Adaptive High-Order Prediction of the Drag Correction Factor for the Upper-Convected Maxwell Fluid

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This paper is dedicated to Vincent Van Kemenade

Abstract

An adaptive high-order finite element method is used to calculate the flow of a viscoelastic fluid around a sphere falling in a cylinder. No corner singularity appears in such a flow, but from a quite complex flow field, one predicts the drag correction factor for the upper-convected Maxwell fluid (UCM). Those properties explain why this problem is used as an benchmark for numerical techniques in rheology.

Accuracy and robustness of the results are demonstrated by p-convergence analysis and by comparison with reference results. Hence, our calculations with high-order interpolations may be considered as reference results for this problem. The Galerkin and the Petrov-Galerkin techniques applied to several formulations (MIX, EVSS, AVSS) are analysed and compared. Error estimation and adaptivity allow us to derive optimal discretizations for each formulation. We observe that both suitable formulation and discretization are critical to obtain a valid prediction.

1 Introduction

Obtaining accurate numerical solutions at high values of the Weissenberg number still remains a challenge in viscoelastic flow simulation. This paper

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presents a further effort towards the calculation of reference results for the sphere problem.

As indicated in Figure 1, let us consider the flow of the upper-convected Maxwell fluid (UCM) around a sphere falling along the axis of a long tube, at a speed V. The radius of the sphere is R, while the tube radius is twice as large. The quantity of interest is the *drag correction factor* Γ, defined as the ratio between the drag D exerted on the sphere by the viscoelastic flow and the drag exerted by a creeping flow of a Newtonian fluid of viscosity η on an identical sphere in an infinite space :

$$\Gamma = \frac{D}{6\pi\eta VR} \quad . \tag{1}$$

Today, it is still difficult to solve this problem. This challenge has long resulted in the failure of numerical simulations to provide solutions at elasticity levels of practical interest. The elastic character of the flow is usually quantified by means of the Weissenberg number We defined as the product of the relaxation time and the characteristic velocity gradient V/R of the flow. It has been shown that one of the major reasons for this High Weissenberg Number Problem (HWNP) is the inaccuracy of the numerical method [1]. Two avenues of research are investigated to develop better numerical techniques for viscoelastic flows.

On one hand, suitable-high order interpolations can reduce numerical inaccuracies or at least limit their dramatic effects on the calculation process. Hence, more and more attention is devoted to spectral methods [2, 3], p-finite element methods [4, 5, 6, 7] and spectral element methods [8, 9, 10, 11]. For problems endorsed with smooth solutions, those methods exhibit an exponential rate of convergence. One the other hand, suitable techniques have been developed in order to take the hyperbolic character of the constitutive equation into account. Typically, the classical Galerkin approach is optimal for elliptic equations, but should, in general, be avoided with hyperbolic problems.

The first numerical solutions for the sphere problem delivered dramatic discrepancies between different numerical methods [12, 1]. Some important improvements are obtained both by using better formulations of viscoelastic problems and by taking advantage of better discretizations.

- In 1993, four different groups [13, 14, 15, 16] obtained the same results using different methods for $We < 1.6$. However, no converged solution was obtained for $We > 1.6$.

- In 1995, Yurun and Crochet [7] took advantage of a p-adaptive finite element discretization and of the Petrov-Galerkin technique (SUPG) for the Elastic Viscous Stress Splitting formulation (EVSS). With such a numerical method, they obtained steady state solutions up to $We = 2.1$. This value was only limited by the available computational resources. Their results are in agreement with the values obtained by Jin et al. [17] using a EEME-SUPG method. They are also in agreement with those obtained by Arigo et al. [18] using a EVSS-SUPG method.

- In 1996, Luo [19] introduced an operator splitting algorithm with a low-order finite element discretization. With 110×10^3 degrees of freedom, he obtained numerical solutions up to $We = 2.8$.

- Recently, Sun et al. [20]. proposed an Adaptive Viscoelastic Stress Splitting formulation (AVSS). Using a streamline integration, a limiting Weissenberg number We of 3.2 was observed. However, unlike other computations, their decoupled iterative calculations predict no levelling off in the drag correction factor, raising the question of the accuracy of the AVSS formulation.

- Finally, Baaijens et al. [21] used the EVSS-like formulation (sometimes referred to DEVSS) proposed by Guénette and Fortin [22] and a discontinuous Galerkin integration. They obtained results up to $We = 2.5$.

Today, it appears that EVSS-AVSS formulations with SUPG-SU techniques are superior to the classical mixed Galerkin approach at high Weissenberg numbers. The objectives of this paper are to present both a cost-effective method and a reference curve for the sphere problem.

In order to provide a reference curve, it seems usual to take advantage of high-order finite element approximations, even if high-order interpolations alone cannot solve the High Weissenberg Number Problem. [4, 5, 6, 7, 9, 10, 11]. However, for smooth problems, high-order interpolations with a suitable formulation provide stable and convergent schemes, a quite unusual fact in viscoelastic flow problems. More accurate solutions with high-order methods can be obtained with a reduced number of degrees of freedom. But, as mentioned by Yurun and Crochet [7], the associated computational cost and memory requirements are still very high.

Therefore, the second objective of this paper is to present a cost effective adaptive high-order approach. The idea is to reduce the approximation

error to a specified level while minimizing the number of unknowns of the problem. The mesh size h and the order of approximation p are adjusted at the elemental level, using error estimates. Such an adaptive hp finite element method leads to very high convergence rates of the approximation error with respect to the number of degrees of freedom, even in the presence of singularities [23]. For the sphere problem, it appears that the most efficient way is to use only p-adaptivity. However, conclusions are different for a problem such as the four-to-one contraction [23].

The numerical results of this paper are summarized in Table 1. Our results are in full agreement with the literature up to $We = 1.6$. For values of the Weissenberg number between 1.6 and 2.0, we observe a very good agreement with the results of Yurun and Crochet [7], of Jin et al. [17] and of Baaijens et al. [21]. However, for larger values of We, Figure 2 shows that published results remain in disagreement. The accuracy of our results up to $We = 2.5$ is demonstrated by p-convergence experiments.

This paper is organized as follows. First, we recall the continuous and the discrete formulations of the viscoelastic flow problem in Sections 2 and 3. In Section 4, we analyse the results obtained with several formulations and interpolations.

2 The Continuous Viscoelastic Problem

We consider the isothermal steady flow of an incompressible viscoelastic fluid in an open domain Ω. We limit ourselves to flows without inertia and body forces, although considering the general case is straightforward. We also assume that no purely viscous component appears in the Cauchy stress tensor.

2.1 Governing equations

The governing partial differential equations are the mass and momentum conservation laws, coupled with constitutive equations characterizing the fluid. Most of the early numerical simulations have been performed with the upper-convected Maxwell model [24]. This model seemed to have the simplest constitutive equations exhibiting most typical viscoelastic effects. However, it has been found that the flow of such fluid is one of the most difficult to simulate among the available constitutive equations, as it may generate more numerical difficulties than apparently more complicated differential models. The upper-convected Maxwell model is therefore often

We	Present results	Previous published results [13]	[7]
0.0	5.9469	5.9475	
0.2	5.6592	5.6597	
0.4	5.1862	5.1868	
0.6	4.8009	4.8021	
0.8	4.5274	4.5299	
1.0	4.3405	4.3405	4.3405
1.2	4.2153	4.2159	
1.4	4.1336	4.1352	
1.6	4.0831	(diverges)	4.0827
1.7	4.0670		
1.8	4.0557		
1.9	4.0487		
2.0	4.0454		4.0420
2.1	4.0451		
2.2	4.0476		(diverges)
2.3	4.0522		
2.4	4.0580		
2.5	4.0660		
2.6	(diverges)		

Table 1: Drag reduction factor for the sphere problem with a Maxwell fluid. Summary of the numerical results.

used as a benchmark model for numerical methods. Hence, a huge amount of numerical results is now available for this fluid. The *classical mixed formulation* (MIX) of the corresponding viscoelastic problem is given by :

Find $(\boldsymbol{T}, \boldsymbol{u}, p)$ *such that*

$$\begin{aligned} \boldsymbol{T} + \lambda \overset{\triangledown}{\boldsymbol{T}} - 2\eta \boldsymbol{\mathcal{D}}(\boldsymbol{u}) &= 0 \ , \\ \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}(\boldsymbol{T}, p) &= 0 \ , \\ \boldsymbol{\nabla} \cdot \boldsymbol{u} &= 0 \ , \end{aligned} \tag{2}$$

where $\boldsymbol{u}$ is the velocity field, p is the pressure, $\boldsymbol{T}$ is the viscoelastic extra-stress tensor, η is the viscoelastic dynamic viscosity and λ is the relaxation time of the fluid. The Cauchy stress tensor $\boldsymbol{\sigma}(\boldsymbol{T}, p)$ and the strain rate tensor $\boldsymbol{\mathcal{D}}(\boldsymbol{u})$ are defined as

$$\begin{aligned} \boldsymbol{\sigma}(\boldsymbol{T}, p) &= -p\boldsymbol{I} + \boldsymbol{T} \ , \\ \boldsymbol{\mathcal{D}}(\boldsymbol{u}) &= \frac{1}{2}(\boldsymbol{\nabla u} + \boldsymbol{\nabla u}^T) \ . \end{aligned}$$

The symbol $\triangledown$ denotes the upper convected derivative given in a steady flow by :

$$\overset{\triangledown}{\boldsymbol{T}} = (\boldsymbol{u} \cdot \boldsymbol{\nabla})\, \boldsymbol{T} - \boldsymbol{\nabla}\, \boldsymbol{u}^T \cdot \boldsymbol{T} - \boldsymbol{T} \cdot \boldsymbol{\nabla}\, \boldsymbol{u} \ .$$

Suitable boundary conditions have to be imposed to lead to a well-posed formulation of the continuous viscoelastic problem [25]. Some components of the extra-stress tensor must be specified at the inflow section. Classical Dirichlet or Neumann boundary conditions for the velocity are also required.

2.2 Stress-splitting formulations

In the limit of Newtonian flows ($\lambda = 0$), the MIX formulation (2) in terms of $(\boldsymbol{T}, \boldsymbol{u}, p)$ is not equivalent to the usual $(\boldsymbol{u}, p)$ formulation of the Stokes equations. When we introduce the discretization, this fact has detrimental numerical implications [1] and can be circumvented with the following change of variables :

$$\boldsymbol{S} = \boldsymbol{T} - 2\eta \boldsymbol{\mathcal{D}}(\boldsymbol{u}) \ .$$

A convected derivative of $\boldsymbol{\mathcal{D}}(\boldsymbol{u})$ is then introduced in the constitutive equation, involving second-order spatial derivatives of $\boldsymbol{u}$. But, these derivatives can be eliminated by considering a $(\boldsymbol{S}, \boldsymbol{u}, p, \boldsymbol{D})$ approximation. In order to define the *Elastic Viscous Split Stress formulation* (EVSS), Rajagopalan et al. [26] introduced the modified stress tensor $\boldsymbol{S}$ and the rate of deformation $\boldsymbol{D}$ as additional unknowns, in the following way :

$$\begin{aligned} \boldsymbol{S} &= \boldsymbol{T} - 2\eta \boldsymbol{D}, \\ \boldsymbol{D} &= \frac{1}{2}(\boldsymbol{\nabla u} + \boldsymbol{\nabla u}^T). \end{aligned} \tag{3}$$

The discretization of the EVSS formulation provides more stable and accurate solutions than the discretization of the MIX formulation. Those good numerical properties are due both to a new elliptic stabilizing term in the momentum equation and to the least-square approximation of $\boldsymbol{\mathcal{D}}(\boldsymbol{u})$ by a new variable $\boldsymbol{D}$.

In order to further improve the numerical properties, Sun et al. [20] proposed a modified version of the EVSS formulation to compute the flow of the upper-convected Maxwell fluid. Their so-called reference viscosity scheme formulation can be written as follows :

$$\boldsymbol{S} = \boldsymbol{T} - 2\beta \boldsymbol{D} \ , \tag{4}$$

where the reference viscosity β is a function of the coordinates and can be much larger than η used in the usual EVSS formulation. In other words, the EVSS formulation is a particular case of this formulation. They also proposed an adaptive procedure to select β so as to obtain a viscous stress $2\beta\boldsymbol{\mathcal{D}}(\boldsymbol{u})$ at least of the same order as the elastic stress $\boldsymbol{S}$. This scheme defines the so-called *Adaptive Viscous Stress Splitting formulation* (AVSS).

In order to introduce a similar elliptic stabilizing term in the momentum equations for other models, Guénette and Fortin [22] considered a modified stress splitting formulation. No change of variable is applied but the momentum equations are modified as follows :

$$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}(\boldsymbol{T}, p) + 2\alpha \boldsymbol{\nabla} \cdot (\boldsymbol{\mathcal{D}}(\boldsymbol{u}) - \boldsymbol{D}) = 0 \ .$$

Obviously, $\boldsymbol{\mathcal{D}}(\boldsymbol{u}) - \boldsymbol{D}$ vanishes in the continuous formulation, but is introduced as a stabilization term in the momentum equations. It acts exactly as the additional term generated by the change of variable of the EVSS formulation. This is sometimes known as the *Discrete Elastic Viscous Stress*

Splitting formulation (DEVSS) or as the extended EVSS formulation of Guénettte and Fortin.

All those formulations are particular cases of the following generic problem :

Find $(\boldsymbol{S}, \boldsymbol{u}, p, \boldsymbol{D})$ *such that*

$$\begin{aligned} \boldsymbol{S} + \lambda(\overset{\triangledown}{\boldsymbol{S}} + 2\beta \overset{\triangledown}{\boldsymbol{D}}) - 2(\eta - \beta)\boldsymbol{\mathcal{D}}(\boldsymbol{u}) &= 0, \\ \boldsymbol{\nabla} \cdot (-p\boldsymbol{I} + \boldsymbol{S} + 2\beta\boldsymbol{\mathcal{D}}(\boldsymbol{u})) + 2\alpha\boldsymbol{\nabla} \cdot (\boldsymbol{\mathcal{D}}(\boldsymbol{u}) - \boldsymbol{D}) &= 0, \\ \boldsymbol{\nabla} \cdot \boldsymbol{u} &= 0, \\ \boldsymbol{D} - \boldsymbol{\mathcal{D}}(\boldsymbol{u}) &= 0. \end{aligned} \tag{5}$$

The MIX formulation (2) corresponds to $\alpha = \beta = 0$ (the field $\boldsymbol{D}$ is then not required and the field $\boldsymbol{S}$ becomes $\boldsymbol{T}$) and the standard EVSS method (3) can be obtained with $\alpha = 0; \beta = \eta$. We obtain the reference viscosity AVSS formulation (4) with $\alpha = 0$ and the extended EVSS formulation of Guénette et al. with $\beta = 0$.

3 The Discrete Viscoelastic Problem

In this section, we derive the discrete formulation associated with the generic mixed formulation (5). We also present how to select the discrete approximation spaces for each field.

3.1 Finite Element Approximations

Following the standard finite element procedure, we begin, as usual, by introducing a family $\mathcal{P}$ of partitions of Ω such that

$$\overline{\Omega} = \bigcup_{K}^{N(\mathcal{P})} \left\{ \overline{\Omega}_K \, ; \, \Omega_K \in \mathcal{P} \right\}, \; \Omega_K \cap \Omega_L = \emptyset, \; K \neq L$$

We suppose the Ω and $\mathcal{P}$ are such that each element Ω_K is the image of a master element $\overline{\widehat{\Omega}} = [-1, 1]^2$ under an invertible map F_K.

In order to develop an adaptive strategy, the following generalizations are introduced. On one hand, h-adaptivity consists of refinements of the

mesh. On the other hand, p-adaptivity consists of using polynomials of differing degrees in the definition of hierarchical basis functions [27, 23, 28]. Continuity of the global basis functions is maintained by enriching the edge function to match highest degree polynomial used on a common interelement boundary. For a scalar field v, those generalizations allow us the contruction of the local and global approximation spaces :

$$\begin{aligned} M_K &= \left\{ v = v(\boldsymbol{x}) = \widehat{v} \circ F_K^{-1},\, \widehat{v} \in \widehat{M}(\widehat{\Omega}) \right\}, \\ M(h,p,\Omega) &= \left\{ v = v(\boldsymbol{x}) \in C^0(\overline{\Omega}) : v_K = v|_{\Omega_K} \in M_K \right\}, \end{aligned}$$

where $\widehat{M}(\widehat{\Omega})$ is the space spanned over $\hat{\Omega}$ by the set of basis functions. The symbols h and p denote here the h-distribution and the p-distribution characterizing a hp-approximation of the scalar field v. The fundamental requirement of this definition is that the global finite element approximation has to be continuous over Ω.

Before developing discrete mixed formulations, it is now necessary to define more general notations. Let us just consider the different fields $\boldsymbol{S}$, $\boldsymbol{u}$, p and $\boldsymbol{D}$ of the mixed formulation. We have to be able to use different h- and p-distributions for each field. In others words, the discrete approximations $\boldsymbol{S}^{hp}$, $\boldsymbol{u}^{hp}$, p^{hp} and $\boldsymbol{D}^{hp}$ belong to the following discrete function spaces :

$$\begin{aligned} \mathcal{S}^{hp} &= ((M(h_S,p_S,\Omega))^3 \cap \mathcal{S}), \\ \mathcal{U}^{hp} &= ((M(h_U,p_U,\Omega))^2 \cap \mathcal{U}), \\ \mathcal{P}^{hp} &= (M(h_P,p_P,\Omega) \cap \mathcal{P}), \\ \mathcal{D}^{hp} &= ((M(h_D,p_D,\Omega))^3 \cap \mathcal{D}), \end{aligned}$$

where $\mathcal{S}, \mathcal{U}, \mathcal{P}$ and $\mathcal{D}$ are the appropriate function spaces defined on Ω for the modified extra-stress tensor $\boldsymbol{S}$, the velocity $\boldsymbol{u}$, the pressure p and the strain rate tensor $\boldsymbol{D}$. For notational convenience, the full sets (h_S, h_U, h_P, h_D) and (p_S, p_U, p_P, p_D) are denoted by h and p. Using the Galerkin technique, the discrete approximations can now be characterized by the following problem :

Find $(\boldsymbol{S}^{hp}, \boldsymbol{u}^{hp}, p^{hp}, \boldsymbol{D}^{hp}) \in \mathcal{S}^{hp} \times \mathcal{U}^{hp} \times \mathcal{P}^{hp} \times \mathcal{D}^{hp}$ *such that*

$$\begin{aligned}
\int_\Omega (\boldsymbol{S}^{hp} + \lambda(\overset{\triangledown}{\boldsymbol{S}}^{hp} + 2\beta \overset{\triangledown}{\boldsymbol{D}}^{hp}) - 2(\eta - \beta)\boldsymbol{\mathcal{D}}(\boldsymbol{u}^{hp})) \cdot \boldsymbol{s}\, d\Omega &= 0 \\
\int_\Omega (-p^{hp}\boldsymbol{I} + \boldsymbol{S}^{hp} + 2(\beta + \alpha)\boldsymbol{\mathcal{D}}(\boldsymbol{u}^{hp})) - 2\alpha \boldsymbol{D}^{hp}) : \boldsymbol{\mathcal{D}}(\boldsymbol{v})\, d\Omega &= 0 \\
\int_\Omega (\boldsymbol{\nabla} \cdot \boldsymbol{u}^{hp})\, q\, d\Omega &= 0, \\
\int_\Omega (\boldsymbol{D}^{hp} - \boldsymbol{\mathcal{D}}(\boldsymbol{u}^{hp})) : \boldsymbol{e}\, d\Omega &= 0,
\end{aligned}$$

$$\forall\ (\boldsymbol{r}, \boldsymbol{v}, q, \boldsymbol{e}) \in \mathcal{S}^{hp} \times \mathcal{U}^{hp} \times \mathcal{P}^{hp} \times \mathcal{D}^{hp}.$$

(6)

When λ increases, the convective term of the first equation in (6) becomes dominant and the Galerkin technique loses its well-known "best approximation property" for elliptic problems [29]. It is known that the consistent Petrov-Galerkin technique (SUPG) [30] is superior to the standard Galerkin technique, with improved stability and accuracy for convection dominated problems. For smooth viscoelastic flow problems, this property has been observed for low- and high-order approximations [7, 13].

Remark 1

In the SUPG technique, we replace the test function $\boldsymbol{s}$ by a modified test function $\boldsymbol{s} + k\boldsymbol{u} \cdot \boldsymbol{\nabla}\boldsymbol{s}$. The factor k is an $\mathcal{O}(h)$ parameter defined in each element. For high-order finite elements, we observe, as did Yurun and Crochet [7], that the best numerical properties are obtained with $k = l/V$ where l is the characteristic size of the element. Such a choice is also made by Lunsmann et al. [14] for low-order finite elements.

3.2 Stable mixed approximations

At this point, we have to select independently and locally the $h-$ and $p-$distributions for all fields in the discrete problem. If we consider the classical velocity-pressure formulation of the Stokes problem, it is well known

that the mixed interpolation must satisfy a compatibility condition, in order to provide stable results, derived by Ladyzhenskaya Babŭska and Brezzi [31]. The popular choice of quadratic velocity interpolation with a linear pressure may be justified by the LBB condition.

There is no general mathematical theory that would give the choice of the approximations of the extra-stresses tensor to be used with the different viscoelastic formulations. It is also known that the approximation of the extra-stres tensor appears to be mainly affected by errors [32]. Some rules to select the discrete functions spaces can be deduced, however, from numerical experiments and some partial theoretical results, but they do not guarantee the suitability of the approximations. Today, the state of the art is the following :

- Fortin and Pierre [33] analysed the stability and convergence properties of the element of Marchal and Crochet [34] for the *three-field formulation* $(\boldsymbol{T}, \boldsymbol{u}, p)$ of the Stokes problem. This formulation can be considered as the Newtonian limit ($\lambda = 0$) of the MIX formulation. The 4×4 element of Marchal and Crochet consists of 16 linear subelements for the extra-stresses, while velocities and pressure are biquadratic and bilinear, respectively. For the three-field formulation of the Stokes problem, Fortin and Pierre proved that a second condition has to be added to the classical LBB condition. They also showed that the element of Marchal and Crochet is stable for this formulation of the Stokes problem, confirming the good numerical properties observed by Marchal and Crochet. A sufficient condition for the mixed approximation to produce results identical to those of the velocity-pressure formulation in Newtonian case is that the discrete extra-stress space $\mathcal{S}^{hp}$ contain the gradient of the discrete velocity space $\mathcal{U}^{hp}$. Such a condition suggests to use a discontinuous approximation of the extra-stress with a continuous approximation of the velocity. It also suggests to increase the number of the degrees of freedoms for a continuous approximation of the extra-stress.

- For a non-vanishing relaxation time, the non-linearity and the coupling between the equations make theoretical analysis much more difficult. In general, it is assumed appropriate to extend to the viscoelastic problem the conditions of Fortin proved for the Stokes problem. The selection of a suitable approximation space for the stresses also depends on whether or not there are singularities inside the domain.

- For high order approximations used in smooth problems (or far away from singularities), the high-order weighted residual methods provide *weakly* $\mathcal{C}^1$ approximations [35]. In other words, it means that the velocity approximation belongs to $\mathcal{C}^1$ when p tends to infinity. Therefore, a continuous extra-stress approximation is not detrimental to the global accuracy and we can use the same discretization in these regions. For creeping flows in smooth geometries, Van Kemenade and Deville [10, 11] observed that the accuracy of the MIX formulation can be improved with a higher order for the approximations of the extra-stresses with $p_S = p_U + 1$. But, Talwar et al. [5] observed that the most efficient discretization for the same formulation is obtained with $p_S = p_U$. For the EVSS formulation, they observed that the best selection seems to be $p_S + 1 = p_U = p_D + 1$. Yurun and Crochet [7] used $p_S = p_U = p_D + 1$ for the EVSS formulation.

- Close to singularities, it is wrong to assume that the high-order weighted residual methods provide *weakly* $\mathcal{C}^1$ approximations. The gradients of the approximation of the velocity may exhibit jumps. A continuous extra-stress approximation must be able to reproduce these jumps, without generating spurious oscillations in the whole computational domain. Moreover, as the regularity of the solution dictates the convergence rate, high-order approximations are not optimal. The element of Marchal and Crochet or a discontinuous approximation of the extra-stress tensor then become mandatory. For the contraction problem, the best efficiency is observed with local h-refinement [36, 37].

The adaptive finite element method brings some insight into this "best mixed approximation problem". Taking advantage of adaptivity, it introduces an overdiscretization of the extra-stress only where it is really necessary. We also observe that local p-enrichment of the approximation of the extra-stress are sufficient to exhibit the fast convergence characteristics of high-order methods. The orders of the velocity and extra-stress approximations are similar in non-critical areas and the local polynomial space of the extra-stresses are increased in critical areas. This can be considered as an adaptive way of selecting the discrete spaces.

For the sphere problem, it is useless to introduce h-refinement if the mesh is well designed. Therefore, we only use p-adaptivity with the following rules for the definitions of the *local* approximation of each field :

- MIX formulation :

$$\begin{array}{lll} p_S - 1 = p_U = p_P + 1, & \text{if} & p_U = 2, \\ p_S - 1 = p_U = p_P + 2, & \text{if} & p_U > 2, \end{array} \tag{7}$$

- EVSS and AVSS formulations :

$$\begin{array}{lll} p_S + 1 = p_U = p_P + 1 = p_D + 1, & \text{if} & p_U = 2, \\ p_S + 1 = p_U = p_P + 2 = p_D + 1, & \text{if} & p_U > 2. \end{array} \tag{8}$$

Note that the above choices, although suboptimal, lead to a stable finite element discretization of the Stokes problem.

3.3 Adaptive strategy

Numerical simulation for viscoelastic flows remains a CPU time consuming task. In this paper, we use a p-adaptive strategy in order to deliver very high rates of convergence with a given number of degrees of freedom. An adaptive strategy can be seen as an optimal control problem, in which the computational effort to minimize must be controlled by an adaptive scheme which orchestrates the discretizations to deliver a pre-set level of accuracy.

First, it is very difficult to derive realistic expressions of this computational effort in terms of the parameters of the discretization [36]. In general, the discretization is adapted in order to equidistribute the rate of error per degree of freedom over the whole mesh. Finally, the strategy is based on *a priori* and *a posteriori* error estimates. As usual, we define a *a priori* and a *a posteriori* estimates, as expressions involving or not the discrete approximations.

We also assume that the local *a priori* error estimates for elliptic problems [38] are also valid for the viscoelastic problem. Therefore, the following *a priori* upper bound for the error is supposed to hold true :

$$|||(\boldsymbol{S}^{error}, \boldsymbol{u}^{error}, p^{error})|||^2_{\Omega_K} \leq \frac{h_K^{2\mu}}{p_K^{2\nu}} \Lambda_K^2, \tag{9}$$

where Λ_K, μ and ν are some unknown constants. The mesh size h_K and the polynomial degree p_K are those characterizing the approximations of the extra-stress tensor that are mainly affected by numerical errors in most simulations. The subscript K denote restrictions to element Ω_K and the energy-like norm $|||.|||_{\Omega_K}$ is defined as

$$|||(\boldsymbol{s}, \boldsymbol{v}, q)|||^2_{\Omega_K} = \int_{\Omega_K} (\boldsymbol{s} \cdot \boldsymbol{s} + 2\eta \boldsymbol{\mathcal{D}}(\boldsymbol{v}) : \boldsymbol{\mathcal{D}}(\boldsymbol{v}) + q^2) d\Omega.$$

In order to guide our adaptive strategy, we also need an *a posteriori* error estimate. In fact, the *a priori* error estimate can provide the rate of convergence, but is useless to quantify the error due to the unknown constant Λ_K in (9). Today, efficient elementwise residual methods to estimate the *a posteriori* error for elliptic problems [39, 40, 41] and the Navier-Stokes equations [42, 43, 28, 44], are available and their accuracy has been theoretically demonstrated [42, 43]. The idea of such cost effective technique consists in solving elementwise boundary-value problems on each element loaded by the FEM residuals. Recently, this technique has been extended to viscoelastic problems [45], but without theoretical results. Numerical experiments, however, show that such a procedure provides reliable and realistic local an global error indicators defined by :

$$\begin{aligned} \theta_K &= |||(\boldsymbol{S}_K^{est},\, \boldsymbol{u}_K^{est},\, p_K^{est})|||_{\Omega_K} \,, \\ \theta^2 &= \sum_{K=1}^{N(\mathcal{P})} \theta_K^2 \,. \end{aligned} \tag{10}$$

Using both *a priori* and *a posteriori* error estimates, the p-adaptive strategy applied to the sphere problem can be defined. Let us recall that this strategy is only a particular example of the global hp-adaptive strategies described in [45, 36].

From a given discretization p on mesh $\mathcal{P}$, we take advantage of the *a priori* and *a posteriori* error estimates to build a new discretization p_{new} on the same mesh. In order to achieve an equidistributed error θ_{target}, we modify the order of the polynomial interpolation of the extra-stress tensor used in each element as follows :

Find $p_{K,new}$ *such that*

$$\left(\frac{p_{K,new}^{2\nu+2}}{p_K^{2\nu}}\right) = \left(\frac{\theta_K^2}{\theta_{target}^2}\right)\left(\sum_{k=1}^{N(\mathcal{P})} p_{K,new}^2\right). \tag{11}$$

This rule is derived by introducing rather strong assumptions for a constitutive equation as the upper-convected Maxwell model. Numerical experiments demonstrate that those assumptions can often provide useful strategies. Let us also recall that, even if the strategy is not theoretically founded,

the obtained solution is still guaranteed by the fast convergence properties of the high-order methods.

First, we asssume that the convergence rate of the true error is well approximated by the rate of a *a priori* error estimate expression (9) in order to eliminate the unknown constant Λ_K. Secondly, we assume that the global and local numbers of degrees of freedom can be estimated by the square of the polynomial order. Finally, we have to approximate the parameter ν. Our rule is approximately similar to the rule for p-enrichment used in the so-called three steps strategy introduced by Oden et al [46, 47, 23, 28].

Remark 2

The major weak point of the adaptive scheme is the selection of ν which has a critical influence on the guidelines produced by the strategy. This, at present, is a heuristic adaptive choice. Details about such procedures can be found in the literature [46, 47]. Our own technique consists of comparing the *a posteriori* error estimate calculated on the new discretization, with the target θ_{target} of the adaptive scheme [36].

Remark 3

The selection of the initial mesh is a key issue for any adaptive strategy. The *a priori* error estimate will deliver suitable predictions of the convergence rate only if the mesh is sufficiently refined to fall in the asymptotic part of the convergence curve.

Since the decrease of local polynomial orders are not allowed by our adaptive strategy, the coarsest mesh satisfying the previous property will provide the best efficiency.

Remark 4

The enrichments of the local polynomial order are applied to the approximation of the extra-stress tensor. The approximations of the other fields are then modified in order to satisfy the stability conditions (7,8). We sometimes use the same order for the approximations of the velocity and the extra-stress tensor ($p_S = p_U = p_D$) in the initial discretization. In elements where the adaptive strategy requires to increase the interpolation order, we then obey the stability conditions (7,8).

4 Results

All our calculations are described in Table 2.

Method	Discretisation	Unknowns	Elements	Limiting We
MIX-GAL	H_1-P_{6-7}	11459	51	0.98
EVSS-GAL	H_1-P_6	22312	51	1.20
	H_1-P_7	29927	51	1.38
	H_4-P_2	21956	510	1.20
	H_1-P_{5-8}	21762	51	1.52
AVSS-GAL	H_2-P_8	51860	79	2.16
AVSS-SUPG	H_2-P_5	19481	79	2.71
	H_2-P_6	28536	79	2.75
	H_2-P_7	39329	79	2.18
	H_2-P_8	51860	79	2.45
	H_4-P_2	21956	510	2.50
	H_3-P_{3-7}	31826	316	2.49

Table 2: Summary of the numerical results.

We investigate several formulations (MIX,EVSS,AVSS) with the Galerkin (GAL) or the consistent Petrov-Galerkin technique (SUPG). The discrete approximations are linear combinations of continuous high-order hierarchical basis functions defined in [27]. The central part of the meshes near the sphere is shown in Figure 3.

In Table 2, the discretization is characterized in the following manner :

- H_i denotes the finite element mesh ($i = 1, \ldots, 4$).

- P_{j-k} denotes the range $[j,k]$ of orders used in the polynomial approximation of the extra-stress tensor.

For each calculation, we also give the global number of degrees of freedom, the number of elements and the limiting Weissenberg number.

4.1 Results with the MIX-GAL method

For completeness, we again report our calculations presented in [23]. In this simulation, the sphere problem is solved with the MIX-GAL method. The approximations of the extra-stress tensor are enriched up to seventh order and we observe a limiting Weissenberg number of 0.98. The same method MIX-GAL with the element of Marchal and Crochet has a similar limiting Weissenberg number for a same CPU cost, but a quite large amount of degrees of freedom. Today, it is clear that a suitable formulation is critical for obtaining results and we consider the EVSS-GAL method.

4.2 Results with the EVSS-GAL method

First, we perform two calculations with a uniform discretization of order six and seven of the extra-stress tensor. Those calculations are reported as (H_1-P_6) (H_1-P_7) in Table 2. The limiting Weissenberg number is 1.2 and 1.38 respectively and the mixed discretizations are ($p_S = 6, p_U = 7, p_P = 5, p_D = 6$) and ($p_S = 7, p_U = 8, p_P = 6, p_D = 7$). Turning to the convergence analysis, we observe the typically fast convergence properties of high order methods. For a uniform low-order discretization with a more refined mesh (H_4-P_2), a limiting Weissenberg number $We = 1.2$ is also observed.

Next, we consider the use of the adaptive strategy. From an initial mesh (H_1-P_5) with uniform discretization ($p_S = 5, p_U = 6, p_P = 4, p_D = 5$), we perform two adaptive steps at $We = 0.75$. In Figure 4, the three discretizations are shown and the estimated error distribution on each meshes is also given. Notice that the higher orders are assigned near the sphere. Shaded elements reflect non-uniform p-distribution in the final mesh. Elements with a high polynomial interpolation and with a high estimated error are filled with darker gray. At this stage, we apply additional local enrichments to obtain the discretization (H_1-P_{5-8}) given in Table 2 and we observe a limiting Weissenber number $We = 1.52$.

In Figure 5, we draw the evolution of the error index as a function of the number of degrees of freedom during the adaptive enrichment performed at $We = 0.75$. The error index is defined as the ratio of the energy-like norm of

the estimated error and the energy-like norm of the solution. It is observed that the adaptive technique exhibits very fast convergence characteristics, compared with a uniform enrichment of the approximations of the extra-stress tensor from order five to six.

In order to illustrate the efficiency of the adaptive strategy, Table 3 provides the CPU times used for each part of the adaptive calculation (H_1-P_{5-8}). The total cost is low compared to that of the non-adaptive calculation (H_1-P_7) that approximatively gives the same accuracy and the same limiting Weissenberg number. The numerical results reported in this work are obtained by a full Newton-Raphson scheme with a continuation technique and a direct frontal solver. Along the continuation path (parametrized by We) of the adaptive calculation (H_1-P_{5-8}), the global error estimate as a function of We is given with the cumulative CPU times at critical steps in Figure 6.

Discretisation	Unknowns	CPU	Number of iterations (Continuation interval)
H_1-P_5	15819	8600	18 $(We = 0 \rightarrow 0.75)$
H_1-P_{5-7}	17392	4400	6 $(We = 0.75)$
H_1-P_{5-8}	20191	5500	5 $(We = 0.75)$
H_1-P_{5-8}	21762	53100	37 $(We = 0.75 \rightarrow 1.52)$
Total		71600	
H_1-P_7		128800	29 $(We = 0 \rightarrow 1.38)$

Table 3: Adaptive strategy and continuation technique.

Very good agreement between results from low- and high-order discretizations demonstrates that robust and reliable solutions can be obtained through the adaptive strategy applied to high-order methods. Numerical results and limiting Weissenberg numbers compare well with those obtained by Yurun and Crochet [7].

4.3 Results with the AVSS-GAL method

In order to increase the limiting Weissenberg number, we now consider the AVSS formulation [20] with a reference viscosity $\beta = 10\eta$ in (4). Applying the Galerkin technique to this formulation, we use the second finite element mesh with a uniform eigth-order approximation of the extra-stresses. Converged solutions are obtained up to $We = 2.16$ confirming the good stabilizing properties of such a formulation.

4.4 Results with the AVSS-SUPG method

Moreover, reliability and accuracy are improved with the consistent Petrov-Galerkin technique. In order to take advantage of the exponential convergence rates of high-order methods, we perform the calculations (H_2-P_5), (H_2-P_6), (H_2-P_7) and (H_2-P_8), with a uniform degree $5, 6, 7$ and 8 for the approximation of the extra-stress tensor. For an order 8, converged solutions are obtained up to $We = 2.45$. The accuracy is demonstrated by p-convergence that can be observed in Figure 7. Therefore, the results obtained with the discretization (H_2-P_8) and the AVSS-SUPG method can be considered as an accurate and robust solution for the sphere problem. For a low-order approximation with a highly refined mesh (H_4-P_2), a limiting Weissenberg number of 2.5 is also reached.

A limiting Weissenberg number 2.49 is observed for a p-adaptive strategy. The adaptive calculation (H_3-P_{3-7}) provides exactly the same values than those obtained with (H_2-P_8). But, the adaptive method is superior both to the uniform high-order method and to the uniform low-order method. On one hand, the accuracy and the robustness of the reference result (H_2-P_8) are obtained by the adaptive approach with 40% fewer degrees of freedom. On the other hand, for a similar CPU cost, the adaptive approach provides more accurate results than the low-order calculation (H_4-P_2). Table 4 gives some insight in the accuracy of the different methods for the drag correction factor at $We = 2.0$. The relative error is estimated from the difference with the value obtained from (AVSS-SUPG H_2-P_8). Notice that those observations may strongly depend on the selection of the intial mesh.

5 Conclusion

We applied an adaptive high-order finite element method to predict the drag correction factor for the upper-convected Maxwell fluid. Accuracy and

Method	Discr.	Unknowns	Drag Cor. Factor	Estimated Error
AVSS-SUPG	H_2-P_8	51860	4.045372	
AVSS-SUPG	H_3-P_{3-7}	31826	4.045048	0.008 %
AVSS-GAL	H_2-P_8	51860	4.040366	0.124 %
AVSS-SUPG	H_4-P_2	21956	4.053510	0.201 %

Table 4: Accuracy of the drag correction factor for $We = 2.0$

robustness of the results are demonstrated by p-convergence analysis and by comparison with results from the literature [7, 12, 15, 19, 20]. It appears that the computation performed with the AVSS-SUPG formulation and the discretizations (H_3-P_{3-7}) and (H_2-P_8) may be considered as a reference result for future benchmarks.

We observe that the combination of a suitable formulation (AVSS), a suitable Petrov-Galerkin technique (SUPG) and a suitable discretization is required to obtain accurate results at high Weissenberg numbers. Error estimate and adaptivity are used to analyse and to develop new formulations, while keeping the CPU cost at a reasonable level. Extension to other viscoelastic models is straightforward.

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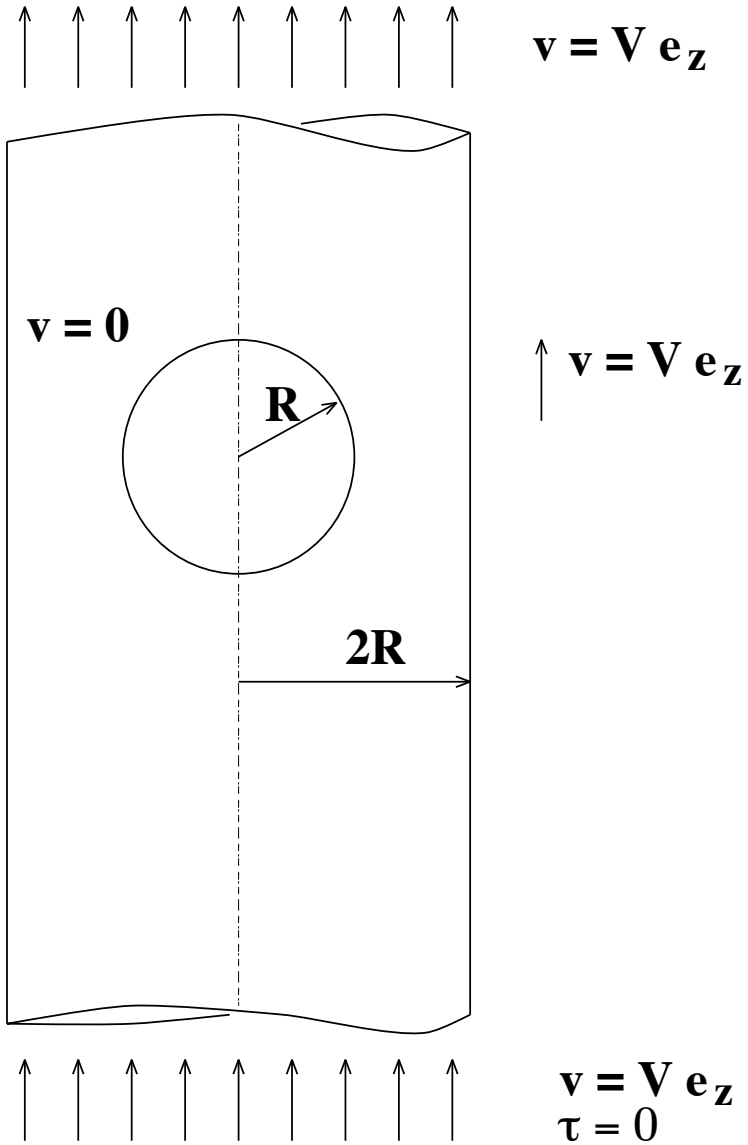

Figure 1: Sphere problem : geometry and boundary conditions.

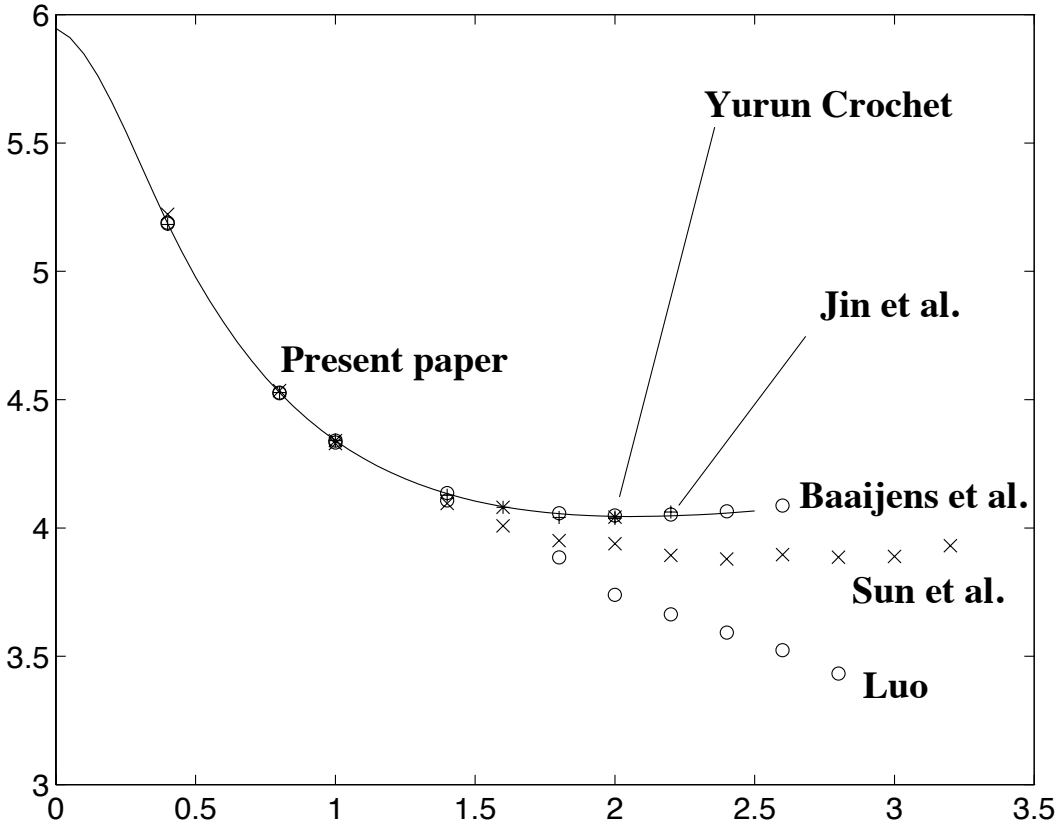

Figure 2: Drag correction factor as a function of the Weissenberg number. Our results are represented by the plain line.

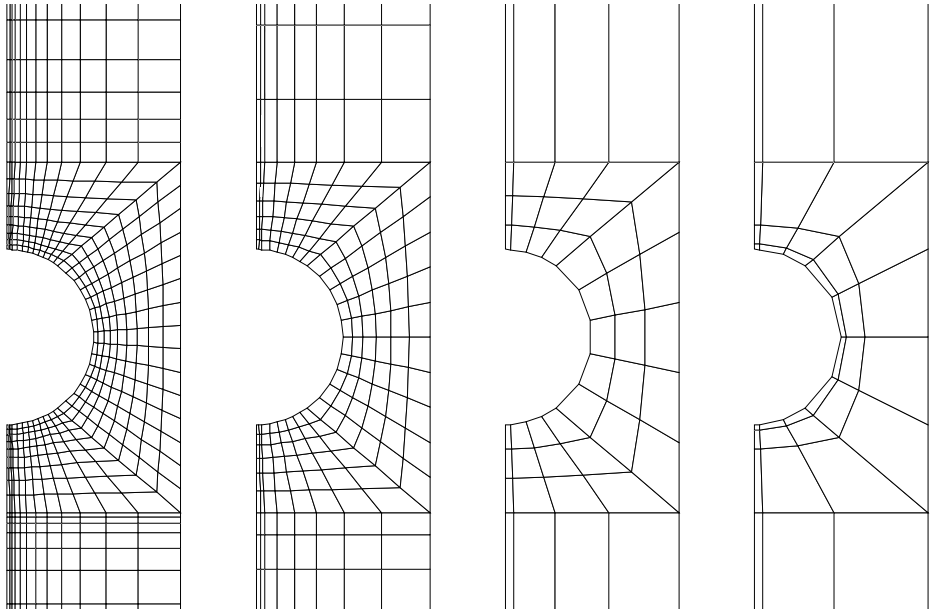

Figure 3: Close-up views of the meshes.

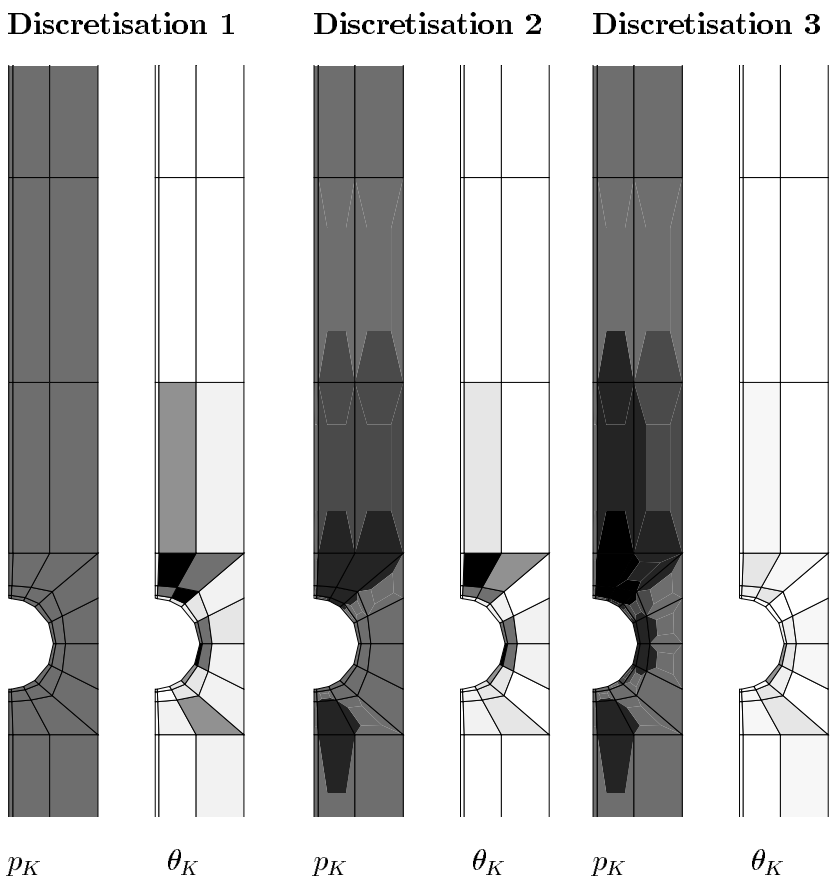

Figure 4: Adaptive strategy at $We = 0.75$. EVSS/GAL formulation. Close-up views of the adaptive discretizations and estimated error.

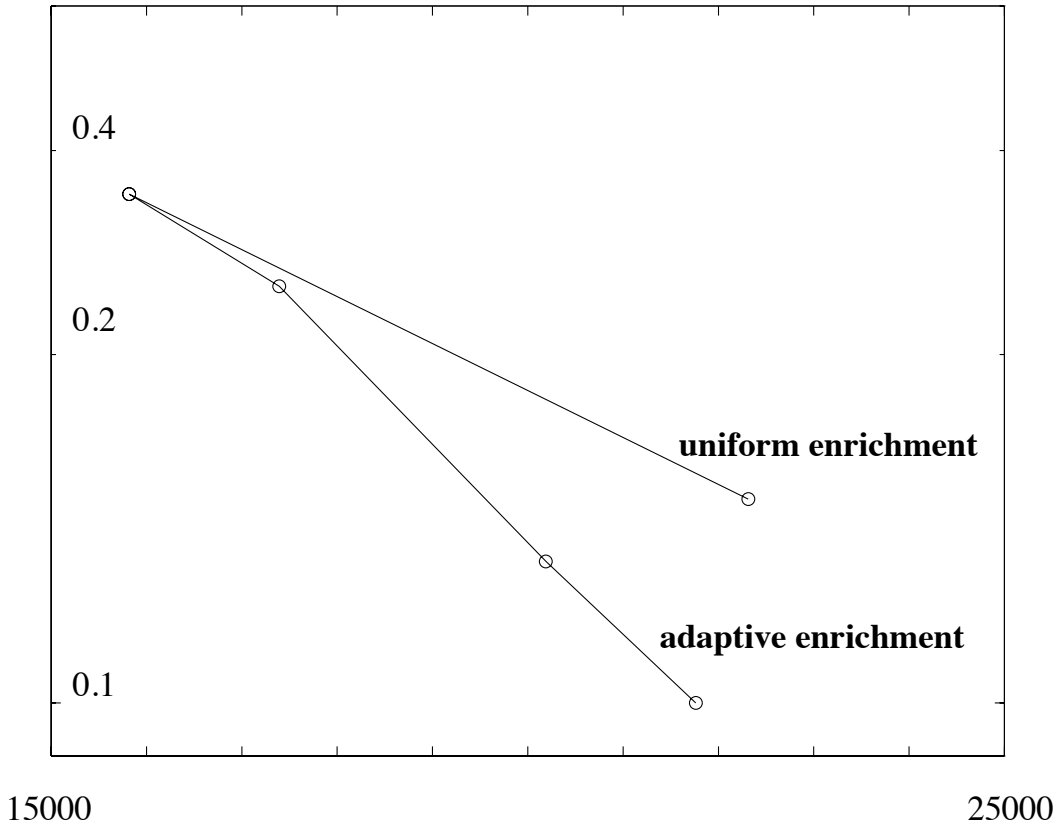

Figure 5: Error index as a function of the number of degrees of freedom. The adaptive strategy is superior to an uniform enrichment of the discretization.

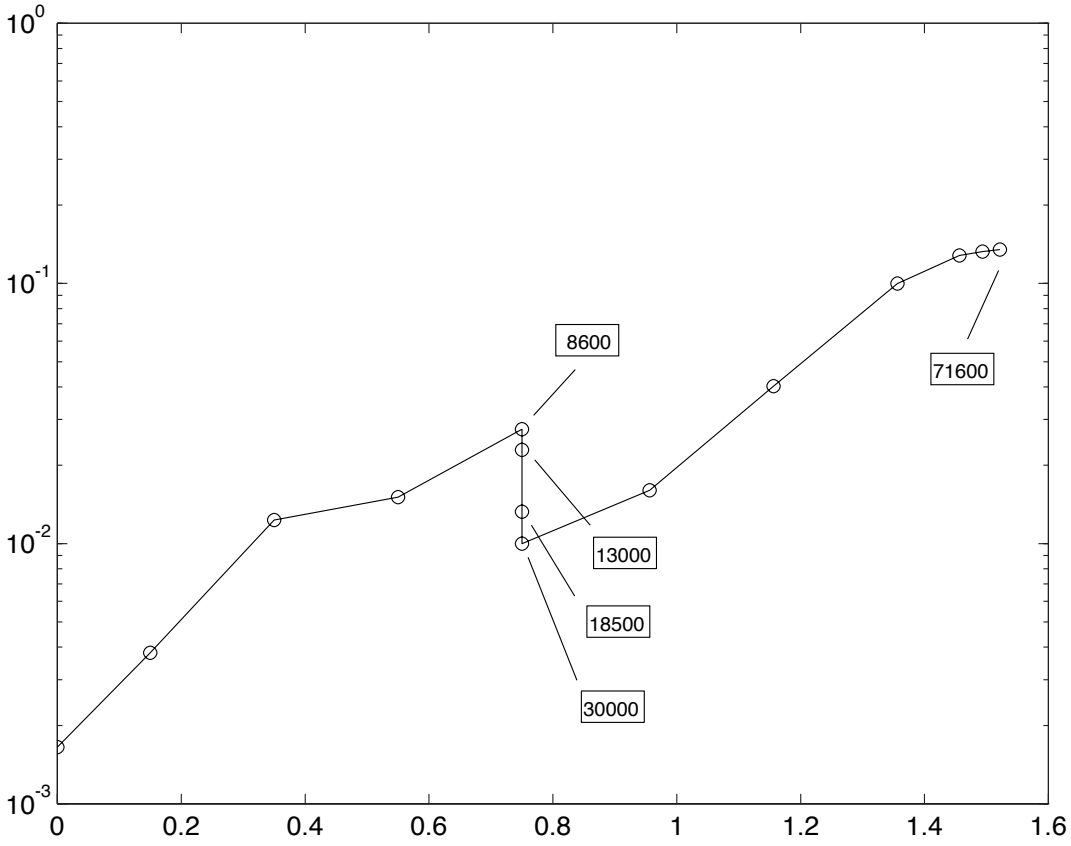

Figure 6: Error estimate as function of We. Adaptive strategy for the continuation scheme. EVSS/GAL formulation. Cumulative CPU time for DEC2000/300 Alpha 50 Mhz workstation.

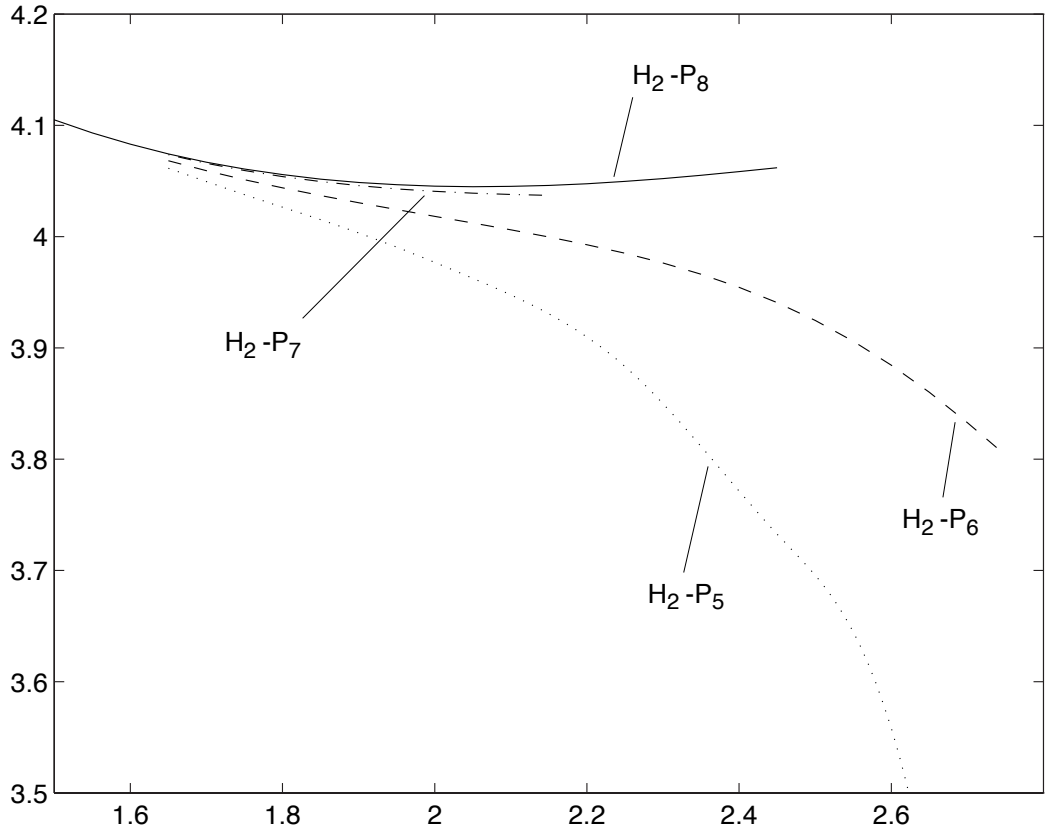

Figure 7: Convergence analysis of the AVSS-SUPG formulation with high-order approximations. Drag correction factor as a function of We.

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