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AN *hp* ADAPTIVE STRATEGY FOR FINITE ELEMENT APPROXIMATION OF THE NAVIER-STOKES EQUATIONS

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Abstract

Recently, a rigorous *a posteriori* error estimate, based on the element residual method, for the steady state Navier–Stokes equations has been derived. In this paper, by using this error estimate, we construct an *h*-*p* adaptive strategy to minimize the total computation costs while achieving a targeted accuracy for steady incompressible viscous flow problems. The basic *h*-*p* adaptive strategy is to solve the approximate problem in three consecutive stages corresponding to three different meshes, i.e. an initial mesh, an intermediate adaptive *h*–mesh, and a final adaptive *h*-*p* mesh. Our numerical result shows that the three–step *h*-*p* adaptive strategy for the incompressible flow problems indeed provides an accurate approximate solution while keeping the computational costs under control.

INTRODUCTION

The goal of *h*-*p* adaptive finite element methods is to obtain an accurate approximate solution within a preset error tolerance at the least possible computational cost, mainly measured in terms of the total CPU time and the total computer memory used. There are two major questions that must be resolved in order to reach this goal. One is how to estimate the accuracy of approximate solutions when exact solutions are not available; the other concerns the control of computational costs to obtain a the user–specified error tolerance. The issue of estimating the error of approximate solution for the steady state Navier–Stokes equations is discussed in [1] [2], while the issue of designing an *h*-*p* adaptive strategy [4] is considered here.

To control computational costs, one needs to develop an efficient *h*-*p* adaptive strategy for obtaining near–optimal adaptive *h*-*p* meshes. The payoff can be considerable: exponential rates of convergence with respect to the computed error can be achieved by using very few degrees of freedom, and this translates into the meshes which deliver targeted accuracies with many fewer degrees of freedom than traditional *h*– or *p*– version methods. On the other hands, the total computational overhead in an unplanned adaptive *h*-*p* strategy can conceivably be greater than the cost for conventional uniform *h*– or *p*– methods. To overcome this potential complication, a prudently designed adaptive strategy is required.

The research in the design of efficient adaptive strategies is still in early stages. The first general *h*-*p* adaptive strategy was proposed by Rachowicz, et al. [3] in 1989, which involved a scheme requiring many steps of solving linear system for different stages of mesh in order to achieve an optimal mesh. Recently Oden, et al.

[4] developed an efficient three–step h-p adaptive strategy in which employs only three stages corresponding to three different meshes. Initial numerical experiments using the three–step h-p adaptive strategy are encouraging when the total CPU time needed is compared with those of uniform h– or p– methods. The current study extends this h-p adaptive strategy to the steady state Navier–Stokes flow problem.

Following this introduction, the Navier-Stokes equations and the basic notations used in this paper will be presented in Section 2. The finite element approximations of the Navier–Stokes equations is described in Section 3. In Section 4, two major theorems on the *a posteriori* error estimation for steady–state Navier–Stokes equations are given. The three–step h-p adaptive strategy is presented in Section 5. Finally, results of numerical experiments are given in Section 6.

THE NAVIER–STOKES EQUATIONS

The steady state Navier–Stokes equations on the bounded Lipschitz domain $\Omega \in \mathbb{R}^n$, n = 2 or 3, is described as follows:

$$\left.\begin{aligned} (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \nabla\cdot\boldsymbol{\sigma}(\boldsymbol{u},p) &= \boldsymbol{f} \quad \text{in } \Omega \\ \nabla\cdot\boldsymbol{u} &= 0 \quad \text{in } \Omega \\ \boldsymbol{u} &= 0 \quad \text{on } \partial\Omega \end{aligned}\right\} \tag{1}$$

where $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x})$, $\boldsymbol{x} = (x_1, \cdots, x_n) \in \Omega$, is the velocity field, and $\boldsymbol{f}$ is the body force. $\boldsymbol{\sigma}(\boldsymbol{u},p)$, the Cauchy stress, is defined as $2\nu\boldsymbol{D}(\boldsymbol{u}) - p\mathbf{1}$ with the kinematic viscosity $\nu > 0$, strain rate tensor $\boldsymbol{D}(\boldsymbol{u}) = \left(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T\right)/2$, pressure p, and the unit tensor $\mathbf{1}$.

To obtain the weak formulation of (1), we introduce the following spaces and norms

$$\begin{aligned} V &= (H_0^1(\Omega))^n \\ H &= \{\boldsymbol{v} \in V : div\boldsymbol{v} = 0\} \\ |\boldsymbol{v}|_1^2 &= \int_\Omega \nabla\boldsymbol{v} : \nabla\boldsymbol{v}\, dx = \int_\Omega \sum_{i,j=1}^{n} \frac{\partial v_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}\, dx \\ Q &= \left\{q \in L^2(\Omega) : \int_\Omega q\, dx = 0\right\} \\ \|q\|_0^2 &= \int_\Omega q^2 dx \end{aligned}$$

where $dx = dx_1 dx_2 \cdots dx_n$ and the trilinear , bilinear, and linear forms,

$$\begin{aligned} c: V\times V\times V &\to \mathbb{R}, & c(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) &= \int_\Omega \boldsymbol{u}\cdot\nabla\boldsymbol{v}\cdot\boldsymbol{w}\, dx \\ a: V\times V &\to \mathbb{R}, & a(\boldsymbol{u},\boldsymbol{v}) &= \int_\Omega 2\nu\boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{v}) dx \\ b: Q\times V &\to \mathbb{R}, & b(q,\boldsymbol{v}) &= \int_\Omega q\nabla\cdot\boldsymbol{v}\, dx \\ f: V &\to \mathbb{R}, & f(\boldsymbol{v}) &= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\, dx \end{aligned}$$

Then the weak formulation of Navier–Stokes equations is

$$\boxed{\begin{array}{l}\text{Find } (\boldsymbol{u},p) \in V \times Q \text{ such that for all} \\ (\boldsymbol{v},q) \in V \times Q, \\ c(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) + a(\boldsymbol{u},\boldsymbol{v}) - b(p,\boldsymbol{v}) = f(\boldsymbol{v}) \\ \qquad\qquad\qquad\qquad b(q,\boldsymbol{u}) = 0 \end{array}} \tag{2}$$

The forms $a(\cdot,\cdot)$, $b(\cdot,\cdot)$, $c(\cdot,\cdot,\cdot)$, and $f(\cdot)$ are continuous and $b(\cdot,\cdot)$ satisfies inf sup condition [5]. Also, we assume that there exists a constant γ such that

$$\gamma = \sup_{\boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in H} \frac{|c(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})|}{|\boldsymbol{u}|_1 |\boldsymbol{v}|_1 |\boldsymbol{w}|_1} \tag{3}$$

the data body force $\boldsymbol{f}$ is defined such that it corresponds to a functional,

$$f \in \left(H^{-1}\right)^n \tag{4}$$

with norm being defined as

$$\|f\|^* = \sup_{\boldsymbol{v} \in H \setminus 0} \frac{|f(\boldsymbol{v})|}{|\boldsymbol{v}|_{1,\Omega}} \tag{5}$$

Under these conditions, the existence and uniqueness of solutions of (2) are given as:

Theorem 1. (i) Under the above definitions and conditions, there exists at least one solution $(\boldsymbol{u},p) \in V \times Q$ to problem (2).

(ii) If, in addition,

$$\|f\|^* < \nu^2/\gamma \tag{6}$$

then the solution $(\boldsymbol{u},p)$ is unique.

Proof: See [6]. ∎

FINITE ELEMENT APPROXIMATIONS

To develop finite element approximations of (2), we introduce a partition $\mathcal{P}$ of Ω into a collection of $N = N(\mathcal{P})$ subdomains Ω_K:

$$\overline{\Omega} = \bigcup_{K=1}^{N(\mathcal{P})} \overline{\Omega}_K, \; \Omega_K \cap \Omega_L = \emptyset \; \forall \; K \neq L$$

We may now write

$$\begin{aligned} a(\boldsymbol{u},\boldsymbol{v}) &= \sum_{K=1}^{N} a_K(\boldsymbol{u},\boldsymbol{v}) \\ a_K(\boldsymbol{u},\boldsymbol{v}) &= \int_{\Omega_K} 2\nu \boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{v}) dx \quad \boldsymbol{u},\boldsymbol{v} \in V_K \\ b(q,\boldsymbol{v}) &= \sum_{K=1}^{N} b_K(q,\boldsymbol{v}) \\ b_K(q,\boldsymbol{v}) &= \int_{\Omega_K} q \nabla \cdot \boldsymbol{v} \, dx \qquad\qquad q \in Q_K, \; \boldsymbol{v} \in V_K \end{aligned}$$

etc., with similar definitions for $c_K(\cdot,\cdot,\cdot)$, $f_K(\cdot)$, where $V_K = V(\Omega_K)$ and $Q_K = \{q \in L^2(K) : q = p|_K,\ p \in Q\}$ denote corresponding local spaces of functions in V and Q, respectively, restricted to Ω_K.

Following the standard finite element approaches, let Ω and $\mathcal{P}$ be constructed such that each subdomain Ω_K is the image of a master element $\widehat{\Omega}$ under an affine invertible map $F_K\colon \widehat{\Omega} \to \Omega_K$, $1 \leq K \leq N$. If $\boldsymbol{\xi} = F_K^{-1}(\boldsymbol{x})$, $\boldsymbol{x} \in \overline{\Omega}_K$, we approximate test functions $\boldsymbol{v} \in V_K$, $q \in Q_K$ by functions $\boldsymbol{v}^h$ and q^h such that $\widehat{v}_i(\boldsymbol{\xi}) = v_i^h \circ F_K^{-1}(\boldsymbol{x})$, $1 \leq i \leq n$, $\widehat{q}(\boldsymbol{\xi}) = q^h \circ F_K^{-1}(\boldsymbol{x})$ are polynomials or products of polynomials in $\boldsymbol{\xi}$. The resulting spaces of functions have the properties $V_K^h \subset V_K$ and $Q_K^h \subset Q_K$.

The finite element approximations of (2) obtained using the spaces V^h and Q^h is characterized by the following discrete problem:

$$\boxed{\begin{array}{l} \text{Find } \left(\boldsymbol{u}^h, p^h\right) \in V^h \times Q^h \text{ such that for} \\ \text{every } \left(\boldsymbol{v}^h, q^h\right) \in V^h \times Q^h, \\ c\left(\boldsymbol{u}^h, \boldsymbol{u}^h, \boldsymbol{v}^h\right) + a\left(\boldsymbol{u}^h, \boldsymbol{v}^h\right) - b\left(p^h, \boldsymbol{v}^h\right) = f\left(\boldsymbol{v}^h\right) \\ \qquad\qquad\qquad\qquad\qquad b\left(q^h, \boldsymbol{u}^h\right) = 0 \end{array}} \tag{7}$$

Under proper conditions, one can construct convergent sequences of solutions to (7). See [6] for details. Moreover, the following result can be established:

Theorem 2. Let $n \leq 3$ and the conditions of Theorem 1 hold. Let $(\boldsymbol{u}, p)$ be the solution of (1). Then, for ν sufficiently large, there exists an h_0 such that for all $h \leq h_0$, (7) has a unique solution $(\boldsymbol{u}^h, p^h) \in V^h \times Q^h$ and

$$\lim_{h \to 0} \left\{ |\boldsymbol{u} - \boldsymbol{u}^h|_1 + \|p - p^h\|_0 \right\} = 0 \tag{8}$$

If, in addition, the solution $(\boldsymbol{u}, p)$ of (3) $\in \left(H^{k+1}(\Omega)^n \cap V\right) \times \left(H^k(\Omega) \cap Q\right)$ for $k \leq \ell$, then a constant $C > 0$ exists, independent of h, such that

$$\left|\boldsymbol{u} - \boldsymbol{u}^h\right|_1 + \|p - p^h\|_0 \leq C h^k \tag{9}$$

Proof: See [6], in particular pp. 317–318. ∎

THE *A POSTERIORI* ERROR ESTIMATE FOR THE STEADY STATE NAVIER–STOKES EQUATIONS

Now we shall construct the *a posteriori* error estimate for the Navier–Stokes equations. Let $(\boldsymbol{u}, p)$ and $(\boldsymbol{u}^h, p^h)$ be the unique solutions to the problems (1) and (7), respectively,

Define two bilinear forms as

$$\begin{aligned} A(\boldsymbol{u}, \boldsymbol{v}) &= \int_\Omega 2\nu \boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{v})\, dx \\ B(p, q) &= \int_\Omega p \cdot q\, dx \end{aligned} \tag{10}$$

and the pair $(\varphi, \psi) \in V \times Q$ which are solutions of

$$\begin{aligned} A(\varphi, v) &= a(e, v) - b(E, v) + c(u, u, v) - c(u^h, u^h, v) \\ B(\psi, q) &= -b(q, e) \\ &\forall\, (v, q) \in V \times Q \end{aligned} \tag{11}$$

where $e = u - u^h$ and $E = p - p^h$.

The existence and uniqueness of the pair (φ, ψ) in (11) follows immediately from the definitions of A and B in (10). Next we define the "star norm" of error to be

$$\|(e, E)\|_*^2 = |\varphi|_A^2 + |\psi|_0^2 \tag{12}$$

with

$$\begin{aligned} |\varphi|_A^2 &= A(\varphi, \varphi) \\ |\psi|_0^2 &= B(\psi, \psi) \end{aligned} \tag{13}$$

The "averaged" approximate flux on the boundary Γ_{KL} is defined as

$$\left\langle n_K \cdot \sigma\left(u^h, p^h\right)\right\rangle = n_K \cdot \left[(1 - \alpha_{KL}(s))\, \sigma_K\left(u^h, p^h\right) + \alpha_{KL}(s)\sigma_L\left(u^h, p^h\right)\right] \tag{14}$$

σ_K is the Cauchy stress in $\overline{\Omega}_K$ at $s \in \Gamma_{KL}$ and σ_L is that in neighboring element $\overline{\Omega}_L$ at s. Thus, (14) defines a linear combination of approximate boundary fluxes on the interelement boundary. Note that if we take $\alpha_{KL} = 1/2$, (14) reduces to a simple average of fluxes. We shall assume hereafter that the parameter functions α_{KL} are constructed in such a way that the element residual and boundary residuals are balanced in the sense of [7, 8], i.e., the residual fluxes are *equilibrated.*

The following theorem confirms that the star norm defined in (12) is actually equivalent to the usual norm used for the Navier-Stokes equation.

Theorem 3. Let the conditions of Theorem 1 holds and Theorem 2 holds for $k > 0$ and, moreover, there exists a constant L such that

$$|u|_1 \leq L < \frac{\nu}{\gamma} \tag{15}$$

Then there exists two positive constants k_1 and k_2 such that as $h \to 0$,

$$k_1\|(e, E)\|_*^2 \leq |e|_1^2 + \|E\|_0^2 \leq k_2\|(e, E)\|_*^2 \tag{16}$$

where k_1 and k_2 are positive constants.

Proof: See [1] [2]. ∎

The *a posteriori* error estimate for the Navier–Stokes equations is as follows:

Theorem 4. Let assumptions on Theorem 4 hold and $A_K(\cdot,\cdot)$ and $B_K(\cdot,\cdot)$ denote the element inner products corresponding to $A(\cdot,\cdot)$ and $B(\cdot,\cdot)$ of (10) and

let $\varphi_K \in V_K$ denote the solution of the local error residual problems,

$$\left.\begin{aligned} A_K(\varphi_K, v_K) &= f_K(v_K) - a_K\left(u_K^h, v_K\right) \\ &\quad + b_K\left(p_K^h, v_K\right) - c_K\left(u_K^h, u_K^h, v_K\right) \\ &\quad + \oint_{\partial\Omega_K/\partial\Omega} \left\langle n_K \cdot \sigma\left(u^h, p^h\right)\right\rangle \cdot v_K\, ds \end{aligned}\right\} \tag{17}$$

for every $v_K \in V_K$, $1 \leq K \leq N$. Then the error (e, E) of the finite element approximations of the Navier-Stokes equations (1) satisfies the following bound:

$$||(e,E)||_*^2 \leq \sum_{K=1}^{N} ||(\varphi_K, div\, u_K^h)||_{*,K}^2 \tag{18}$$

Proof: see [1]. ∎

THE THREE–STEP *h-p* ADAPTIVE STRATEGY

Here, we employ the three-step scheme introduced in [4] and exploited in [10] and elsewhere. For completeness, we record its basic properties. The goal of the three–step adaptive strategy is to reach a preset target error of the problem and to minimize the computational effort required.

To develop the scheme, we suppose that a global *a priori* error estimate exists in the star norm [9] :

$$||(e,E)||_*^2 \leq \sum_{K=1}^{N(\mathcal{P})} \frac{h_K^{2\mu_K}}{p_K^{2\nu_K}} \Lambda_K^2 \tag{19}$$

where h_K, p_K are respectively the size and the order of the element K, and Λ_K is a local unknown constant. We also define an error indicator θ and an error index η:

$$\begin{aligned} \theta &= \sqrt{\sum_{K=1}^{N} (\theta_K)^2}, \qquad \theta_K = \| \left(\varphi_K, div\, u_K^h\right) \|_{*,K} \\ \eta &= \frac{||\,(e,E)\,||_*^2}{||\,(u,p)\,||_*^2} \end{aligned} \tag{20}$$

Now, we introduce several major assumptions. This asymptotic estimate is treated as an equality and the actual error is approximated by the *a posteriori* error estimate θ. By setting, respectively, the unknown exponents μ_K and ν_K to given μ,ν, and then passing to the element level, we write the basic equation of the scheme :

$$\theta_K \approx \frac{h_K^{\mu}}{p_K^{\nu}} \Lambda_K \tag{21}$$

Now, we assign a target error index η^{tgt} and we are able to describe the three steps as follows :

- Introduce an initial mesh $\mathcal{P}^0$ of N^0 elements sufficiently fine to fall in the asymptotic part of the convergence curve for h-refinements. Solve the problem on this mesh. Calculate a local *a posteriori* error indicator θ_K^0 to estimate the local error.

  From the orthogonality of the error to the space of approximation, we can estimate both the energy like norm of the solution and the initial error index

$$||(u,p)||_*^2 \;\approx\; ||(u^0,p^0)||_*^2 \;=\; ||(u^{h\;0},p^{h\;0})||_*^2 + (\theta^0)^2$$

$$\eta^0 \;\approx\; \frac{\theta^0}{||(u^0,p^0)||_*}$$

Select η^{int} such that $\eta^0 \leq \eta^{int} \leq \eta^{tgt}$.

- Calculate the number n_K of new sub-elements required in each element of $\mathcal{P}^0$ in order to obtain an optimal mesh $\mathcal{P}^1$ of N^1 elements achieving the required error index η^{int}.

  For uniform refinements, the number of sub-elements can be correlated to their main size $n_K = (h_K^0/h_K^1)^{2/\beta}$ where $\beta = 2/n$ and n is the dimension of the problem. From (21), we can find the following system which allows us to compute n_K :

$$\left.\begin{aligned} n_K &= \left[N^1 \frac{(\theta_K^0)^2}{(\theta^{int})^2}\right]^{\frac{1}{\beta\mu}} \\ \sum_{K=1}^{N^0} n_K &= N^1 \end{aligned}\right\} \qquad (22)$$

  where the global error θ^{int} is predicted by $\eta^{int}\ ||(u^0, p^0, h^0)||_*$. Having n_K, we introduce h refinements to construct $\mathcal{P}^1$.

  Now, solve the problem on this second mesh and compute the local *a posteriori* error indicators θ_L^1.

- The third mesh $\mathcal{P}^2$ is constructed by calculating a distribution of polynomial degrees p_L for each element of $\mathcal{P}^1$ to reach the target error index η^{tgt}. From (21), we can calculate the final order of each element in order to reach an equidistributed target error indicator on the next mesh :

$$p_L = \left[N^1 \frac{(\theta_L^1)^2 (p_L^0)^{2\nu}}{(\theta^{tgt})^2}\right]^{\frac{1}{2\nu}} \qquad (23)$$

  where the global error θ^{tgt} is predicted by $\eta^{tgt}\ ||(u^1, p^1, h^1)||_*$.

  Now, enrich p on each element to obtain $\mathcal{P}^2$. Solve the problem on $\mathcal{P}^2$ and compute an estimate of the final error index η^2. If $\eta^2 \leq \eta^{tgt}$ the computation is terminated; otherwise the whole procedure is repeated.

This technique seems to be a good compromise between the cost of the adaptivity and the quality of the final mesh. In fact, it leads to good (but suboptimal) meshes and exhibits very fast convergence characteristics with respect to CPU time.

NUMERICAL RESULTS

Now, the results of the backstep channel solved with the three-steps adaptive strategy is presented. We consider the steady motion of an isothermal incompressible Newtonian fluid. We impose no-slip conditions on the walls and a fully developed profile in the entry section. The lengths of both channels are respectively 2 and 16 lengths of the outflow section. In order to compare our results with [11], we select an inflow section equal to 0.51485 and a Reynolds number of 300. The Reynolds number is defined by $Re = UL/\nu$ where U and L denote the average inflow velocity and the inflow length. The geometry features of the problem are defined in Fig. 1.

From an initial mesh of 877 scalar degrees of freedom and a quadratic interpolation, we calculate an estimated error index of 0.14. Then, the three-step strategy

Mesh	CPU for the solution (Number of iterations)		CPU for the error estimates	
			(α)	(0.5)
$\mathcal{P}^0$	12246	(21)	1283	866
$\mathcal{P}^1$	3333	(4)	2073	1171
$\mathcal{P}^2$	9264	(5)	3845	2787
Total	24843		7201	4824
	100%		28%	19%

Table 1: CPU time accounting for the backstep problem

Reattachement lengths	Reference results [11]	Present results
L^1	4.96	4.95
L^2	4.05	4.13
L^3	7.55	7.32

Table 2: Backstep problem : reattachement lengths

is used with an intermediate error index $\eta^{int} = 0.10$ and a target error index of $\eta^{tgt} = 0.08$. The final mesh is shown in Fig. 2. Computed pressure is shown in Fig. 3. Closeup views of the three meshes and the error index evolution are shown in Fig. 4. It is expected that the elements are h-refined near the singularity and that orders of $p = 4$ and $p = 3$ are assigned near this point. However, the adaptive strategy also leads to refinements and enrichments in other areas. The local *equilibrated* error estimates ((α)-Estimated Error) are plotted on Fig. 5. In order to illustrate the cost of the adaptive strategy, Table 1. contains the CPU time used for each part of the calculation. The numerical results reported in this work are obtained by a full Newton-Raphson scheme. A continuation technique is used for obtaining the solution on the first mesh. For each Newton's step, a direct frontal solver is used. The total number of iterations to reach the solution on each mesh (relative variation 10^{-9}) is also provided.

Table 2 contains results seen to be in excellent agreement with the literature. With 1530 scalar degrees of freedoms, values for the reattachement lengths are obtained which are agree with with those calculated with 12870 d.o.f.'s in [11].

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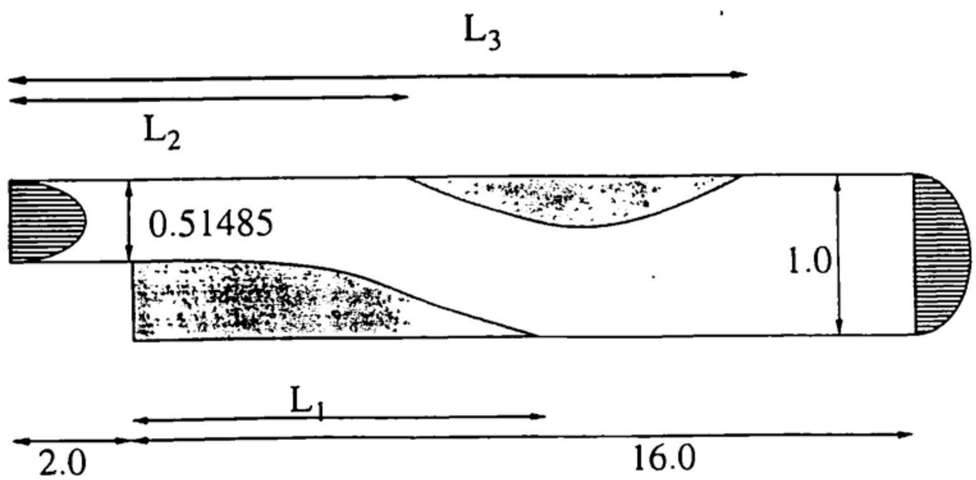

Figure 1: Geometry for the backstep problem.

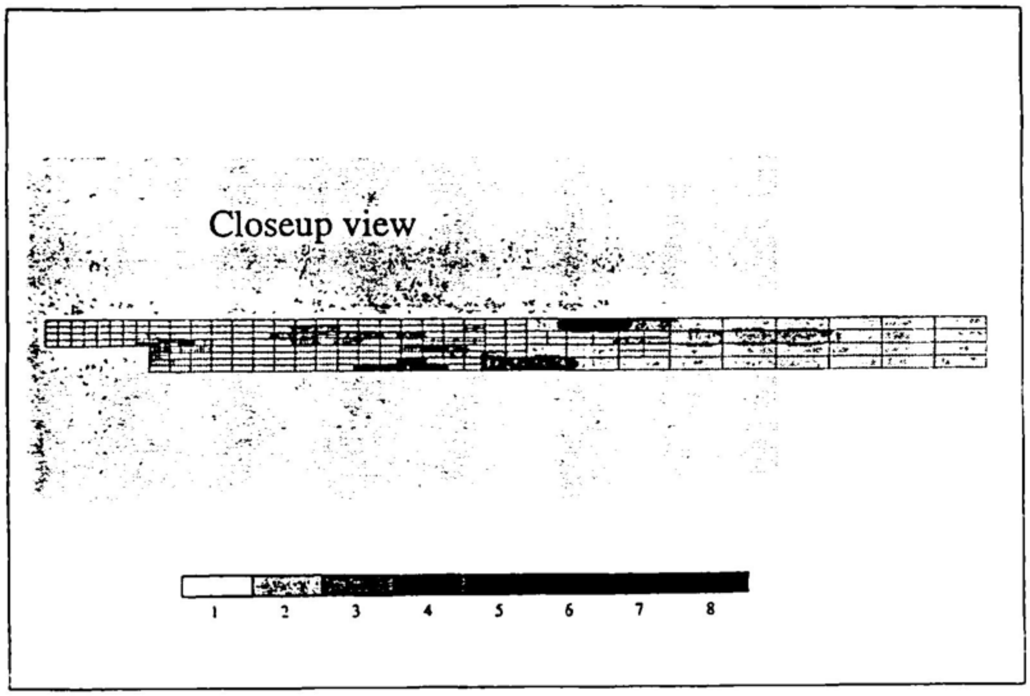

Figure 2: Backstep channel problem (Re=300), Newtonian fluid. Shaded elements reflect nonuniform p-distribution in final mesh.

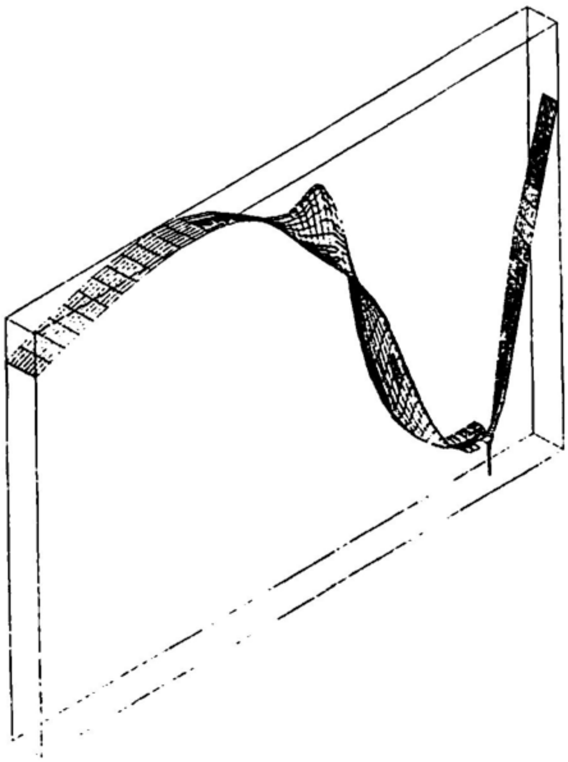

Figure 3: Backstep channel problem (Re=300), Newtonian fluid. 3D plot of the pressure

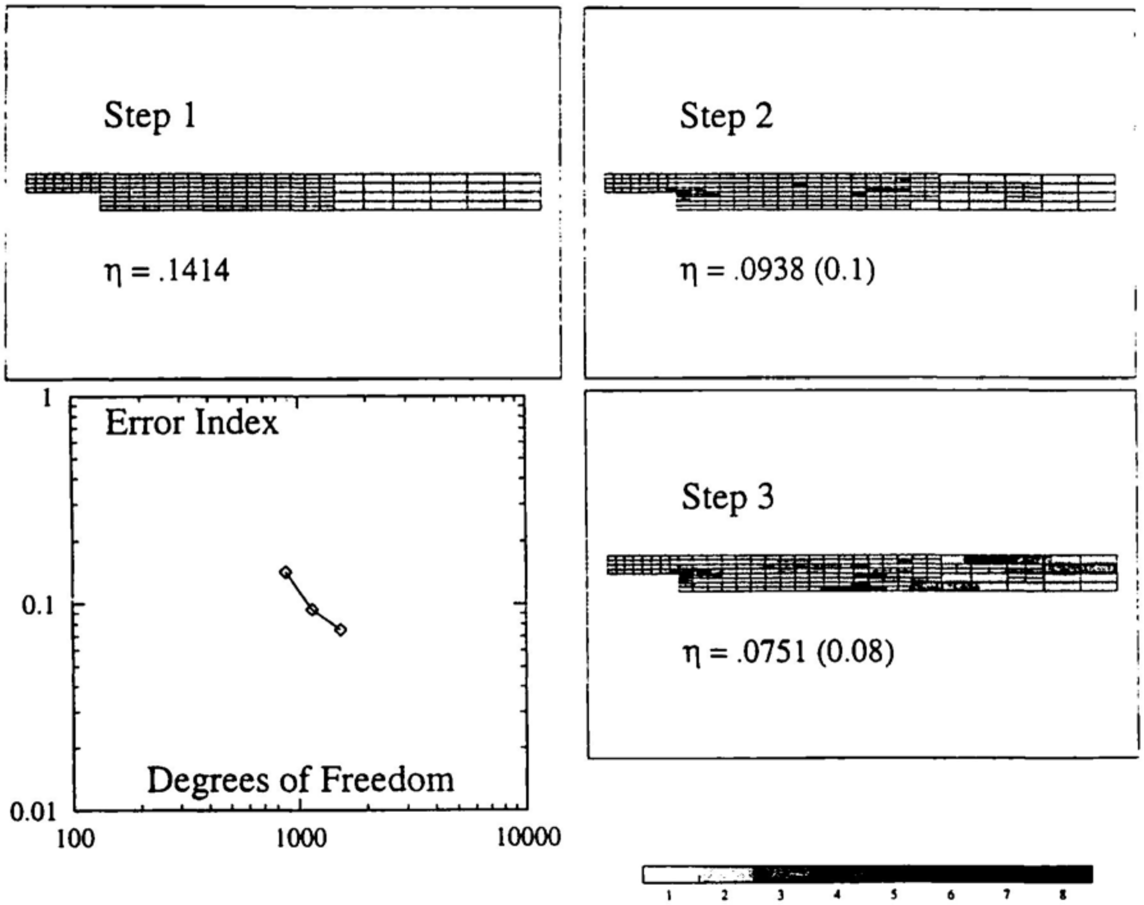

Figure 4: Backstep channel problem (Re=300), Newtonian fluid. Closeup views of the 3 adaptive meshes.

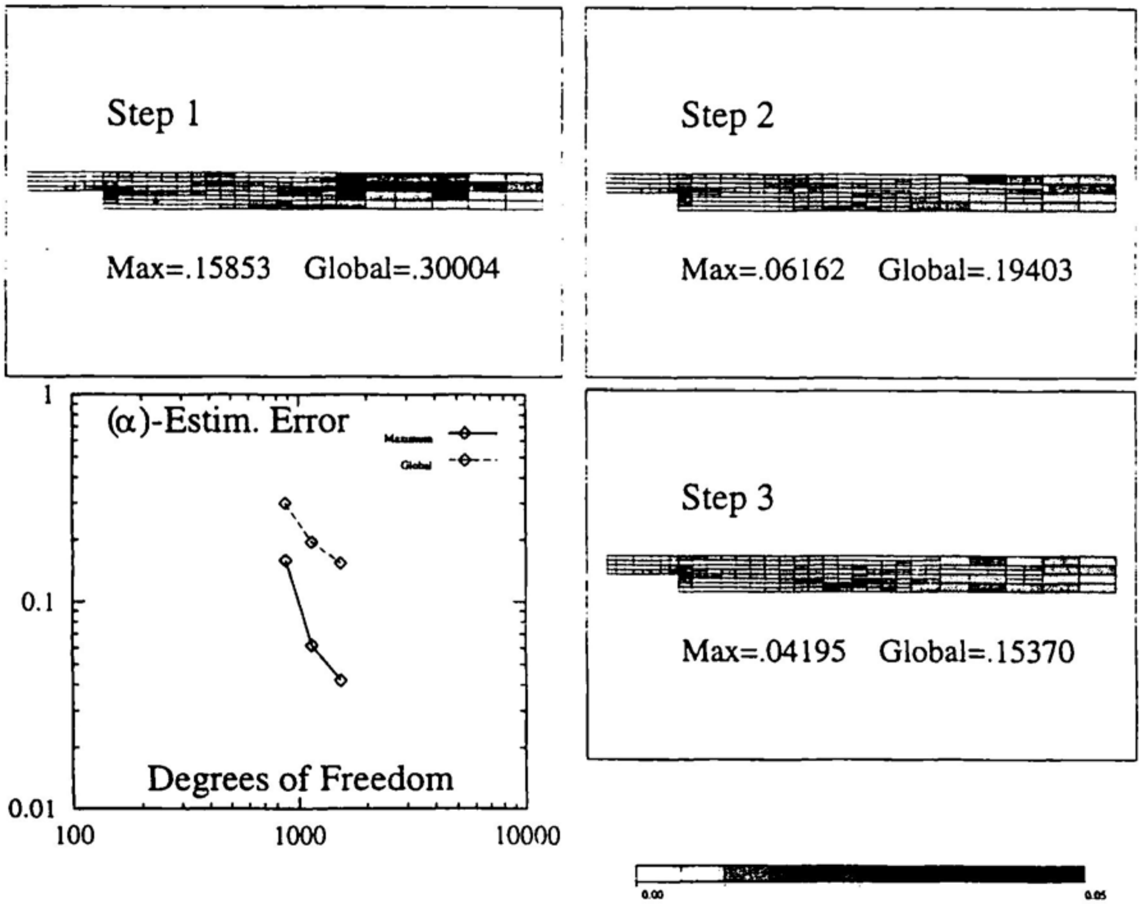

Figure 5: Backstep channel problem (Re=300), Newtonian fluid. Equilibrated estimated error.