Adaptive *hp*-Finite Element Viscoelastic Flow Calculations

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Abstract

Accuracy and stability remain key issues in viscoelastic flow simulation. Classical low-order finite element techniques fail to converge when the elasticity of the fluid is increased. In this paper, an adaptive *hp*-finite element method is used to solve differential viscoelastic flow problems. An *a posteriori* error estimator, based on some recent rigorous results of Oden and Wu for the Navier-Stokes equations ([1]) is also used. Starting from an initial mesh, local refinements (*h*-adaptivity) or enrichments (*p*-adaptivity) are applied in the spirit of the strategy proposed in [2]. The approximation error is reduced to a given level of accuracy with a minimal set of additional degrees of freedom. Numerical results on two 2D model problems illustrate both the validity of the error estimator presented and the efficiency of the adaptive procedure.

1 Introduction

Accurate and stable numerical methods are particularly important in viscoelastic flow simulation. Most of the techniques used in the literature are based upon a coupled stress-velocity-pressure formulation, leading to a mixed hyperbolic-elliptic set of equations. A characteristic number for this type of problem is the non dimensional Deborah number (De), which measures the elasticity of the fluid. Viscoelastic flow problems at high values of De has worked out to be extremely difficult to solve, keeping numerical simulations

far away from realistic problems. One of the major reasons are the inaccuracies developed by the numerical methods ([3]).

More and more attention has therefore been devoted to spectral ([4], [5], [6]) and high order methods ([7]). For smooth problems, those methods exhibit an exponential rate of convergence and an improved robustness.

In order to extend these properties to problems exhibiting singularities, we use an adaptive *hp*-finite element method. Local h-refinements and p-enrichments are particularly attractive in problems where complex geometries and singularities are involved. Hierarchical shape functions and 1-irregular meshes are introduced to allow local mesh adaptivity ([8]). The element size h and the order of approximation p are adjusted at the elemental level by different adaptive strategies, based on *a priori* and *a posteriori* error estimates. We obtain very high rates of convergence of the approximation error, even in the presence of singularities, so that the computer time required to achieve a target error can significantly be reduced. When the viscoelastic solver is imbedded in a continuation scheme, further savings in CPU time are obtained by careful adaptivity of the finite element mesh at intermediate steps of the evolution.

In Section 2, governing equations for viscoelastic flow simulation are detailed. The mathematical properties of the constitutive equations are reviewed and a weak formulation of the problem is given. The spatial discretization is briefly described in Section 3, leading to an *hp* discrete problem. The error estimating post-processor is described in Section 4 and a mesh adapting procedure is presented. Numerical experiments are performed on two model problems : the flow around a sphere falling in a cylinder and the flow through an abrupt 4:1 axisymmetric contraction. Lots of data are available for both problems, which have been extensively solved as benchmark problems in the viscoelastic flow community. Although no general conclusions can be drawn yet, the results presented in Section 5 give good indications of how the geometry and the fluid model affect the efficiency of the adaptive strategies. Comparisons are made for each problem with classical finite element results, taken from the literature.

2 The Viscoelastic Problem

We consider the steady flow of a viscoelastic fluid in an open bounded Lipschitz domain $\Omega \in R^n$. the governing partial differential are the mass and momentum conservation laws, coupled with constitutive equations characterizing the viscoelastic fluid.

The conservation laws can be written as follow :

$$\left.\begin{array}{rcl} -\boldsymbol{\nabla}\cdot\boldsymbol{\sigma} &=& \boldsymbol{f} \\ \boldsymbol{\nabla}\cdot\boldsymbol{u} &=& 0 \end{array}\right\} \tag{1}$$

where $\boldsymbol{u}$ is the velocity field, $\boldsymbol{f}$ the body force and $\boldsymbol{\sigma}$ the Cauchy stress tensor. We neglect the effects of inertia in the flow.

Many constitutive equations have been proposed to model the behavior of viscoelastic fluids. An important class of model constitutive equations can be written in the form

$$\boldsymbol{\sigma} \;=\; \boldsymbol{\tau}_N + \boldsymbol{\tau}_V - p\boldsymbol{I} \tag{2}$$

where p is the pressure, $\boldsymbol{\tau}_N$ is the Newtonian part of the stress tensor and $\boldsymbol{\tau}_V$ the viscoelastic extra-stress tensor. In particular, we consider a family of differential viscoelastic models ([9]), with the following constitutive equations.

$$\left.\begin{array}{rcl} \boldsymbol{\tau}_N &=& 2\eta_N \boldsymbol{D}(\boldsymbol{u}) \\ \boldsymbol{\tau}_V + \lambda(tr(\boldsymbol{\tau}_V))\overset{\triangledown}{\boldsymbol{\tau}}_V &=& 2\eta_V \boldsymbol{D}(\boldsymbol{u}) \end{array}\right\} \tag{3}$$

where the strain rate tensor $\boldsymbol{D}(\boldsymbol{u})$ is defined as

$$\boldsymbol{D}(\boldsymbol{u}) \;=\; (\boldsymbol{\nabla}\,\boldsymbol{u} + \boldsymbol{\nabla}^T\boldsymbol{u})/2 \tag{4}$$

λ is the relaxation time of the fluid, η_N and η_V are kinematic viscosities associated respectively with $\boldsymbol{\tau}_N$ and $\boldsymbol{\tau}_V$. The symbol $\triangledown$ denotes the upper convected derivative. For a steady flow, we have

$$\overset{\triangledown}{\boldsymbol{\tau}} = \;(\boldsymbol{u}\cdot\boldsymbol{\nabla})\,\boldsymbol{\tau} - \boldsymbol{\nabla}\,\boldsymbol{u}^T\cdot\boldsymbol{\tau} - \boldsymbol{\tau}\cdot\boldsymbol{\nabla}\,\boldsymbol{u} \tag{5}$$

When λ is a constant, we obtain the classical Oldroyd-B model.

$$\left.\begin{array}{rcl} \boldsymbol{\tau}_N &=& 2\eta_N \boldsymbol{D}(\boldsymbol{u}) \\ \boldsymbol{\tau}_V + \lambda\,\overset{\triangledown}{\boldsymbol{\tau}}_V &=& 2\eta_V \boldsymbol{D}(\boldsymbol{u}) \end{array}\right\} \tag{6}$$

Another interesting model of the family (3) is the Modified Upper Convected Maxwell (MUCM) model ([10, 11]). The associated relaxation time λ is given by

$$\lambda(tr(\boldsymbol{\tau}_V)) = \frac{\lambda_0}{1 + \left(\frac{tr(\boldsymbol{\tau}_V) F(\lambda_0 \dot{\gamma}_0)}{\eta_V \dot{\gamma}_0}\right)^{\alpha - 1}} \tag{7}$$

where $\dot{\gamma}_0$ is a given characteristic shear rate inside Ω. The constants λ_0, α and the function F are given parameters of the model. Generally, they are chosen in order to fit the MUCM model to experimental data.

The well-posedness of the finite element problem is guaranteed, as the stress tensor $\boldsymbol{\tau}_V$ remains square integrable. If F is kept fixed at a non zero value, λ goes to zero in region of high stresses, leading to a Newtonian behavior of the fluid. This is why Apelian et al. [10] proposed such a model. If F vanishes, we derive the classical Maxwell-B and Oldroyd-B models, which may exhibit non integrable extra-stresses.

This set of equations (1)(2)(3) is completed by appropriate boundary conditions [12].

- The mixed type (elliptic-hyperbolic) of the constitutive equations implies that some components of the stresses should be imposed at the inflow section. In fact, we consider that the flow is fully developed and we impose

$$\boldsymbol{\tau}_V = \hat{\boldsymbol{\tau}}_V \quad on\ \partial\Omega_{Inflow} \tag{8}$$

- Classical Dirichlet or Neumann boundary conditions on each velocity component are required by the momentum equations. We impose pure Dirichlet conditions at inflow and outflow sections and along rigid walls, i.e.

$$\boldsymbol{u} = \hat{\boldsymbol{u}} \quad on\ \partial\Omega_D \tag{9}$$

and Robin conditions along axis of symmetry,

$$\begin{aligned} \boldsymbol{u}.\boldsymbol{n} &= 0 \\ \boldsymbol{\sigma}.\boldsymbol{n} - \boldsymbol{n}\boldsymbol{\sigma}.\boldsymbol{n}.\boldsymbol{n} &= \boldsymbol{0} \quad on\ \partial\Omega_R \end{aligned} \tag{10}$$

where $\hat{\boldsymbol{u}}$ is a prescribed velocity and $\boldsymbol{n}$ is the outward unit normal vector to the boundary.

The characteristic number for this type of flow is the Deborah number which compares a characteristic time of the fluid with a characteristic time of the experiment. Here, we have

$$De = \lambda_0 \dot{\gamma}_0$$

The weak formulation of the viscoelastic problem can be written as follow

Given body forces $\boldsymbol{f} \in V^*$,
find $(\boldsymbol{\tau}_V, \boldsymbol{u}, p) \in S \times V \times Q$ *such that*

$$\begin{aligned}
\int_\Omega (\boldsymbol{\tau}_V \cdot \boldsymbol{s} + \lambda (tr(\boldsymbol{\tau}_V)) \overset{\triangledown}{\boldsymbol{\tau}}_V \cdot \boldsymbol{s}) dx &= \int_\Omega 2\eta_V \boldsymbol{D}(\boldsymbol{u}) \cdot \boldsymbol{s} dx \\
\int_\Omega (\boldsymbol{\sigma}(\boldsymbol{\tau}_V, \boldsymbol{u}, p) : \boldsymbol{D}(\boldsymbol{v}) - \boldsymbol{f} \cdot \boldsymbol{v}\ ) dx &= 0 \\
\int_\Omega q \boldsymbol{\nabla} \cdot \boldsymbol{u}\, dx &= 0 \\
&\forall\ \boldsymbol{s} \in S, \forall \boldsymbol{v} \in V, \forall q \in Q.
\end{aligned}$$

$$\begin{aligned}
\textit{where} \quad S &= \{\boldsymbol{\tau}_V \in (H^1(\Omega))^m : \boldsymbol{\tau}_V = \hat{\boldsymbol{\tau}}_V \textit{ on } \partial\Omega_{Inflow}\} \\
V &= \{\boldsymbol{u} \in (H^1(\Omega))^n : \boldsymbol{u} = \hat{\boldsymbol{u}} \textit{ on } \partial\Omega_D, \boldsymbol{u} \cdot \boldsymbol{n} = 0 \textit{ on } \partial\Omega_R\} \\
Q &= \{p \in L^2(\Omega) : \int_\Omega p dx = 0\}
\end{aligned}$$

(11)

n and m are the number of components of the velocity vector and the extra-stress tensor, respectively.

3 Spatial Discretization of the Problem

We describe first the discretization of a scalar variable u. We recall here the basic features of the *hp*-finite element method. For a full description, the reader is referred to [8, 13, 14].

3.1 The *hp*-finite element approximation space

The domain Ω is decomposed into N non overlapping subdomains Ω_K, so that each Ω_K is the image of a master element $\hat{\Omega} = [-1,1]^n$ under an invertible map $\mathcal{F}_K$.

For the one-dimensional case, the approximate solution is built using the following shape functions.

- At the vertex node i, we use the standard linear shape functions

$$\hat{\chi}_i(\xi) = \frac{1}{2}(1 \pm \xi) \qquad i = 1,2 \tag{12}$$

- At the center of the element, we use

$$\hat{\zeta}_j(\xi) = \sqrt{\frac{2j-1}{2}} \int_{-1}^{s} P_{j-1}(t)dt \qquad j = 2,...,p \tag{13}$$

  where P_{j-1} is the Legendre polynomial of degree $j-1$.

It is straightforward to obtain two and three dimensional shape functions with tensor product extensions of the one-dimensional case.

With this polynomial basis, the approximation inside an element can no more be determined by a single integer p. In the two-dimensional case, a set of 6 integers is needed. This set is defined as

$$\bar{p}_K = \{p_{K,1}, ...p_{K,6}\}$$

with

$$p_K = \min_{p_{K,i} \in \bar{p}_K} \{p_{K,i}\}$$

The distribution of these sets over the whole mesh will be noted

$$p = \{\bar{p}_K : K = 1...N\} \tag{14}$$

In the same way, we define the distribution of the maximal diagonal lengths h_K inside each element Ω_K

$$h = \{h_K : K = 1...N\} \tag{15}$$

The local hp approximation space for a scalar variable u over element Ω_K can be defined as

$$M_K^{hp} = \{u = u(x) = \hat{u} \circ \mathcal{F}_K^{-1},\ \hat{u} \in \widehat{M}(\hat{\Omega}, \bar{p}_K)\} \tag{16}$$

where $\widehat{M}(\hat{\Omega}, \bar{p}_K)$ is the space spanned by the basis of n-dimensional shape functions described above, with a given set of exponent $\bar{p}_K$.

The global hp approximation space built under the classical continuity assumptions as in [8] is written as

$$S(h, p, \Omega) = \{u = u(x) \in \mathcal{C}^0(\overline{\Omega}) : u_K = u|_{\Omega_K} \in M_K^{hp}\} \tag{17}$$

Inside element Ω_K, the following interpolation property holds (see [8])

$$\left\| u - \tilde{u}^{hp} \right\|_{s,\Omega_K} \leq C\, \frac{h_K^{\min(p_K+1-s,\, r-s)}}{p_K^{r-s}} \|u\|_{r,\Omega_K} \tag{18}$$

where $u \in H^r(\Omega), r > s$ and $\tilde{u}^{hp}$ is an appropriate approximation of u inside $S(h, p, \Omega)$, The symbol $\|\ \ \|_{s,\Omega_K}$ denotes the usual Sobolev norm.

3.2 The *hp* discrete problem

We present now the approximation spaces associated with the full viscoelastic problem (11). Our mixed formulation involves different fields and we plan to be able to use different hp-interpolations for each of them. In particular, it means that we define h_S, h_V, h_Q and p_S, p_V, p_Q for the h- and p-distributions of each field. By notational convenience, h and p will denote the full set (h_S, h_V, h_Q) and (p_S, p_V, p_Q) respectively.

We select a finite dimensional subspace $T(h, p, \Omega)$ of $S \times V \times Q$ and rewrite

the discrete form of the weak problem using the classical Galerkin's method.

Given body forces $\boldsymbol{f} \in V^*$,
find $(\boldsymbol{\tau}_V^{hp},\, \boldsymbol{u}^{hp},\, p^{hp}) \in T(h,p,\Omega)$ *such that*

$$
\begin{aligned}
\int_\Omega (\boldsymbol{\tau}_V^{hp} \cdot \boldsymbol{s} + \lambda\, \overset{\triangledown}{\boldsymbol{\tau}}_V^{hp} \cdot \boldsymbol{s}) dx &= \int_\Omega 2\eta_V \boldsymbol{D}(\boldsymbol{u}^{hp}) \cdot \boldsymbol{s} dx \\
\int_\Omega (\boldsymbol{\sigma}(\boldsymbol{\tau}_V^{hp}, \boldsymbol{u}^{hp}, p^{hp}) : \boldsymbol{D}(\boldsymbol{v}) - \boldsymbol{f} \cdot \boldsymbol{v}\ ) dx &= 0 \\
\int_\Omega q\, \boldsymbol{\nabla} \cdot \boldsymbol{u}^{hp}\, dx &= 0 \\
&\forall\, (\boldsymbol{s}\,,\, \boldsymbol{v}\,,\, q) \in T(h,p,\Omega)
\end{aligned}
$$

$$
\begin{aligned}
\textit{where}\ \ T(h,p,\Omega) = \{(\boldsymbol{\tau}_V^{hp},\, \boldsymbol{u}^{hp},\, p^{hp}) :\quad & \boldsymbol{\tau}_V^{hp} \in (S(h_S, p_S, \Omega))^m \cap S \\
& \boldsymbol{u}^{hp} \in (S(h_V, p_V, \Omega))^n \cap V \\
& p^{hp} \in S(h_Q, p_Q, \Omega) \cap Q \qquad \}
\end{aligned}
$$

(19)

The exponents sets distributions associated with each variable cannot be chosen independently. Let us just mention that the Ladyzhenskaya-Brezzi-Babuška (LBB) condition imposes p_Q to be constrained with respect to p_V for the Stokes problem.

The numerical simulation of viscoelastic flows has always been suffering from the difficulty of reaching high Deborah numbers. This is certainly related both to the choice of the approximation spaces and to the selection of the mixed formulation. Such topic has been discussed in a large number of recent papers in the literature ([5, 15]). We just want to mention that the general *hp*-finite element discretization presented here allow us to modify *separately* and *locally* the distributions p_S, p_V, p_Q and h_S, h_V, h_Q. We will keep the velocity and extra-stress distributions close to each other in non-critical regions and enrich or refine the extra-stresses only where needed.

4 Error Estimator and Adaptive Strategy

In this section, we describe a numerical procedure to obtain reliable and realistic upper bounds to the approximation error. Our estimation technique is based upon some recent rigorous results for elliptic problems ([16, 17], for the Stokes ([18]) and the Navier-Stokes ([1]) equations. The adaptive strategies are some generalized versions of the procedure given in [2] and exploited in [14].

First, let us define the true approximation errors :

$$\begin{aligned} \boldsymbol{\tau}^{error} &= \boldsymbol{\tau} - \boldsymbol{\tau}^{hp}, \\ \boldsymbol{u}^{error} &= \boldsymbol{u} - \boldsymbol{u}^{hp}, \\ p^{error} &= p - p^{hp}, \end{aligned}$$

and the following norms:

$$\begin{aligned} |\boldsymbol{s}|_S^2 &= \sum_{K=1}^{N^h} \int_{\Omega_K} \boldsymbol{s} \cdot \boldsymbol{s} dx, \\ |\boldsymbol{v}|_V^2 &= \sum_{K=1}^{N^h} \int_{\Omega_K} 2\eta_N \boldsymbol{D}(\boldsymbol{v}) : \boldsymbol{D}(\boldsymbol{v}) dx, \\ |q|_Q^2 &= \sum_{K=1}^{N^h} \int_{\Omega_K} q^2 dx. \end{aligned}$$

The error will be estimated in an energy-like norm defined on $S \times V \times Q$ by:

$$|||(\boldsymbol{s}, \boldsymbol{v}, q)|||^2 = |\boldsymbol{s}|_S^2 + |\boldsymbol{v}|_V^2 + |q|_Q^2. \tag{20}$$

4.1 *A posteriori* error estimation

As in all element residual methods, a local error estimator for the approximation error is given by the peculiar norm of the solution of a boundary-value problem built at the elemental level. No rigorous results have been obtained so far for the viscoelastic problem, but we assume that we can extend the

local set of equations used in [18] for the Stokes problem to the following one:

Given $(\boldsymbol{\tau}_K^{hp},\, \boldsymbol{u}_K^{hp},\, p_K^{hp})$ *solution of the discrete problem (19),*
find $(\boldsymbol{\tau}_K^{est},\, \boldsymbol{u}_K^{est},\, p_K^{est}) \in S_K \times V_K \times Q_K$ *such that*

$$\int_{\Omega_K} \boldsymbol{\tau}_K^{est} \cdot\ \boldsymbol{s}_K dx =$$

$$\begin{aligned} & \int_{\Omega_K} (\boldsymbol{\tau}_K^{hp} \cdot \boldsymbol{s}_K + \lambda \overset{\triangledown}{\boldsymbol{\tau}}_K^{hp} \cdot \boldsymbol{s}_K) dx \\ - & \int_{\Omega_K} 2\eta_V \boldsymbol{D}(u_K^{hp}) \cdot \boldsymbol{s}_K dx \end{aligned}$$

$$\int_{\Omega_K} 2\eta_N \boldsymbol{D}(\boldsymbol{u}_K^{est}) : \boldsymbol{D}(\boldsymbol{v}_K) dx =$$

$$\begin{aligned} & \int_{\Omega_K} -\boldsymbol{\sigma}_K(\boldsymbol{\tau}_K^{hp}, \boldsymbol{u}_K^{hp}, p_K^{hp}) : \boldsymbol{D}(\boldsymbol{v}_K) dx \\ + & \int_{\partial\Omega_K} \left\langle \boldsymbol{n}_K \cdot \boldsymbol{\sigma}^{est}(\boldsymbol{\tau}_K^{hp}, \boldsymbol{u}_K^{hp}, p_K^{hp}) \right\rangle \cdot \boldsymbol{v}_K ds \\ + & \int_{\Omega_K} \boldsymbol{f} \cdot \boldsymbol{v}_K\ dx \end{aligned}$$

$$\int_{\Omega_K} p_K^{est}\ q_K\ dx \quad = \quad \int_{\Omega_K} q_K\, \boldsymbol{\nabla} \cdot \boldsymbol{u}_K^{hp}\, dx$$

$$\forall\ \boldsymbol{s}_K \in S_K, \forall\ \boldsymbol{v}_K \in V_K, \forall\ q_K \in Q_K,\ 1 \leq K \leq N^h.$$

(21)

where $\langle \boldsymbol{n}_K \cdot \boldsymbol{\sigma}^{est}(\boldsymbol{\tau}_K^{hp},\, \boldsymbol{u}_K^{hp},\, p_K^{hp}) \rangle$ is a flux term appearing in the local problem, as the continuity constraints have been relaxed at the interelement boundaries. The spaces S_K, V_K and Q_K are the restrictions of S, V and Q to the element Ω_K.

We approximate $(\boldsymbol{\tau}_K^{est},\, \boldsymbol{u}_K^{est},\, p_K^{est})$ with a polynomial expansion of higher degree than the expansion used for $(\boldsymbol{\tau}_K^{hp},\, \boldsymbol{u}_K^{hp},\, p_K^{hp})$.

The local error estimators are exploited in order to obtain the global error indicator θ_i for a given mesh $\mathcal{P}_i^h$.

$$\theta_{i,K} \quad = \quad |||(\boldsymbol{\tau}_K^{est},\, \boldsymbol{u}_K^{est},\, p_K^{est})|||_K \tag{22}$$

$$\theta_i^2 \quad = \quad \sum_{K=1}^{N_i^h} \theta_{i,K}^2 \tag{23}$$

where $|||\ |||_K$ is the energy-like norm for Ω_K.

The reliability and efficiency of the estimation procedure described above cannot be guaranteed by a mathematical proof. However, numerical examples will show the excellent agreement of our error estimator with some other error measurements.

4.2 *A priori* error estimation

Considering the interpolation property (18), we suppose that a local *a priori* estimator of the approximation error for each field is available :

$$|||(\boldsymbol{\tau}^{error}, \boldsymbol{u}^{error}, p^{error})|||_K^2 \leq \frac{h_K^{2\mu_K}}{p_K^{2\nu_K}} \Lambda_K^2, \tag{24}$$

where Λ_K, μ_K and ν_K are local unknown constants and have to be estimated.

To generalize this expression to the viscoelastic problem, we have to define correct values for h_K and p_K. As a well-known observation in all viscoelastic flows ([3]), we expect the extra-stresses to be mainly affected by errors. Such inaccuracies are due to the convective term in the constitutive equations (3), the stress concentration near geometrical singularities and the appearance of very thin stress boundary layers due to the presence of normal stresses. A natural choice is therefore to take $p_K = p_{S,K}$ as the reference order of approximation and $h_K = h_{S,K}$ as the reference mesh size inside element Ω_K. Numerical results presented in [19] clearly illustrate the validity of this assumption.

4.3 Adaptive strategy

Practically, we set μ_K and ν_K to given μ and ν (constant over the whole domain Ω). The error estimator (23) is computed on uniformly h-refined or p-enriched meshes, so that μ and ν can be obtained easily from linear regression of the error curve in a log-log scale. More sophisticated techniques can be used to estimate μ_K and ν_K at the elemental level, improving the efficiency of the adaptive strategy as proposed in [20]. However, this is not the purpose of the present paper. The Λ_K are estimated by means of the *a posteriori* error estimates $\theta_{i,K}$.

Starting with an initial set of meshes $\mathcal{P}_h$, let us choose θ^{tgt} as preset level of accuracy. We assume that we have computed the solutions of problems

(19) and (21) and that the actual local error estimators $\theta_{i,K}$ are available. If we apply pure h-refinements to build the new meshes $\mathcal{P}_h^{tgt}$, the reference mesh size $h_{S,K}^{tgt}$ inside each element of $\mathcal{P}_h^{tgt}$ has to satisfy

$$\frac{h_{S,K}}{h_{S,K}^{tgt}} = \left[N^{tgt} \frac{(\theta_{i,K})^2}{(\theta^{tgt})^2} \right]^{\frac{1}{2\mu}}. \tag{25}$$

In this paper, the same mesh size distribution is taken for all variables, so that

$$h_{V,K}^{tgt} = h_{Q,K}^{tgt} = h_{S,K}^{tgt}. \tag{26}$$

The corresponding pure p-refinements are obtained directly from the following expression.

$$\frac{p_{S,K}^{tgt}}{p_{S,K}} = \left[N \frac{(\theta_{i,K})^2}{(\theta^{tgt})^2} \right]^{\frac{1}{2\nu}}. \tag{27}$$

The rule chosen to select the p-distribution for pressure and velocities is related to both the aforementioned LBB condition and numerical experiments presented in [19]. We have

$$p_{V,K}^{tgt} = p_{S,K}^{tgt} - 1 \text{ and } p_{Q,K}^{tgt} = p_{V,K}^{tgt} - \beta, \tag{28}$$

where β is equal to 1 or 2, respectively for $p_{V,K}^{tgt} = 2$ or $p_{V,K}^{tgt} > 2$.

For the Stokes problem, it has been demonstrated ([21, 22]) that this particular choice is simultaneously stable with respect to the mesh size h and to the polynomial degree p. Stenberg and Suri recently proposed several stable families of mixed finite elements for the Stokes problem. Some of these should be more suitable than (28) in the framework of uniform p-distributions. As we are using local p-enrichments in our approximations, no immediate generalizations are available. We choose therefore the suboptimal family (28) to ensure the stability of the mixed finite element method.

Several numerical experiments have shown that abrupt increase of the order of approximation between two neighboring elements can be detrimental for the accuracy of the solution. We will prevent these jumps in the p-distribution by adding extra-elements to the list given by Equation (27).

5 Numerical Results

To illustrate our adaptive procedure, we consider the steady motion of an isothermal, incompressible viscoelastic fluid in two model problems: the flow around a sphere falling in a cylinder and the flow through an axisymmetric contraction. The results reported are obtained by a full Newton-Raphson scheme, coupled with a suitable continuation technique.

5.1 Sphere problem

We consider the flow around a sphere falling in a cylinder at a constant velocity V. The geometry of the problem and the boundary conditions are given in Figure 1. The radius of the sphere is half the radius of the cylinder and our calculation domain extends from $-15R$ to $30R$, with the sphere centered at the origin. The geometrical parameters are such that $\dot{\gamma}_0 = V/R = 4/3$.

The *drag correction factor* Γ is defined by the ratio between the drag exerted on the sphere and the drag D^0 exerted by a creeping Newtonian flow on an identical sphere in an infinite space. We have

$$\Gamma = \frac{D}{D^0} = \frac{D}{6\pi(\eta_N + \eta_V)VR} \quad . \tag{29}$$

Debae et al. [15] give reference values for Γ at increasing De for a Maxwell-B fluid. We check the accuracy of the error estimation procedure by the comparison of the *drag correction factor* computed by the *hp*-adaptive method with the reference. This *drag based error* is presented on Figure 2 with the *a posteriori* estimated error for three different *hp*-meshes. The fluid is described by the constitutive equations (3)(7) with $F = 0$, $\lambda_0 = 0.3$, $\eta_N = 0$ and $\eta_V = 1$, corresponding to a Maxwell-B model. It can be pointed out that the error estimator provides a similar evolution with mesh enrichments as the *drag based error*. Closeup views of the meshes produced by the automatic adaptation procedure are presented on Figure 2, with the corresponding error distributions. Their number of degrees of freedom are given in Table 1. The error index, obtained by dividing the error indicator by the current norm of the solution, is also presented in Table 1, with mass and momentum balances and the computed *drag correction factor*. In this case, it appears to be not

optimal to apply h-refinements, as local p-enrichments lead to higher rates of convergence.

The same procedure is applied with a MUCM fluid, characterized by $F \neq 0$, $\lambda_0 = 0.45$, $\eta_N = 0$ and $\eta_V = 1$. The results are presented on Figure 3. From the numerical values given in Table 2, we see that both mass and momentum balances decrease with the error index.

Finally, we compare the drag correction factors obtained at increasing De with several low- and high-order finite element methods for a Maxwell-B fluid. These results are reported on Figure 4. With a classical low-order method (taken from [15]), numerical inaccuracies developed during the calculations prevent the continuation scheme to reach De numbers larger than 0.8. The stresses are interpolated with 4×4 linear subelements, while velocities and pressure are, respectively, biquadratic and bilinear. A very fine mesh is used with 510 elements and 38644 degrees of freedom. A p-finite element method with 3 times less degrees of freedom, allow us to reach the same De number. The order of approximation for stresses, velocities and pressure are set, respectively, to 6,5 and 3 throughout the mesh. Using a purely p-adaptive finite element method, local enrichments of the stress field up to order 7 bring the critical De number to 0.975. Only 3% extra degrees of freedom were needed for this computation, if compared with the p-method. The mesh used for extra-stress interpolation in the p-adaptive method is plotted in Figure 5 with the 4×4 h-finite element mesh.

5.2 Axisymmetric contraction problem

The second problem considered is the steady motion of a viscoelastic fluid through an abrupt 4:1 contraction. The geometry and boundary conditions are given in Figure 6. At the inflow, we impose $\boldsymbol{v} = \hat{\boldsymbol{v}}$ and $\boldsymbol{\tau}_V = \hat{\boldsymbol{\tau}}_V$, assuming that these fields are fully developed. We suppose that the fluid sticks to the wall. Only velocities are imposed at the exit section, which is taken long enough to insure a fully developed profile. This profile is chosen to achieve global mass conservation in Ω. The lengths of the entry and exit sections are equal to 20 downstream radii. The characteristic shear rate evaluated at the wall of the exit section is $\dot{\gamma}_0 = 1$.

The Couette correction is the nondimensional pressure loss due to the

contraction and is defined as follows :

$$C = \frac{\delta p - \nabla p_U L_U - \nabla p_D L_D}{2\tau_w} \ , \tag{30}$$

where ∇p_U and ∇p_D are the upstream and downstream fully developed pressure gradients and τ_w is the fully developed wall shear stress at the outflow. The global pressure difference between the entry and the exit section is denoted by δp.

Starting with an initial mesh containing 35 elements and 1129 degrees of freedom, we compute Couette correction values at increasing De, in the particular case of a Maxwell-B fluid ($F = 0$, $\eta_N = 0$, $\eta_V = 1$). Mixed h- and p-refinements are applied at $De = 0$ and $De = 1.4$. Using reference results from [15] for the Couette correction, we calculate a *Couette correction based error*. The difference between the computed Couette correction and the reference result is scaled by a characteristic Couette correction of 0.2, as classical relative values would suffer from numerical inaccuracies around $C = 0$. An estimated error index is obtained by dividing the error indicator θ given by Equations (21)(23) by the current norm of the solution. Figure 7 exhibits a very good agreement between the error index and the *Couette correction based error*. This shows that the global error estimation computed is a reliable tool to monitor the accuracy of a given hp-finite element discretization.

For this problem, the particular form of Equation (3) may lead to non-integrable extra-stresses at the reentrant corner, for Maxwell-B and Oldroyd-B models. We compare several adaptive strategies for an Oldroyd-B fluid ($F = 0$, $\lambda_0 = 2$, $\eta_V/(\eta_N+\eta_V) = 0.875$) in order to analyze the behavior of the solution with a singularity. The convergence curves of the estimated error are plotted in Figure 8 for pure h-refinements, pure p-enrichments and a mixed hp-adaptivity of the original mesh. As expected, with pure p-enrichments, the error converges algebraically, at the same rate as the one obtained with pure h-refinements. The hp-mesh is built through a first pure h-refinement step followed by pure p-enrichments, leading to a higher global convergence rate of the method. The numerical results are summarized in Table 3, where h stands for a pure h-refinement step and p for a pure p-enrichment step. For each mesh, the number of degrees of freedom is given, with the error index and the mass and momentum balances. In this case, $\boldsymbol{\tau}_K^{est}$ clearly dominates $\boldsymbol{u}_K^{est}$ and p_K^{est} in (23), so that the mass and momentum balances do not exactly

follow the error index evolution.

Now using a MUCM model ($F \neq 0$, $\lambda_0 = 4$, $\eta_V/(\eta_N + \eta_V) = 0.875$), Equation (7) leads to a Newtonian behavior of the fluid close to the singularity. We present in Figure 9 the results obtained with the three adaptive strategies described above. The influence of the model chosen on the convergence rate of the methods is well observed on the convergence curves. The stresses remain square integrable and pure p-enrichments give the highest rate of convergence. The mass and momentum balances presented in Table 4 are found to follow the same evolution as the computed error index.

To evaluate the cost reduction due to our adaptive hp-interpolation, we present in Table 5 the CPU time needed to obtain the flow of a MUCM fluid ($(F \neq 0$, $\lambda_0 = 2.8$, $\eta_V/(\eta_N + \eta_V) = 1.0)$ through an axisymmetric contraction. Starting from a Stokes flow (i.e $De = 0$), a suitable evolution scheme is used to reach $De = 2.8$. A first solution is obtained on a coarse mesh $\mathcal{P}_0$ at $De = 0.1$. A new mesh $\mathcal{P}_1$ is then built through h-refinements and p-enrichments, as defined in Section 4. Using this improved mesh, we compute the solution up to $De = 1.6$. The final mesh $\mathcal{P}_2$ is generated at this stage of the evolution and the flow at $De = 2.8$ can be calculated. As shown in Table 5, the evolution needed to obtain the solution at the required De number is therefore much faster than a complete evolution on mesh $\mathcal{P}_2$. It can be noted that the cost of adaptivity is really negligible (less than 0.5%) as compared to the time spent to solve the viscoelastic discrete problem.

Finally, we compare our results on this problem with those obtained in [15] for a Maxwell-B fluid. Figure 10 shows the Couette correction computed at increasing De. A very good agreement is found between the four methods. The hp-adaptive finite element method exhibits an improved robustness when compared with MIX-1 (biquadratic interpolation for the stresses and velocities, bilinear interpolation for the pressure), but the 4×4 sublinear stress interpolation is still the most robust method for this problem. The hypothesis (26) taken in the adaptive scheme to select h_V and h_Q is clearly responsible for this lack of robustness, and particular attention will be devoted in future works to this aspect of the mesh-adapting procedure.

6 Concluding Remarks

A general adaptive *hp*-finite element is used to solve viscoelastic flow problems in complex geometries. As in classical finite element approaches, a local approximation is defined at the elemental level, using polynomial basis functions. The global approximation spaces are defined so as to allow local *h*-refinements and *p*-enrichments in the mesh.

The adaptive procedure is based upon both *a priori* and *a posteriori* error estimators. The actual error is approximated by a global error indicator, obtained by solving an elementwise boundary-value problem. It will allow us to better understand the behavior of several numerical methods frequently used in viscoelastic flow simulation and to monitor their efficiency.

Experiments which illustrate the validity of the estimator have been performed on the 4:1 contraction and on the sphere problem, taking respectively the Couette correction and the drag correction factor as relevant parameters. Lots of degrees of freedom are saved, taking advantage of the local character of the refinements.

Numerical results obtained show how the domain geometry and the set of constitutive equations affect the rate of convergence of the approximation error on both test problems. It allows us to select the best adaptive strategy. Further investigations on other benchmark problems will however be necessary to build a general mesh-adapting procedure for viscoelastic flow problems. In the case of Maxwell-B and Oldroyd-B fluids, the proposed strategy seems to be not so effective in the presence of singularities. Independent *h*-refinements for extra-stresses and velocities will therefore be investigated in future works for this kind of fluids.

Acknowledgements

This paper presents research results of the Belgian Programme on Interuniversity Poles of Attraction, initiated by the Belgian State, Prime Minister's Office for Science, Technology and Culture. The scientific responsibility rests with its authors.

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Adaptive step performed	D.O.F.	Error index	Mass balance	Momentum balance	Drag correction factor
-	2002	0.2208	0.585	0.0869	5.5546
p	2351	0.1678	0.587	0.0763	5.4926
p-p	2445	0.1317	0.567	0.0687	5.4198

Table 1: Sphere problem (Re=0), Maxwell-B fluid. Numerical results

Adaptive step performed	D.O.F.	Error index	Mass balance	Momentum balance
-	3907	0.0950	0.9535	0.0496
p	4595	0.0652	0.3135	0.0001
p-p	5325	0.0515	0.2623	0.0001

Table 2: Sphere problem (Re=0), MUCM fluid. Numerical results

Adaptive step performed	D.O.F.	Error index	Mass balance	Momentum balance
-	2289	0.1298	0.0770	0.1251
p	2437	0.0766	0.0653	0.0054
h	3787	0.0675	0.0426	0.0044
p-p	2717	0.0560	0.0413	0.0028
h-p	4047	0.0462	0.0349	0.0038
h-h	5005	0.0440	0.0299	0.0034

Table 3: Contraction problem (Re=0), Oldroyd-B fluid. Numerical results

Adaptive step performed	D.O.F.	Error index	Mass balance	Momentum balance
-	2289	0.1069	0.1050	0.0147
h	2454	0.0779	0.0679	0.0130
h-h	2728	0.0720	0.0565	0.0087
p	2461	0.0652	0.0669	0.0130
p-p	2690	0.0598	0.0517	0.0072
h-p	2664	0.0529	0.0449	0.0146

Table 4: Contraction problem (Re=0), MUCM fluid. Numerical results

	CPU for the solution (nr of iterations)		CPU for the error estimation
$\mathcal{P}_0$ ($De = 0.1$)	1660	(5)	426
$\mathcal{P}_1$ ($De = 0.1 \rightarrow 1.6$)	39378	(37)	1411
$\mathcal{P}_2$ ($De = 1.6 \rightarrow 2.8$)	363347	(38)	1126
Total hp-adaptive method	404385		2963
	63.8%		0.5%
$\mathcal{P}_2$ ($De = 0.1 \rightarrow 2.8$)	633586	(62)	1126
	100%		0.2%

Table 5: Contraction problem (Re=0), MUCM fluid. CPU time

Figure 1: Sphere problem : geometry and boundary conditions

Figure 2: Sphere problem (Re=0), Maxwell-B fluid. Closeup views of the adaptive meshes and local error indicator distribution.

Figure 3: Sphere problem (Re=0), MUCM fluid. Closeup views of the adaptive meshes and local error indicator distribution.

Figure 4: Sphere problem (Re=0), Maxwell-B fluid. Drag correction factor *vs De* number.

Figure 5: Sphere problem. Extra-stress interpolation. Closeup view of classical 4x4 and *p*-adaptive finite element mesh.

Figure 6: Contraction problem : geometry and boundary conditions

Figure 7: Contraction problem (Re=0), Maxwell-B fluid. *Couette correction based error* and Error index *vs De*.

Figure 8: Contraction problem (Re=0), Oldroyd-B fluid. Closeup views of the adaptive meshes and global error indicator.

Figure 9: Sphere problem (Re=0), MUCM fluid. Closeup views of the adaptive meshes and global error indicator.

Figure 10: Contraction problem (Re=0), Maxwell-B fluid. Couette correction *vs De* number.