Introduction to Multigrid Methods

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Suggested readings


• Hackbusch and Trottenburg, “Multigrid Methods, Springer-Verlag, 1982”


First observation toward multigrid

Many relaxation schemes have the smoothing property, where oscillatory modes of the error are eliminated effectively, but smooth modes are damped very slowly.

- Error after 35 iteration sweeps:
Reason 1 for using coarse grids: Nested Iteration

This might seem like a limitation, but by using coarse grids we can use the smoothing property to good advantage

Coarse grids can be used to compute an improved initial guess for the fine-grid relaxation. This is advantageous because:

- Relaxation on the coarse-grid is much cheaper (1/2 as many points in 1D, 1/4 in 2D, 1/8 in 3D)

- Relaxation on the coarse grid has a marginally better convergence rate.
Idea !: Nested Iteration

... 

Relax on $A_{4h}u_{4h} = b_{4h}$ to obtain initial guess $v_{2h}$

Relax on $A_{2h}u_{2h} = b_{2h}$ to obtain initial guess $v_{h}$

Relax on $A_{h}u_{h} = b_{h}$ to obtain ... final solution?

**Question 1:** what is $A_{4h}u_{4h} = b_{4h}$?

**Question 2:** what if error has smooth components when on the final grid?
Reason 2 for using coarse grids: smooth error on a fine grid is relatively more oscillatory on coarse grid

On the coarse grid, the smooth error appears to be relatively higher in frequency.

Relaxation will be more effective on this mode if done on the coarser grid!!

\[ w_{k,2j}^h = \sin \left( \frac{2jk\pi}{N} \right) = \sin \left( \frac{jk\pi}{N/2} \right) = w_{k,j}^{2h}. \]

For \( k = 1, 2, \ldots, N/2 \), the \( k \)th mode is preserved (still alive) on the coarse grid.

\[ w_{N/2}^h = 0. \]

For \( k > N/2 \), \( w_k^h \) is invisible on the coarse grid (aliasing). It appears as the \( N - k \)th mode on the coarse grid

\[ w_{k,2j}^h = \sin \left( \frac{2j\pi k}{N} \right) = -\sin \left( \frac{2j\pi(N-k)}{N} \right) = -w_{N-k,j}^{2h}. \]
• **A smooth function:**

- Can be represented by linear interpolation from a coarser grid:

• **Can be represented by linear interpolation from a coarser grid:**

- Relaxation will be more effective on this mode if done on the coarser grid!!
For $k > N/2$, $w_k$ is invisible on the coarse grid: aliasing!!

- For $k > N/2$, the $k$th mode on the fine grid is aliased and appears as the $(N-k)$th mode on the coarse grid.

$k=9$ mode, $N=12$ grid

$k=3$ mode, $N=12$ grid
Multigrid Methods

Smooth errors can be approximated on coarser grids and seen as more oscillatory. Relaxation schemes are able to get those.

Non smooth errors are not seen on coarser grids and cannot therefore be killed there.

The residual correction idea: Let \( v \) be an approximation to the solution of \( Au = b \), where the residual \( r = b - Av \). The the error \( e = u - v \) satisfies

\[
Ae = r.
\]

After relaxing on \( Au = b \) on the fine grid, the error will be smooth. On the coarse grid, however, this error appears more oscillatory, and relaxation will be more effective.

Therefore we go to a coarse grid and relax on the residual equation \( Ae = r \), with an initial guess of \( e = 0 \).
Coarse-grid correction

Relax on $A_h u_h = b_h$ to obtain an approximation $v_h$,

Compute $r_h = A_h v_h - b_h$

Compute $r_{2h} = I_{2h}^h r_{2h}$

Relax on $A_{2h} e_{2h} = r_{2h}$ to obtain an approximation to the error $e_{2h}$,

Correct the approximation $v_h = v_h + I_{2h}^h e_{h2}$.

Clearly, we have to define the mappings $I_{2h}^h$ and $I_{2h}^h$. 
Prolongation

Prolongation operator: the mapping from the coarse grid to the fine grid $I^h_{2h}$.

Example: values at points on the coarse grid map unchanged to the fine grid and values at fine-grid points NOT on the coarse grid are the averages of their coarse-grid neighbors.
Example, linear interpolation, $N = 7$,

$$I_{2h}^h = \begin{bmatrix} 1/2 & 0 & 0 \\ 1 & 0 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 1 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 0 & 1 \\ 0 & 0 & 1/2 \end{bmatrix}$$

$I_{2h}^h$ has full rank.
Restriction operator: the mapping from the fine grid to the coarse grid $I^2_h$.

Example, restriction by injection:
Restriction, full weighting

Let \( v, h \) be defined on \( v, h \). Then

\[
I_{2h}^h = \frac{1}{4} \begin{bmatrix}
1 & 2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 2 & 1
\end{bmatrix} = \frac{1}{2} (I_{2h}^h)^T.
\]
Now, let’s put all these ideas together

- Nested Iteration (effective on smooth error modes)
- Relaxation (effective on oscillatory error modes)
- Residual equation (i.e., residual correction)
- Prolongation and Restriction
First, do some smoothing sweeps to annihilate high frequency terms of the error. Let $S$ be our smoother:

$$x \mapsto Sx.$$ 

We aim at solving

$$A_hu_h = b_h$$

the $h$ subscript being related to the grid size. We start with $u_h^0$. and apply $\nu$ times the smoothing procedure:

$$\tilde{u}_h = S^\nu u_h^k$$

After the application of a few smoothing sweeps, we obtain an approximation $\tilde{u}_h$ whose error $\tilde{e}_h = u_h - \tilde{u}_h$ is smooth. Then $\tilde{e}_h$ can be approximated on a coarser space.
The 2-Grid Scheme, start with $u_h^{(0)}$

1. **Pre-smoothing steps**: $u_h^{(l)} = S u_h^{(l-1)}$, $l = 1, \ldots, \nu_1$

2. **Compute the residual**: $r_h = b_h - A_h u_h^{(\nu_1)}$

3. **Restriction of the residual**: $r_H = I_H^h r_h$

4. **Solution of the coarse grid problem**: $e_H = (A_H)^{-1} r_H$

5. **Coarse grid correction**: $u_h^{(\nu_1+1)} = u_h^{(\nu_1)} + I_H^h e_H$

6. **Post-smoothing steps**: $u_h^{(l)} = S u_h^{(l-1)}$, $l = \nu_1 + 2, \ldots, \nu_1 + \nu_2 + 1$. 
Coarse-grid Correction

Relax on

\[ A^h u^h = f^h \]
\[ r^h = f^h - A^h u^h \]

Compute

\[ u^h \leftarrow u^h + e^h \]

Correct

Restrict

\[ r^{2h} = I_h^{2h} r^h \]

Interpolate

\[ e^h \approx I^{2h}_h e^{2h} \]

Solve

\[ A^{2h} e^{2h} = r^{2h} \]
\[ e^{2h} = (A^{2h})^{-1} r^{2h} \]
The 2-Grid Scheme

**Property:** The iteration matrix of the 2-grid scheme is

\[ G_{TG} = S_h^{\nu_2} \left( I_h - I_H^h (A_H)^{-1} I_h^H A_h \right) S_h^{\nu_1}. \]

where \( I_h \) is the identity matrix and \( S_h \) is the smoothing iteration matrix.

This property is straightforward to demonstrate.

For the model problem, it is possible to estimate the spectral radius of \( G_{TG} \).

Consider the damped Jacobi smoother with \( \omega = 1/2 \). Assume that \( I_H^h \) is piecewise linear interpolation, and \( I_h^H \) is restriction by weighting such that

\[ (r_H)_{i,j} = \frac{(r_h)_{i-1,j} + (r_h)_{i+1,j} + (r_h)_{i,j-1} + (r_h)_{i,j+1} + 4(r_h)_{i,j}}{8}, \]

for \( i, j = 2, 4, 6, \ldots \).
The 2-Grid Scheme

In this case the following theorem is proved using discrete Fourier analysis in W. Hackbusch, *Multi-Grid Methods and Applications*. Springer-Verlag, Heidelberg, 1985.

**Property:** Let the 2-Grid scheme with \( \nu = \nu_1 + \nu_2 \geq 1 \) The spectral radius of the iteration matrix \( G_{MT} \) is bounded by

\[
\rho(G_{MT}) \leq \max\{\xi(1 - \xi)\nu + (1 - \xi)\xi^\nu, \quad 0 \leq \xi \leq 1/2\}
\]

Obviously, \( \rho(G_{MT}) < 1 \) so that the iteration is globally convergent.
The Multi-Grid Scheme

In the 2-Grid scheme, the size of the coarse grid is twice larger than the fine one.

The coarse problem has the same form as the residual problem on the fine level.

Therefore one can use the 2-Grid method to determine $\bar{e}_H$ of

$$A_H \bar{e}_H = r_H.$$ 

So we can introduce a further coarse-grid problem and this process can be repeated recursively until a coarsest grid is reached where the corresponding residual equation is inexpensive to solve.

This is, roughly speaking, the qualitative description of the Multi-Grid method.
How do we "solve" the coarse-grid residual equation?

Recursion!

\[ u^h \leftarrow G^h(A^h, f^h) \]
\[ f^{2h} \leftarrow I^{2h}_h(f^h - A^h u^h) \]
\[ u^{2h} \leftarrow G^{2h}(A^{2h}, f^{2h}) \]
\[ f^{4h} \leftarrow I^{4h}_{2h}(f^{2h} - A^{2h} u^{2h}) \]
\[ u^{4h} \leftarrow G^{4h}(A^{4h}, f^{4h}) \]
\[ f^{8h} \leftarrow I^{8h}_{4h}(f^{4h} - A^{4h} u^{4h}) \]
\[ u^{8h} \leftarrow G^{8h}(A^{8h}, f^{8h}) \]

\[ e^H = (A^H)^{-1} f^H \]
The Multi-Grid Scheme for solving $A_k u_k = b_k$

1. If $k = 1$ solve $u_k = A_k^{-1}b_k$ directly

2. *Pre-smoothing steps:* $u_k^{(l)} = S u_k^{(l-1)}$, $l = 1, \ldots, \nu_1$

3. *Compute the residual:* $r_k = b_k - A_k u_k^{(\nu_1)}$

4. *Restriction of the residual:* $r_{k-1} = I_{k-1}^{k-1} r_k$

5. Set $u_{k-1} = 0$

6. Call $\mu$ times the Multi-Grid scheme to solve $A_{k-1} u_{k-1} = b_{k-1}$

7. *Coarse grid correction:* $u_k^{(\nu_1+1)} = u_k^{(\nu_1)} + I_{k-1}^k u_{k-1}$

8. *Post-smoothing steps:* $u_k^{(l)} = S u_k^{(l-1)}$, $l = \nu_1 + 2, \ldots, \nu_1 + \nu_2 + 1$. 
The Multi-Grid Scheme for solving $A_k u_k = b_k$

The number $\mu$ refers to the **cycling strategy**.

Typical values: $\mu = 1$ (V cycle) and $\mu = 2$ (W cycle).