $\sigma < 0$

TRACTION $\sigma > 0$

$$\int_a^b \frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho v^2)\,dx = \int_a^b \rho g + \frac{\partial \sigma}{\partial x}\,dx \qquad \forall a,b$$

$$\boxed{\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho v^2) = \rho g + \frac{\partial \sigma}{\partial x}}$$

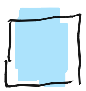
FORME LOCALE

$\rho \quad v \quad \sigma$

Deux équations
Trois inconnues

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}(\rho v) = 0$$

$$\frac{\partial}{\partial t}\rho v + \frac{\partial}{\partial x}(\rho v^2) = \frac{\partial \sigma}{\partial x} + \rho g$$



FORME CONSERVATIVE

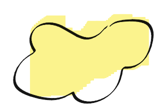
$$\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho v^2) = \rho \underbrace{\left[\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}\right]}_{\rho \frac{Dv}{Dt}} + v \underbrace{\left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v)\right]}_{=0}$$

$$\underbrace{\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x}}_{\frac{D\rho}{Dt}} + \rho\frac{\partial v}{\partial x} = 0$$

FORME NON CONSERVATIVE

$$\frac{D\rho}{Dt} + \rho\frac{\partial v}{\partial x} = 0$$

$$\rho\frac{Dv}{Dt} = \frac{\partial \sigma}{\partial x} + \rho g$$



Il faut une équation de comportement pour clore le modèle !

$$\boxed{pV = nRT}$$

LOI DES GAZ IDEAUX

$$\rho = \frac{n}{V} M$$

$$\sigma = -p + \text{“TERMES DE FRICTION”}$$

$$T = K \rho^{\gamma - 1}$$

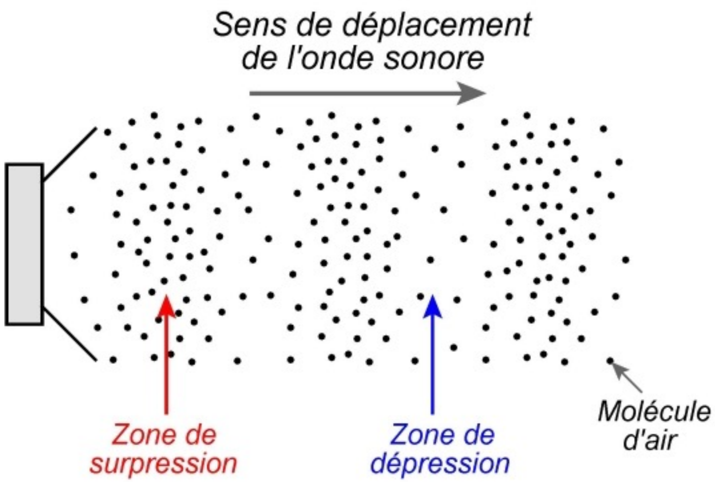
$$\boxed{\gamma \underset{\text{AIR}}{=} 7/5}$$

$$\underbrace{-\sigma}_{p} = \rho R_* T$$

$$= \underbrace{\rho R_* K \rho^{\gamma-1}}_{A\rho^\gamma}$$

$$\boxed{p = A\rho^\gamma}$$

Vitesse du son ?



$$\text{MACH} = \frac{U}{\sqrt{\underbrace{\frac{c_p}{c_v}}_{\gamma} R_* T}} \Big\} \text{ VITESSE DU SON}$$

Wikipedia est mon ami !

Modèle de gaz idéal

$$\boxed{\widehat{\rho}(p, T) \;=\; \frac{p}{R_* T}}$$

Constante du gaz

Un exemple d'équation d'état pour la masse volumique

Ecoulements compressibles

Propagation des sons au sein de l'air : c'est un effet de la compressibilité de l'écoulement.
Caractérisation par le nombre de Mach

Presque comme en thermo…

Concentration molaire [mole/m³]

$$\boxed{\rho} = \boxed{c}\boxed{M}$$

Masse volumique [kg/m³]

Masse molaire [kg/mole]

$$pV = nRT$$

$$c = \frac{n}{V}$$

$$c = \frac{p}{RT}$$

Constante des gaz

$$\rho = \frac{p}{R_* T}$$

Constante du gaz

$$R = 8.314 \; [J/moleK]$$

$$R_{*,air} = \frac{R}{M_{air}} = 287 \; [m^2/s^2K]$$

Nombre de Mach

$$Ma = \frac{U}{\sqrt{\frac{c_p}{c_v} R_* T}}$$

Born: 18 Feb 1838, Turas, Moravia

Died: 19 Feb 1916, Munchen, Germany

caractérise un écoulement d'un fluide !

Vitesse caractéristique du fluide

Vitesse caractéristique de propagation du son

Modèle de Newton

T = constante !

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}(\rho v) = 0$$

$$\frac{\partial}{\partial t}\rho v + \frac{\partial}{\partial x}(\rho v^2) = -\frac{\partial p}{\partial x}$$

$$p = \underbrace{R_* T}_{\text{FIXE}}\, \rho$$

$$\underbrace{\frac{\partial \rho'}{\partial t}}_{O(\varepsilon)} + (\rho_0 + \underbrace{\rho'}_{O(\varepsilon)})\underbrace{\frac{\partial v'}{\partial x}}_{O(\varepsilon)} + \underbrace{v'}_{O(\varepsilon)}\underbrace{\frac{\partial \rho'}{\partial x}}_{O(\varepsilon)} = 0$$

$$\frac{\partial \rho'}{\partial t} + \rho_0 \frac{\partial v'}{\partial x} = 0$$

$$\rho_0 \frac{\partial v'}{\partial t} = -R_* T \frac{\partial \rho'}{\partial x}$$

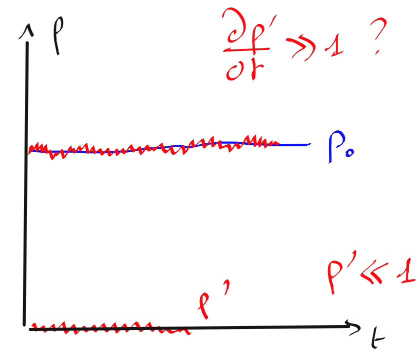
MODELE EN PETITES PERTURBATIONS

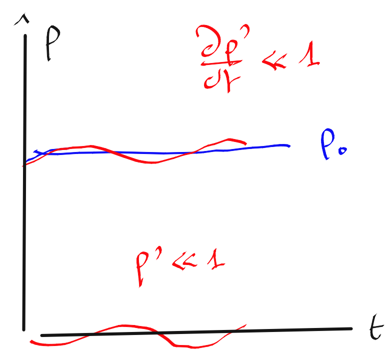
$$\rho(x,t) = \rho_0 + \rho'(x,t)$$

$$p(x,t) = p_0 + p'(x,t)$$

$$v(x,t) = v'(x,t)$$

$$\underbrace{\rho'(x,t),\; p'(x,t),\; v'(x,t)}_{O(\varepsilon)}$$





Modèle linéaire pour de petites perturbations

Modèle de Newton
T = constante !

$$\frac{\partial \rho'}{\partial t} + \rho_0 \frac{\partial v'}{\partial x} = 0$$

$$\rho_0 \frac{\partial v'}{\partial t} = -R_* T \frac{\partial \rho'}{\partial x}$$

MODELE EN PETITES PERTURBATIONS

$$\frac{\partial^2 \rho}{\partial t^2} = -\rho_0 \frac{\partial^2 v}{\partial x \partial t}$$

$$\rho_0 \frac{\partial^2 v}{\partial x \partial t} = -R_* T \frac{\partial^2 \rho}{\partial x^2}$$

$$\frac{\partial^2 \rho}{\partial t^2} = \underbrace{R_* T}_{c^2} \frac{\partial^2 \rho}{\partial x^2}$$

$$c = \sqrt{R_* T}$$

Et boum !

On a pas le bon résultat !

Calcul de la vitesse du son : l'erreur de Newton !

$$
\begin{aligned}
\rho(x,t) &= \rho_0 + \rho'(x,t) \\
v(x,t) &= \cancel{v_0} + v'(x,t) \\
p(x,t) &= p_0 + \underbrace{p'(x,t)}_{\mathcal{O}(\epsilon)}
\end{aligned}
$$

Petites pertubations de vitesse, pression et de densité.
Les effets visqueux sont négligeables.
L'air est un gaz idéal.

Que devient la conservation de la masse ?

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\Big(\rho v\Big) = 0$$

$$\frac{\partial \rho'}{\partial t} + \rho_0 \frac{\partial v'}{\partial x} + \underbrace{\cancel{\rho' \frac{\partial v'}{\partial x}} + \cancel{v' \frac{\partial \rho'}{\partial x}}}_{\mathcal{O}(\epsilon^2)} = 0$$

Equation linéarisée en termes de petites perturbations

Par paresse de notations, nous noterons désormais les perturbations sans apostrophe :-)

Modèle 1 :
Ecoulement isotherme :-(

$$\left\{\begin{array}{rcl}\dfrac{\partial \rho}{\partial t} + \rho_0 \dfrac{\partial v}{\partial x} & = & 0 \\ \rho_0 \dfrac{\partial v}{\partial t} & = & -\dfrac{\partial p}{\partial x}\end{array}\right.$$

$$\boxed{p = \rho R_* T} \quad \boxed{T = cst}$$

$$\frac{\partial^2 \rho}{\partial t^2} = \frac{\partial^2 p}{\partial x^2}$$

$$\frac{\partial^2 p}{\partial x^2} = R_* T \frac{\partial^2 \rho}{\partial x^2}$$

Modèle 1 : Ecoulement isotherme :-(

$$\boxed{p = \rho R_* T} \qquad \boxed{T = cst}$$

$$\frac{\partial^2 \rho}{\partial t^2} = R_* T \frac{\partial^2 \rho}{\partial x^2}$$

La vitesse du son ainsi prédite ne correspond pas aux valeurs mesurées expérimentalement ...

$$c = \sqrt{R_* T}$$

Modèle 2 :
Ecoulement adiabatique :-)

$$\frac{\partial^2 \rho}{\partial t^2} = \frac{\partial^2 p}{\partial x^2}$$

Petites pertubations de vitesse, pression et de densité.
Les effets visqueux sont négligeables : pas de dissipation.
L'air est un gaz idéal.

Il s'agit donc d'un écoulement adiabatique réversible ou encore d'un <u>écoulement isentropique</u>

$$\left\{ \begin{array}{rcl} \dfrac{\partial \rho}{\partial t} + \rho_0 \dfrac{\partial v}{\partial x} & = & 0 \\ \rho_0 \dfrac{\partial v}{\partial t} & = & -\dfrac{\partial p}{\partial x} \\ \rho c_p \dfrac{dT}{dt} & = & \dfrac{dp}{dt} \end{array} \right.$$

$$\boxed{p = \rho R_* T} \qquad \boxed{\frac{\partial T}{\partial x} = 0}$$

L'air est un mauvais conducteur !
C'est même un bon isolant !

Un peu d'algèbre fastidieuse

$$\boxed{p = \rho R_* T} \qquad \boxed{\frac{\partial T}{\partial x} = 0}$$

$$\rho c_p \frac{dT}{dt} = \frac{dp}{dt}$$

$$\downarrow$$

$$\frac{p}{R_* T} c_p \frac{dT}{dt} = \frac{dp}{dt}$$

$$\frac{d}{dt}\left(\ln\left(T^{c_p}\right)\right) = \frac{c_p}{T}\frac{dT}{dt} = \frac{c_p - c_v}{p}\frac{dp}{dt} = \frac{d}{dt}\left(\ln\left(p^{c_p - c_v}\right)\right)$$

$$\frac{d}{dt}\left(\ln\left(\frac{p^{c_p - c_v}}{T^{c_p}}\right)\right) = 0$$

$$\frac{p^{c_p - c_v}}{T^{c_p}} = C_1$$

$$\boxed{\frac{\rho^\gamma}{p} = C_2}$$

Il faut bien imposer que l'écoulement est parfaitement isentropique !

On retrouve une relation bien connue !

Modèle 2 :
Ecoulement adiabatique :-)

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \rho_0 \dfrac{\partial v}{\partial x} &= 0 \\ \rho_0 \dfrac{\partial v}{\partial t} &= -\dfrac{\partial p}{\partial x} \end{cases}$$

$$\boxed{p = A\rho^\gamma}$$

$$\boxed{p = \rho R_* T} \qquad \boxed{\frac{\partial T}{\partial x} = 0}$$

$$\frac{\partial^2 \rho}{\partial t^2} = \frac{\partial^2 p}{\partial x^2}$$

$$\frac{\partial p}{\partial x} = A\gamma\rho^{\gamma-1}\frac{\partial \rho}{\partial x} = \gamma R_* T \frac{\partial \rho}{\partial x}$$

$$\frac{\partial^2 p}{\partial x^2} = \gamma R_* T \frac{\partial^2 \rho}{\partial x^2} + \gamma R_* \cancel{\frac{\partial T}{\partial x}} \frac{\partial \rho}{\partial x}$$

Modèle 2 :
Ecoulement adiabatique :-)

$$\boxed{p = \rho R_* T}$$

$$\boxed{\frac{\partial T}{\partial x} = 0}$$

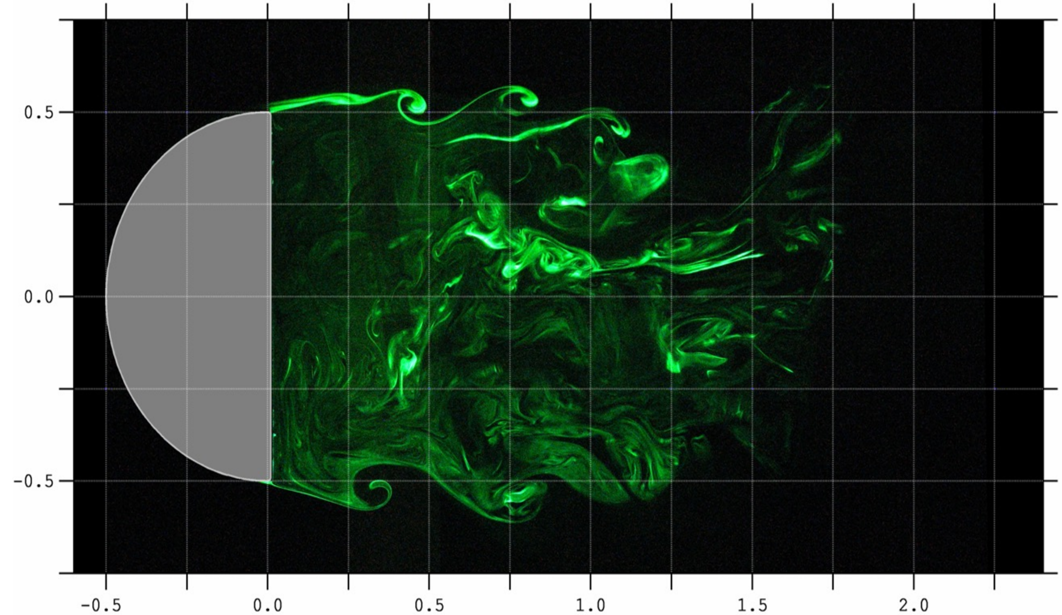
$$\frac{\partial^2 \rho}{\partial t^2} = \gamma R_* T \frac{\partial^2 \rho}{\partial x^2}$$

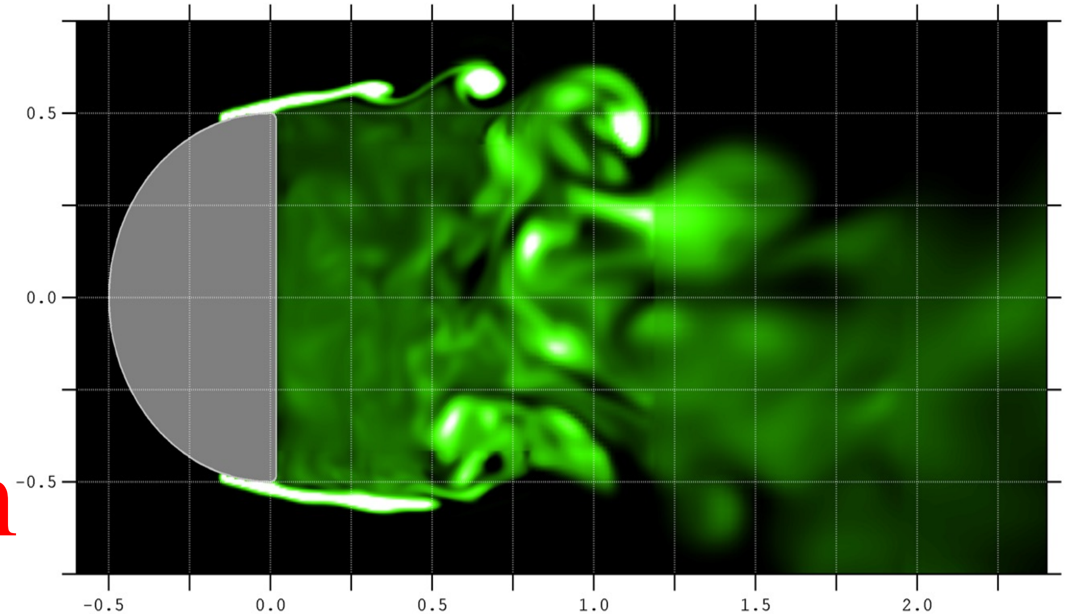
La vitesse du son ainsi prédite <u>correspond</u> bien aux valeurs mesurées expérimentalement ...

$$c = \sqrt{\gamma R_* T} = 342[m/s]$$

Expérience…

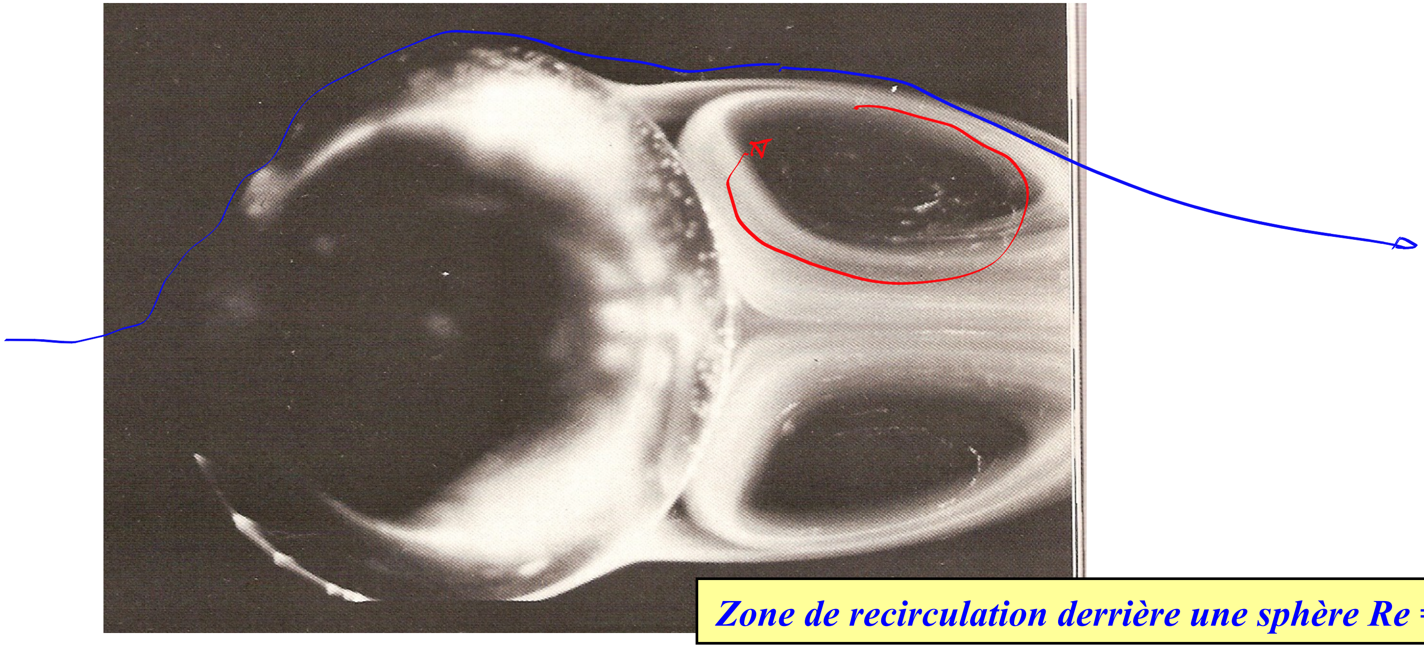
Experimental visualization of the flow past the hemisphere at Re=3,000 in a towing tank. Fluorescein is injected in the boundary layer at the front of the hemisphere.





…et simulation

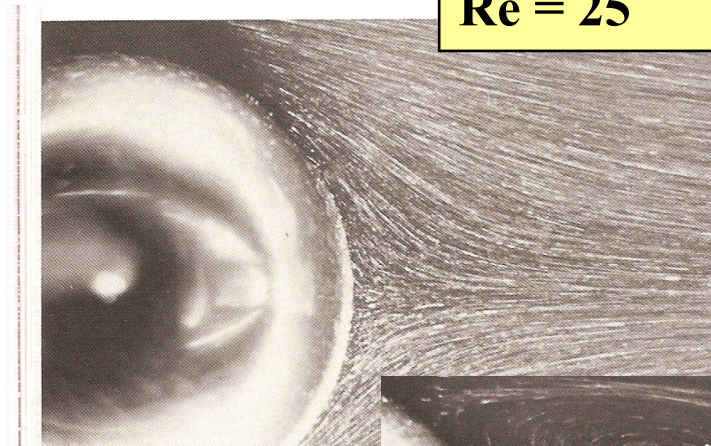
Ecoulement laminaire : Re = 104



Zone de recirculation derrière une sphère Re = 104

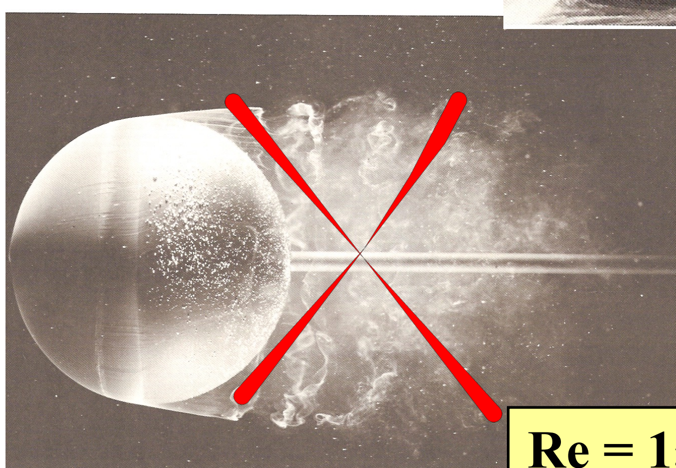
Taneda 1956
(from An Album of Fluid Motion, Van Dyke)

Re = 25



Ecoulement laminaire ?

Re = 118



Re = 15000

(Van Dyke, 1982)

Nombre de Reynolds

caractérise un écoulement d'un fluide !

$$Re = \frac{\rho_0 u_0 L}{\mu}$$

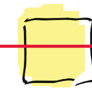
Born: 23 Aug 1842 in Belfast, Ireland

Died: 21 Feb 1912 in Watchet, Somerset, England

En 3D !

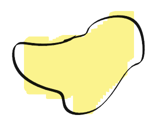
$$\frac{\partial \rho}{\partial t} + \underset{\sim}{\nabla} \cdot (\rho \underset{\sim}{v}) = 0$$

$$\frac{\partial}{\partial t}(\rho \underset{\sim}{v}) + \underset{\sim}{\nabla} \cdot (\rho \underset{\sim}{v}\, \underset{\sim}{v}) = \underset{\sim}{\nabla} \cdot \underset{\approx}{\sigma} + \rho \underset{\sim}{g}$$



$$\frac{D\rho}{Dt} + \rho\, \underset{\sim}{\nabla} \cdot \underset{\sim}{v} = 0$$

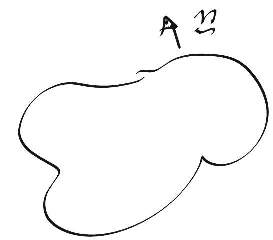
$$\rho \frac{D\underset{\sim}{v}}{Dt} = \underset{\sim}{\nabla} \cdot \underset{\approx}{\sigma} + \rho \underset{\sim}{g}$$



NOTATIONS TENSORIELLES

$\perp$ SYST COORDONNEES

CARTESIENNES
CYLINDRIQUES
SPHERIQUES
CURVILIGNES

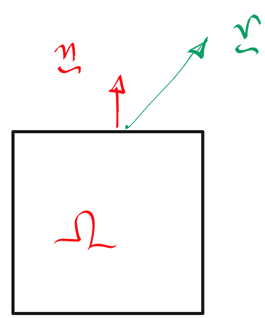


$$\int_{\partial\Omega} \rho \underset{\sim}{v} \cdot \underset{\sim}{n} = \int_{\Omega} \underset{\sim}{\nabla} \cdot (\rho \underset{\sim}{v})$$

ECOULEMENT INCOMPRESSIBLE

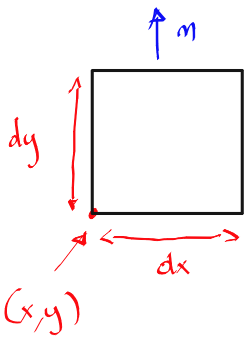
$$\rho = 0$$

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{v} = 0$$



$$\frac{d}{dt}\int_{\Omega} \rho = -\int_{\partial\Omega} \rho \underset{\sim}{v} \cdot \underset{\sim}{n}$$

Théorème d'Ostrogradsky



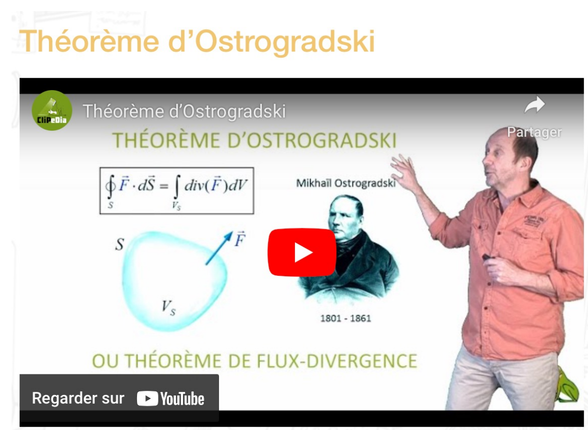
$$\underline{v} = \begin{bmatrix} u \\ v \end{bmatrix}_{xy} = \begin{bmatrix} v_x \\ v_y \end{bmatrix}_{xy}$$

$$\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho \underline{v}) = 0$$

$$\underline{\nabla} = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \end{bmatrix} \qquad \rho \underline{v} = \begin{bmatrix} \rho u \\ \rho v \end{bmatrix}$$

$$dx\,dy\,\frac{\partial \rho}{\partial t} = \left[\rho\, u(x,y) - \rho\, u(x+dx,y)\right] dy + \left[\rho\, v(x,y) - \rho\, v(x,y+dy)\right] dx$$

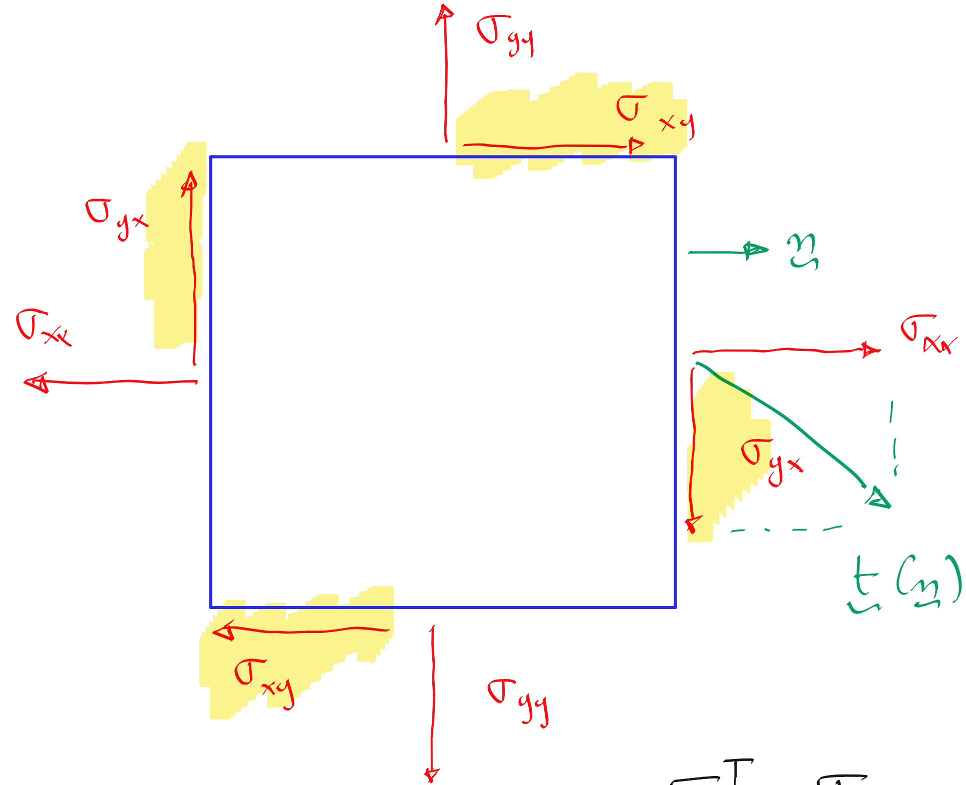
$$\frac{\partial \rho}{\partial t} = \underbrace{\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v)}_{\underline{\nabla} \cdot (\rho \underline{v})}$$



https://clipedia.be/videos/theoreme-d-ostrogradski

Tenseur des contraintes

$$\underline{\underline{\sigma}} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix}$$



$$\underset{\sim}{t}(\underset{\sim}{n}) = \underline{\underline{\sigma}}^T \cdot \underset{\sim}{n}$$

$$\underline{\underline{\sigma}}^T = \underline{\underline{\sigma}}$$

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yy} & \sigma_{yy} \end{bmatrix}^T \cdot \underbrace{\begin{bmatrix} n_x \\ n_y \end{bmatrix}}_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{xy} \end{bmatrix}$$

Le modèle mathématique du fluide newtonien…

$$\underline{\underline{d}} = \frac{1}{2}\left(\nabla \underline{v} + \nabla \underline{v}^T\right) = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ & \frac{\partial v}{\partial y} \end{bmatrix}$$

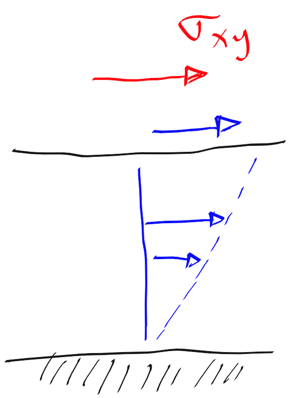
$$\nabla \underline{v} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$$

$$\underline{\underline{\sigma}} = -p\,\underline{\underline{\delta}} + 2\mu\,\underline{\underline{d}}$$

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} d_{xx} & d_{xy} \\ d_{xy} & d_{yy} \end{bmatrix}$$

VISCOSITE

TENSEUR DES TAUX DE DEFORMATIONS



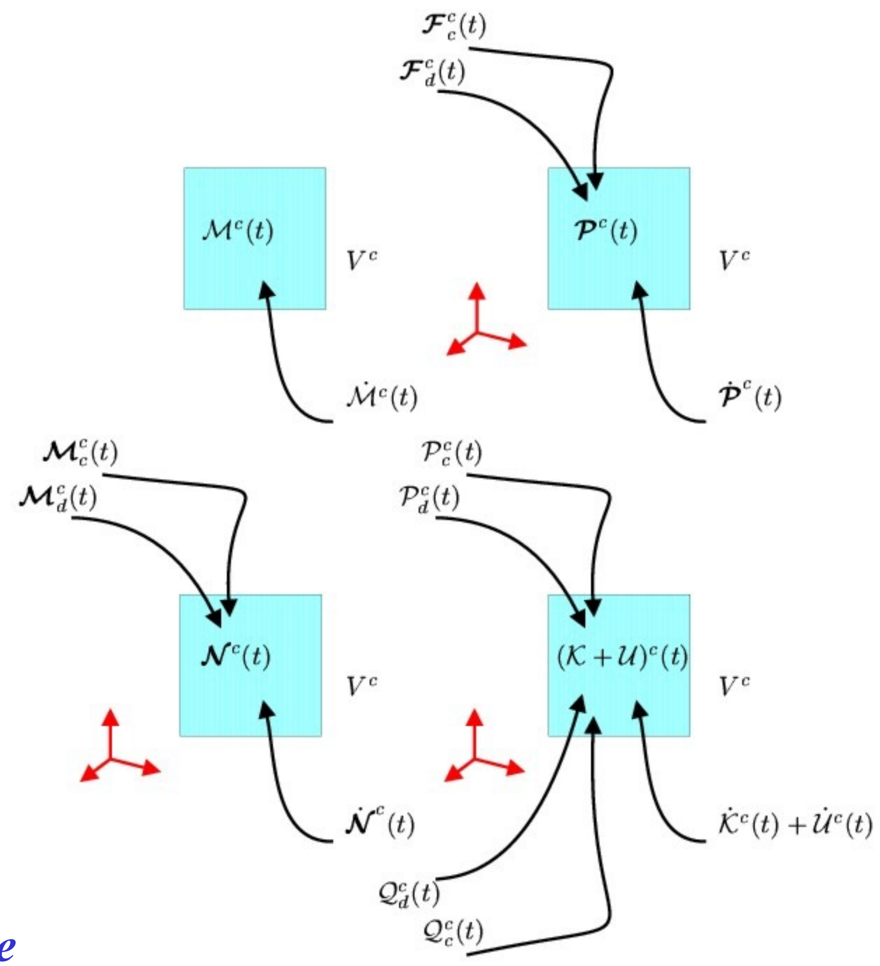
$$u(y) = \dot{\gamma}\, y$$

$$\frac{\partial u}{\partial y} = \dot{\gamma}$$

$$\sigma_{xy} = \mu \frac{\partial u}{\partial y}$$

…enfin !

Lois de conservation…



Forme globale
pour des volumes de controle

…dont on peut déduire des formes locales

$$\begin{aligned} \mathbf{t}(\mathbf{n}) &= \boldsymbol{\sigma}^T \cdot \mathbf{n} \\ \boldsymbol{\sigma} &= \boldsymbol{\sigma}^T \\ q(\mathbf{n}) &= -\mathbf{q} \cdot \mathbf{n} \end{aligned}$$	$$\begin{aligned} \frac{D\rho}{Dt} + \rho \boldsymbol{\nabla} \cdot \mathbf{v} &= 0, \\ \rho \frac{D\mathbf{v}}{Dt} &= \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \rho \mathbf{g} \\ \rho \frac{DU}{Dt} &= \boldsymbol{\sigma} : \mathbf{d} + r - \boldsymbol{\nabla} \cdot \mathbf{q} \end{aligned}$$

Forme locale
dite non-conservative

$$\begin{aligned} \mathbf{t}(\mathbf{n}) &= \boldsymbol{\sigma}^T \cdot \mathbf{n} \\ \boldsymbol{\sigma} &= \boldsymbol{\sigma}^T \\ q(\mathbf{n}) &= -\mathbf{q} \cdot \mathbf{n} \end{aligned}$$	$$\begin{aligned} \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{v}) &= 0 \\ \frac{\partial (\rho \mathbf{v})}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{v}\mathbf{v}) &= \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \rho \mathbf{g} \\ \frac{\partial (\rho U)}{\partial T} + \boldsymbol{\nabla} \cdot (\rho \mathbf{v} U) &= \boldsymbol{\sigma} : \mathbf{d} + r - \boldsymbol{\nabla} \cdot \mathbf{q} \end{aligned}$$

Forme locale
dite conservative

Viscosité de volume

Viscosité de cisaillement

$$\boldsymbol{\sigma} = -p\boldsymbol{\delta} + 3\boxed{\hat{\kappa}(p,T)}\mathbf{d}^s + 2\boxed{\hat{\mu}(p,T)}\mathbf{d}^d,$$

$$\mathbf{q} = -\boxed{\hat{k}(p,T)}\boldsymbol{\nabla}T,$$

Conductibilité thermique

$$\rho = \hat{\rho}(p,T),$$
$$H = \hat{H}(p,T),$$
$$\boxed{S = \hat{S}(p,T).}$$

L'équation de comportement pour l'entropie n'est utile que pour vérifier que le second principe est bien satisfait !

$$TdS = dH - \frac{dp}{\rho} = dU - \frac{pd\rho}{\rho^2},$$

$$k \geq 0,$$
$$\kappa \geq 0,$$
$$\mu \geq 0.$$

Contraintes à respecter pour satisfaire Clausius-Duhem

Modèle du fluide visqueux newtonien

$$
\begin{aligned}
\boldsymbol{\sigma} &= -p\boldsymbol{\delta} + 3\hat{\kappa}(p,T)\mathbf{d}^s + 2\hat{\mu}(p,T)\mathbf{d}^d, \\
\mathbf{q} &= -\hat{k}(p,T)\boldsymbol{\nabla}T, \\
\rho &= \hat{\rho}(p,T), \\
H &= \hat{H}(p,T), \\
S &= \hat{S}(p,T).
\end{aligned}
$$

Le compte est bon !

conservation locale de la masse	ρ	1
conservation locale de la quantité de mouvement	$\mathbf{v}$	3
conservation locale de l'énergie	T	1
constitution pour les contraintes	$\boldsymbol{\sigma}$	6
constitution pour le flux calorifique	$\mathbf{q}$	3
constitution pour la masse volumique	p	1
constitution pour l'enthalpie	H	1
constitution pour l'entropie	S	1

<u>Remarque</u> : si une équation de comportement pour l'enthalpie est donnée… on en déduit automatique l'énergie interne et vice-versa.

$$U = -\frac{p}{\rho} + H$$

Une piscine de fluide non-newtonien…

Voilà un fluide bien compliqué à modéliser !
C'est bien autre chose que simple écoulement d'eau ou d'air !

Comprendre les fluides complexes : c'est la science de la rhéologie !
Même si 99% des fluides sont de l'air et de l'eau, tout le reste est aussi intéressant !