An introduction to mesh generation
Part I : Introduction

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Historically, the principal approaches in engineering design are

- Experimental and empirical techniques
- Analytical methods

Recently, those traditional tools have been completed by computer aided design.
There are several specialized tools that can be classified in the domain of scientific computing:

- The art of modeling, i.e. finding the relevant set of equations that describe a given process,
- Geometrical modeling and mesh generation tools,
- Discretization techniques, numerical methods,
- Numerical linear algebra, resolution of large systems,
- Post-processing techniques and visualization of simulation data’s.

Scientific computing consist in the numerical simulation of various processes: physical, chemical, biological, financial... It enables:

- To span a wide range of parameters,
- To model very complex geometries,
- To couple multiple phenomenon together (multiphysics).
Scientific computing is now considered as a common tool in engineering design offices because of its high benefit/cost ratio. Moreover, scientific computing enables to

- Enlarge the range of parameters that can be used in a laboratory,
- Is not limited to empirical relations or analytical tools,
- Enables to understand the physics of phenomenon for which it is impossible to do any experience (climate, quantum mechanics, astrophysics...)

Moreover, scientific computing heavily relies on computer power for which the “price per flop” is decreasing rapidly.

Yet, scientific computing does not make other methods obsolete (yet) and usually requires deep knowledge in the processes that are modeled (no black box).
Benefits for engineering

Scientific computing technologies provide reliable tools for

- Enhancing the quality of products,
- Lowering the cost of development,
- Accelerate the design cycle,
- Enabling an efficient control on fabrication processes,
- Predicting damages and their consequences (damage tolerant structures).
- ...
Historical Background (1940’s)


1943

- Whirlwind program simulating aircraft performance developed at MIT from 1943 to 1950.
- Richard Courant studies St. Venant torsion problem using FE concepts: proposed breaking a continuum problem into triangular regions replacing the fields with piecewise approximations within the triangular regions.
Historical Background (1940’s)

1945  Electronic digital computer gains exposure.

1946  Analysis Lab at Caltech builds analog computer to solve problems in engineering mechanics.

1947  W. Prager and J. L. Synge develop hypercircle method, in connection with classical elasticity theory (Q. Applied Math, Approximation in Elasticity Based on the Concept of Function Space).

1948  Transistor invented (Jack S. Kilby, MIT, Nobel Price of Physics, 2000).
Introduction

Historical Background (1950’s)

1950  Computer size and speed increase, making it possible to handle FEA, available only to large aircraft.

1952

- CEA (Computer Engineering Associates) used FE modeling with a couple of hundred degrees of freedom.
- B. Langefors recognizes global behavior through assimilation of local behavior (J. Aeronautical Science, Analysis of Elastic Structures by Matrix Transformation with Special Regard to Semi-monologue Structures).

1953  N. J. Turner at Boeing’s Structural Dynamics Unit tests plane stress plate models, which was published in 1956 (independent of Argyris).
1954 John H. Argyris (University of Stuttgart in Germany) publishes series on linear structural analysis with solutions suitable for digital computation.


1957 Ray W. Clough develops Matrix Algebra Program for IBM 701 (1951) and 704 (1954) that solves linear equations.

1959 J. Greenstadt’s work allows for irregularly shaped cell meshes (IBM, J. Research Development, On the Reduction of Continuous Problems to Discrete Form).
1960

- R. J. Melosh addresses convergence of GEM and variational method (e.g., Ritz-Galerkin method).


1963  SADSAM (Structural Analysis by Digital Simulation of Analog Methods) developed by Richard MacNeal and Robert Schwendler, forerunner of MSC/NASTRAN.

1964  IBM 7094, Univac 1107, etc., used for American aircraft FE programs.
Introduction

Historical Background (1960’s)

1965

- ASKA (automatic system for kinematics analysis) commercial software product based on Argyris’ work.
- Variational form develops, opening FEM to all fields of application; O. C. Zienkiewicz and Y. K. Cheung give it broader interpretation in article (Engineer, Finite Elements in the Solution of Field Problems).
- Portable, desktop and microcomputers marketed.
- NASTRAN (NASA Structural Analysis Program) developed by MacNeal-Schwendler Corp., as early effort to consolidate structural mechanics into a single program (proprietary version announced in 1971).

1967

- M. J. L. Hussey applies theory of cyclical symmetry to FEM (ASCE, Journal of Structural Division)
- Two-dimensional analysis of the Concorde wing with approximately 1,000 degrees of freedom was made on an analog computer in France.
1970

- ANSYS (from ANSYS, Inc., formerly Swanson Analysis Systems Inc.) grew out of nuclear industry.
- Automotive industry begins to use FEM.
- FE software still limited to mainframe computers, with transition from pre and post-processing mostly being done by hand to interactive pre-and post-processing programs becoming generally available.
- First public release of NASTRAN. NASTRAN Systems Management Office est. at NASA Langley Research Center. Dr. J. P. Raney, Head.

1971

- INF-SUP condition identified and proved (also called Babuska-Brezzi condition).
- First NASTRAN Colloquium held at NASA Langley Research Center. Over 300 attendees. Papers presented covering many applications of NASTRAN.
1972
- 1,004 papers published during year.
- First version of MARC program introduced (general-purpose nonlinear FEA program), developed by Brown University researchers.

1974  Prof. R. D. Cook writes textbook for University of Wisconsin FE class (Concepts and Applications in Finite Element Analysis, Wiley).

1975  Interactive FE programs on small computers leads to rapid growth of CAD systems.

1976  CRAY supercomputer developed, which can handle flow analysis.
1980  Trends include electromagnetic applications (e.g., high-frequency power supplies), thermal analysis, greater emphasis on model generation.

1983  Fluid Dynamics introduces FIDAP for industrial fluid flow and heat and mass transfer.

1989  

- CFD (Computational Fluid Dynamics) and other nonstructural applications increase routine use of FE.
- Fluid flow analysis becomes more widespread, extending beyond work by large aerospace firms, and turbulent flow analysis expected to develop.
Scope of the presentation

- History of mesh generation,
- Classification of meshes, definitions,
- Structured mesh generation techniques (elliptic, hyperbolic),
- Unstructured techniques (delaunay and frontal),
- Hybrid meshes,
- Surface mesh generation,
- Data structures,
- Adaptive meshes.
In early 70’s, the Finite Element Method (FEM) was starting to become a major actor in engineering design offices.

CAD systems are the basis of engineering design. They represent parts within manufacturing tolerances.

The FEM requires an alternative representation of the parts.

Very quickly, it was clear that some kind of automatization was required in mesh generation procedures.
The plane-2D mesh generation problem was solved within 5 years.

Planty of robust methods/codes/algorithms have been produced. Triangular and quadrangular algorithms are available.

The 3D mesh generation is at least on level of magnitude more complex.
The tetrahedron is the 3D simplex.
- 4 vertices, 6 edges, 4 faces,
- more numbers later.

Three categories of algorithm are available:
1. Delaunay (P.-L. George, N. Weatherhill...)
2. Frontal (J. Peraire, Bassi & Rebay, ...).
3. Octree (M.S. Shephard).

Only few teams/researchers can pretend to develop a “usable” 3D mesh generator.

Softwares
- Commercial: Ghs3D (INRIA), MeshSim (Simmetrix)...
- OpenSource: Gmsh, NetGen, TetGen.
The hexaedron is the 3D cube.

- 8 vertices, 12 edges, 6 faces,

Three are many algorithms available:

1. Medial axis,
2. Whisker weaving, plastering, paving...

The quest of the Graal: an automating conforming hex-mesh generator.

Softwares

- Hexpress (NUMECA), TrueGrid,
- CUBIT (Sandia), ...
## Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>G</td>
<td>the geometric model</td>
</tr>
<tr>
<td>M</td>
<td>the mesh model</td>
</tr>
<tr>
<td>T/H</td>
<td>tetrahedral/hexahedral mesh</td>
</tr>
<tr>
<td>$\Omega$</td>
<td>domain associated with the model $V$, $V = G, M$</td>
</tr>
<tr>
<td>$V_i^d$</td>
<td>the $i$th entity of dimension $d$ in a model $V$</td>
</tr>
<tr>
<td>${V^d}$</td>
<td>unordered group of topological entities of dimension $d$ in model $V$</td>
</tr>
<tr>
<td>$[V^d]$</td>
<td>ordered group of topological entities of dimension $d$ in model $V$</td>
</tr>
<tr>
<td>$\phi{V^d}$</td>
<td>set of mesh entities of dimension $d$ that are adjacent or contained in a model $V$. $\phi$ may be a single entity, a group of entities or a complete model.</td>
</tr>
<tr>
<td>$V_i^d{V^q}$</td>
<td>the unordered group of topological entities of dimension $q$ that are adjacent to the entity $V_i^d$ of model $V$</td>
</tr>
<tr>
<td>$V_i^d{V^q}_j$</td>
<td>the $j$th member of the adjacency list $V_i^d{V^q}$</td>
</tr>
<tr>
<td>$\text{dim} V_i^d{V^q}$</td>
<td>the number of elements in the adjacency list $V_i^d{V^q}$</td>
</tr>
<tr>
<td>$\square$</td>
<td>classification symbol used to indicate the association of one or more mesh entities from the mesh model, $M$ with an entity of the geometric model $G$, $G$</td>
</tr>
<tr>
<td>$I$</td>
<td>the incidence matrix of the mesh.</td>
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</table>
A mesh $M$ is a geometrical discretization of a domain $\Omega$ that consists of

- A collection of mesh entities $M_i^d$ of controlled size and distribution;
- Topological relationships or adjacencies forming the graph of the mesh.

The mesh $M$ covers $\Omega$ without neither overlap nor hole.

A mesh is a mechanism that organizes the structuration of the domain $\Omega$ with the aim of doing computation.
A cartesian mesh is characterized by

- A template that repeats itself,
- The adjacencies in the mesh are known implicitly: $\dim \mathcal{V}_i^d \{ \mathcal{V}_q \}$ is constant for all $p$ and $q$, $p \neq q$,
- Mesh lines are aligned with coordinates,
- A cartesian mesh does not comply with curved boundaries.
Structured meshes

A structured mesh is characterized by

- A template that repeats itself,
- The adjacencies in the mesh are known implicitly: \( \dim V_i^d \{ V^q \} \) is constant for all \( p \) and \( q \), \( p \neq q \).
- Mesh lines are aligned with the geometry of the body,
- A structured mesh does comply with curved boundaries
Unstructured meshes

An unstructured mesh is characterized by

- No template, the structure of the mesh is irregular,
- The adjacencies in the mesh have to be given explicitly (in a file): \( \dim V_i^d(V^q) \) varies for \( p < q \).
- An unstructured mesh does comply with curved boundaries.

Unstructured mesh generation techniques have the advantage to be more general (they apply to general complex domains) and more automatic (reduction of human interaction).
Example

The classical NACA012 wing
- The geometry of the wing is contained in ./examples/naca12_2d.geo
- The geometry has to be completed, it only contains the definition of curves (B-Splines).
Hybrid meshes contain

- Structured/Cartesian parts
- Unstructured parts

Hybrid meshes are often used in CFD, especially for boundary layer meshes
Example: Flow past a sphere

Setup

- Computational mesh (10^6 nodes) with resolved boundary layer (32 CPU ≈ 24h)
- Kinematic viscosity \( \nu = 0.1 \text{m}^2/\text{s} \)
- Free Stream Velocity \( U = 30 \text{m/s} \)
- Sphere diameter \( D = 1 \text{m} \)
**Example: Flow past a sphere**

**Setup**

- Computational mesh (10⁶ nodes) with resolved boundary layer (32 CPU ≈ 24h)
- Kinematic viscosity $\nu = 0.1 m^2/s$
- Free Stream Velocity $U = 30 m/s$
- Sphere diameter $D = 1 m$
<table>
<thead>
<tr>
<th>Method</th>
<th>$C_D$</th>
<th>$C_L$</th>
<th>$St$</th>
</tr>
</thead>
<tbody>
<tr>
<td>LP + stab. MASS</td>
<td>0.667</td>
<td>0.052</td>
<td>0.134</td>
</tr>
<tr>
<td>PSPG</td>
<td>0.663</td>
<td>0.071</td>
<td>0.140</td>
</tr>
<tr>
<td>Ploumhans et al.</td>
<td>0.683</td>
<td>0.061</td>
<td>0.135</td>
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<tr>
<td>Johnson et al. (Exp.)</td>
<td>0.629</td>
<td>−</td>
<td>0.148−0.165</td>
</tr>
<tr>
<td>Georges et al. (Cen. flux)</td>
<td>0.661</td>
<td>0.066</td>
<td>0.134</td>
</tr>
<tr>
<td>Jindal et al.</td>
<td>0.835</td>
<td>−</td>
<td>0.153</td>
</tr>
<tr>
<td>Constantinescu et al</td>
<td>0.655</td>
<td>0.065</td>
<td>0.136</td>
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